Ant-Inspired Density Estimation via Random Walks

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Abstract

Many ant species employ distributed population density estimation in applications ranging from quorum sensing [Pra05], to task allocation [Gor99], to appraisal of enemy colony strength [Ada90]. It has been shown that ants estimate density by tracking encounter rates – the higher the population density, the more often the ants bump into each other [Pra05, GPT93].

We study distributed density estimation from a theoretical perspective. We show that a group of anonymous agents randomly walking on a grid are able to estimate their density d to within a multiplicative factor $1 \pm \epsilon$ with probability $1 - \delta$ in just $\tilde{O}\left(\frac{\log(1/\delta)\log(1/d\epsilon)}{d\epsilon^2}\right)$ steps by measuring their encounter rates with other agents. Despite dependencies inherent in the fact that nearby agents may collide repeatedly (and, worse, cannot recognize when this happens), this bound nearly matches what is required to estimate d by independently sampling grid locations.

From a biological perspective, our work helps shed light on how ants and other social insects can obtain relatively accurate density estimates via encounter rates. From a technical perspective, our analysis provides new tools for understanding complex dependencies in the collision probabilities of multiple random walks. We bound the strength of these dependencies using *local mixing properties* of the underlying graph. Our results extend beyond the grid to more general graphs and we discuss applications to social network size estimation, density estimation by robot swarms, and random walked-based sampling of sensor networks.

1 Introduction

The ability to sense local population density is an important tool used by many ant species. When a colony must relocate to a new nest, scouts search for potential nest sites, assess their quality, and recruit other scouts to high quality locations. A high enough density of scouts at a potential new nest (a quorum threshold) triggers those ants to decide on the site and transport the rest of the colony there [Pra05]. When neighboring colonies compete for territory, a high relative density of a colony's ants in a contested area will cause those ants to attack enemies in the area, while a low relative density will cause the colony to retreat [Ada90]. Varying densities of ants successfully performing certain tasks such as foraging or brood care can trigger other ants to switch tasks, maintaining proper worker allocation within in the colony [Gor99, SHG06].

It has been shown that ants estimate density in a distributed manner, by measuring encounter rates [Pra05, GPT93]. As ants randomly walk around an area, if they bump into a larger number of other ants, this indicates a higher population density. By tracking encounters with specific types of ants, e.g. successful foragers or enemies, ants can estimate more specific densities. This strategy allows each ant to obtain an accurate density estimate and requires very little communication – ants must simply detect when they collide and do not need to perform any higher level data aggregation.

1.1 Density Estimation on the Grid

We study distributed density estimation from a theoretical perspective. We model a colony of ants as a set of anonymous agents randomly distributed on a two-dimensional grid. Computation proceeds in rounds, with each agent stepping in a random direction in each round. A *collision* occurs when two agents reach the same position in the same round and encounter rate is measured as the number of collisions an agent is involved in during a sequence of rounds divided by the number of rounds. Aside from collision detection, the agents have no other means of communication.

The intuition that encounter rate tracks density is clear. It is easy to show that, for a set of randomly walking agents, the *expected* encounter rate measured by each agent is exactly the density d – the number of agents divided by the grid size (see Lemma 3). However, it is unclear if encounter rate actually gives a good density estimate – i.e., if it concentrates around its expectation.

Consider agents positioned not on the grid, but on a complete graph. In each round, each agent steps to a uniformly random position and in expectation, the number of other agents they collide with in this step is d. Since each agent chooses its new location uniformly at random in each step, collisions are essentially independent between rounds. The agents are effectively taking independent Bernoulli samples with success probability d, and by a standard Chernoff bound, within $O\left(\frac{\log(1/\delta)}{d\epsilon^2}\right)$ rounds obtain a $(1 \pm \epsilon)$ multiplicative approximation to d with probability $1 - \delta$.

On the grid graph, the picture is significantly more complex. If two agents are initially located near each other on the grid, they are more likely to collide via random walking. After a first collision, due to their proximity, they are likely to collide repeatedly in future rounds. The agents cannot recognize repeat collisions since they are anonymous and even if they could, it is unclear that it would help. On average, compared to the complete graph, agents collide with fewer individuals and collide multiple times with those individuals that they do encounter, causing an increase in encounter rate variance and making density estimation more difficult.

Mathematically speaking, on a graph with a *fast mixing time* [Lov93], like the complete graph, each agent's location is only weakly correlated with its previous locations. This ensures that collisions are also weakly correlated between rounds and encounter rate serves as a very accurate estimate of density. The grid graph on the other hand is *slow mixing* – agent positions and hence collisions are highly correlated between rounds. This correlation increases encounter rate variance.

1.2 Our Contributions

Surprisingly, despite this increased variance, encounter rate-based density estimation on the grid is nearly as accurate as on the complete graph. After just $O\left(\frac{\log(1/\delta)\log\log(1/\delta\epsilon)}{d\epsilon^2}\right)$ rounds, each agent's encounter rate is a $(1 \pm \epsilon)$ approximation to d with probability $1 - \delta$ (see Theorem 2).

Technically, to bound accuracy on the grid, we obtain moment bounds on the number of times that two randomly walking agents repeatedly collide over a set of rounds. These bounds also apply to the number of equalizations (returns to starting location) of a single walk. While *expected* random walk hitting times, return times, and collision rates are well understood [Lov93, ES09], higher moment bounds and high probability results are much less common. We hope our bounds are of general use in the theoretical study of random walks and random-walk based algorithms.

Our moment bounds show that, while the grid graph is slow mixing, it has sufficiently strong *local mixing* to make random walk-based density estimation accurate. Random walks tend to spread quickly over a local area and not repeatedly cover the same nodes. Significant work has focused on showing that random walk sampling is nearly as good as independent sampling for fast mixing expander graphs [Gil98, CLLM12]. We are the first to extend this type of analysis to slowly mixing graphs, showing that strong local mixing is sufficient in many applications.

Beyond the grid, we show how to generate moment bounds from a bound on the probability that two random walks re-collide (or analogously, that a single random walk equalizes) after a certain number of steps, and demonstrate application of this technique to d-dimensional grids, regular expanders, and hypercubes. We discuss applications of our results to social network size estimation via random walk [KLS11], obtaining significant improvements over known work for networks with relatively slow global mixing times, but strong local mixing. We also discuss connections to density estimation by robot swarms and random walk-based sensor network sampling [AB04, LB07].

1.3 Road Map

In Section 2 we overview our theoretical model for distributed density estimation on the grid.

In Section 3, as a warm up, we give a simple density estimation algorithm that does not employ random walks, is easy to analyze, but is not biologically plausible.

In Section 4 we give our main technical results on random walk-based density estimation.

In Section 5 we show how to extend our bounds to a number of graphs other than the grid.

In Section 6 we discuss applications of our results to social network size estimation, robot swarm, and sensor network algorithms.

2 Theoretical Model for Density Estimation

We consider a two-dimensional torus with A nodes (dimensions $\sqrt{A} \times \sqrt{A}$) populated with identical anonymous agents. We assume that A is large – larger than the area agents traverse over the runtimes of our algorithms. We feel that this torus model successfully captures the dynamics of density estimation on a surface, while avoiding complicating factors of boundary behavior.

Initially each agent is placed independently at a uniform random node in the torus. Computation proceeds in discrete, synchronous rounds. In each round an agent may either remain in its current location or step to any of its four neighboring grid squares. Formally, each agent has an ordered pair *position* which it may update in each round by adding step $s \in \{(0,1), (0,-1), (1,0), (-1,0), (0,0)\}$.

A randomly walking agent chooses s uniformly at random from $\{(0,1), (0,-1), (1,0), (-1,0)\}$ in each round. Of course, in reality ants do not move via pure random walk – observed encounter rates seem to actually be lower than predicted by a pure random walk model [GPT93, NTD05]. However, we feel that our model sufficiently captures the highly random movement of ants while remaining tractable to analysis and applicable to ant-inspired random walk-based algorithms (Section 6).

Aside from the ability to move in each round, agents can sense the number of agents other than themselves at their position at the end of each round, formally through calling count(position). We say that two agents collide in round r if they have the same position at the end of the round. Outside of collision counting, agents have no means of communication. They are anonymous (cannot uniquely identify each other) and all execute identical density estimation routines.

Density Estimation Problem Let (n+1) be the number of agents and define population density as $d \stackrel{\text{def}}{=} n/A$. Each agent's goal is to estimate d to $(1 \pm \epsilon)$ accuracy with probability $1 - \delta$ for $\epsilon, \delta \in (0,1)$ – i.e., to return an estimate \tilde{d} with $\mathbb{P}\left[\tilde{d} \in [(1-\epsilon)d,(1+\epsilon)d]\right] \geq 1-\delta$. As a technicality, with n+1 agents we define d=n/A instead of d=(n+1)/A for convenience of calculation. In the natural case, when n is large, the distinction is minor.

3 Density Estimation via Simulation of Independent Sampling

As discussed, the challenge in analyzing random walk-based density estimation arises from increased variance due to repeated collisions of nearby agents. Here we show that, if not restricted to random walking, agents can avoid collision correlations by splitting into 'stationary' and 'mobile' groups and only counting collisions between members of different groups. This allows them to essentially simulate independent sampling of gird locations to estimate density. This algorithm is not 'natural' in a biological sense, however it is easy to analyze, and demonstrates the feasibility of density estimation by anonymous agents on the grid. We give pseudocode in Algorithm 1.

Algorithm 1 Independent Sampling-Based Density Estimation

```
input: runtime t
Set c := 0 and with probability 1/2, state := walking, else state := stationary.

for r = 1, ..., t do

if state := walking then

position := position + (0, 1)
end if

c := c + count(position)
c := c + count(position)
end for

end for

end for
```

3.1 Independent Sampling Accuracy Bound

We now present our main accuracy bound for the independent sampling algorithm.

Theorem 1 (Independent Sampling Accuracy Bound). After running for t rounds, assuming $t < \sqrt{A}$ and $d \le 1$, Algorithm 1 returns \tilde{d} such that, for any $\delta > 0$, with probability $\ge 1 - \delta$, $\tilde{d} \in [(1 - \epsilon)d, (1 + \epsilon)d]$ for $\epsilon = O\left(\sqrt{\frac{\log(1/\delta)}{td}}\right)$. In other words, for any $\epsilon, \delta \in (0,1)$ if $t = \Theta\left(\frac{\log(1/\delta)}{d\epsilon^2}\right)$, \tilde{d} is a $(1 \pm \epsilon)$ multiplicative estimate of d with probability $\ge 1 - \delta$.

Proof. Our analysis is from the perspective of an agent with state = walking. By symmetry, the distribution of \tilde{d} is identical for walking and stationary agents, so considering this case is sufficient.

Initially, assume that no two walking agents start in the same location. Given this assumption, we know that a walking agent never collides with another walking agent – by assumption they all start in different positions and update these positions identically in each round. In the written implementation agents always step up, however any fixed pattern (e.g. a spiral) suffices.

In t steps, a walking agent visits t unique squares. Each of the n other agents is located in this set of squares and stationary with probability $\frac{t}{2A}$. Further, each of these events is entirely independent from the rest, as the agents are positioned and choose their state independently. So, for a walking agent, c is just a sample of n independent random coin flips, each with success probability $\frac{t}{2A}$. Clearly, $\mathbb{E}\,c = n \cdot \frac{t}{2A} = \frac{td}{2}$ so $\mathbb{E}\,\tilde{d} = \mathbb{E}\,\frac{2}{c}t = d$. Further, by a Chernoff bound, for any $\epsilon \in (0,1)$, the probability that \tilde{d} is not a $(1 \pm \epsilon)$ multiplicative estimate of d is:

$$\delta = \mathbb{P}\left[|\tilde{d} - d| \ge \epsilon d \right] = \mathbb{P}\left[|c - \mathbb{E}\,c| \ge \epsilon\,\mathbb{E}\,c \right] \le 2e^{-\epsilon^2\,\mathbb{E}\,c/3} \le 2e^{-\epsilon^2td/6}.$$

This gives: $\log(1/\delta) \ge \epsilon^2 t d/6$ so $\epsilon = O\left(\sqrt{\frac{\log(1/\delta)}{td}}\right)$, yielding the result.

