

Integrating Temporal Information to Spatial Information in a Neural Circuit

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Abstract

In this paper, we consider networks of deterministic spiking neurons, firing synchronously at discrete times; such spiking neural networks are inspired by networks of neurons and synapses that occur in brains. We consider the problem of translating temporal information into spatial information in such networks, an important task that is carried out by actual brains.

Specifically, we define two problems: “First Consecutive Spikes Counting (FCSC)” and “Total Spikes Counting (TSC)”, which model spike and rate coding aspects of translating temporal information into spatial information respectively. Assuming an upper bound of T on the length of the temporal input signal, we design two networks that solve these two problems, each using $O(\log T)$ neurons and terminating in time $T + 1$. We also prove that these bounds are tight.

1 Introduction

Algorithms in the brain are inherently distributed. Although each neuron has relatively simple dynamics, as a distributed system, a network of neurons shows strong computational power. There have been many attempts to model the brain computationally. At a single-neuron level, theoretical neuroscientists were able to model the dynamics of a single neuron to high accuracy with the Hodgkin-Huxley model [HH52]. At a circuit level, to make the analysis tractable, neuroscientists approximated detailed dynamics of neurons with simplified models such as the nonlinear integrate-and-fire model [FTHvVB03] and the spiking response model [WWvJ97]. Recently, there is increasing interest on using biologically plausible spiking neuronal dynamics to solve different computational problems [LMP17b, LMP17a, LM18, TLD17, CNC18, JHM14]. These models vary in their assumptions about spike/rate code, deterministic/stochastic response, and continuous/discrete time. In this paper, we consider a network of spiking neurons with a deterministic synchronous firing rule in discrete time to simplify the analysis and focus on the computational principles.

One of the most important questions in neuroscience is how humans integrate information over time. Sensory inputs such as visual and auditory stimulus are inherently temporal; however, brains are able to integrate the temporal information to a single concept, such as a moving object in a visual scene, or an entity in a sentence. There are two kinds of neuronal codings: rate coding and temporal coding. Rate coding is a neural coding scheme assuming most of the information is coded in the firing rate of the neurons. It is most commonly seen in muscle in which the higher firing rates of motor neurons correspond to higher intensity in muscle contraction [AZ26]. On the other hand, rate coding cannot be the only neural coding brains employ. A fly is known to react to new stimuli and change its direction of flight within 30-40 ms. There is simply not enough time for neurons to compute averages [RWdRvSB96]. Therefore, neuroscientists have proposed the idea of

temporal spike coding, assuming the information is coded in the specific temporal firing patterns. One of the popular spike codings is the first-to-spike coding. It has been shown that the timing of the first spike encodes most information of an image in retinal cells [GM08]. We propose two toy problems to model how brains extract information from different coding. “First consecutive spikes counting” (FCSC) counts the first consecutive interval of spikes, which is equivalent to counting the distance between the first two spikes, a prevalent spike coding scheme in sensory cortex. “Total spikes counting” (TSC) counts the number of the spikes over an arbitrary interval, which is an example of rate coding. In particular, TSC contains an interesting difficulty: there are conflicting objectives between maintaining the count when no spike arrives and updating the count when a spike arrives. To overcome this difficulty, we allow the network to enter an unstable intermediate state which carries the information of the count. The intermediate state then converges to a stable state that represents the count after a computation step without inputs.

In this paper, we design two networks that solve the above two problems by translating temporal information into spatial information in time $T + 1$ with $O(\log T)$ neurons. We also show that our time bounds are tight. We would like to remark that although our problems are biologically inspired, the optimal solutions we propose are not biologically plausible. Our networks are not noise tolerant, whereas the neuronal dynamics are highly noisy and it is hard to conceive that the brain uses binary representation as a neuronal representation. However, we hope that our proofs can demonstrate an important computational principle: unstable intermediate states can carry temporal information and then converge to a stable representation efficiently. The organization of this paper is as follows. In Section 2, we present the definition of a network of spiking neurons and the problem statements. In Section 3, we present the FCSC network that counts consecutive spikes in binary. In Section 4, we generalize Section 3 and present the TSC network that counts spikes over arbitrary intervals. In Section 5, we present the time lower bound of FCSC and TSC problem and show our networks are optimal. In Section 6, we discuss our model assumptions and their implications along with possible future directions.

2 Problem Statements/Goals

In this section, we cover the model definition and the following two problems: First Consecutive Spikes Counting (FCSC) and Total Spikes Counting (TSC).

2.1 Model

In this paper, we consider a network of spiking neurons with deterministic synchronous firing at discrete times. Formally, a neuron z consists of the following data with $t \geq 1$

$$z^{(t)} = \Theta\left(\sum_{y \in P_z} w_{yz} y^{(t-1)} - b_z\right)$$

where $z^{(t)}$ is the indicator function of neuron z firing at time t . b_z is the threshold (bias) of neuron z . P_z is the set of presynaptic neurons of z , w_{yz} is the strength of connection from neuron y to neuron z and Θ is a nonlinear function. Here we take Θ as the Heaviside function given by $\Theta(x) = 1$ if $x > 0$ and 0 otherwise. At $t = 0$, we let $z^{(0)} = 0$ if z is not one of the input neurons.

For the rest of the paper, we fix an input neuron x and m output neurons $\{y_i\}_{0 \leq i < m}$ in a network.

2.2 First Consecutive Spikes Counting(T) (FCSC(T))

Given an input neuron x and the max input length T , we consider any input firing sequence such that for all $t \geq T$, $x^{(t)} = 0$. Define L_x in terms of this firing sequence as follows: if $x^{(t)} = 1$ for some t , then there must exist integers $\hat{t}, L$ such that for all $t, t < \hat{t}$ we have $x^{(t)} = 0$, for all $i, 0 \leq i < L$ we have $x^{(\hat{t}+i)} = 1$, and $x^{(\hat{t}+L)} = 0$. Define $L_x = L$. (i.e., L is the length of the first consecutive spikes interval in the sequence.) Otherwise, that is if for all $t \geq 0$, $x^{(t)} = 0$, then define $L_x = 0$.

Then we say a network of neurons solves FCSC(T) in time t' with m' neurons if there exists an injective function $F : \{0, \dots, T\} \rightarrow \{0, 1\}^m$ such that for all x and for all $t, t \geq t'$ we have $y^{(t)} = F(L_x)$ and the network has m' total neurons.

Intuitively, FCSC serves as a toy model for encoding distance between spikes, a prevalent spike coding in sensory cortex. For mathematical convenience, we model the problem as counting the distance between non-spikes which is mathematically equivalent as counting the distance between spikes in our model.

2.3 Total Spikes Counting(T) (TSC(T))

Given an input neuron x and the max input length T , we consider any input firing sequence such that for all $t \geq T$, $x^{(t)} = 0$. Define $L_x = |\{t : x^{(t)} = 1, 0 \leq t < T\}|$ as the total number of spikes in the sequence. Then we say a network of neurons solves TSC(T) in time t' with m' neurons if there exists an injective function $F : \{0, \dots, n\} \rightarrow \{0, 1\}^m$ such that for all x and for all $t, t \geq t'$ we have $y^{(t)} = F(L_x)$ and the network has m' total neurons.

Intuitively, TSC serves as a toy model for rate coding implemented by spiking neural networks because the network is able to extract the rate information by counting the number of spikes over arbitrary intervals.

2.4 Main Theorems

Our contributions in this paper are to design networks that solve these two problems respectively with matching lower bounds in numbers of neurons.

Theorem 2.1. *There exists a network with $O(\log T)$ neurons that solves FCSC(T) problem in time $T + 1$.*

Theorem 2.2. *There exists a network with $O(\log T)$ neurons that solves TSC(T) problem in time $T + 1$.*

It is easy to see that we also have the corresponding information-theoretical lower bound on the number of neurons all being $\Omega(\log T)$ by the requirements of the tasks.

