Feedback Design for Multi-contact Push Recovery via LMI Approximation of the Piecewise-Affine Quadratic Regulator

Weiqiao Han and Russ Tedrake

***Abstract*— To recover from large perturbations, a legged robot must make and break contact with its environment at various locations. These contact switches make it natural to model the robot as a hybrid system. If we apply Model Predictive Control to the feedback design of this hybrid system, the on/off behavior of contacts can be directly encoded using binary variables in a Mixed Integer Programming problem, which scales badly with the number of time steps and is too slow for online computation. We propose novel techniques for the design of stabilizing controllers for such hybrid systems. We approximate the dynamics of the system as a discrete-time Piecewise Affine (PWA) system, and compute the state feedback controllers across the hybrid modes offline via Lyapunov theory. The Lyapunov stability conditions are translated into Linear Matrix Inequalities. A Piecewise Quadratic Lyapunov function together with a Piecewise Linear (PL) feedback controller can be obtained by Semidefinite Programming (SDP). We show that we can embed a quadratic objective in the SDP, designing a controller approximating the Piecewise-Affine Quadratic Regulator. Moreover, we observe that our formulation restricted to the linear system case appears to always produce exactly the unique stabilizing solution to the Discrete Algebraic Riccati Equation. In addition, we extend the search from the PL controller to the PWA controller via Bilinear Matrix Inequalities. Finally, we demonstrate and evaluate our methods on a few PWA systems, including a simplified humanoid robot model.**

I. Introduction

Local stabilization of a fixed point or a trajectory of a nonlinear system, such as a humanoid robot, can normally be achieved by means of linearizing the dynamics and designing a Linear Quadratic Regulator (LQR) controller [1]. However, many critical tasks, such as recovery from a large external push, require a humanoid robot to make and break contact with its environment at multiple locations. For the purpose of such tasks, the humanoid robot is best modeled as a hybrid dynamic system. Unfortunately, there is a surprising lack of principled design techniques for such systems.

Despite this lack of generally applicable techniques, humanoid push recovery has been studied extensively in recent years. Strategies based on the zero moment point (ZMP) are often used to balance the biped robot [2], [3]. In [4], three strategies were proposed for large disturbance recovery, including center of pressure (CoP) balancing, centroidal moment point (CMP) balancing, and stepping. N-step capturability, the ability of a legged system to come to a stop without falling by taking N or fewer steps, was studied recently [5]. While previous work on humanoid push recovery has only looked at foot contacts, we are considering recovery strategies where the robot can also reach out with its hand and push on the surrounding environment.

Computer Science and Artificial Intelligence Laboratory, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA. weiqiaoh,russt@mit.edu

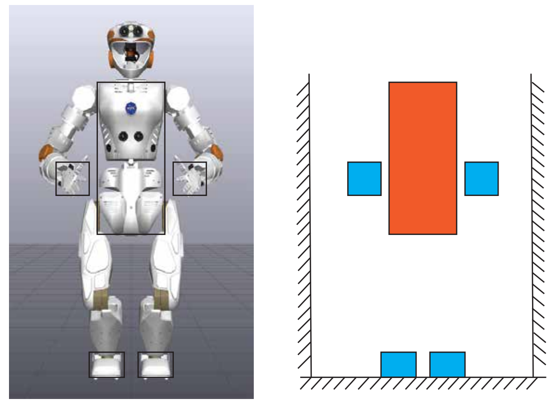

Fig. 1. Left: The Valkyrie bipedal robot. Right: The simplified model for Valkyrie. The hands may push against the walls.

To address these issues, we propose to globally approximate the nonlinear hybrid system by a time-stepping Piecewise Affine (PWA) system. Such a system can be obtained by performing a first order Taylor expansion of the nonlinear dynamics at many different points. PWA systems for short are defined by partitioning the state-input space into polyhedral regions and associating with each region a different affine state update equation [6]. We adopt the terminology in the hybrid system literature and call each region a "mode". Approximating a nonlinear system by a PWA system is actually ubiquitous, and in theory any general nonlinear system can be globally approximated by a PWA system with arbitrary accuracy [7]. We are interested in stabilizing the resulting PWA system.