We now remove the assumption that no two walking agents start in the same location. We slightly modify the algorithm – each agent sets $c := c \pmod{t}$ before returning $\tilde{d} = \frac{2c}{t}$. If an agent starts alone and is involved in < t collisions, this operation has no effect – the above bound holds.

If a walking agent is involved in < t 'true collisions' but starts in the same position as $w \ge 1$ other walking agents, the agents move in lockstep throughout the algorithm and are involved in $w \cdot t$ 'spurious collisions' (w in each round). Setting $c := c \pmod{t}$ exactly corrects for these spurious collisions and since c now only includes collisions with stationary agents, the bound above holds.

Finally, if an agent is involved in $\geq t$ true collisions, this modification cannot worsen their estimate. If $c \ge t$ and the agent does not set $c := c \pmod{t}$, they compute $\tilde{d} \ge \frac{2t}{t} \ge 2$. For $\epsilon < 1$, the agent fails since $d \leq 1$. So setting $c := c \pmod{t}$ can only increase probability of success.

Density Estimation via Random Walk Collision Rates 4

In Algorithm 1, each pair of agents can only collide once, at a specific location – the starting position of the stationary agent in the pair. Collisions are independent and it is easy to show that the number of collisions (and hence the density estimate) concentrates around its expectation. However, as discussed, independence of collisions is unnecessary! Algorithm 2 describes a simple random walk-based approach that gives a nearly matching bound.

Algorithm 2 Random Walk Encounter Rate-Based Density Estimation

```
input: runtime t
  c := 0
  for r = 1, ..., t do
      position := position + rand\{(0,1), (0,-1), (1,0), (-1,0)\}
                                                                                        ▶ Random walk step.
      c := c + count(position)
                                                                                        ▶ Update collision count.
  end for
  return d = \frac{c}{t}
```

4.1 Random Walk-Based Density Estimation Analysis

Our main result follows; its proof appears at the end of Section 4 after some preliminary lemmas.

Theorem 2 (Random Walk Sampling Accuracy Bound). After running for t rounds, assuming t < A, Algorithm 2 returns \tilde{d} such that, for any $\delta>0$, with probability $\geq 1-\delta$, $\tilde{d}\in[(1-\epsilon)d,(1+\epsilon)d]$ for $\epsilon = \sqrt{\frac{\log(1/\delta)\log(t)}{td}}. \text{ In other words, for any } \epsilon, \delta \in (0,1) \text{ if } t = \Theta\left(\frac{\log(1/\delta)\log\log(1/\delta)\log(1/d\epsilon)}{d\epsilon^2}\right), \ \tilde{d} \text{ is a } (1 \pm \epsilon) \text{ multiplicative estimate of } d \text{ with probability } \geq 1 - \delta.$

Throughout our analysis, we take the viewpoint of a single agent executing Algorithm 2, referred to as 'agent a'. To start, we show that the encounter rate \tilde{d} is an unbiased estimator of d:

Lemma 3 (Unbiased Estimator). $\mathbb{E} \tilde{d} = d$.

Proof. We can decompose c as the sum of collisions with different agents over different rounds. Specifically, give the n other agents arbitrary ids 1, 2, ..., n and let $c_j(r)$ equal 1 if agent a collides with agent j in round r, and 0 otherwise. By linearity of expectation: $\mathbb{E} c = \sum_{j=1}^{n} \sum_{r=1}^{t} \mathbb{E} c_{j}(r)$. Since each agent is initially at a uniform random location and after any number of steps, is still at uniform

random location, for all $j, r, \mathbb{E} c_j(r) = 1/A$. Thus, $\mathbb{E} c = nt/A = dt$ and $\mathbb{E} d = \mathbb{E} c/t = d$.

With Lemma 3 in place, we now must show that the encounter rate is close to its expectation with high probability and hence provides a good estimate of density.

4.2 Bounding the Effects of Repeat Collisions

Let $c_j = \sum_{r=1}^t c_j(r)$ be the total number of collisions with agent j. Due to the initial uniform distribution of the agents, the c_j 's are all independent and identically distributed.

Each c_j is the sum of highly correlated random variables – due to the slow mixing of the grid, if two agents collide at round r, they are much more likely to collide in successive rounds. However, by bounding the strength of this correlation, we are able to give strong bounds on the moments of the distribution of each c_j , showing that it is sub-exponential. It follows that $\tilde{d} = \frac{1}{t} \sum_{j=1}^{n} c_j$, is also sub-exponential and hence concentrates strongly around its expectation, the true density d.

We first bound the probability of a re-collision in round r + m, assuming a collision in round r:

Lemma 4 (Re-collision Probability Bound). Consider two agents a_1 and a_2 randomly walking on a two-dimensional torus of dimensions $\sqrt{A} \times \sqrt{A}$. If a_1 and a_2 collide again in round r, for any $m \geq 0$, the probability that a_1 and a_2 collide in round r + m is $\Theta\left(\frac{1}{m+1}\right) + O\left(\frac{1}{A}\right)$.

Proof. From round r to round r + m, a_1 and a_2 take 2m random steps in total. Let M_x be the total number of steps they take in the x direction and M_y be the total number in the y direction. $M_x + M_y = 2m$.

We start by computing the probability that the agents collide in round r+m conditioned on the values of M_x and M_y . All steps are chosen independently, so we can consider movement in the x and y directions separately. Specifically, let \mathcal{C} be the event that the a_1 and a_2 collide in round r+m, \mathcal{C}_x be the event that they have the same x position, and \mathcal{C}_y be the event that they have the same y position. We have:

$$\mathbb{P}\left[\mathcal{C}|M_x = m_x, M_y = m_y\right] = \mathbb{P}\left[\mathcal{C}_x|M_x = m_x\right] \cdot \mathbb{P}\left[\mathcal{C}_y|M_y = m_y\right]. \tag{1}$$

We first consider $\mathbb{P}\left[\mathcal{C}_x|M_x=m_x\right]$. All bounds will hold symmetrically for the y dimension. We split our analysis into two cases. Let \mathcal{C}_x^1 be the event that the two agents have the same x position after round r+m and have identical displacements from their starting locations. Let \mathcal{C}_x^2 be the event that the two agents have the same x position after round r+m but do not have identical displacements. This requires that the agents 'wrap' around the torus, ending at the same position despite moving different amounts in the x direction. We have $\mathbb{P}[\mathcal{C}_x|M_x=m_x]=\mathbb{P}[\mathcal{C}_x^1|M_x=m_x]+\mathbb{P}[\mathcal{C}_x^2|M_x=m_x]$.

 $\mathbb{P}[\mathcal{C}_x^1|M_x=m_x]$ is identical to the probability that a single random walk takes m_x steps and has 0 overall displacement – i.e. takes an equal number of clockwise and counterclockwise steps. It is given by:

$$\mathbb{P}[\mathcal{C}_x^1 | M_x = m_x] = \binom{m_x}{m_x/2} \left(\frac{1}{2}\right)^{m_x} = \frac{m_x!}{(\frac{m_x}{2}!)^2} \cdot \left(\frac{1}{2}\right)^{m_x}. \tag{2}$$

Above we assume m_x is even – otherwise C_x^1 cannot occur. By Stirling's approximation for any n > 0, $n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right)$. Plugging this into 2:

$$\mathbb{P}[\mathcal{C}_{x}^{1}|M_{x} = m_{x}] = \frac{m_{x}!}{(\frac{m_{x}}{2}!)^{2}} \cdot \left(\frac{1}{2}\right)^{m_{x}} = \frac{\sqrt{2\pi m_{x}} \left(\frac{m_{x}}{e}\right)^{m_{x}} \left(1 + O\left(\frac{1}{m_{x}}\right)\right)}{\pi m_{x} \left(\frac{m_{x}/2}{e}\right)^{m_{x}} \left(1 + O\left(\frac{1}{m_{x}/2}\right)\right)^{2}} \cdot \left(\frac{1}{2}\right)^{m_{x}} = \Theta\left(\frac{1}{\sqrt{m_{x}+1}}\right).$$

(We use $m_x + 1$ instead of m_x in the denominator so that the bound holds in the case when $m_x = 0$.)

 $\mathbb{P}\left[C_x^2|M_x=m_x\right]$ is the probability that two agents have the same x position after round r+m but have different total displacements. It is identical to the probability that a single m_x step random walk has overall displacement $\pm c\sqrt{A}$ for some integer $c\geq 1$ (and so 'wraps around' the torus, ending at its starting location). Roughly, we bound the probability of this event by the probability that the random walk ends at any other location on the torus. There are \sqrt{A} such locations, so the probability is bounded by $O\left(\frac{1}{\sqrt{A}}\right)$. We have:

$$\mathbb{P}\left[\mathcal{C}_x^2 \middle| M_x = m_x\right] = 2 \cdot \left(\frac{1}{2}\right)^{m_x} \cdot \sum_{c=1}^{\left\lfloor \frac{m_x}{\sqrt{A}} \right\rfloor} \binom{m_x}{\frac{m_x - c\sqrt{A}}{2}}.$$
 (3)

where the extra factor of 2 comes from the fact that the displacement may be either clockwise or counterclockwise. (Note that if $\frac{m_x - c\sqrt{A}}{2}$ is not an integer we just define the binomial coefficient to equal 0.) For $i \in [1, ..., \sqrt{A} - 1]$, let \mathcal{D}_x^i be the event that a single random walk is i steps clockwise from its starting location after taking M_x steps. We have:

$$\mathbb{P}[\mathcal{D}_{x}^{i}|M_{x} = m_{x}] = \left(\frac{1}{2}\right)^{m_{x}} \cdot \sum_{c=-\left\lfloor\frac{m_{x}+i}{\sqrt{A}}\right\rfloor}^{\left\lfloor\frac{m_{x}-i}{\sqrt{A}}\right\rfloor} \binom{m_{x}}{\frac{m_{x}+i+c\sqrt{A}}{2}} \ge \left(\frac{1}{2}\right)^{m_{x}} \cdot \sum_{c=-\left\lfloor\frac{m_{x}+i}{\sqrt{A}}\right\rfloor}^{-1} \binom{m_{x}}{\frac{m_{x}+i+c\sqrt{A}}{2}} \\
\ge \left(\frac{1}{2}\right)^{m_{x}} \cdot \sum_{c=1}^{\left\lfloor\frac{m_{x}}{\sqrt{A}}\right\rfloor} \binom{m_{x}}{\frac{m_{x}+i-c\sqrt{A}}{2}}.$$
(4)