In terms of time bound, we also show that our networks are optimal for FCSC and TSC problem in the following sense:

Theorem 2.3. *There does not exist a network with $o(T)$ neurons that solves FCSC(t) problem in time t for all $0 \leq t \leq T$.*

Theorem 2.4. *There does not exist a network with $o(T)$ neurons that solves TSC(t) problem in time t for all $0 \leq t \leq T$.*

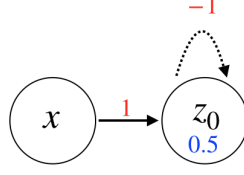


Figure 1: mod 2 Base Network

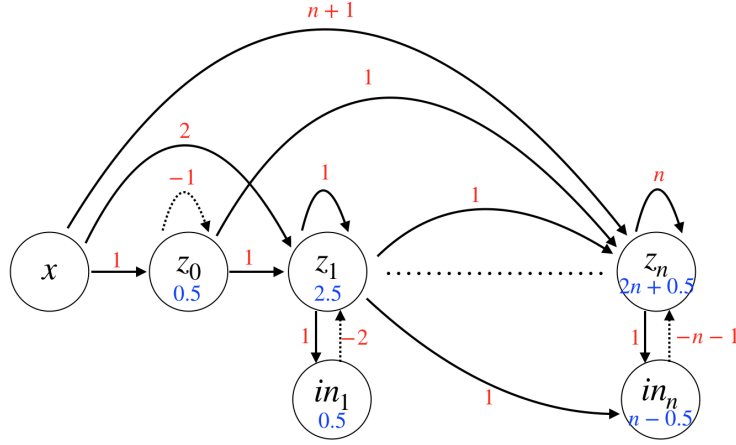


Figure 2: First Stage

3 First Consecutive Spikes Counting

We present the constructions in two stages. At the first stage, we count consecutive spikes in binary transiently. At the second stage, we transform the transient firing into persistent firing. By composing the two stages, we get our desired network.

First stage: The network contains neurons $z_0, \dots, z_n, in_1, \dots, in_n$ and we build the network inductively. To construct mod 2 Base Network which counts mod 2, we have

$$w_{xz_0} = 1, w_{z_0z_0} = -1, b_{z_0} = 0.5$$

By noticing that for $t \geq 1$, $z_0^{(t)} = 1$ if and only if $x^{(t-1)} = 1$ and $z_0^{(t-1)} = 0$, we have the following lemma

Lemma 3.1. *For the mod 2 base network, given $t \geq 0$ if for all t' such that $0 \leq t' \leq t$ we have $x^{(t')} = 1$, then at time t , $z_0^{(t)} = t \bmod 2$.*

Now we iteratively build the network where $1 \leq i \leq n$ on top of the mod 2 base network with the following rule:

$$w_{xz_i} = i + 1, w_{z_jz_i} = 1, \forall j, 0 \leq j < i, w_{z_kin_i} = 1, \forall k, 0 < k \leq i, w_{in_iz_i} = -i - 1, w_{z_iz_i} = i$$

$$b_{z_i} = 2i + 0.5, b_{in_i} = i - 0.5$$

This completes the construction. From the construction, we can deduce the following lemma

Lemma 3.2. *For $i > 0$, neurons z_i, in_i fire according to the following rules:*

1. $z_i^{(t)} = 1$ if and only if $x^{(t-1)} = 1$, $in_i^{(t-1)} = 0$, and (either for all $j, 0 \leq j < i$ we have $z_j^{(t-1)} = 1$ or $z_i^{(t-1)} = 1$)

2. $in_i^{(t)} = 1$ if and only if for all $j, 1 \leq j \leq i$ we have $z_j^{(t-1)} = 1$

Proof. Case (1): The potential of $z_i^{(t)}$ is

$$w_{xz_i}x^{(t-1)} + \sum_{j=0}^{i-1} w_{z_j z_i} z_j^{(t-1)} + w_{in_i z_i} in_i^{(t-1)} + w_{z_i z_i} z_i^{(t-1)} = (i+1)x^{(t-1)} + \sum_{j=0}^{i-1} z_j^{(t-1)} - (i+1)in_i^{(t-1)} + iz_i^{(t-1)}$$

Only if: Let's show the only if direction for the firing rule of $z_i^{(t)}$ by proving the contrapositive. If $x^{(t-1)} = 0$, then the potential of $z_i^{(t)}$ is

$$\sum_{j=0}^{i-1} x_j^{(t-1)} - (i+1)in_i^{(t-1)} + iz_i^{(t-1)} \leq 2i < 2i + 0.5 = b_{z_i}$$

If $in_i^{(t-1)} = 1$, then the potential of $z_i^{(t)}$ is

$$(i+1)x^{(t-1)} + \sum_{j=0}^{i-1} z_j^{(t-1)} - (i+1) + iz_i^{(t-1)} \leq 2i < 2i + 0.5 = b_{z_i}$$

If there exists $\hat{j}, 0 \leq \hat{j} < i$ such that $z_{\hat{j}}^{(t-1)} = 0$ and $z_i^{(t-1)} = 0$, then the potential of $z_i^{(t)}$ is

$$\sum_{j \neq \hat{j}, 0 \leq j \leq i-1} z_j^{(t-1)} + (i+1)x^{(t-1)} - (i+1)in_i^{(t-1)} \leq 2i < 2i + 0.5 = b_{z_i}$$

In all three cases, we have $z_i^{(t)} = 0$.

If: For the if direction, if $x^{(t-1)} = 1$, $in_i^{(t-1)} = 0$ and for all $j, 0 \leq j < i$ we have $z_j^{(t-1)} = 1$, then the potential of $z_i^{(t)}$ is

$$(i+1) + \sum_{j=0}^{i-1} 1 + iz_i^{(t-1)} \geq 2i + 1 > 2i + 0.5 = b_{z_i}$$

If $x^{(t-1)} = 1$, $in_i^{(t-1)} = 0$ and $z_i^{(t-1)} = 1$, then the potential of $z_i^{(t)}$ is

$$(i+1) + \sum_{j=0}^{i-1} z_j^{(t-1)} + i \geq 2i + 1 > 2i + 0.5 = b_{z_i}$$

In both cases, we have $z_i^{(t)} = 1$.

Case (2): The firing rule of $in_i^{(t)}$ can be analyzed similarly.

The potential of $in_i^{(t)}$ is

$$\sum_{j=1}^i w_{z_j in_i} z_j^{(t-1)} = \sum_{j=1}^i z_j^{(t-1)}$$

Only If: For the only if direction, if there exists $\hat{j}, 1 \leq \hat{j} \leq i$ such that $x_{\hat{j}}^{(t-1)} = 0$, then the potential of $in_i^{(t)}$ is

$$\sum_{j \neq \hat{j}, 1 \leq j \leq i} z_j^{(t-1)} \leq i - 1 < i - 0.5 = b_{in_i}$$

We have $in_i^{(t)} = 0$.

If: For the if direction, if for all $j, 1 \leq j \leq i$ we have $z_j^{(t-1)} = 1$, then the potential of $in_i^{(t)}$ is

$$\sum_{j=1}^i 1 = i > i - 0.5 = b_{in_i}$$

We have $in_i^{(t)} = 1$ as desired. □

Using the above lemma, we can verify that indeed the network at the first stage fires in binary, with z_i encoding the i th digit in the binary representation.

Theorem 3.3. *Given $i \geq 1$ and $t \geq 0$, if for all t' such that $0 \leq t' \leq t$ we have $x^{(t')} = 1$, then*

1. $z_i^{(t)} = a_i$ for $t = \sum_{j=0}^{\infty} a_j 2^j$ where $a_j \in \{0, 1\}$.
2. $in_i^{(t)} = 1$ if and only if $t \bmod 2^{i+1} = 2^{i+1} - 1$ or 0.