A state-of-the-art planning technique that can be used to tackle the problem of stabilizing PWA systems is based on explicit Model Predictive Control (MPC) and Mixed Integer Quadratic Programming (MIQP) [8]. This approach enumerates all possible mode switch sequences over a certain number of time steps. For example, $\{$mode 2 at time 1, mode 2 at time 2, mode 1 at time 3, ..., mode 1 at time $T\}$ is one such mode switch sequence. For each mode switch sequence the approach solves a multi-parametric QP, and by comparing the cost of all such solutions it finds the optimal solution. The optimal controller is known to be piecewise affine and the cost-to-go function piecewise quadratic [9]. This approach performs only a few time steps of look-ahead and scales badly with the number of time steps. Different from the previous approach, we mainly seek a natural analogue of the

LQR control design procedure for the PWA system, which shall be more scalable.

The study of the stability and stabilization of PWA systems goes back decades. Hassibi *et al.* studied the stabilization and control of PWA systems via a Linear Matrix Inequality (LMI) approach [10]. Solving an LMI amounts to solving a feasibility Semidefinite Program (SDP). A Piecewise Quadratic (PWQ) Lyapunov function was used for the stability proof. An ellipsoidal approximation to the state cells was used to reduce the conservativeness of the LMI. However, they considered continuous-time systems instead of discrete-time systems. Rodrigues studied the state feedback and output feedback controller synthesis of continuous-time PWA systems, in which the Lyapunov stability conditions were formulated as Bilinear Matrix Inequalities (BMI), and methods for solving this particular type of BMIs were developed [11]. Özkan *et al.* studied MPC for discrete-time PWA systems [12]. To reduce the complexity, they assume the mode switch sequence is known. For controller synthesis, both Hassibi *et al.* and Özkan *et al.* use a common quadratic Lyapunov function, which is conservative.

In this paper, we study the Piecewise Affine Quadratic Regulator (PWAQR) problem, the design of the optimal controller for PWA systems with quadratic cost. We adopt an LMI approach, similar to [10], [12]. We solve an SDP to get a PWQ Lyapunov function, a Piecewise Linear (PL) feedback controller, and an upper bound on the cost of any trajectory. However, the SDP depends on the initial state of the system. In order to be general, we consider a few variants of the SDP. We observe that one of the variants is more natural than the others in that when it is applied to the linear system, it appears to always produce the unique stabilizing solution to the Discrete Algebraic Riccati Equation (DARE), hence making it a true generalization of LQR. Since our controller synthesis rule does not produce the optimal controller that generates the minimum cost trajectory for all initial states, we call it an *approximation* of the PWAQR controller. Furthermore, we extend our search from the PL controller to the PWA controller by means of a PWQ Lyapunov function with linear and constant terms. In contrast to the case of searching for a PL controller, searching for a PWA controller requires a formulation involving BMIs, a much harder problem. Finally, we demonstrate and evaluate our methods on a few PWA systems, including a simplified humanoid model (Figure 1).

We choose the discrete-time instead of continuous-time system modeling so that we do not have to deal with measure differential inclusions as in [13], while discrete-time systems still provide good approximations to real systems [14]. In addition, for discrete-time systems, the PWQ Lyapunov function does not have to be continuous on the boundaries of the polyhedron regions, and we do not have to consider sliding modes as in the case of continuous-time systems [11].

We introduce the Lyapunov-based controller synthesis for discrete-time PWA systems in Section II. Our main contributions are designing the approximation of the PWAQR controller (Section III), extending it from designing PL controllers to designing PWA controllers (Section IV), and applying the rule to humanoid push recovery where the robot makes and breaks multiple contacts with the environment (Section V). Compared with [4], [5], our approach applied to humanoid push recovery has the following advantages: (1) works on non-flat terrain; (2) does not assume that the CoM moves on a plane; (3) can incorporate richer swing leg dynamics; (4) can handle a multi-contact scenario. Compared with the approach based on explicit MPC and MIQP, our method does not require enumeration of mode switch sequences and hence scales better.

II. Stabilization of the PWA System

In this section, we introduce the PWA system and the Lyapunov-based controller synthesis for the PWA system.