For any $i \in [1, ..., \sqrt{A} - 1]$, and any $c \ge 1$, $\frac{m_x + i - c\sqrt{A}}{2}$ is closer to $\frac{m_x}{2}$ than $\frac{m_x - c\sqrt{A}}{2}$ is, so

$$\begin{pmatrix} m_x \\ \frac{m_x + i - c\sqrt{A}}{2} \end{pmatrix} > \begin{pmatrix} m_x \\ \frac{m_x - c\sqrt{A}}{2} \end{pmatrix}$$
(5)

as long as $\frac{m_x+i-c\sqrt{A}}{2}$ is an integer. This allows us to lower bound $\mathbb{P}[\mathcal{D}_x^i|M_x=m_x]$ using $\mathbb{P}\left[\mathcal{C}_x^2|M_x=m_x\right]$. Let $\mathcal{E}_{i,c}$ equal 1 if $\frac{m_x+i-c\sqrt{A}}{2}$ is an integer and 0 otherwise. Since \mathcal{C}_x^2 and each \mathcal{D}_x^i are disjoint events:

$$\mathbb{P}\left[\mathcal{C}_{x}^{2}|M_{x}=m_{x}\right]+\sum_{i=1}^{\sqrt{A}-1}\mathbb{P}\left[\mathcal{D}_{x}^{i}|M_{x}=m_{x}\right]\leq1$$

$$\mathbb{P}\left[\mathcal{C}_{x}^{2}|M_{x}=m_{x}\right]+\left(\frac{1}{2}\right)^{m_{x}}\cdot\sum_{i=1}^{\sqrt{A}-1}\left(\sum_{c=1}^{\left\lfloor\frac{m_{x}}{\sqrt{A}}\right\rfloor}\binom{m_{x}}{\frac{m_{x}+i-c\sqrt{A}}{2}}\right)\leq1 \qquad \text{(applying (4))}$$

$$\mathbb{P}\left[\mathcal{C}_{x}^{2}|M_{x}=m_{x}\right]+\left(\frac{1}{2}\right)^{m_{x}}\cdot\sum_{c=1}^{\left\lfloor\frac{m_{x}}{\sqrt{A}}\right\rfloor}\left(\binom{m_{x}}{\frac{m_{x}-c\sqrt{A}}{2}}\cdot\sum_{i=1}^{\sqrt{A}-1}\mathcal{E}_{i,c}\right)\leq1 \qquad \text{(by (5) and switching summations)}$$

$$\mathbb{P}\left[\mathcal{C}_{x}^{2}|M_{x}=m_{x}\right]\cdot\Theta(\sqrt{A})\leq1.$$

The last step follows from combining (3) with the fact that $\sum_{i=1}^{\sqrt{A}-1} \mathcal{E}_{i,c} = \Theta\left(\sqrt{A}\right)$ for all c since $\frac{m_x + i - c\sqrt{A}}{2}$ is integral for half the possible $i \in [1, ..., \sqrt{A} - 1]$. Rearranging, we have $\mathbb{P}\left[\mathcal{C}_x^2 | M_x = m_x\right] = O\left(\frac{1}{\sqrt{A}}\right)$.

Combining our bounds for C_x^1 and C_x^2 , $\mathbb{P}[C_x|M_x=m_x]=\Theta\left(\frac{1}{\sqrt{m_x+1}}\right)+O\left(\frac{1}{\sqrt{A}}\right)$. Identical bounds hold for the y direction and by (1) we have:

$$\mathbb{P}\left[\mathcal{C}|M_{x}=m_{x}, M_{y}=m_{y}\right] = \Theta\left(\frac{1}{\sqrt{(m_{x}+1)(m_{y}+1)}}\right) + O\left(\frac{1}{\sqrt{A(m_{x}+1)}} + \frac{1}{\sqrt{A(m_{y}+1)}}\right) + O\left(\frac{1}{A}\right). \tag{6}$$

Our final step is to remove the conditioning on M_x and M_y . Since direction is chosen independently and uniformly at random for each step, $\mathbb{E} M_x = \mathbb{E} M_y = m$. By a standard Chernoff bound:

$$\mathbb{P}[M_x \le m/2] \le 2e^{-(1/2)^2 \cdot m/2} = O\left(\frac{1}{m+1}\right).$$

(Again using m+1 instead of m to cover the m=0 case). An identical bound holds for M_y , and so, except with probability $O\left(\frac{1}{m+1}\right)$ both are $\geq m/2$. Plugging into (6) this gives us:

$$\mathbb{P}\left[\mathcal{C}\right] = \Theta\left(\frac{1}{m+1}\right) + O\left(\frac{1}{\sqrt{A(m+1)}}\right) + O\left(\frac{1}{A}\right) = \Theta\left(\frac{1}{m+1}\right) + O\left(\frac{1}{A}\right).$$

We note that the techniques of Lemma 4 also apply to bounding the probability that a single random walk returns to its origin (equalizes) after m steps.

Corollary 5 (Equalization Probability Bound). Consider agent a_1 randomly walking on a two-dimensional torus of dimensions $\sqrt{A} \times \sqrt{A}$. If a_1 is located at position p after round r, for any even $m \geq 0$, the probability that a_1 is again at position p after round r + m is $\Theta\left(\frac{1}{m+1}\right) + O\left(\frac{1}{A}\right)$.

Proof. The analysis of Lemma 4 treats the two walks of a_1 and a_2 as a single walk with 2m total steps. An identical analysis where 2m is replaced by m yields the corollary.

Roughly, assuming as in Theorem 2 that $t \leq A$, by Lemma 4, in t rounds, a expects to re-collide with any agent it encounters $\sum_{m=0}^{t-1} \Theta\left(\frac{1}{m+1}\right) = \Theta(\log t)$ times. By Lemma 3, a expects to be involved in dt = nt/A total collisions. So accounting for re-collisions, it expects to collide with $\Theta\left(\frac{nt}{A\log t}\right)$ unique individuals. This is formalized in Lemma 6.

Lemma 6 (First Collision Probability). Assuming $t \leq A$, for all $j \in [1, ..., n]$, $\mathbb{P}[c_j \geq 1] = \Theta\left(\frac{t}{A \log t}\right)$.

Proof. Using the fact that c_j is identically distributed for all j,

$$\mathbb{E}\,\tilde{d} = d = \frac{1}{t} \cdot \mathbb{E}\sum_{i=1}^{n} c_i = \frac{n}{t} \cdot \mathbb{E}\,c_j = \frac{n}{t} \cdot \mathbb{P}\left[c_j \ge 1\right] \cdot \mathbb{E}\left[c_j | c_j \ge 1\right]$$
$$\frac{n}{A} = \frac{n}{t} \cdot \mathbb{P}\left[c_j \ge 1\right] \cdot \mathbb{E}\left[c_j | c_j \ge 1\right].$$

Rearranging gives:

$$\mathbb{P}\left[c_j \ge 1\right] = \frac{t}{A \cdot \mathbb{E}\left[c_i \middle| c_i \ge 1\right]}.\tag{7}$$

To compute $\mathbb{E}[c_j|c_j \geq 1]$, we use Lemma 4 and linearity of expectation. Since $t \leq A$, the $O\left(\frac{1}{A}\right)$ term in Lemma 4 is absorbed into the $\Theta\left(\frac{1}{m+1}\right)$. Let $r \leq t$ be the first round that the two agents collide. We have:

$$\mathbb{E}[c_j|c_j \ge 1] = \sum_{m=0}^{t-r} \Theta\left(\frac{1}{m+1}\right) = \Theta\left(\log(t-r)\right). \tag{8}$$

After any round the agents are located at uniformly and independently chosen positions, so collide with probability exactly 1/A. So, the probability of the *first* collision between the agents being in a given round can only decrease as the round number increases. So, at least 1/2 of the time that $c_j \geq 1$, there is a collision in the first t/2 rounds. So, overall, by (8), $\mathbb{E}[c_j|c_j \geq 1] = \Theta\left(\log(t - t/2)\right) = \Theta\left(\log t\right)$. Using (7), $\mathbb{P}\left[c_j \geq 1\right] = \Theta\left(\frac{t}{A \cdot \log t}\right)$, completing the proof.

We now give our main technical lemma – a strong moment bound on the distribution of c_j . Intuitively, not only does an agent *expect* to collide at most $O(\log t)$ times with any other agent it encounters, but this bound extends to the higher moments of the collision distribution, and so holds with high probability. In this sense, the grid has strong *local mixing* – random walks spread quickly over a local area and do not cover the same nodes too many times.

Lemma 7. (Collision Moment Bound) For all $j \in [1,...,n]$, let $\bar{c}_j \stackrel{\text{def}}{=} c_j - \mathbb{E} c_j$. For all $k \geq 2$, assuming $t \leq A$, $\mathbb{E}\left[\bar{c}_j^k\right] = O\left(\frac{t}{A} \cdot k! \log^{k-1} t\right)$.

Proof. We expand $\mathbb{E}[\bar{c}_j^k] = \mathbb{P}[c_j \ge 1] \cdot \mathbb{E}[\bar{c}_j^k|c_j \ge 1] + \mathbb{P}[c_j = 0] \cdot \mathbb{E}[\bar{c}_j^k|c_j = 0]$, and so by Lemma 6:

$$\mathbb{E}\left[\bar{c}_{j}^{k}\right] = O\left(\frac{t}{A \log t} \cdot \mathbb{E}\left[\bar{c}_{j}^{k} | c_{j} \geq 1\right] + \mathbb{E}\left[\bar{c}_{j}^{k} | c_{j} = 0\right]\right).$$

 $\mathbb{E}\left[\bar{c}_{j}^{k}|c_{j}=0\right]=\left(\mathbb{E}\,c_{j}\right)^{k}=\left(t/A\right)^{k}\leq\frac{t}{A}k!\log^{k-1}t\text{ for all }k\geq2.\text{ Further, }\mathbb{E}\left[\bar{c}_{j}^{k}|c_{j}\geq1\right]\leq\mathbb{E}\left[c_{j}^{k}|c_{j}\geq1\right],\text{ since }\mathbb{E}\,c_{j}=\frac{t}{A}\leq1.\text{ So to prove the lemma, it just remains to show that }\mathbb{E}\left[c_{j}^{k}|c_{j}\geq1\right]=O\left(k!\log^{k}t\right).$

Conditioning on $c_j \geq 1$, we know the agents have an initial collision in some round $t' \leq t$. We split c_j over rounds: $c_j = \sum_{r=t'}^t c_j(r) \leq \sum_{r=t'}^{t'+t-1} c_j(r)$. To simplify notation we relabel round t' round 1 and so round t' + t - 1 becomes round t. Expanding c_j^k out fully using the summation:

$$\mathbb{E}\left[c_{j}^{k}\right] = \mathbb{E}\left[\sum_{r_{1}=1}^{t}\sum_{r_{2}=1}^{t}...\sum_{r_{k}=1}^{t}c_{j}(r_{1})c_{j}(r_{2})...c_{j}(r_{k})\right] = \sum_{r_{1}=1}^{t}\sum_{r_{2}=1}^{t}...\sum_{r_{k}=1}^{t}\mathbb{E}\left[c_{j}(r_{1})c_{j}(r_{2})...c_{j}(r_{k})\right].$$

 $\mathbb{E}\left[c_{r_1}(j)c_{r_2}(j)...c_{r_k}(j)\right]$ is just the probability that the two agents collide in each of rounds $r_1, r_2, ... r_k$. Assume w.l.o.g. that $r_1 \leq r_2 \leq ... \leq r_k$. By Lemma 4 this is: $O\left(\frac{1}{r_1(r_2-r_1+1)(r_3-r_2+1)...(r_k-r_{k-1}+1)}\right)$. So we can rewrite, by linearity of expectation:

$$\mathbb{E}\left[c_{j}^{k}\right] = k! \sum_{r_{1}=1}^{t} \sum_{r_{2}=r_{1}}^{t} \dots \sum_{r_{k}=r_{k-1}}^{t} O\left(\frac{1}{r_{1}(r_{2}-r_{1}+1)(r_{3}-r_{2}+1)\dots(r_{k}-r_{k-1}+1)}\right).$$

The k! comes from the fact that in this sum we only have *ordered* k-tuples and so need to multiple by k! to account for the fact that the original sum is over *unordered* k-tuples. We can bound:

$$\sum_{r_k=r_{k-1}}^t \frac{1}{r_k-r_{k-1}+1} = 1 + \frac{1}{2} + \ldots + \frac{1}{t} = O(\log t)$$

so rearranging the sum and simplifying gives:

$$\begin{split} \mathbb{E}\left[c_{j}^{k}\right] &= k! \sum_{r_{1}=1}^{t} \frac{1}{r_{1}} \sum_{r_{2}=r_{1}+1}^{t} \frac{1}{r_{2}-r_{1}} \dots \sum_{r_{k}=r_{k-1}+1}^{t} \frac{1}{r_{k}-r_{k-1}} \\ &= k! \sum_{r_{1}=1}^{t} \frac{1}{r_{1}} \sum_{r_{2}=r_{1}}^{t} \frac{1}{r_{2}-r_{1}+1} \dots \sum_{r_{k-1}=r_{k-2}}^{t} \frac{1}{r_{k-2}-r_{k-1}+1} \cdot O(\log t). \end{split}$$

We repeat this simplification for each level of summation replacing $\sum_{r_i=r_{i-1}}^t \frac{1}{r_i-r_{i-1}+1}$ with $O(\log t)$. Iterating through the k levels gives $\mathbb{E}\left[c_j^k\right] = O(k!\log^k t)$ giving the lemma.