Proof. First, let's verify that the claim is true for z_0 . Since for all $t', 0 \leq t' \leq t$ we have $x^{(t')} = 1$, $z_0^{(t')} = 1$ if and only if $z_0^{(t'-1)} = 0$. This implies exactly $z_0^{(t)} = t \bmod 2$ as desired (for all the modular arithmetic at this paper, we choose the smallest nonnegative number from the equivalence class). Now let's do the induction on t and we will verify the induction by checking z_i, in_i fires in according to the induction hypothesis for all $i \geq 1$. When $t = 1$, the induction statement is trivially satisfied for all $i \geq 1$. Fix i , we have the following cases:

1. $0 < t \bmod 2^{i+1} < 2^i, z_i^{(t-1)} = 0$:
This implies that $0 \leq t - 1 \bmod 2^i < 2^i - 1$. By induction hypothesis, not all $z_j^{(t-1)} = 1$ for $0 \leq j < i$. Now by Lemma 3.2, we have $z_i^{(t)} = 0 = a_i, in_i^{(t)} = 0$ as desired.
2. $t \bmod 2^{i+1} = 2^i, z_i^{(t-1)} = 0, in_i^{(t-1)} = 0$:
This implies that $t - 1 \bmod 2^i = 2^i - 1$. By induction hypothesis, for all $j, 0 \leq j < i$ we have $z_j^{(t-1)} = 1$. Now by Lemma 3.2, we have $z_i^{(t)} = 1 = a_i, in_i^{(t)} = 0$ as desired.
3. $2^i < t \bmod 2^{i+1} < 2^{i+1} - 1, z_i^{(t-1)} = 1, in_i^{(t-1)} = 0$:
This implies that $0 \leq t - 1 \bmod 2^i < 2^i - 2$. By induction hypothesis, not all $j, 1 \leq j < i$ we have $z_j^{(t-1)} = 1$. Now by Lemma 3.2, we have $z_i^{(t)} = 1 = a_i, in_i^{(t)} = 0$ as desired.
4. $t \bmod 2^{i+1} = 2^{i+1} - 1, z_i^{(t-1)} = 1, in_i^{(t-1)} = 0$:
This implies that $t - 1 \bmod 2^i = 2^i - 2$. By induction hypothesis, for all $j, 1 \leq j < i$ we have $z_j^{(t-1)} = 1$. Now by Lemma 3.2, we have $z_i^{(t)} = 1 = a_i, in_i^{(t)} = 1$ as desired.
5. $t \bmod 2^{i+1} = 0, z_i^{(t-1)} = 1, in_i^{(t-1)} = 1$:
This implies that $t - 1 \bmod 2^i = 2^i - 1$. By induction hypothesis, for all $j, 1 \leq j < i$ we have $z_j^{(t-1)} = 1$. Now by Lemma 3.2, we have $z_i^{(t)} = 0 = a_i, in_i^{(t)} = 1$ as desired.

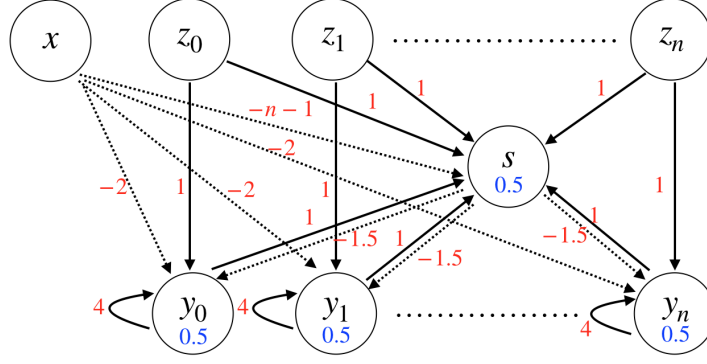


Figure 3: Second Stage

This completes the induction. $\square$

Second stage: Now the second stage is a simple “capture network” with input neurons x, z_i for all $i, 0 \leq i \leq n$, output neurons y_i for $0 \leq i \leq n$ and an auxiliary neuron s . Intuitively, the network persistently captures the state of z_i for all $i, 0 \leq i \leq n$ into y_i for all $i, 0 \leq i \leq n$. We will specify the timing of the states of z_i being captured later. The network is defined as the following:

$$w_{xy_i} = -2, w_{xs} = -n-1, w_{ss} = n+2, w_{y_i y_i} = 4, w_{z_i y_i} = 1, w_{z_i s} = w_{y_i s} = 1, w_{sy_i} = -1.5, \forall 0 \leq i \leq n$$

and

$$b_{y_i} = b_s = 0.5$$

Notice that the above weight ensures the following one step firing rule:

Lemma 3.4. For $0 \leq i \leq n$, neurons $y_i^{(t)}, s^{(t)}$ fire according to the following rules:

1. $y_i^{(t)} = 1$ if and only if $y_i^{(t-1)} = 1$, or $(y_i^{(t-1)} = 0, x^{(t-1)} = 0, s^{(t-1)} = 0$ and $z_i^{(t-1)} = 1)$
2. $s^{(t)} = 1$ if and only if $s^{(t-1)} = 1$, or (there exists i, i' such that $z_i^{(t-1)} = 1$ or $y_{i'}^{(t-1)} = 1$, and $x^{(t-1)} = 0$)

Proof. **Case (1):** The potential of $y_i^{(t)}$ is

$$w_{xy_i}x^{(t-1)} + w_{y_i y_i}y_i^{(t-1)} + w_{z_i y_i}z_i^{(t-1)} + w_{z_i z_i}z_i^{(t-1)} + w_{sy_i}s^{(t-1)} = -2x^{(t-1)} + 4y_i^{(t-1)} + z_i^{(t-1)} - 1.5s^{(t-1)}$$

Only If: Let's show the only if direction for the firing rule of $y_i^{(t)}$ first. If $y_i^{(t-1)} = 0, x^{(t-1)} = 1$, the potential of $y_i^{(t)}$ is

$$-2 + z_i^{(t-1)} - 1.5s^{(t-1)} \leq -1 < 0.1 = b_{y_i}$$

If $y_i^{(t-1)} = 0, s^{(t-1)} = 1$, the potential of $y_i^{(t)}$ is

$$-2x^{(t-1)} + z_i^{(t-1)} - 1.5 \leq -0.5 < 0.1 = b_{y_i}$$

If $y_i^{(t-1)} = 0, z_i^{(t-1)} = 0$, the potential of $y_i^{(t)}$ is

$$-2x^{(t-1)} - 1.5s^{(t-1)} \leq 0 < 0.1 = b_{y_i}$$

In all three cases, we have $y_i^{(t)} = 0$.

If: For the if direction, if $y_i^{(t-1)} = 1$, then the potential of $y_i^{(t)}$ is

$$-2x^{(t-1)} + 4 + z_i^{(t-1)} - 1.5s^{(t-1)} \geq 0.5 > 0.1 = b_{y_i}$$

If $y_i^{(t-1)} = 0, x^{(t-1)} = 0, s^{(t-1)} = 0, z_i^{(t-1)} = 1$, the potential of $y_i^{(t)}$ is

$$4y_i^{(t-1)} + 1 \geq 1 > 0.1 = b_{y_i}$$

In both cases, we have $y_i^{(t)} = 1$.

Case (2): The potential of $s^{(t)}$ is

$$\sum_{j=0}^n w_{z_j s} z_j^{(t-1)} + \sum_{j=0}^n w_{y_j s} y_j^{(t-1)} + w_{x s} x^{(t-1)} + w_{s s} s^{(t-1)} = \sum_{j=0}^n z_j^{(t-1)} + \sum_{j=0}^n y_j^{(t-1)} - (n+1)x^{(t-1)} + (n+2)s^{(t-1)}$$

Only If: For the only if direction, if $s^{(t-1)} = 0$ and for all $j, 0 \leq j \leq n$ we have $y_j^{(t-1)} = z_j^{(t-1)} = 0$, then the potential of $s^{(t)}$ is

$$-(n+1)x^{(t-1)} \leq 0 < 0.5 = b_s$$

If $s^{(t-1)} = 0, x^{(t-1)} = 1$, the potential of $s^{(t)}$ is

$$\sum_{j=0}^n z_j^{(t-1)} + \sum_{j=0}^n y_j^{(t-1)} - (n+1) \leq 0 < 0.5 = b_s$$

In both cases, we have $s^{(t)} = 0$.