A. The Piecewise Affine System

Discrete-time Piecewise Affine (PWA) systems are described by the state-space equations:

$$x_{k+1} = A_i x_k + B_i u_k + a_i, \text{ for } x_k \in \mathcal{X}_i \subseteq \mathbb{X} \tag{1}$$

where the state set $\mathbb{X} \subset \mathbb{R}^n$ is a polyhedron containing the origin, $\{\mathcal{X}_i\}_{i=1}^s$ is a polyhedron partition of $\mathbb{X}$, and $u_k \in \mathbb{R}^m$ is the control input. $\mathcal{I} = \{1, \ldots, s\}$ is the set of indices of the state space cells. $\mathcal{I}_0 \subseteq \mathcal{I}$ is the set of indices of the state space cells that contain the origin (There can be cases where the origin is on the boundaries of several cells.). Let $\mathcal{I}_1 = \mathcal{I} \backslash \mathcal{I}_0$. Let $\mathcal{S} = \{(i,j) \in \mathcal{I} \times \mathcal{I} : \exists\ x_k \in \mathcal{X}_i, x_{k+1} \in \mathcal{X}_j\}$ be the set of all ordered pairs (i, j) of indices denoting the possible switches from cell i to cell j.

Assume $\mathcal{X}_i = \{x \mid E_i x_i \geq e_i\}$. For later use, it is convenient to outer approximate each cell $\mathcal{X}_i$ with a union of ellipsoids $\mathcal{E}_{ip} = \{x \mid ||F_{ip} x + f_{ip}|| \leq 1\}$, $p = 1, \ldots, n_i$:

$$\mathcal{X}_i \subseteq \bigcup_{p=1}^{n_i} \mathcal{E}_{ip}.$$

We assume $x = 0$ is an equilibrium of the system (1), and $a_i = 0$ for all $i \in \mathcal{I}_0$.

B. State Feedback Synthesis

We consider the synthesis of a PL state feedback controller

$$u_k = K_i x_k, \forall x_k \in \mathcal{X}_i, \tag{2}$$

for the PWA system (1) that stabilizes the origin, certified by a PWQ Lyapunov function

$$V(x) = x^\top P_i x, P_i > 0, \forall x \in \mathcal{X}_i. \tag{3}$$

By Lyapunov theory, a sufficient condition for stability is that

$$\Delta V(x_{k+1}, x_k) = V(x_{k+1}) - V(x_k) < 0 \tag{4}$$

for any $x_k \in \mathbb{X}$.

In the following, to simplify the notations, we denote $A_i + B_i K_i$, the closed-loop state matrix, by $A_{cl,i}$. Since

$$\Delta V(x_{k+1}, x_k) = (A_{cl,i} x_k + a_i)^\top P_j (A_{cl,i} x_k + a_i) - x_k^\top P_i x_k,$$

for $x_k \in \mathcal{X}_i$, the condition (4) is equivalent to looking for the matrices $P_i > 0$ and K_i, $\forall i \in \mathcal{I}$ such that

$$\begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top \begin{bmatrix} A_{cl,i}^\top P_j A_{cl,i} - P_i & A_{cl,i}^\top P_j a_i \\ a_i^\top P_j A_{cl,i} & a_i^\top P_j a_i \end{bmatrix} \begin{bmatrix} x_k \\ 1 \end{bmatrix} < 0, \quad \forall x_k \in \mathcal{X}_i, \forall (i,j) \in \mathcal{S}. \tag{5}$$

A sufficient condition for (5) to hold is

$$\begin{bmatrix} A_{cl,i}^\top P_j A_{cl,i} - P_i & A_{cl,i}^\top P_j a_i \\ a_i^\top P_j A_{cl,i} & a_i^\top P_j a_i \end{bmatrix} < 0, \forall (i,j) \in \mathcal{S}. \tag{6}$$

By Schur complement, (6) is equivalent to

$$\begin{bmatrix} P_i & * & * \\ 0 & 0 & * \\ A_i + B_i K_i & a_i & P_j^{-1} \end{bmatrix} > 0, \forall (i,j) \in \mathcal{S}. \tag{7}$$

From now on, for symmetric matrices, we sometimes omit the symmetric halves and simply use stars $*$ to represent their entries. Introducing variables $W_i = P_i^{-1}, Y_i = K_i W_i$, and multiplying (7) with $\begin{bmatrix} W_i & 0 \\ 0 & I \end{bmatrix}$ from both sides, (7) is equivalent to

$$\begin{bmatrix} W_i & * & * \\ 0 & 0 & * \\ A_i W_i + B_i Y_i & a_i & W_j \end{bmatrix} > 0, \forall (i,j) \in \mathcal{S}, \tag{8}$$

which is an LMI in (W, Y).