As with Lemma 4, the techniques used in Lemmas 6 and 7 can be applied to a single walk. We give two bounds that may be of independent interest.

Corollary 8 (Random Walk Visits Moment Bound). Consider an agent a_1 randomly walking on a twodimensional $\sqrt{A} \times \sqrt{A}$ torus that is initially located at a uniformly random location and takes $t \leq A$ steps. Let c_j be the number of times that a_1 visits node j. For all $j \in [1,...A]$ and all $k \geq 2$,

$$\mathbb{E}\left[\bar{c}_{j}^{k}\right] = O\left(\frac{t}{A} \cdot k! \log^{k-1} t\right).$$

Proof. This follows from noting that the expected number of visits to a given node is t/A and so Lemma 6 can be used in conjunction with Corollary 5 to show that $\mathbb{P}[c_j \geq 1] = \Theta\left(\frac{t}{A \log t}\right)$. We can then just follow the proof of Lemma 7, using Corollary 5 where needed to obtain the result.

Corollary 9 (Equalization Moment Bound). Consider an agent a_1 randomly walking on a two-dimensional $\sqrt{A} \times \sqrt{A}$ torus. If a_1 takes $t \leq A$ steps and c is the number of times it returns to its starting position (the number of equalizations), for all $k \geq 2$,

$$\mathbb{E}\left[\bar{c}^k\right] = O\left(k! \log^k t\right).$$

Proof. This follows directly from the proof of the moment bound given in Lemma 7 for the number of collisions between two agents that are assumed to collide at least once: $\mathbb{E}[c_j^k|c_j \geq 1] = O(k!\log^k t)$. By replacing the application of Lemma 4 with Corollary 5 we obtain the result.

4.3 Concentration of Encounter Rate-Based Density Estimate

Armed with the moment bound of Lemma 7 we can finally show that $\sum_{j=1}^{n} \bar{c}_{j}$ concentrates strongly about its expectation. Since $\sum_{j=1}^{n} \bar{c}_{j}$ is just a mean-centered and scaled version of $\tilde{d} = \frac{1}{t} \sum_{j=1}^{n} c_{j}$, this is enough to prove the accuracy of encounter rate-based density estimation. We start by showing that $\sum_{j=1}^{n} \bar{c}_{j}$ is a sub-exponential random variable.

Corollary 10 $(\sum_{j=1}^n \bar{c}_j \text{ is sub-exponential})$. Assuming $t \leq A$, $\sum_{j=1}^n \bar{c}_j \text{ is sub-exponential with parameters}$ $b = \Theta(\log t)$ and $\sigma^2 = \Theta(td \log t)$. Specifically, for any λ with $|\lambda| < \frac{1}{b} \mathbb{E}\left[e^{\lambda \sum_{j=1}^n \bar{c}_j}\right] \leq e^{\frac{\sigma^2 \lambda^2}{2}}$.

Proof. By Lemma 7, for $\sigma^2 = \Theta(\frac{t \log t}{A})$ and $b = \Theta(\log t)$, \bar{c}_j satisfies the Bernstein condition:

$$\mathbb{E}\left[\bar{c}_{j}^{k}\right] \leq \frac{1}{2}k!\sigma^{2}b^{k-2}.$$

This implies that \bar{c}_j is sub-exponential with parameters $\sigma^2 = \Theta(\frac{t \log t}{A})$ and $b = \Theta(\log t)$ (see [Wai15], Chapter 2). Since each \bar{c}_j is independent, this gives us, for all λ with $|\lambda| \leq \frac{1}{b}$:

$$\mathbb{E}\left[e^{\lambda \sum_{j=1}^{n} \bar{c}_{j}}\right] = \prod_{j=1}^{n} \mathbb{E}\left[e^{\lambda \bar{c}_{j}}\right] \leq e^{n \cdot \frac{\lambda^{2}}{2} \cdot \Theta\left(\frac{t \log t}{A}\right)} = e^{\Theta(\lambda^{2} t d \log t)}.$$

This completes the proof by the definition of a sub-exponential random variable.

We finally apply a standard sub-exponential tail bound [Wai15] to prove our main result.

Lemma 11 (Sub-exponential tail bound). Suppose that X is sub-exponential with parameters (σ^2, b) . Then, for any $\Delta \leq \frac{\sigma^2}{b}$, $\mathbb{P}[|X - \mathbb{E} X| \geq \Delta] \leq 2e^{-\frac{\Delta^2}{2\sigma^2}}$.

Proof of Theorem 2. Since \bar{c}_j is just a mean-centered version of c_j , $\sum_{j=1}^n \bar{c}_j$ deviates from its mean exactly the same amount as $\sum_{j=1}^n c_j$. Further, \tilde{d} is just equal to $\frac{1}{t} \sum_{j=1}^n c_j$, so the probability that it falls within an ϵ multiplicative factor of its mean is the same as the probability that $\sum_{j=1}^n c_j$ falls within an ϵ multiplicative factor of its mean. By Corollary 10 and Lemma 11:

$$\delta = \mathbb{P}\left[\left|\sum_{j=1}^{n} c_{j} - \mathbb{E}\left[\sum_{j=1}^{n} c_{j}\right]\right| \ge \epsilon \,\mathbb{E}\left[\sum_{j=1}^{n} c_{j}\right]\right] = \mathbb{P}\left[\left|\sum_{j=1}^{n} c_{j} - td\right| \ge \epsilon td\right] \le 2e^{\Theta\left(-\frac{\epsilon^{2} td}{\log t}\right)}.$$

$$\frac{\epsilon^2 t d}{\log t} = \Theta\left(\log(1/\delta)\right)$$
 and so $\epsilon = \Theta\left(\sqrt{\frac{\log(1/\delta)\log t}{t d}}\right)$, yielding the theorem.

5 Extensions to Other Topologies

We now extend our results to a broader set of graph topologies, demonstrating the generality of the local mixing analysis discussed above. We illustrate divergence between local and global mixing properties, which can have significant effects on random walk-based algorithms.

5.1 From Repeat Collision Bounds to Estimation Accuracy

Our proofs are largely independent of graph structure, using just a re-collision probability bound (Lemma 4) and the regularity (uniform node degrees) of the grid, so agents remain uniformly distributed on the nodes in each round. Hence, extending our results to other regular graphs primarily involves obtaining re-collision probability bounds for these graphs.

We consider agents on a graph with A nodes that execute analogously to Algorithm 2, stepping to a random neighbor in each round. Again, we focus on the multi-agent case but similar bounds (resembling Corollaries 8 and 9) hold for single random walk. We start with a general lemma, giving density estimation accuracy in terms of re-collision probability.

Lemma 12 (Re-collision Probability to Density Estimation Accuracy). Consider a regular graph with A nodes such that, if two randomly walking agents a_1 and a_2 collide in round r, for any $0 \le m \le t$, the probability that they collide again in round r+m is $\Theta\left(\beta(m)\right)$ for some non-increasing function $\beta(m)$. Let $B(t) \stackrel{\text{def}}{=} \sum_{m=0}^{t} \beta(m)$. After running for $t \le A$ steps, Algorithm 2 returns \tilde{d} such that, for any $\delta > 0$, with $probability \ge 1 - \delta$, $\tilde{d} \in [(1 - \epsilon)d, (1 + \epsilon)d]$ for $\epsilon = O\left(\sqrt{\frac{\log(1/\delta)B(t)}{td}}\right)$.

Note that in the special case of the grid, by Lemma 4, we can set $\beta(m) = 1/(m+1)$ and hence $B(t) = \Theta(\log t)$, yielding Theorem 2.

Proof. $\mathbb{E}\tilde{d} = d$ (Lemma 3) still holds as the regularity of the graph ensures that agents remain uniformly distributed on the nodes in every round (the stable distribution of any regular graph is the uniform distribution). Lemma 6 is also analogous except that (8) becomes:

$$\mathbb{E}[c_j|c_j \ge 1] = \Theta\left(\sum_{m=0}^{t-r} \beta(m)\right)$$

and using the fact that at least 1/2 the time that $c_j \ge 1$, there is a collision in the first t/2 rounds and that $\beta(m)$ is non-increasing, $\mathbb{E}[c_j|c_j \ge 1] = \Theta\left(\sum_{m=0}^{t/2} \beta(m)\right) = \Theta\left(B(t)\right)$. This gives:

$$\mathbb{P}[c_j \ge 1] = \Theta\left(\frac{t}{A \cdot B(t)}\right).$$

Following the moment calculations in Lemma 7, $\mathbb{E}[c_i^k|c_j \geq 1] = O\left(k!B(t)^k\right)$ and hence:

$$\mathbb{E}[\bar{c}_j^k] = O\left(\frac{t}{A} \cdot k! B(t)^{k-1}\right).$$

As in Corollary 10, this gives that \bar{c}_j is sub-exponential with parameters $b = \Theta(B(t))$ and $\sigma^2 = \Theta\left(\frac{tB(t)}{A}\right)$ so $\sum_{j=1}^n \bar{c}_k$ is sub-exponential with parameters $b = \Theta(B(t))$ and $\sigma^2 = \Theta\left(n \cdot \frac{tB(t)}{A}\right) = \Theta\left(tdB(t)\right)$. Plugging into Lemma 11 gives: $\frac{\epsilon^2 td}{B(t)} = \Theta(\log(1/\delta))$. Rearranging yields the result.

Applying the above bound requires a constant factor approximation to the re-collision probability – the probability is $\Theta(\beta(m))$. Sometimes however, it is much easier to give just an upper bound – so the probability is $O(\beta(m))$. In this case a slightly weaker bound holds:

Lemma 13 (Re-collision Probability Upper Bound to Density Estimation Accuracy). Consider a regular graph with A nodes such that, if two randomly walking agents a_1 and a_2 collide in round r, for any $0 \le m \le t$, the probability that they collide again in round r+m is $O(\beta(m))$ for some non-increasing function $\beta(m)$. Let $B(t) \stackrel{\text{def}}{=} \sum_{m=0}^{t} \beta(m)$. After running for $t \le A$ steps, Algorithm 2 returns \tilde{d} such that, for any $\delta > 0$, with probability $\geq 1 - \delta$ $\tilde{d} \in [(1 - \epsilon)d, (1 + \epsilon)d]$ for $\epsilon = O\left(\sqrt{\frac{\log(1/\delta) \cdot B(t)^2}{td}}\right)$.

Proof. The proof is identical to that of Lemma 12 except that, we can only show $\mathbb{P}[c_j \geq 1] = \Theta\left(\frac{t}{A}\right)$. Therefore, our moment bound becomes:

$$\mathbb{E}[\bar{c}_j^k] = O\left(\frac{t}{A} \cdot k! B(t)^k\right).$$

This gives that \bar{c}_j^k is sub-exponential with parameters $b = \Theta(B(t))$ and $\sigma^2 = \Theta(\frac{tB(t)^2}{A})$. Following Lemma 12 we therefore have $\frac{\epsilon^2 td}{B(t)^2} = \Theta(\log(1/\delta))$. Rearranging yields the proof.