If: For the if direction, if there exists $i, 0 \leq i \leq n$ such that $y_i^{(t-1)} = 1$ and $x^{(t-1)} = 0$, then the potential of $s^{(t)}$ is

$$\sum_{j=0}^n z_j^{(t-1)} + \sum_{j \neq i, 0 \leq j \leq n} y_j^{(t-1)} + 1 + (n+2)s^{(t-1)} \geq 1 > 0.5 = b_s$$

If there exists $i, 0 \leq i \leq n$ such that $z_i^{(t-1)} = 1$ and $x^{(t-1)} = 0$, the potential of $s^{(t)}$ is

$$\sum_{j=0}^n y_j^{(t-1)} + \sum_{j \neq i, 0 \leq j \leq n} z_j^{(t-1)} + 1 + (n+2)s^{(t-1)} \geq 1 > 0.5 = b_s$$

If $s^{(t-1)} = 1$, the potential of $s^{(t)}$ is

$$\sum_{j=0}^n z_j^{(t-1)} + \sum_{j=0}^n y_j^{(t-1)} - (n+1)x^{(t-1)} + (n+2) \geq 1 > 0.5 = b_s$$

In all three cases, we have $s^{(t)} = 1$ as desired. □

Now we can describe the behaviors of the capture network in the following theorem. The network persistantly captures the state of z_i for all $i, 0 \leq i \leq n$ at the first time point such that $x = 0$ and there exists some $\hat{i}$ such that $z_{\hat{i}} = 1$ into y_i for all $i, 0 \leq i \leq n$.

Theorem 3.5. *For the network at the second stage, let $t' \geq 0$ be such that $x^{(t')} = 0$ and there exists $\hat{j}$ such that $z_{\hat{j}}^{(t')} = 1$, and for all $t, 0 \leq t < t'$, either $x^{(t)} = 1$ or for all $i, 0 \leq i \leq n$ we have $z_i^{(t)} = 0$. Then for all i, t such that $0 \leq i \leq n, t > t'$ we have $y_i^{(t)} = z_i^{(t')}$.*

Proof. First by Lemma 3.4, for all $t, 0 < t \leq t'$ and for all $i, 0 \leq i \leq n$ we have $y_i^{(t)} = s^{(t)} = 0$. Now at time $t' + 1$, by Lemma 3.4, we see that $y_i^{(t'+1)} = z_i^{(t')}, \forall i, 0 \leq i \leq n$ and $s^{(t'+1)} = 1$. Now by Lemma 3.4, we know that for all $t, t > t'$ we have $s^{(t)} = 1$. Now by Lemma 3.4 again, if $y_i^{(t'+1)} = 0$, then since for all $t, t > t'$ we have $s^{(t)} = 1$, for all $t > t'$ we have $y_i^{(t)} = 0$; and if $y_i^{(t'+1)} = 1$, then we also have for all $t, t > t', y_i^{(t)} = 1$ as desired. $\square$

Now we are ready to prove the main Theorem 2.1 by setting $n = m = \lceil \log T' \rceil$

Proof. We are going to prove the main theorem by composing the networks from stage one and two together. If for all $t, 0 \leq t \leq T$ we have $x^{(t)} = 0$, then the network satisfies the criterion trivially since for all $0 \leq t \leq T, y_i^{(t)} = 0$. If not, then there exists $\hat{t} \geq 0, L_x > 0$ such that for all $t, 0 \leq t < \hat{t}$ we have $x^{(t)} = 0$, for all $i, 0 \leq i < L_x$ we have $x^{(\hat{t}+i)} = 1$, and $x^{(\hat{t}+L_x)} = 0$ where L_x is the length of the first consecutive spikes interval. Let $L_x = \sum_{j=0}^{\infty} a_j 2^j$; then by Theorem 3.3 and Lemma 3.1, for all $i, 0 \leq i \leq n$, we have $z_i^{(\hat{t}+L_x-1)} = a_i$. Now because $L_x > 0$, we know there exists $\hat{j}$ such that $z_{\hat{j}}^{(\hat{t}+L_x)} = 1$ by Theorem 3.3. And by Lemma 3.2, we know for all i, t such that $0 \leq t \leq \hat{t}, 0 \leq i \leq n$, we have $z_i^{(t)} = 0$. Now the assumption of Theorem 3.5 is satisfied with $t' = \hat{t} + L_x$. By Theorem 3.5, we get for all t, i such that $0 \leq i \leq n, t \geq \hat{t} + L_x$ we have $y_i^{(t)} = a_i$ and $T + 1 \geq \hat{t} + L_x$ as desired. This shows that the above network solves FCSC(T) problem in time $T + 1$ with $O(\log T)$ neurons. $\square$

Notice that in fact by the proof above, FCSC network enjoys an early convergence property. The network actually converges at time $\hat{t} + L_x$. Therefore we have the following stronger version of Theorem 2.1.

Corollary 3.6. *For all $t, 0 \leq t \leq T$, FCSC network with $O(\log T)$ neurons solves FCSC(t) problem in time $t + 1$.*

4 Total Spikes Counting

To count the total number of spikes in an arbitrary interval requires persistence of neurons without external spikes. Notice that on FCSC network, each neuron toggles itself according to binary representation without delay. However, persistence of neurons and toggles without delays are conflicting objectives; persistence of neurons stabilizes the network while toggling without delays changes the firing patterns of the network. For example, we use self-inhibition to count mod 2 but if we use self inhibition to count mod 2, the neuron cannot maintain the count during intervals with no inputs. In this section, we circumvent this difficulty by allowing the network to enter an unstable intermediate state that still stores the information of the count when the spikes arrive; however, the network will converge to a *clean state* that according to binary representation after one step of computation without external signals, and this *clean state* is stable in an arbitrary interval with no input.

In this section, because the self-inhibition used in Section 3 to count mod 2 cannot induce persistence, we build a network of four neurons to count mod 4 to replace the function of z_0, z_1 in

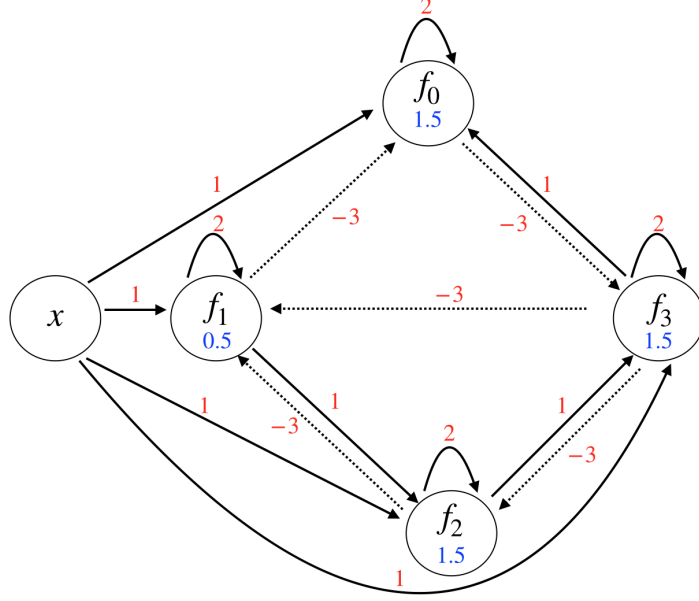


Figure 4: mod 4 Counter Network

Section 3. We then iteratively build the rest of the network that approximately fires in binary on top of the mod 4 counter network.