However, since the inequality in (8) is strict but the matrix on the left-hand side has a principal minor equal to 0, (8) has no solution. We will see later that the PWAQR approximation design procedure resolves this issue automatically. Now we look for a better sufficient condition for (5) that can be turned into a feasible LMI.

The sufficient condition (6) is conservative, because it requires the inequality in (5) to hold for all $x_k \in \mathbb{R}^n$, while we only need it hold for all $x_k \in \mathcal{X}_i$. Unfortunately, this latter condition cannot be translated directly into semidefinite constraints. However, we can still reduce conservativeness while retaining an SDP formulation by outer-approximating each of the state cells with a union of ellipsoids, $\mathcal{X}_i \subseteq \bigcup \mathcal{E}_{ip}$. We want the inequality in (5) to hold for all $x_k \in \bigcup \mathcal{E}_{ip}$ i.e., for all x_k satisfying

$$\begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top \begin{bmatrix} F_{ip}^\top F_{ip} & * \\ f_{ip}^\top F_{ip} & f_{ip}^\top f_{ip} - 1 \end{bmatrix} \begin{bmatrix} x_k \\ 1 \end{bmatrix} < 0, 1 \le p \le n_i.$$

By S-procedure, a sufficient condition for (5) becomes

$$\begin{bmatrix} A_{cl,i}^\top P_j A_{cl,i} - P_i & A_{cl,i}^\top P_j a_i \\ a_i^\top P_j A_{cl,i} & a_i^\top P_j a_i \end{bmatrix} - \lambda_{ip} \begin{bmatrix} F_{ip}^\top F_{ip} & * \\ f_{ip}^\top F_{ip} & f_{ip}^\top f_{ip} - 1 \end{bmatrix} < 0, \quad \lambda_{ip} > 0, 1 \le p \le n_i, (i,j) \in \mathcal{S}. \tag{9}$$

By some algebraic manipulations, including Schur complement, similar matrix transformations, and the identities:

$$(I - E^\top E)^{-1} = I + E^\top (I - EE^\top)^{-1} E$$
$$E(I - E^\top E)^{-1} = (I - EE^\top)^{-1} E$$

where E can be a matrix of any size, (9) is equivalent to

$$\begin{bmatrix} W_i & * & * \\ A_i W_i + B_i Y_i & W_j + \beta_{ip} a_i a_i^\top & * \\ F_{ip} W_i & \beta_{ip} f_{ip} a_i^\top & \beta_{ip}(f_{ip} f_{ip}^\top - I) \end{bmatrix} > 0, \quad \beta_{ip} > 0, 1 \le p \le n_i, (i,j) \in \mathcal{S}, \tag{10}$$

which is an LMI in (W, Y, β), where $\beta_{ip} = \lambda_{ip}^{-1}$.

Notice that (10) requires $f_{ip} f_{ip}^\top - I > 0$, which does not hold for $i \in \mathcal{I}_0$. For $i \in \mathcal{I}_0$, we simply require

$$A_{cl,i}^\top P_j A_{cl,i} - P_i < 0,$$

which is equivalent to

$$\begin{bmatrix} W_i & * \\ A_i W_i + B_i Y_i & W_j \end{bmatrix} > 0, (i,j) \in \mathcal{S}, \tag{11}$$

In summary, a PL state feedback controller that stabilizes the origin can be obtained by solving the feasibility SDP:

$$\begin{aligned} &\text{find } W, Y, \beta \\ &\text{subject to } W_i > 0, i \in \mathcal{I}, \\ &\qquad (11) \text{ if } i \in \mathcal{I}_0, \\ &\qquad (10) \text{ if } i \in \mathcal{I}_1, \end{aligned}$$

and computing the state feedback matrices $K_i = Y_i W_i^{-1}, i \in \mathcal{I}$.