5.2 k-Dimensional Tori

We first consider k-dimensional tori for general k. As k increases, local mixing becomes stronger, fewer re-collisions occur, and density estimation becomes easier. In fact, for $k \geq 3$, although the torus still mixes slowly, density estimation is as accurate as on the complete graph! We first give the 1-dimensional case:

5.2.1 Density Estimation on the Ring

Lemma 14 (Re-collision Probability Bound – Ring). If two randomly walking agents a_1 and a_2 are located on a 1-dimensional torus (a ring) with A nodes, and collide in round r, for any $m \ge 0$, the probability that a_1 and a_2 collide again in round r + m for $k \ge 1$ is $\Theta\left(\frac{1}{\sqrt{m+1}}\right) + O\left(\frac{1}{A}\right)$.

Proof. We have already shown this re-collision bound in the proof of Lemma 4. It is identical to $\mathbb{P}[\mathcal{C}_x|M_x=m]$ on an $A\times A$ grid, which is bounded by $\Theta\left(\frac{1}{\sqrt{m+1}}\right)+O\left(\frac{1}{A}\right)$.

For $m \leq A$, the $O\left(\frac{1}{A}\right)$ is absorbed into the $\Theta\left(\frac{1}{\sqrt{m+1}}\right)$ term. We can estimate the sum of repeat collision probabilities by using the fact that

$$\frac{1}{\sqrt{x}} \le \frac{2}{\sqrt{x} + \sqrt{x - 1}} = \frac{2(\sqrt{x} - \sqrt{x - 1})}{(\sqrt{x} + \sqrt{x - 1})(\sqrt{x} - \sqrt{x - 1})} = 2(\sqrt{x} - \sqrt{x - 1}).$$

So:

$$\sum_{t=0}^{t} \frac{1}{\sqrt{m+1}} \le 1 + 2\left(\sqrt{2} - \sqrt{1}\right) + \left(\sqrt{3} - \sqrt{2}\right) + \dots + \left(\sqrt{t+1} - \sqrt{t}\right) = 2\sqrt{t+1} - 1.$$

Similarly,

$$\frac{1}{\sqrt{x-1}} \ge \frac{2}{\sqrt{x} + \sqrt{x-1}} = 2(\sqrt{x} - \sqrt{x-1})$$

and so:

$$\sum_{m=0}^{t} \frac{1}{\sqrt{m+1}} \ge 2\left(\sqrt{2} - \sqrt{1}\right) + \left(\sqrt{3} - \sqrt{2}\right) + \dots + \left(\sqrt{t+2} - \sqrt{t+1}\right) = 2\sqrt{t+2} - 2.$$

So, overall $\sum_{m=0}^{t} \frac{1}{\sqrt{m+1}} = \Theta(\sqrt{t})$. Plugging into Lemma 12, on a ring, random walk-based density estimation gives: $\epsilon = O\left(\sqrt{\frac{\log(1/\delta)\sqrt{t}}{td}}\right) = O\left(\sqrt{\frac{\log(1/\delta)}{\sqrt{t}d}}\right)$. Rearranging, $t = \Theta\left(\left(\frac{\log(1/\delta)}{\epsilon^2 d}\right)^2\right)$ rounds are necessary to obtain a $1 \pm \epsilon$ approximation with probability $\geq 1 - \delta$. Local mixing on the ring is much worse than on the torus—we expect to see $\Theta(\sqrt{t})$ rather than $\Theta(\log t)$ repeat collisions with every agent interacted with. Hence, density estimation is much more difficult, requiring t to be quadratic rather than linear in 1/d.

5.2.2 Density Estimation on Higher Dimensional Tori

We now cover $k \geq 3$. While global mixing time is on the order of $A^{2/k}$ and so is slow if k << A, local mixing is so strong that our accuracy bounds actually match those of independent sampling! Throughout this section, we assume that k is a small constant and hide it in our asymptotic notation.

Lemma 15 (Re-collision Probability Bound – High-Dimensional Torus). If two randomly walking agents a_1 and a_2 are located on a k-dimensional torus with A nodes, and collide in round r, for any constant $k \geq 3$, $m \geq 0$, the probability that a_1 and a_2 collide in round r + m is $\Theta\left(\frac{1}{(m+1)^{k/2}}\right) + O\left(\frac{1}{A}\right)$.

Proof. We closely follow the proof of Lemma 4. In total, a_1 and a_2 take 2m steps: M_i in each dimension for $i \in [1, ..., k]$. Let C_i be the event that the agents have the same position in the i^{th} dimension in round r + m. By the analysis of Lemma 4,

$$\mathbb{P}[\mathcal{C}_i|M_i=m_i] = \Theta\left(\frac{1}{\sqrt{m_i+1}}\right) + O\left(\frac{1}{A^{1/k}}\right).$$

So,

$$\mathbb{P}[\mathcal{C}|M_1 = m_1, ..., M_k = m_k] = \left[\Theta\left(\frac{1}{\sqrt{m_1 + 1}}\right) + O\left(\frac{1}{A^{1/k}}\right)\right] \cdot \cdot \left[\Theta\left(\frac{1}{\sqrt{m_k + 1}}\right) + O\left(\frac{1}{A^{1/k}}\right)\right]. \quad (9)$$

In expectation, $M_i = 2m/k$. So by a Chernoff bound,

$$\mathbb{P}[M_i \le m/k] \le 2e^{-(1/2)^2 \cdot 2m/3k} = O\left(\frac{1}{(m+1)^{k/2}}\right)$$

again assuming k is a small constant. Union bounding over all k dimensions, we have $M_i \ge m/k$ for all i except with probability $O\left(\frac{1}{(m+1)^{k/2}}\right)$ and hence by (9):

$$\mathbb{P}[\mathcal{C}] = O\left(\frac{1}{(m+1)^{k/2}}\right) + \left[\Theta\left(\frac{1}{\sqrt{m/k+1}}\right) + O\left(\frac{1}{A^{1/k}}\right)\right]^k = \Theta\left(\frac{1}{(m+1)^{k/2}}\right) + O\left(\frac{1}{A}\right)$$

giving the lemma (again, asymptotic notation hides multiplicative factors in k since it is constant).

We can plug the above bound into Lemma 13. For $t \leq A$ and $k \geq 3$, $\sum_{m=0}^{t} \left(\frac{1}{(m+1)^{k/2}} + \frac{1}{A}\right) < 1 + \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k/2}} = O(1)$. So we can set B(t) = 1 and have $\epsilon = O\left(\frac{\sqrt{\log(1/\delta)}}{td}\right)$. Rearranging, we require $t = \Theta\left(\frac{\log(1/\delta)}{\epsilon^2 d}\right)$. This matches the performance of independent sampling up to constants.

5.3 Regular Expanders

When a graph *does* mix well globally, it also mixes well locally. The number of repeat collisions is low and accurate density estimation is possible. The most obvious example is the complete graph, on which random-walk based density estimation is equivalent to density estimation via independent sampling. We generalize this intuition to any regular expander.

Lemma 16 (Re-collision Probability Bound – Regular Expander). Let G be a k-regular expander with A nodes and adjacency matrix \mathbf{M} . Let $\mathbf{W} = \frac{1}{k} \cdot \mathbf{M}$ be its random walk matrix, with eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_A$. Let $\lambda = \max\{|\lambda_2|, |\lambda_A|\}$. If two randomly walking agents a_1 and a_2 collide in round r, for any $m \geq 0$, the probability that they collide again in round r + m is at most $\lambda^m + 2/A$.

Proof. Suppose that a_1 and a_2 collide at node i in round r. The probability they re-collide at round r + m is $||\mathbf{W}^m \mathbf{e}_i||_2^2$, since for each j, $\mathbf{W}_{i,j}^m = (\mathbf{W}^m \mathbf{e}_i)_j$ is the probability an agent is at node j after round r + m given that it is at node i after round r. We bound this norm using the following lemma on how rapidly an expander random walk converges to its stable distribution:

Lemma 17 ([Lov93]). Let G be a k-regular expander with A nodes, adjacency matrix \mathbf{M} , and random walk matrix $\mathbf{W} = \frac{1}{k} \cdot \mathbf{M}$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_A$ be the eigenvalues of \mathbf{W} and $\lambda = \max\{|\lambda_2|, |\lambda_A|\}$. For each $1 \leq i \leq n$.

$$\left| (\mathbf{W}^m \cdot e_i)_j - \frac{1}{A} \right| \le \lambda^m.$$

Now we can bound $||\mathbf{W}^m \mathbf{e}_i||_2^2$ by:

$$||\mathbf{W}^m \mathbf{e}_i||_2^2 = \sum_{i=1}^A (\mathbf{W}^m \mathbf{e}_i)_j^2 = \sum_{i=1}^A \left(\frac{1}{A} + \chi_j\right)^2$$

where $\chi_j \stackrel{\text{def}}{=} (\mathbf{W}^m \cdot \mathbf{e}_i)_j - \frac{1}{A}$ so that $\chi_j \in [-1/A, \lambda^m]$ due to Lemma 17. We have $\sum_j \chi_j = \sum_j (\mathbf{W}^m \mathbf{e}_i)_j - A \cdot (1/A) = 0$. Therefore,

$$||\mathbf{W}^{m}\mathbf{e}_{i}||_{2}^{2} = \sum_{j=1}^{A} \left(\frac{1}{A} + \chi_{j}\right)^{2}$$

$$\leq \sum_{j=1}^{A} \left(\left(\frac{1}{A}\right)^{2} + \frac{2\chi_{j}}{A} + \chi_{j}^{2}\right) = \frac{1}{A} + \sum_{j=1}^{A} \chi_{j}^{2}.$$

 $\begin{array}{l} \sum_{j}\chi_{j}^{2} \text{ is maximized when the number of possible } j \text{ with } \chi_{j} = \lambda^{m} \text{ is maximized. Let } S \subset [1,A] \text{ be the indices} \\ j \text{ with } \chi_{j} = \lambda^{m}. \text{ Since } \sum_{j}\chi_{j} = 0, \text{ we have } \sum_{j \in S}\lambda^{m} + \sum_{j \notin S}\chi_{j} = 0. \text{ Therefore, } |S| \cdot \lambda^{m} \leq -\sum_{j \notin A}\chi_{j} \leq \\ 1 - |S|/A \text{ and so } |S| \leq \frac{1}{\lambda^{m} + 1/A}. \text{ Therefore,} \end{array}$

$$\sum_{j=1}^{A} \chi_j^2 \le \sum_{j \in S} \lambda^{2m} + \sum_{j \notin S} \chi_j^2$$

$$\le \frac{\lambda^{2m}}{\lambda^m + 1/A} + \frac{A - |S|}{A^2} \le \lambda^m + 1/A.$$

Thus, $||\mathbf{W}^m \mathbf{e}_i||_2^2 \leq \lambda^m + 2/A$, giving the lemma.

We now apply Lemma 13, with $B(t) = \sum_{m=0}^{t} \beta(m) \leq \frac{1}{1-\lambda} + 2t/A$. Assuming t = O(A),

$$\epsilon = O\left(\sqrt{\frac{\log(1/\delta)(1/(1-\lambda) + 2t/A)^2}{td}}\right) = O\left(\sqrt{\frac{\log(1/\delta)}{td(1-\lambda)^2}}\right).$$

Rearranging, $t = \Theta\left(\frac{\log(1/\delta)}{\epsilon^2 d(1-\lambda)^2}\right)$, matching independent sampling up to a factor of $O(1/(1-\lambda)^2)$.

5.4 k-Dimensional Hypercube

Finally, we give bounds for a k-dimensional hypercube. Such a graph has $A=2^k$ vertices mapped to the elements of $\{\pm 1\}^k$, with an edge between any two vertices that differ by hamming distance 1. The hypercube is relatively fast mixing. Its adjacency matrix eigenvalues are [-k, -k+2, -k+4, ..., k-4, k-2, k]. Since it is bipartite, we can effectively ignore the negative eigenvalues and apply Lemma 16 with $\lambda = \Theta(1-2/k) = \Theta(1-1/\log A)$. This yields $t = \Theta\left(\frac{\log(1/\delta)\log^2(A)}{\epsilon^2 d}\right)$. However, it is possible to remove the dependence on A via a more refined analysis – while the global mixing time of the graph increases as A grows, local mixing becomes stronger!