The construction of the mod 4 counter network is the following:

$$w_{xf_i} = 1, w_{f_i f_i} = 2, 0 \leq i \leq 3, w_{f_{j+1} f_j} = -3, 0 \leq j \leq 2, w_{f_1 f_2} = w_{f_2 f_3} = w_{f_3 f_0} = 1, w_{f_0 f_3} = w_{f_3 f_1} = -3$$

and

$$b_{f_1} = 0.5, b_{f_i} = 1.5, i \neq 1$$

We have the following lemma to specify the firing rules of f_i :

Lemma 4.1. *For all t, i such that $t \geq 1, 0 \leq i < 4$, neurons $f_i^{(t)}$ fire according to the following rules:*

1. $f_1^{(t)} = 1$ if and only if $f_2^{(t-1)} = f_3^{(t-1)} = 0$, and $(x^{(t-1)} = 1$ or $f_1^{(t-1)} = 1)$
2. For $i \neq 1$ we have $f_i^{(t)} = 1$ if and only if $f_{(i+1) \bmod 4}^{(t-1)} = 0$, and $(x^{(t-1)} = 1, f_{(i-1) \bmod 4}^{(t-1)} = 1$ or $f_i^{(t-1)} = 1)$

Proof. Case (1): The potential of $f_1^{(t)}$ is

$$w_{xf_1} x^{(t-1)} + w_{f_1 f_1} f_1^{(t-1)} + w_{f_2 f_1} f_2^{(t-1)} + w_{f_3 f_1} f_3^{(t-1)} = x^{(t-1)} + 2f_1^{(t-1)} - 3f_2^{(t-1)} - 3f_3^{(t-1)}$$

Only If: Let's show the only if direction for the firing rule of $f_1^{(t)}$ first. If $f_2^{(t-1)} = 1$, then the potential of $f_1^{(t)}$ is

$$x^{(t-1)} + 2f_1^{(t-1)} - 3 - 3f_3^{(t-1)} \leq 0 < 0.5 = b_{f_1}$$

If $f_3^{(t-1)} = 1$, then the potential of $f_1^{(t)}$ is

$$x^{(t-1)} + 2f_1^{(t-1)} - 3f_2^{(t-1)} - 3 \leq 0 < 0.5 = b_{f_1}$$

If $f_1^{(t-1)} = 0, x^{(t-1)} = 0$, then the potential of $f_1^{(t)}$ is

$$-3f_2^{(t-1)} - 3f_3^{(t-1)} \leq 0 < 0.5 = b_{f_1}$$

In all three cases, we have $f_1^{(t)} = 0$.

If: For the if direction, if $f_2^{(t-1)} = f_3^{(t-1)} = 0, x^{(t-1)} = 1$, then the potential of $f_1^{(t)}$ is

$$1 + 2f_1^{(t-1)} \geq 1 > 0.5 = b_{f_1}$$

If $f_2^{(t-1)} = f_3^{(t-1)} = 0, x^{(t-1)} = 1, f_1^{(t-1)} = 1$, then the potential of $f_1^{(t)}$ is

$$x^{(t)} + 2 \geq 2 > 0.5 = b_{f_1}$$

In both cases, we have $f_1^{(t)} = 1$.

Case (2): For $i \neq 1$, The potential of $f_i^{(t)}$ is

$$\begin{aligned} w_{xf_i}x^{(t-1)} + w_{f_i f_i}f_i^{(t-1)} + w_{f_{(i-1) \bmod 4} f_i}f_{(i-1) \bmod 4}^{(t-1)} + w_{f_{(i+1) \bmod 4} f_i}f_{(i+1) \bmod 4}^{(t-1)} = \\ x^{(t-1)} + 2f_i^{(t-1)} + f_{(i-1) \bmod 4}^{(t-1)} - 3f_{(i+1) \bmod 4}^{(t-1)} \end{aligned}$$

Only If: For the only if direction, if $f_{(i+1) \bmod 4}^{(t-1)} = 1$, then the potential of $f_i^{(t)}$ is

$$x^{(t-1)} + 2f_i^{(t-1)} + f_{(i-1) \bmod 4}^{(t-1)} - 3 \leq 1 < 1.5 = b_i$$

If $x^{(t-1)} = 0, f_i^{(t-1)} = 0$, then the potential of $f_i^{(t)}$ is

$$f_{(i-1) \bmod 4}^{(t-1)} - 3f_{(i+1) \bmod 4}^{(t-1)} \leq 1 < 1.5 = b_i$$

If $f_{(i-1) \bmod 4}^{(t-1)} = 0, f_i^{(t-1)} = 0$, then the potential of $f_i^{(t)}$ is

$$x^{(t-1)} - 3f_{(i+1) \bmod 4}^{(t-1)} \leq 1 < 1.5 = b_i$$

In all three cases, we have $f_i^{(t)} = 0$.

If: For the if direction, if $f_{(i+1) \bmod 4}^{(t-1)} = 0, x^{(t-1)} = 1, f_{(i-1) \bmod 4}^{(t-1)} = 1$, then the potential of $f_i^{(t)}$ is

$$1 + 2f_i^{(t-1)} + 1 \geq 2 > 1.5 = b_i$$

If $f_{(i+1) \bmod 4}^{(t-1)} = 0, f_i^{(t-1)} = 1$, then the potential of $f_i^{(t)}$ is

$$x^{(t-1)} + 2 + f_{(i-1) \bmod 4}^{(t-1)} \geq 2 > 1.5 = b_i$$

In both cases, we have $f_i^{(t)} = 1$ as desired. $\square$

For $0 \leq i < 4$, define a *clean state* with value i at time t' of the mod 4 counter network to be a state in which $f_i^{(t')} = 1$ and for all $j, j \neq i$ we have $f_j^{(t')} = 0$. By Lemma 4.1, it is trivial to see that if for all $t, t \geq t'$ we have $x^{(t)} = 0$, then for all $t, t \geq t'$ and for all $i, 0 \leq i < 4$ we have $f_i^{(t)} = f_i^{(t')}$. Using Lemma 4.1, we have the following lemma describing the behaviors of mod 4 counter network. Intuitively, when a new input arrives, the network enters an intermediate state in which both neurons represent the old count and the new count fire; when there is no input, the neuron that represents the new count will inhibit the neuron that represents the old count to stabilize the network in a *clean state*.

Lemma 4.2. *Let the mod 4 counter network be at a clean state with value $\hat{i}$ at time t' . Fix a positive integer L . For all $i, 0 \leq i < L$, let $x^{(t'+i)} = 1$ and $x^{(t'+L)} = 0$. Then, at time $t, t' < t < t' + L + 1$, we have the state of the network being*

$$f_{(\hat{i}+t-t') \bmod 4}^{(t)} = f_{(\hat{i}+t-t'-1) \bmod 4}^{(t)} = 1, f_{(\hat{i}+t-t'-2) \bmod 4}^{(t)} = f_{(\hat{i}+t-t'-3) \bmod 4}^{(t)} = 0$$

Furthermore, the network will be at a clean state again at time $t' + L + 1$ with $f_{(\hat{i}+L) \bmod 4}^{(t'+L+1)} = 1$.

Proof. First, let's use induction on t to prove at time $t, t' < t < t' + L + 1$, we have the state of the network be

$$f_{(\hat{i}+t-t') \bmod 4}^{(t)} = f_{(\hat{i}+t-t'-1) \bmod 4}^{(t)} = 1, f_{(\hat{i}+t-t'-2) \bmod 4}^{(t)} = f_{(\hat{i}+t-t'-3) \bmod 4}^{(t)} = 0$$

Base Case: By Lemma 4.1, we have

$$f_{(\hat{i}+1) \bmod 4}^{(t'+1)} = f_{(\hat{i}+t-t') \bmod 4}^{(t'+1)} = 1, f_{(\hat{i}-1) \bmod 4}^{(t'+1)} = f_{(\hat{i}-2) \bmod 4}^{(t'+1)} = 0$$

for the base case.

Inductive Step: Now assume the induction hypothesis is true for $t = k$, since we have $x^{(k)} = 1$ by Lemma 4.1, we indeed have

$$f_{(\hat{i}+k+1-t') \bmod 4}^{(k+1)} = f_{(\hat{i}+k+1-t'-1) \bmod 4}^{(k+1)} = 1, f_{(\hat{i}+k+1-t'-2) \bmod 4}^{(k+1)} = f_{(\hat{i}+k+1-t'-3) \bmod 4}^{(k+1)} = 0$$

This completes the induction.