III. PL Approximation of PWAQR Controller

In this section, we shift from merely stabilizing the PWA system to additionally trying to minimize a quadratic cost function, i.e., finding an approximate solution to the Piecewise-Affine Quadratic Regular (PWAQR) problem. The controller in consideration is still PL as in (2) and the PWQ Lyapunov function is still of the form (3).

A. Quadratic Objective and Its Upper Bound

We consider a quadratic objective (cost function) for the controller synthesis of the PWA system. Define the cost matrices $Q_i \ge 0, R_i > 0$ for the cell $\mathcal{X}_i, i \in \mathcal{I}$. The quadratic cost function is

$$\sum_{k=0}^{\infty} x_k^\top Q_{i(k)} x_k + u_k^\top R_{i(k)} u_k, \tag{12}$$

where $i(k) \in \mathcal{I}$ is the index such that $[x_k^\top, u_k^\top]^\top \in \mathcal{X}_{i(k)}$.

Lemma 1. If there are matrices $P_i > 0$ and $K_i, i \in \mathcal{I}$, satisfying

$$\Delta V(x_{k+1}, x_k) + x_k^\top (Q_{i(k)} + K_{i(k)}^\top R_{i(k)} K_{i(k)}) x_k \le 0, \forall x_k, \tag{13}$$

then the PL controller (2) stabilizes the origin asymptotically, and the PWQ Lyapunov function $V(x) = x^\top P_i x$ proves the bound

$$\sum_{k=0}^{\infty} x_k^\top Q_{i(k)} x_k + u_k^\top R_{i(k)} u_k \le x_0^\top P_{i(0)} x_0. \tag{14}$$

□

Proof: Since $\Delta V(x_{k+1}, x_k) < 0$ for all $x_k \neq 0$, the controller asymptotically stabilizes the origin.

Since $x_k^\top (Q_{i(k)} + K_{i(k)}^\top R_{i(k)} K_{i(k)}) x_k \leq -\Delta V(x_{k+1}, x_k)$. Summing over the trajectory $\{x_k\}_{k=0}^\infty$,

$$\begin{aligned}
&\sum_{k=0}^{\infty} x_k^\top Q_{i(k)} x_k + u_k^\top R_{i(k)} u_k \\
=& \sum_{k=0}^{\infty} x_k^\top (Q_{i(k)} + K_{i(k)}^\top R_{i(k)} K_{i(k)}) x_k \\
\leq& \sum_{k=0}^{\infty} -\Delta V(x_{k+1}, x_k) \\
=& \sum_{k=0}^{\infty} -(V(x_{k+1}) - V(x_k)) \\
=& V(x_0).
\end{aligned}$$

In the last step, the series $\sum_{k=0}^\infty -(V(x_{k+1}) - V(x_k))$ converges to $V(x_0)$, because the partial sum $\sum_{k=0}^K -(V(x_{k+1}) - V(x_k)) = V(0) - V(K+1)$ and $V(K+1) \to 0$ as $K \to \infty$ by asymptotic stability. □

Since $\Delta V(x_{k+1}, x_k) = (A_{cl,i} x_k + a_i)^\top P_j (A_{cl,i} x_k + a_i) - x_k^\top P_i x_k$, for $x_k \in \mathcal{X}_i$, (13) is equivalent to

$$\begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top \begin{bmatrix} \begin{pmatrix} A_{cl,i}^\top P_j A_{cl,i} - P_i \\ + Q_i + K_i^\top R_i K_i \end{pmatrix} & * \\ a_i^\top P_j A_{cl,i} & a_i^\top P_j a_i \end{bmatrix} \begin{bmatrix} x_k \\ 1 \end{bmatrix} \leq 0, \quad \forall x_k \in \mathcal{X}_i, \forall (i,j) \in \mathcal{S}. \tag{15}$$

A sufficient condition for (15) is

$$\begin{bmatrix} A_{cl,i}^\top P_j A_{cl,i} - P_i + Q_i + K_i^\top R_i K_i & * \\ a_i^\top P_j A_{cl,i} & a_i^\top P_j a_i \end{bmatrix} \leq 0, \quad \forall (i,j) \in \mathcal{S}. \tag{16}$$