Lemma 18 (Re-collision Probability Bound – k-Dimensional Hypercube). If two randomly walking agents a_1 and a_2 are located on a k-dimensional hypercube with $A=2^k$ vertices and collide in round r, for any $m \geq 0$, the probability that a_1 and a_2 collide in round r+m is at most $(7/10)^m + O\left(\frac{1}{\sqrt{A}}\right)$.

Proof. A node of the hypercube can be represented as a k-bit string and each random walk step seen as choosing one of the bits uniformly at random and flipping it. If a_1 and a_2 collide, for each of the bit, the total number of times a_1 and a_2 chose that bit must be even. The total number of possible ways for re-collision to occur at round r + m is exactly the number of ways 2m flips can be placed into k buckets, where each bucket has even number of elements. This quantity is:

$$\sum_{\substack{a_1+\ldots+a_k=2m\\(a_i\mod 2)\equiv 0}}\frac{(2m)!}{a_1!\cdot\ldots\cdot a_k!}.$$

This value is equal to the coefficient of x^{2m} in the exponential generating function

$$(2m)! \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)^k = (2m)! \left(\frac{e^x + e^{-x}}{2}\right)^k = \frac{(2m)!}{2^k} \sum_{i=0}^k {k \choose i} e^{x(2i-k)}.$$

By differentiating 2m times, we find that the coefficient of x^{2m} is:

$$\frac{1}{2^k} \sum_{i=0}^k \binom{k}{i} (2i-k)^{2m} = \sum_{i=0}^k \left(\binom{k}{i} / 2^k \right) \cdot (2i-k)^{2m}.$$

This summation is exactly $\mathbb{E}[X^{2m}]$, where X is a sum of k i.i.d. random variables each equal to 1 with probability 1/2 and -1 otherwise. For any $c \in (0,1]$, we can split the expectation:

$$\mathbb{E}[X^{2m}] = \mathbb{E}[X^{2m}||X| \ge ck] \cdot \mathbb{P}[|X| \ge ck] + \mathbb{E}[X^{2m}||X| \le ck] \cdot \mathbb{P}[|X| \le ck]$$
$$\le k^{2m} \cdot \mathbb{P}[|X| \ge ck] + (ck)^{2m}.$$

To bound the return probability bound, we this count by the total number of possible paths taken by a_1 and a_2 in m steps, k^{2m} , giving an upper bound of:

$$\mathbb{P}[|X| \ge ck] + c^{2m}.$$

By a Hoeffding bound, $\mathbb{P}[|X| \ge ck] \le 2e^{-c^2k/2}$. If we set $c = \sqrt{\ln A/k} = \sqrt{\ln 2}$ then $\mathbb{P}[|X| \ge ck] \le 1/\sqrt{A}$. So our final probability bound is:

$$\mathbb{P}[|X| \ge ck] + c^{2m} \le \frac{1}{\sqrt{A}} + (\sqrt{\ln 2})^{2m} < \frac{1}{\sqrt{A}} + (7/10)^m$$

yielding the lemma. Note that, by adjusting c, it is possible trade off the terms in the above bound, giving stronger inverse dependence on A at the expense of slower exponential decay in m.

Plugging into Lemma 13, we have $B(t) = \sum_{m=0}^{t} \beta(m) \leq \frac{10}{3} + t/\sqrt{A}$. If we assume $t = O(\sqrt{A})$, this gives $\epsilon = O\left(\sqrt{\frac{\log(1/\delta)}{td}}\right)$ and so $t = \Theta\left(\frac{\log(1/\delta)}{\epsilon^2 d}\right)$, matching independent sampling.

6 Applications

We conclude by discussing algorithmic applications of our ant-inspired density estimation algorithm (Algorithm 2), variations on this algorithm, and the analysis techniques we develop.

6.1 Social Network Size Estimation

Random walk-based density estimation is closely related to work on estimating the size of social networks and other massive graphs using random walks [KLS11, KBM12, LL12, LW14]. In these applications, one does not have access to the full graph (so cannot exactly count the nodes), but can simulate random walks by following links between nodes [MMG⁺07, GKBM09]. One approach is to run a single random walk and count repeat node visits [LL12, KBM12]. Alternatively, [KLS11] proposes running multiple random walks and counting their collisions. This can be significantly more efficient since the dominant cost is typically in link queries to the network. With multiple random walks, this cost can be trivially distributed to multiple servers simulating walks independently.

Walks are first run for a 'burn-in period' so that their locations are distributed approximately by the stable distribution of the network. The walks are then halted, and the number of collisions in this final round are counted. The collision count gives an estimate of the walks' density. Since the number of walks is known, this yields an estimate for network size.

We show that ant-inspired algorithms can give runtime improvements over this method. After burn-in, instead of halting the walks, it is possible to run the random walks for *multiple rounds*, recording encounter rates as in Algorithm 2. This allows the use of fewer random walks, decreasing total burn-in cost, and leading to faster runtimes when mixing time is relatively slow, as is common in social network graphs [MYK10].

6.1.1 Random Walk-Based Algorithm for Network Size Estimation

Consider an undirected, connected, non-bipartite graph G = (V, E). Let S be the set of vertices of G that are 'known'. Initially, $S = \{v\}$ where v is a seed vertex. We can access G by looking up the neighborhood $\Gamma(v_i)$ of any vertex $v_i \in S$ and adding $\Gamma(v_i)$ to S.

To compute the number of nodes |V| in the network, we could scan S, looking up the neighbors of each vertex and adding them to the set. After querying all nodes in S we will have S = V and will know the network size. However, the number of queries required equals |V|. The goal is to estimate network size using a significantly more efficient random-walk based approach.

A number of challenges are introduced by this application. While we can simulate many random walks on G, we can no longer assume these random walks start at randomly chosen nodes, as we do not have the ability to uniformly sample nodes from the network. Instead, we must allow the random walks to run for a burn-in phase of length proportional to the mixing time of G. After this phase, the walks are distributed approximately according to the stable distribution of G.

In general, G is not regular. In the stable distribution, a random walk is located at a vertex with probability proportional to its degree. Hence, collisions tend to occur more at higher degree vertices. To correct for this bias, we count a collision at vertex v_i with weight $1/\deg(v_i)$. As we will see, with this modification, we must adjust our final estimate by the average degree of the graph, which we must estimate.

Our results depend on a natural generalization of re-collision probability. For any $i, j \in |V|$, let $p(v_i, v_j, m)$ be the probability of an m step random walk starting at v_i ending at v_j . Define:

$$\beta(m) \stackrel{\text{def}}{=} \frac{\max_{i,j} p(v_i, v_j, m)}{\deg(v_j)}$$

Intuitively, this is the maximum m step collision probability, weighted by degree since higher degree vertices are more likely to be visited in the stable distribution. Let $B(t) = \sum_{m=1}^{t} \beta(m)$. Note that this weighted B(t) is upper bounded by the unweighted B(t) used in Lemmas 12 and 13.

For simplicity, we initially ignore burn-in and assume that our walks start distributed exactly by the stable distribution of G. A walk starts at vertex v_i with probability $p_i \stackrel{\text{def}}{=} \frac{\deg(v_i)}{\sum_i \deg(v_i)} = \frac{\deg(v_i)}{2|E|}$ and initial locations are independent. We also assume knowledge of the average degree $\overline{\deg} = 2|E|/|V|$. We later rigorously analyze burn-in and show to estimate $\overline{\deg}$, completing our analysis.

Algorithm 3 Random Walk-Based Network Size Estimation

input: step count t, average degree $\overline{\deg}$, n random starting locations $[w_1, ..., w_n]$ distributed according to the network's stable distribution

```
\begin{split} &[c_1,...,c_n] := [0,0,...,0] \\ &\textbf{for } r = 1,...,t \ \textbf{do} \\ &\forall j, \text{ set } w_j := randomElement(\Gamma(w_j)) \\ &\forall j, \text{ set } c_j := c_j + \frac{count(w_j)}{\deg(w_j)} \\ &\textbf{end } \underbrace{for}_{C} := \frac{\deg \sum c_j}{\cdot n(n-1)t} \\ &\textbf{return } \tilde{A} = 1/C \end{split}
```

Theorem 19. If
$$n^2t = \Theta\left(\frac{B(t)\overline{\deg}}{\epsilon^2\delta} \cdot |V|\right)$$
, then with probability at least $1 - \delta$, Algorithm 3 returns $\tilde{A} \in [(1 - \epsilon)|V|, (1 + \epsilon)|V|]$.

Throughout this section, we work directly with the weighted total collision count C, showing that it is close to its expectation with high probability and hence giving the accuracy bound for \tilde{A} . As in the density estimation case, we start by showing that C is correct in expectation.

Lemma 20. $\mathbb{E} C = 1/|V|$.

Proof. Let $c_j(r)$ be the number of collisions, weighted by inverse vertex degree, walk j expects to be involved in at round r. In each round all walks are at vertex v_i with probability $p_i = \frac{\deg(v_i)}{2|E|}$, so:

$$\mathbb{E} c_j(r) = \sum_{i=1}^{|V|} \left[\frac{\deg(v_i)}{2|E|} \cdot \frac{(n-1)\deg(v_i)}{2|E|} \cdot \frac{1}{\deg(v_i)} \right] = \frac{n-1}{4|E|^2} \sum_{i=1}^{|V|} \deg(v_i) = \frac{n-1}{2|E|}.$$

By linearity of expectation:
$$\mathbb{E} c_j = \frac{t(n-1)}{2|E|}$$
, $\mathbb{E} \sum c_j = \frac{tn(n-1)}{2|E|}$ and hence, $\mathbb{E} C = \frac{\overline{\deg}}{2|E|} = 1/|V|$.

We now need to show concentration of C about its expectation. Let $c_{i,j}$ be the weighted collision count between walks w_i and w_j where $i \neq j$. It is possible to closely follow the moment bound proof of Lemma 7 and show that $c_{i,j}$ is sub-exponential. However, unlike in the case of regular graphs, we will not be able to claim that the different $c_{i,j}$'s are independent. Hence, we will not be able to use the same sub-exponential tail bounds employed in Section 4.3.

Instead, we bound the second moment (the variance) of each $c_{i,j}$ and obtain our concentration results via Chebyshev's inequality. This leads to a linear rather than logarithmic dependence on the failure probability $1/\delta$. However, we note that we can simply perform $\log(1/\delta)$ estimates each with failure probability 1/3 and return the median, which will be correct with probability $1-\delta$.

Lemma 21. (Degree Weighted Collision Variance Bound) For all $i, j \in [1, ..., n]$ with $i \neq j$, let $\bar{c}_{i,j} \stackrel{\text{def}}{=} c_{i,j} - \mathbb{E}[\bar{c}_{i,j}^2] = O\left(\frac{tB(t)}{|E|}\right)$.