Now since $x^{(t'+L)} = 0$, by Lemma 4.1 we can derive the state of the network at time $t' + L + 1$

$$f_{(\hat{i}+L) \bmod 4}^{(t'+L+1)} = 1, f_j^{(t'+L+1)} = 0, \forall j \neq (\hat{i} + L) \bmod 4$$

as desired. $\square$

Now we iteratively build the network with the following rule on top of the mod 4 counter network,

$$w_{f_3 z_i} = w_{f_3 in_i} = 3, w_{f_0 z_i} = w_{f_0 in_i} = -1, w_{x z_i} = w_{x in_i} = 1, \\ w_{z_j z_i} = w_{z_j in_i} = 1, \forall j, 2 \leq j < i, w_{in_i z_i} = -i - 3, w_{z_i in_i} = 1, w_{z_i z_i} = i + 3$$

and

$$b_{z_i} = i + 1.5, b_{in_i} = i + 2.5$$

In the full construction of the TSC network, intuitively, we replace the function of z_0, z_1 in Section 3 with a mod 4 counter network. We design the weights coming from f_3, f_0 such that they will induce proper carry in an approximate binary representation at $z_i, i \geq 2$, and we use a similar idea as the mod 4 counter network to make TSC network converge to an exact binary representation in one computation step without input.

The following lemma specifies the firing rules of z_i, in_i for $i \geq 2$:

Lemma 4.3. *For $i \geq 2$, neurons $z_i^{(t)}, in_i^{(t)}$ fire according to the following rules:*

1. $z_i^{(t)} = 1$ if and only if $in_i^{(t-1)} = 0$, and either $(f_3^{(t-1)} = 1, f_0^{(t-1)} = 0, x^{(t-1)} = 1$ and for all $j, 2 \leq j < i$ we have $z_j^{(t-1)} = 1)$ or $z_i^{(t-1)} = 1$.

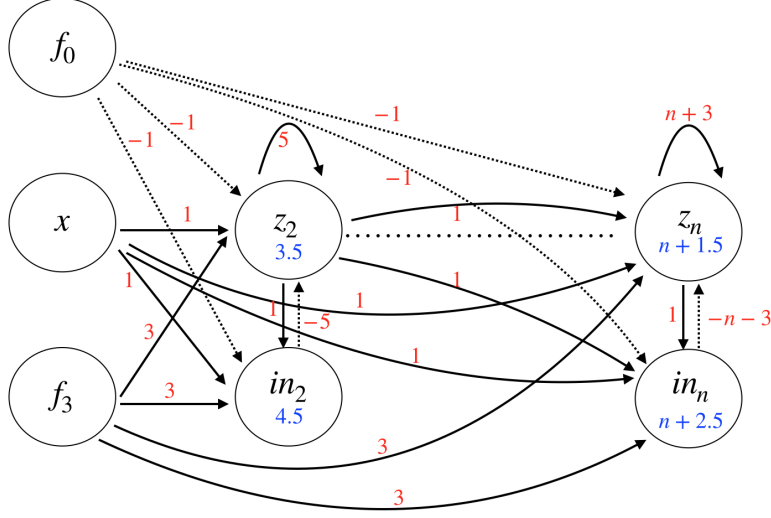


Figure 5: Total spikes counting (TSC) Network

2. $in_i^{(t)} = 1$ if and only if $z_i^{(t-1)} = 1, f_3^{(t-1)} = 1, f_0^{(t-1)} = 0, x^{(t-1)} = 1$ and for all $j, 2 \leq j < i$ we have $z_j^{(t-1)} = 1$.

Proof. Case 1: The potential of $z_i^{(t)}$ is

$$\begin{aligned} w_{f_3 z_i} f_3^{(t-1)} + w_{f_0 z_i} f_0^{(t-1)} + \sum_{j=2}^{i-1} w_{z_j z_i} z_j^{(t-1)} + w_{z_i z_i} z_i^{(t-1)} + w_{in_i z_i} in_i^{(t-1)} + w_{x z_i} x^{(t-1)} \\ = 3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} + (i+3)z_i^{(t-1)} - (i+3)in_i^{(t-1)} + x^{(t-1)} \end{aligned}$$

Only If: Let's show the only if direction for the firing rule of $z_i^{(t)}$ first. If $in_i^{(t-1)} = 1$, the potential of $z_i^{(t)}$ is

$$3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} + (i+3)z_i^{(t-1)} - (i+3) + x^{(t-1)} \leq i+1 < i+1.5 = b_{z_i}$$

If $f_3^{(t-1)} = 0, z_i^{(t-1)} = 0$, the potential of $z_i^{(t)}$ is

$$-f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} - (i+3)in_i^{(t-1)} + x^{(t-1)} \leq i-1 < i+1.5 = b_{z_i}$$

If $f_0^{(t-1)} = 1, z_i^{(t-1)} = 0$, the potential of $z_i^{(t)}$ is

$$3f_3^{(t-1)} - 1 + \sum_{j=2}^{i-1} z_j^{(t-1)} - (i+3)in_i^{(t-1)} + x^{(t-1)} \leq i+1 < i+1.5 = b_{z_i}$$

If $x^{(t-1)} = 0, z_i^{(t-1)} = 0$, the potential of $z_i^{(t)}$ is

$$3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} - (i+3)in_i^{(t-1)} \leq i+1 < i+1.5 = b_{z_i}$$

If $z_i^{(t-1)} = 0$ and there exists $\hat{j}, 2 \leq \hat{j} < i$ such that $z_{\hat{j}}^{(t-1)} = 0$, the potential of $z_i^{(t)}$ is

$$3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j \neq \hat{j}, 2 \leq j < i} z_j^{(t-1)} - (i+3)in_i^{(t-1)} + x^{(t-1)} \leq i+1 < i+1.5 = b_{z_i}$$

In all cases, we have $z_i^{(t)} = 0$.

If: For the if direction, if $in_i^{(t-1)} = 0, f_3^{(t-1)} = 1, f_0^{(t-1)} = 0, x^{(t-1)} = 1$ and for all $j, 2 \leq j < i$ we have $z_j^{(t-1)} = 1$, then the potential of $z_i^{(t)}$ is

$$3 + \sum_{j=2}^{i-1} 1 + (i+3)z_i^{(t-1)} + 1 \geq i+2 > i+1.5 = b_{z_i}$$

If $in_i^{(t-1)} = 0, z_i^{(t-1)} = 1$, the potential of $z_i^{(t)}$ is

$$3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} + (i+3) + x^{(t-1)} \geq i+2 > i+1.5 = b_{z_i}$$

In both cases, we have $z_i^{(t)} = 1$.

Case 2: the potential of $in_i^{(t)}$ is

$$\begin{aligned} w_{f_3 in_i} f_3^{(t-1)} + w_{f_0 in_i} f_0^{(t-1)} + \sum_{j=2}^{i-1} w_{z_j in_i} z_j^{(t-1)} + w_{z_i in_i} z_i^{(t-1)} + w_{x in_i} x^{(t-1)} \\ = 3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} + z_i^{(t-1)} + x^{(t-1)} \end{aligned}$$

Only If: For the only if direction, if $z_i^{(t-1)} = 0$, then the potential of $in_i^{(t)}$ is

$$3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} + x^{(t-1)} \leq i+2 < i+2.5 = b_{in_i}$$

If $f_3^{(t-1)} = 0$, the potential of $in_i^{(t)}$ is

$$-f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} + z_i^{(t-1)} + x^{(t-1)} \leq i < i+2.5 = b_{in_i}$$

If $f_0^{(t-1)} = 1$, the potential of $in_i^{(t)}$ is

$$3f_3^{(t-1)} - 1 + \sum_{j=2}^{i-1} z_j^{(t-1)} + z_i^{(t-1)} + x^{(t-1)} \leq i+2 < i+2.5 = b_{in_i}$$

If $x^{(t-1)} = 0$, the potential of $in_i^{(t)}$ is

$$3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j=2}^{i-1} z_j^{(t-1)} + z_i^{(t-1)} \leq i + 2 < i + 2.5 = b_{in_i}$$

If there exists $\hat{j}, 2 \leq \hat{j} < i$ such that $z_{\hat{j}} = 0$, the potential of $in_i^{(t)}$ is

$$3f_3^{(t-1)} - f_0^{(t-1)} + \sum_{j \neq \hat{j}, 2 \leq j < i} z_j^{(t-1)} + z_i^{(t-1)} + x^{(t-1)} \leq i + 2 < i + 2.5 = b_{in_i}$$

In all cases, $in_i^{(t)} = 0$.

If: For the if direction, if $z_i^{(t-1)} = 1, f_3^{(t-1)} = 1, f_0^{(t-1)} = 0, x^{(t-1)} = 1$ and for all $j, 2 \leq j < i$ we have $z_j^{(t-1)} = 1$, then the potential of $in_i^{(t)}$ is

$$3 + \sum_{j=2}^{i-1} 1 + 1 + 1 \leq i + 3 > i + 2.5 = b_{in_i}$$

We have $in_i^{(t)} = 1$ as desired. □

Define a clean state at time t' of TSC network with value X stored be one in which

1. $f_{X \bmod 4}^{(t')} = 1, f_j^{(t')} = 0, \forall j \neq X \bmod 4$ (i.e., the mod 4 counter subnetwork is clean with value $X \bmod 4$)
2. For $X = \sum_{i=0}^{\infty} a_i 2^i, a_i \in \{0, 1\}, z_k^{(t')} = a_k, \forall k \geq 2$
3. $in_i^{(t')} = 0$ if $X \bmod 2^{i+1} = 2^{i+1} - 1$

So being at a clean state for TSC network with value X stored implies being at a clean state with value $X \bmod 4$ for its mod 4 counter subnetwork with z_i in binary representation for $i \geq 2$. By Lemma 4.3, it is trivial to see that if for all $t \geq t'$ we have $x^{(t)} = 0$, then for all $i \geq 2$ and for all $t, t \geq t'$ we have $f_i^{(t)} = f_i^{(t')}$. Using Lemma 4.3, we have the following lemma describing the behaviors of the TSC network.

Lemma 4.4. *Let TSC network be at a clean state at time t' with value X stored. Fix a positive integer L . For all i such that $0 \leq i < L$, let $x^{(t'+i)} = 1$ and $x^{(t'+L)} = 0$. Then, at $t, t' < t < t' + L + 1$, z_i, in_i fire with the following rules for all $i \geq 2$:*

1. for $1 = X + t - t' \bmod 2^{i+1} < 2^i, z_i^{(t)} = 0$
2. for $1 < X + t - t' \bmod 2^{i+1} < 2^i, z_i^{(t)} = in_i^{(t)} = 0$
3. for $X + t - t' \bmod 2^{i+1} \geq 2^i$, we have $z_i^{(t)} = 1, in_i^{(t)} = 0$
4. for $X + t - t' \bmod 2^{i+1} = 0$, we have $z_i^{(t)} = 1, in_i^{(t)} = 1$

Furthermore, the network will be at a clean state with value $X + L$ stored at time $t' + L + 1$.

Proof. Just like the mod 4 counter network case, we want to deduce the behaviors of network at $t, t' < t < t' + L + 1$ using induction first.

Base Case: Fix i , for $t = t' + 1$, we have the following cases

1. $0 < X + 1 \bmod 2^{i+1} < 2^i$:
This implies that $0 \leq X \bmod 2^{i+1} < 2^i - 1$. This shows that not all $j, j < i$ we have $z_j^{(t-1)} = 1$ or $f_3^{(t-1)} = 0$ or $f_0^{(t-1)} = 1$. By Lemma 4.3, we have $z_i^{(t)} = in_i^{(t)} = 0$
2. $X + 1 \bmod 2^{i+1} \geq 2^i$:
This implies that $2^i - 1 \leq X \bmod 2^{i+1} < 2^{i+1} - 1$. This shows that either for all $j, j < i$ we have $f_3^{(t-1)} = 1, f_0^{(t-1)} = 0, z_j^{(t-1)} = 1$ or $z_i^{(t-1)} = 1$ but not both. By Lemma 4.3, we have $z_i^{(t)} = 1, in_i^{(t)} = 0$
3. $X + 1 \bmod 2^{i+1} = 0$:
This implies that $X \bmod 2^{i+1} = 2^{i+1} - 1$. This shows that $f_3^{(t-1)} = 1, f_0^{(t-1)} = 0$ and for all $j \leq i$ we have $z_j^{(t-1)} = 1$ and by the definition of a clean state, we have $in_i^{(t-1)} = 0$. Now by Lemma 4.3, we have $z_i^{(t)} = 1, in_i^{(t)} = 1$.

Inductive Step: Assume the induction hypothesis is accurate for $t = k$. We have the following cases

1. $1 = X + k + 1 - t' \bmod 2^{i+1} < 2^i$:
This implies that $X + k - t' \bmod 2^{i+1} = 0$. Now by induction hypothesis and Lemma 4.2, we know that $f_3^{(k)} = 1, f_0^{(k)} = 0$ and for all $j, i \geq j \geq 2$ we have $z_j^{(k)} = 1, in_j^{(k)} = 1$. By Lemma 4.3, we have $z_i^{(k+1)} = 0, in_i^{(k+1)} = 1$.
2. $1 < X + k + 1 - t' \bmod 2^{i+1} < 2^i$:
This implies that $1 \leq X + k - t' \bmod 2^{i+1} < 2^i - 1$. By induction hypothesis and Lemma 4.2, this shows that not all $j, j < i$ we have $z_j^{(k)} = 1$ or $f_3^{(k)} = 0$ or $f_0^{(k)} = 1$. By Lemma 4.3, we have $x_i^{(k+1)} = in_i^{(k+1)} = 0$
3. $X + k + 1 - t' \bmod 2^{i+1} \geq 2^i$:
This implies that $2^i - 1 \leq X + k - t' \bmod 2^{i+1} < 2^{i+1} - 1$. By induction hypothesis and Lemma 4.2, this shows that either for all $j, j < i$ we have $f_3^{(k)} = 1, f_0^{(k)} = 0, z_j^{(k)} = 1$ or $z_i^{(k)} = 1$ but not both. By Lemma 4.3, we have $z_i^{(k+1)} = 1, in_i^{(k+1)} = 0$
4. $X + k + 1 - t' \bmod 2^{i+1} = 0$:
This implies that $X + k - t' \bmod 2^{i+1} = 2^{i+1} - 1$. By induction hypothesis and Lemma 4.2, this shows that all $f_3^{(k)} = 1, f_0^{(k)} = 0, in_i^{(k)} = 0$ and for all $j, j \leq i$ we have $z_j^{(k)} = 1$. Now by Lemma 4.3, we have $z_i^{(t)} = 1, in_i^{(t)} = 1$.

This completes the induction.

Now we just need to show that at time $t' + L + 1$ the network is at a clean state with value $X + L$ stored. We have the following cases:

1. $1 = X + L \bmod 2^{i+1} < 2^i$:
By above induction, we have for $j, j \leq i$, $z_j^{(t'+L)} = 0$. No matter what the value of $in_i^{(t'+L)}$ is, by Lemma 4.3 we have $z_i^{(t'+L+1)} = in_i^{(t'+L+1)} = 0$.
2. $1 < X + L \bmod 2^{i+1} < 2^i, z_i^{(t)} = in_i^{(t)} = 0$:
By above induction, we have $z_i^{(t'+L)} = in_i^{(t'+L)} = 0$. By Lemma 4.3, we have $z_i^{(t'+L+1)} = in_i^{(t'+L+1)} = 0$

3. $X + L \bmod 2^{i+1} \geq 2^i$, we have $z_i^{(t'+L)} = 1, in_i^{(t'+L)} = 0$. By Lemma 4.3, we have $z_i^{(t'+L+1)} = in_i^{(t'+L+1)} = 0$
4. $X + L \bmod 2^{i+1} = 0$, we have $z_i^{(t'+L)} = 1, in_i^{(t'+L)} = 1$. By Lemma 4.3, we have $z_i^{(t'+L+1)} = 0, in_i^{(t'+L+1)} = 1$

which is exactly a clean state with value $X + L$ stored combining with Lemma 4.2. $\square$

Now we are ready for the main proof of Theorem 2.2 by setting $n = \lceil \log T' \rceil$ and let $f_i, z_j, 0 \leq i \leq 3, 2 \leq j \leq n$ be our output neurons.