Proof. We can split $\mathbb{E} \bar{c}_{i,j}^2$ over rounds as:

$$\mathbb{E}\left[\left(\sum_{r=1}^{t} c_{i,j}(r)\right)^{2}\right] = \sum_{r=1}^{t} \mathbb{E}\left[\bar{c}_{i,j}(r)^{2}\right] + 2\sum_{r=1}^{t-1} \sum_{r'=r+1}^{t} \mathbb{E}\left[\bar{c}_{i,j}(r)\bar{c}_{i,j}(r')\right] \\
= t\sum_{i=1}^{|V|} \left(\frac{\deg(v_{i})^{2}}{(2|E|)^{2}} \cdot \frac{1}{\deg(v_{i})^{2}}\right) + 2\sum_{r=1}^{t-1} \sum_{r'=r+1}^{t} \left(\sum_{i=1}^{|V|} \left(\frac{\deg(v_{i})^{2}}{(2|E|)^{2}} \cdot \frac{1}{\deg(v_{i})} \cdot \sum_{j=1}^{|V|} \frac{p(v_{i}, v_{j}, r - r')^{2}}{\deg(v_{j})}\right)\right) \\
\leq \frac{t|V|}{4|E|^{2}} + 2t\sum_{m=1}^{t-1} \left(\sum_{i=1}^{|V|} \left(\frac{\deg(v_{i})}{(2|E|)^{2}} \cdot \beta(m) \sum_{j=1}^{|V|} p(v_{i}, v_{j}, m)\right)\right) \\
\leq \frac{t}{4|E|} + 2t\sum_{m=1}^{t-1} \frac{\beta(m)}{2|E|} = O\left(\frac{tB(t)}{|E|}\right).$$

Lemma 22 (Total Collision Variance Bound). Let $\overline{C} = \frac{\overline{\deg \sum_j \bar{c}_j}}{n(n-1)t}$. $\mathbb{E}\left[\bar{C}^2\right] = O\left(\frac{1}{n^2t} \cdot \frac{B(t)|E|}{|V|^2}\right)$.

Proof. $\sum_{j=1}^n \bar{c}_j = \sum_{i,j \in [1,\dots,n], i \neq j} \bar{c}_{i,j}$. We closely follow the variance calculation in [KLS11]:

$$\mathbb{E}\left[\left(\sum_{i,j\in[1,\dots,n],i\neq j}\bar{c}_{i,j}\right)^{2}\right] = \sum_{i,j\in[1,\dots,n],i\neq j}\left[\sum_{i',j'\in[1,\dots,n],i\neq j}\bar{c}_{i,j}\cdot\bar{c}_{i',j'}\right]$$

$$= 2\binom{n}{2}\mathbb{E}\left[\bar{c}_{i,j}^{2}\right] + 4!\binom{n}{4}\left(\mathbb{E}\,\bar{c}_{i,j}\right)^{2} + 2\cdot 3!\binom{n}{3}\mathbb{E}\,\bar{c}_{i,j}\bar{c}_{i,k}.$$

The first term corresponds to the cases when i=i' and j=j'. The second corresponds to $i\neq i'$ and $j\neq j'$, in which case $\bar{c}_{i,j}$ and $\bar{c}_{i',j'}$ are independent and identically distributed. The $4!\binom{n}{4}$ multiplier is the number of ways to choose an ordered set of four distinct indices. The last term corresponds to all cases when either i=i' or j=j'. There are $3!\binom{n}{3}$ ways to choose an ordered set of three distinct indices, multiplied by two to

account for the repeated index being in either the first or second position. Using $\mathbb{E}\bar{c}_{i,j} = 0$ and the bound on $\mathbb{E}[\bar{c}_{i,j}^2]$ from Lemma 21:

$$\mathbb{E}\left[\left(\sum_{i,j\in[1,\dots,n],i\neq j}\bar{c}_{i,j}\right)^{2}\right] = O\left(\frac{n^{2}tB(t)}{|E|}\right) + 0 + 2\cdot 3!\binom{n}{3}\mathbb{E}\,\bar{c}_{i,j}\bar{c}_{i,k}.\tag{10}$$

When $j \neq k$, $\bar{c}_{i,j}$ and $\bar{c}_{i,k}$ are independent and identically distributed conditioned on the path that walk w_i traverses. Let Ψ_i be the t step path chosen by w_i .

$$\mathbb{E}\left[\bar{c}_{i,j}\bar{c}_{i,k}\right] = \sum_{\psi_i} \mathbb{P}\left[\Psi_i = \psi_i\right] \cdot \mathbb{E}\left[\bar{c}_{i,j}|\Psi_i = \psi_i\right] \cdot \mathbb{E}\left[\bar{c}_{i,k}|\Psi_i = \psi_i\right]
= \sum_{\psi_i} \mathbb{P}\left[\Psi_i = \psi_i\right] \cdot \mathbb{E}\left[\bar{c}_{i,j}|\Psi_i = \psi_i\right]^2
= \sum_{\psi_i} \mathbb{P}\left[\Psi_i = \psi_i\right] \cdot \left(\mathbb{E}\left[c_{i,j}|\Psi_i = \psi_i\right] - \mathbb{E}\left[c_{i,j}\right]\right)^2.$$
(11)

 $\mathbb{E}\left[c_{i,j}|\Psi_i=\psi_i\right] = \sum_{r=1}^t \frac{\deg(\psi_i(r))}{2|E|} \cdot \frac{1}{\deg(\psi_i(r))} = \frac{t}{2|E|} = \mathbb{E}\left[c_{i,j}\right]. \text{ That is, the expected number of collisions is identical for every path of } w_i. \text{ Plugging into (11), } \mathbb{E}\left[\bar{c}_{i,j}\bar{c}_{i,k}\right] = 0.$

So finally, plugging back into equation (10), $\mathbb{E}\left[\left(\sum_{i,j\in[1,...,n],i\neq j}\bar{c}_{i,j}\right)^2\right]=O\left(\frac{n^2tB(t)}{|E|}\right)$ and thus:

$$\mathbb{E}\left[\overline{C}^2\right] = O\left(\frac{n^2 t B(t)}{|E|} \cdot \left(\frac{\overline{\deg}}{n(n-1)t}\right)^2\right) = O\left(\frac{1}{n^2 t} \cdot \frac{B(t)|E|}{|V|^2}\right).$$

With this variance bound in place, we can now finally prove Theorem 19.

Proof of Theorem 19. Note that $\bar{C} = C - \mathbb{E}C$. By Chebyshev's inequality Lemma 22 gives:

$$\mathbb{P}[|C - \mathbb{E}C| \ge \epsilon \,\mathbb{E}C] \le \frac{1}{\epsilon^2 n^2 t} \cdot B(t)|E|.$$

Rearranging gives us that, in order to have $\bar{C} \in \left[\frac{(1-\epsilon)}{|V|}, \frac{1+\epsilon}{|V|}\right]$ with probability δ , we must have:

$$n^2 t = \Theta\left(\frac{1}{\epsilon^2 \delta} B(t) |E|\right).$$

As long as we have $\epsilon < 1/4$ then this immediately gives $\tilde{A} \in [(1-2\epsilon)|V|, (1+2\epsilon)|V|]$, giving the lemma after adjusting constants on ϵ .

6.1.2 Estimating The Average Degree

We now show how to estimate the value of $\overline{\deg}$ used in Algorithm 3. Specifically, we need a $(1 \pm \epsilon)$ approximation to $\frac{1}{\deg}$. If we then substitute this into the formula $\tilde{A} = \frac{\sum_j c_j}{\deg n(n-1)t}$, we still have a $(1 \pm O(\epsilon))$ approximation to the true network size. We use the algorithm and analysis of [KLS11], which gives a simple approximation via inverse degree sampling.

Algorithm 4 Average Degree Estimation

input: n random starting locations $[w_1,...,w_n]$ distributed according to the network's stable distribution.

$$\forall j, \text{ set } d_j := \frac{1}{\deg(w_j)}$$

$$\mathbf{return} \ D := \frac{\sum d_j}{n}$$

Theorem 23 (Average Degree Estimation). If $n = \Theta\left(\frac{1}{\epsilon^2 \delta} \cdot \frac{\overline{\deg}}{\operatorname{deg}_{\min}}\right)$, Algorithm 4 returns D such that, with probability at least $1 - \delta$, $D \in \left[\frac{1 - \epsilon}{\operatorname{deg}}, \frac{1 + \epsilon}{\operatorname{deg}}\right]$.

Proof.

$$\mathbb{E} D = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} d_j = \frac{1}{n} \cdot n \cdot \sum_{i=1}^{|V|} \left(\frac{\deg(v_i)}{2|E|} \cdot \frac{1}{\deg(v_i)} \right) = \frac{|V|}{2|E|} = \frac{1}{\overline{\deg}}.$$

Since each d_j is independent, letting $\bar{D} = D - \mathbb{E} D$,

$$\mathbb{E}\left[\bar{D}^{2}\right] = \frac{1}{n} \mathbb{E}[\bar{d}_{j}^{2}] \leq \frac{1}{n} \mathbb{E}[d_{j}^{2}] = \frac{1}{n} \sum_{i=1}^{|V|} \frac{\deg(v_{i})}{(2|E|)} \frac{1}{\deg(v_{i})^{2}} \leq \frac{1}{n} \cdot \frac{|V|}{2|E| \deg_{\min}} = \frac{1}{n \deg_{\min}} \cdot \frac{1}{\overline{\deg}}.$$

Applying Chebyshev's inequality: $\mathbb{P}\left[\left|D - \frac{1}{\deg}\right| \leq \frac{\epsilon}{\deg}\right] \leq \frac{\overline{\deg}}{\epsilon^2 n \deg_{min}}$ and rearranging, to succeed with probability at least $1 - \delta$ it suffices to set:

$$n = \Theta\left(\frac{1}{\epsilon^2 \delta} \cdot \frac{\overline{\deg}}{\deg_{\min}}\right).$$

6.1.3 Handling Burn-In Error

Finally, we remove our assumption that walks start distributed exactly according to the network's stable distribution, rigorously bounding the length of burn-in required before running Algorithm 3.

Let $\mathcal{D}^* \in \mathbb{R}^{|V|^n}$ be a vector representing the true stable distribution of n random walks on G and $\mathcal{D}_t \in \mathbb{R}^{|V|^n}$ be a vector representing the distribution of the walks after running for t burn-in steps. Specifically, each walk $w_1, ..., w_n$ is initialized at a single seed vertex v. For t rounds we then update the location of each walk independently by moving to a randomly chosen neighbor. Both vectors are probability distributions: they have all entries in [0,1] and $\|\mathcal{D}^*\|_1 = \|\mathcal{D}\|_1 = 1$.

Let $\Delta = \mathcal{D}^* - \mathcal{D}_t$ and assume that $\|\Delta\|_1 \leq \delta$. We can consider two equivalent algorithms: draw an initial set of locations $W = w_1, ..., w_n$ from \mathcal{D}^* , run Algorithm 3, and then artificially fail with probability $\max\{0, \Delta(W)\}$. Alternatively, draw $W = w_1, ..., w_n$ from \mathcal{D}_t , run Algorithm 3, and then artificially fail with probability $\max\{0, -\Delta(W)\}$. These algorithms are clearly equivalent. The first obtains a good estimator with probability $1 - 2\delta$ - probability δ that Algorithm 3 fails when initialized via the stable distribution \mathcal{D}^* by Theorem 19 plus an artificial failure probability of $\leq \|\Delta\|_1 \leq \delta$. The second then clearly also fails with probability 2δ . This can only be higher than if the we did not perform the artificial failure after running Algorithm 3. Therefore, running Algorithm 3 with a set of random walks initially distributed according to \mathcal{D}_t yields success probability $\geq 1 - 2\delta$.