Proof. Let $f_i, z_j, 0 \leq i \leq 4, 2 \leq j \leq n$ be our output neurons. Let there be X spikes in T time steps. Let $[t_0, t_0 + X_0 - 1], \dots, [t_k, t_k + X_k - 1]$ be the disjoint maximal intervals of spikes ordered by time (i.e., $x^{(t)} = 1$ if $t \in [t_i, t_i + X_i - 1]$ for some $0 \leq i \leq k$ and $[t_i, t_i + X_i] \cap [t_j, t_j + X_j] = \emptyset$ for all $i \neq j$ and $t_0 < t_1 < \dots < t_k, \sum_{i=0}^k X_i = X$). Now I claim that at time $t_i + X_i + 1$, the network is at a clean state with value $\sum_{j=0}^i X_j$ stored. We will prove the claim with induction on i . For $i = 0$, apply Lemma 4.4, we get that the network is at a clean state with value X_0 stored. Assume the network is at a clean state with value $\sum_{j=0}^i X_j$ stored at time $t_i + X_i + 1$. Then apply Lemma 4.4 again, we get at time $t_{i+1} + X_{i+1} + 1$, the network is at a clean state with value $\sum_{j=0}^{i+1} X_j$ stored at time $t_{i+1} + X_{i+1} + 1$. So at time $t_k + X_k + 1 \leq T + 1$, the network is at a clean state with value $\sum_{j=0}^k X_j = X$ stored as desired. This shows that the above network solves TSC(T) problem in time $T + 1$ with $O(\log T)$ neurons. $\square$

Notice that in fact by the proof above, TSC network enjoys an early convergence property. The network actually converges at time $t_k + X_k + 1$. Therefore we have the following stronger version of Theorem 2.2.

Corollary 4.5. *For all $t, 0 \leq t \leq T$, TSC network with $O(\log T)$ neurons solves FCSC(t) problem in time $t + 1$.*

5 Time Lower Bound for FCSC and TSC

In Section 4, we mentioned that there is a conflicting objective between stabilizing the output and toggling without delays. We therefore introduced the idea of carrying information of the count at an unclean state and then converging to a clean state, which introduces one time step of delay. In this Section, we are going to show that this delay is unavoidable.

Intuitively, the proof of the time lower bound uses the fact that if the network has to solve the problem without delay, the network must stabilize immediately at each time step. Therefore, the neurons that fire at the last round will stay firing. By injectivity of the representation, we can conclude that the network can at most count up to the network size.

The proof of Theorem 2.3 is the follows. The proof of Theorem 2.4 is identical.

Proof. Consider the following input sequence such that for all $0 \leq t < T$ we have $x^{(t)} = 1$ and for all $t \geq T$ we have $x^{(t)} = 0$. Let X be the collections of all neurons in the network. Assume for all $0 \leq t \leq T$, the network solves FCSC(t) at time t . For all $0 \leq j \leq T$, let $S_j = \{y_i : y_i^{(j)} = 1, 1 \leq i \leq m\}$. We would like to show that $S_T \supsetneq S_{T-1} \supsetneq \dots \supsetneq S_0$. To prove this by induction on t , we strengthen our induction hypothesis to become $S_t \supsetneq S_{t-1} \supsetneq \dots \supsetneq S_0$ and for all $y_j \in S_{t-1}$ we have $w_{xy_j} > 0$.

Base Case: When $t = 1$, notice that $S_0 = \emptyset$ by construction. Now by injectivity of the counter representation, we have $S_1 \supsetneq S_0$ and for $y_j \in S_0$, $w_{xy_j} > 0$ is vacuously true.

Induction Step: Now assume $S_t \supsetneq S_{t-1} \supsetneq \dots \supsetneq S_1$ and $w_{xy_j} > 0$ for $y_j \in S_{t-1}$. At time step $t + 1$, since the network solves FCSC(t) at time t , the neurons in y is stabilized even without the input from x . This means that

$$\sum_{z \in X/\{x\}} w_{zy_j} z^{(t)} - b_{y_j} > 0 \text{ if } y_j \in S_t$$

Now since $w_{xy_j} > 0$, we know that neurons in S_{t-1} will keep firing at time $t + 1$. For neurons in S_t/S_{t-1} , since those neurons fire at time t , we have

$$w_{xy_j} + \sum_{z \in X/\{x\}} w_{zy_j} z^{(t-1)} - b_{y_j} > 0 \text{ if } y_j \in S_t/S_{t-1}$$

And since the network solves FCSC(t-1) at time $t - 1$, we also have

$$\sum_{z \in X/\{x\}} w_{zy_j} z^{(t-1)} - b_{y_j} \leq 0 \text{ if } y_j \in S_t/S_{t-1}$$

Subtract two equations we get

$$w_{xy_j} > 0 \text{ if } y_j \in S_t/S_{t-1}$$

And hence $S_{t+1} \subset S_t$. By injectivity of the count representation, we have $S_{t+1} \supsetneq S_t$ as desired.

Now we have $S_T \supsetneq S_{T-1} \supsetneq \dots \supsetneq S_2 \supsetneq S_1$, but we only have $o(T)$ output neurons. Contradiction. $\square$

6 Discussion and Future Direction

In this paper, we have shown that networks of neurons are capable of integrating temporal information to solve two different tasks with temporal inputs efficiently. Out of the spiking neural networks literature, Hitron and Parter [HP19] tackled a similar problem. Their neural counter problem is our TSC problem. Our works differ in three ways. First, our network has time bound $T + 1$ while theirs is $T + O(\log T)$. Second, we provide a time lower bound result and show our time bound is optimal. Third, they consider an approximate version of the problem while we focus on the exact version of the problem.

Our paper follows similar approaches to Lynch et al. [LMP17a, LMP17b, LM18] by treating neurons as static circuits to explore the computational power of neural circuits. There are three noteworthy points about our model. First, instead of a stochastic model, we use a deterministic one. However, it should be noted that all the results in this paper would still hold under the randomized model of Lynch et al. [LMP17a, LMP17b, LM18] with high probability. Second, we use a model which resets the potential at every round. Therefore, to retain temporal information, many self-excitation connections are employed in our networks. At the other extreme, we could have a model in which the potential does not decay from past rounds. In that model, temporal information can be stored in potentials, but it might require different mechanisms to translate the information from potentials to spikes. The two models thus could lead to different possible computational principles in brains. Third, we used a discrete time model instead of a continuous time model, which would be more biologically plausible. However, this might not be a concern since we could use Maass's synchronization module [Maa96] to simulate our discrete time model from a continuous time model.

This paper mainly deals with the exact versions of the problems. One possible extension is to consider the approximate versions of the problems. By introducing noise into our models, we might be able to solve the approximate versions of the problems more efficiently. For example, for approximate counting, we aim to output some firing patterns corresponding to a number $\tilde{X}$ such that

$$P(|\tilde{X} - X| > \epsilon X) < \delta$$

is small. The lower bound for this question is $\Omega(\log \log T)$ and finding a matching upper bound can be an interesting future direction. However, approximate versions of the questions are tricky with temporal inputs because the network inevitably reuses random bits if they are stored inside the weights. A possible approach is to use a small number of random bits to generate a large family of k -wise independent random functions within neurons.

Another aspect of the temporal input we have not exploited is the time-scale invariance of the problem. In biology, many problems are time-scale invariant. A person who says “apple” fast can be understood as well as a person who says “apple” slowly. If we exploit this invariance, we might be able to reduce the networks’ complexity further.

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