How long must the burn-in period be to ensure $\|\mathcal{D}^* - \mathcal{D}_t\|_1 \leq \delta$? Let **W** be the random walk matrix of G. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_A$ be the eigenvalues of **W** and $\lambda = \max\{|\lambda_2|, |\lambda_{|V|}|\}$. Let $\mathcal{C}_t \in \mathbb{R}^{|V|}$ denote the location distribution for a single random walk after burn-in and $\mathcal{C}^* \in \mathbb{R}^{|V|}$ denote the stable distribution of a single random walk. If we have, for all i, $|\mathcal{C}_t(v_i) - \mathcal{C}^*(v_i)| \leq \delta/n \cdot \mathcal{C}^*(v_i)$ then we have for any W:

$$|\mathcal{D}_{t}(W) - \mathcal{D}^{*}(W)| = \left| \prod_{i=1}^{n} \mathcal{C}_{t}(w_{i}) - \prod_{i=1}^{n} \mathcal{C}^{*}(w_{i}) \right|$$

$$\leq \prod_{i=1}^{n} (\mathcal{C}^{*}(w_{i}) + \delta/n \cdot \mathcal{C}^{*}(w_{i})) - \prod_{i=1}^{n} \mathcal{C}^{*}(w_{i})$$

$$< \mathcal{D}^{*}(W) \sum_{i=1}^{n} \binom{n}{i} (\delta/n)^{i} \leq \mathcal{D}^{*}(W) \sum_{i=1}^{n} \delta^{i} \leq 2\delta \cdot \mathcal{D}^{*}(W)$$

as long as $\delta < 1/2$. This multiplicative bound gives $\|\mathcal{D}^* - \mathcal{D}_t\|_1 \leq 2\delta$. By standard mixing time bounds ([Lov93], Theorem 5.1), $|\mathcal{C}_t(v_i) - \mathcal{C}^*(v_i)| \leq \frac{\delta}{n|E|} \cdot \mathcal{C}^*(v_i)$ for all i after $O\left(\frac{\log(n|E|/\delta)}{1-\lambda}\right) = O\left(\frac{\log(|E|/\delta)}{1-\lambda}\right)$ burn-in steps (since n < |E| or else we could have scanned the full graph.)

6.1.4 Overall Runtime and Comparision to Previous Work

Let $M = O\left(\frac{\log(|E|/\delta)}{1-\lambda}\right)$ denote the burn-in time required before running Algorithm 3. In order to obtain a $(1 \pm \epsilon)$ estimate of network size with probability $1 - \delta$ we must run n random walks for M + t steps, making n(M + t) link queries, where by Theorems 19 and 23:

$$n = \Theta\left(\max\left\{\frac{\overline{\deg}}{\deg_{\min}\epsilon^2\delta}, \sqrt{\frac{|V| \cdot B(t)\overline{\deg}}{t \cdot \epsilon^2\delta}}\right\}\right). \tag{12}$$

Typically, the second term dominates since $\overline{\deg} \ll |V|$. Hence, by increasing t, we are able to use fewer random walks, significantly decreasing the number of link queries required if M is large.

In the special case with t = 1 we obtain a somewhat simpler bound. Instead of using the more general analysis of Lemma 21 we can directly calculate:

$$\mathbb{E}\left[\bar{c}_{i,j}^{2}\right] = \sum_{i=1}^{|V|} \frac{\deg(v_{i})^{2}}{(2|E|)^{2}} \frac{1}{\deg(v_{i})^{2}} = \frac{1}{\overline{\deg} \cdot 2|E|}.$$

Plugging this into Lemma 22 gives $\mathbb{E}[\overline{C}^2] = \frac{n^2}{\overline{\deg} \cdot 2|E|} \cdot \left(\frac{\overline{\deg}}{n(n-1)}\right)^2 = O\left(\frac{\overline{|E|}}{n^2|V|^2}\right)$. So, to have, $C \in [(1-\epsilon)\mathbb{E}\,C, (1+\epsilon)\mathbb{E}\,C]$ with probability $1-\delta$ by Chebyshev's inequality, we need: $n^2 = O\left(\frac{|E|}{\epsilon^2\delta}\right) = O\left(\frac{\overline{\deg}}{\epsilon^2\delta} \cdot |V|\right)$. So overall require:

$$n = \Theta\left(\max\left\{\frac{\overline{\deg}}{\deg_{\min}\epsilon^2\delta}, \sqrt{\frac{|V| \cdot \overline{\deg}}{\epsilon^2\delta}}\right\}\right). \tag{13}$$

[KLS11] also halts random walks after burn-in (uses t = 1), but uses a different estimator, tracking average degree, average inverse degree, and unweighted collisions. They require:

$$n = \Theta\left(\max\left\{\sqrt{\frac{1}{\epsilon^2 \delta \sum p_i^2}}, \frac{\sum p_i^3}{\epsilon^2 \delta \left(\sum p_i^2\right)^2}, \frac{\sum 1/p_i}{\epsilon^2 \delta |V|^2}\right\}\right). \tag{14}$$

Where $p_i = \frac{\deg(v_i)}{2|E|}$. Their third term comes from $\overline{\deg}$ estimation and can be upper bounded by our first, although, by giving a tighter analysis in Theorem 23 we match this term. Their first term can be rewritten:

$$\sqrt{\frac{1}{\epsilon^2 \delta \sum_{i=1}^{|V|} p_i^2}} = \sqrt{\frac{2|E|}{\epsilon^2 \delta \sum_{i=1}^{|V|} \deg(v_i)^2/(2|E|)}} = \sqrt{\frac{2|E|}{\sum_{i=1}^{|V|} \deg(v_i)^2}} \cdot \sqrt{\frac{|V| \cdot \overline{\deg}}{\epsilon^2 \delta}}.$$

This will always be somewhat smaller than our second term as $2|E| \leq \sum_{i=1}^{|V|} \deg(v_i)^2$. Their second term is a bit harder to compare, however, can be easily upper bounded by:

$$\frac{\sum p_i^3}{\epsilon^2 \delta \left(\sum p_i^2\right)^2} \leq \frac{\deg_{\max}^3}{\overline{\deg}^4 |V|^2} \cdot \frac{2|E|}{\epsilon^2 \delta} = \frac{\deg_{\max}^3}{\overline{\deg}^3} \cdot \frac{1}{\epsilon^2 \delta |V|}.$$

Assuming $\deg_{\max}/\overline{\deg}$ is not too large, this term will be small. However, a few very high degree nodes in an otherwise sparse graph can make it very large.

In sum, while our bounds are not directly comparable to those of [KLS11], in the t=1 case, assuming reasonably bounded degrees, they are of the same order of magnitude. Further, 12 gives an important tradeoff for graphs with relatively slow mixing time – we can increase the number of steps in our random walks, decreasing the total number of walks.

As an illustrative example, consider a k-dimensional torus graph for $k \geq 3$ (for k = 2 mixing time is $\Theta(|V|)$ so we might as well census the full graph). For any $\delta > 0$, the mixing time required for $\|\mathcal{D}^* - \mathcal{D}_M\|_1 = O(\delta)$

(see Section 6.1.3) is $M = \Theta(\log(|V|/\delta)|V|^{2/k})$. All nodes have degree 2k, and using the bounds above, to obtain a $(1 \pm \epsilon)$ estimate of |V|, the algorithm of [KLS11] requires:

$$n = \Theta\left(\max\left\{\frac{1}{\epsilon^2\delta}, \sqrt{\frac{|V|}{\epsilon^2\delta}}\right\}\right),$$

Assuming |V| is large so the second term dominates, they require $M \cdot n = \Theta\left(\frac{\log(|V|/\delta)}{\epsilon\sqrt{d}} \cdot |V|^{(k+4)/(2k)}\right)$ link queries to obtain a size estimate. In contrast, we require:

$$n = \Theta\left(\max\left\{\frac{\overline{\deg}}{\deg_{\min}\epsilon^2\delta}, \sqrt{\frac{|V| \cdot B(t)\overline{\deg}}{t \cdot \epsilon^2\delta}}\right\}\right) = \Theta\left(\max\left\{\frac{1}{\epsilon^2\delta}, \sqrt{\frac{|V|}{t \cdot \epsilon^2\delta}}\right\}\right).$$

since by Lemma 15, B(t) = O(1/k) and $\overline{\deg} = \deg_{\min} = k$. If we set $t = \Theta(M)$, and again assume that the second term dominates since |V| is large, the total number of link queries we need is:

$$n(M+t) = O\left(\frac{\sqrt{\log(|V|/\delta)}}{\epsilon\sqrt{d}} \cdot |V|^{(k+1)/2k}\right).$$

This beats [KLS11] by decreasing the polynomial in |V| and the logarithmic burn-in term. Ignoring error dependences, if k = 3, [KLS11] requires $\Theta(n^{7/6})$ queries which is more expensive than fully censusing the graph, whereas we require $O(n^{2/3})$ queries, which is sublinear in the graph size.

We leave it as an open question to compare our bounds with those of [KLS11] on more natural classes of graphs, and to determine either experimentally or theoretically, typical values of B(t) on these graphs. As used above, any of the bounds obtained in Section 5 apply since the degree weighted B(t) is upper bounded by the unweighted B(t) used in our regular graph analysis. However, bounds for real networks or popular random graph models used to study these networks would be very interesting.

6.2 Distributed Density Estimation by Robot Swarms

Algorithm 2 can be directly applied as a simple and robust density estimation algorithm for robot swarms. Additionally, the algorithm can be used to estimate the frequency of certain properties within the swarm. Let d be the overall population density and d_P be the density of agents with some property P. Let $f_P = d_P/d$ be the relative frequency of P.

Assuming that agents with property P are distributed uniformly in population and that agents can detect this property (through direct communication or some other signal), then they can separately track encounters with these agents. They can compute an estimate \tilde{d} of d and \tilde{d}_P of d_P . By Theorem 2, after running for $t = \Theta\left(\frac{\log(1/\delta)\log\log(1/\delta)\log(1/d\epsilon)}{d_P\epsilon^2}\right)$ steps, with probability $1-2\delta$, $\tilde{d}_P/\tilde{d} \in \left[\left(\frac{1-\epsilon}{1+\epsilon}\right)f_P, \left(\frac{1+\epsilon}{1-\epsilon}\right)f_P\right] = \left[(1-O(\epsilon))f_P, (1+O(\epsilon))f_P\right]$ for small ϵ .

In a biological setting, properties may include if an ant has recently completed a successful food foraging trip [Gor99], or if an ant is a nestmate or enemy [Ada90]. In a robotics setting, properties may include whether a robot is part of a certain task group, whether a robot has completed a certain task, or whether a robot has detected a certain event or environmental property.

6.3 Random Walk-Based Sensor Network Sampling

Finally, we believe our moment bounds for a single random walk (Corollaries 8 and 9) can be applied to random walk-based distributed algorithms for sensor network sampling. We leave obtaining rigorous bounds in this domain to future work.

Random walk-based sensor network sampling [LB07, AB04] is a technique in which a query message (a 'token') is initially sent by a base station to some sensor. The token is relayed randomly between sensors, which are connected via a grid network, and its value is updated appropriately at each step to give an answer to the query. This scheme is robust and efficient - it easily adapts to node failures and does not require setting up or storing spanning tree communication structures.

However, if attempting to estimate some quantity, such as the percentage of sensors that have recorded a specific condition, as in density estimation, unless an effort is made to record which sensors have been previously visited, additional variance is added due to repeat sensor visits. Recording previous visits introduces computational burden – either the token message size must increase or nodes themselves must remember which tokens they have seen. We are hopeful that our moment bounds can be used to show that this is unnecessary – due to strong local mixing, the number of repeat sensor visits will be low, and increased variance due to random walking will be limited.

6.4 Generalization to Data Aggregation

We note that, estimating the percentage of sensors in a network or the density of robots in a swarm with a property that is uniformly distributed is a special case of a more general data aggregation problem: each agent or sensor holds a value v_i drawn independently from some distribution \mathcal{D} . The goal is to estimate some statistic of \mathcal{D} - e.g. its expectation. In the case of density estimation v_i is simply an indicator random variable which is 1 with probability d and 0 otherwise.

Ideally, to estimate the expectation of \mathcal{D} , one would independently sample the agents and compute the sample mean. As in the case of density estimation, if instead, one simply takes an average of values encountered via a random walk, additional variance is added due to repeated encounters with many agents. However, it is likely that the effect of this variance can be bounded, as we have done in the special case of density estimation. We believe extending our results to more general data aggregation problems via random walk is an interesting future direction.

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