By Schur complement, (16) is equivalent to

$$\begin{bmatrix} W_i & * & * & * & * \\ 0 & 0 & * & * & * \\ A_i W_i + B_i Y_i & a_i & W_j & * & * \\ Q_i^{1/2} W_i & 0 & 0 & I & * \\ R_i^{1/2} Y_i & 0 & 0 & 0 & I \end{bmatrix} \geq 0, W_i > 0, \forall (i,j) \in \mathcal{S}, \tag{17}$$

Here we have non-strict inequality. The previous issue raised by the strict inequality is automatically resolved. Moreover, even if $i \in \mathcal{I}_0$ and $a_i = 0$, we can still use (17).

We want to minimize the upper bound (14) on the cost function. This leads to the following lemma.

Lemma 2. Let $V(x) = x^\top P_i x, \forall x \in \mathcal{X}_i$. A stable PL state feedback that asymptotically stabilizes the origin with initial state x_0 can be found by solving the SDP (18) for γ, W_i and Y_i. K_i is then given by $K_i = Y_i W_i^{-1}$. The cost of any trajectory $\{x_k\}_{k=0}^\infty$ is bounded by γ.

$$\begin{aligned} &\min_{\gamma, W_i, Y_i} \gamma \\ &\text{subject to} \begin{bmatrix} \gamma & x_0^\top \\ x_0 & W_{i(0)} \end{bmatrix} \geq 0, W_i > 0, \text{ and (17)}. \end{aligned} \tag{18}$$

□

Notice that the SDP (18) depends on the initial state x_0. This is not in the spirit of LQR for linear systems, and would necessitate solving SDPs online, which is impractical. We want an SDP that is independent of the initial state. This will be discussed in detail later.

B. Ellipsoid Approximation

As before, using outer ellipsoid approximation, a sufficient condition for (15) is

$$\begin{bmatrix} W_i & * & * & * \\ A_i W_i + B_i Y_i & W_j & * & * \\ Q_i^{1/2} W_i & 0 & I & * \\ R_i^{1/2} Y_i & 0 & 0 & I \end{bmatrix} \geq 0, i \in \mathcal{I}_0, (i,j) \in \mathcal{S}, \tag{19}$$

$$\begin{bmatrix} W_i & * & * & * & * \\ A_i W_i + B_i Y_i & W_j + \beta_{ip} a_i a_i^\top & * & * & * \\ F_{ip} W_i & \beta_{ip} f_{ip} a_i^\top & \beta_{ip}(f_{ip} f_{ip}^\top - I) & * & * \\ Q_i^{1/2} W_i & 0 & 0 & I & * \\ R_i^{1/2} Y_i & 0 & 0 & 0 & I \end{bmatrix} \geq 0, \beta_{ip} > 0, \forall p, i \in \mathcal{I}_1, (i,j) \in \mathcal{S}. \tag{20}$$

So a counterpart for SDP (18) is the following SDP:

$$\begin{aligned} &\min_{\gamma, W_i, Y_i} \gamma \\ &\text{subject to} \begin{bmatrix} \gamma & x_0^\top \\ x_0 & W_{i(0)} \end{bmatrix} \geq 0, W_i > 0, \\ &\text{and (19), (20)}. \end{aligned} \tag{21}$$

C. Variants of the Objective

As mentioned earlier, the SDP (18) depends on the initial state x_0. Now we discuss some possible variants of the objective that are independent of the initial state x_0.

Since we want to minimize the quantity $x_0^\top P_{i(0)} x_0$ for generic x_0, it is natural to minimize trace($P_{i(0)}$), or more generally, minimize trace($\sum_i P_i$). Since $P = W^{-1}$, it is natural to maximize trace($\sum_i W_i$). If we do not care about the upper bound on the cost and only want to find a feasible controller, we can simply solve a feasibility problem. Another possible objective arises if we let $P_i = \gamma W_i^{-1}, \forall i$, and minimize $x_0^\top P_{i(0)} x_0$ as in [12]. The inequality $\gamma \geq x_0^\top P_{i(0)} x_0$ becomes the LMI

$$\begin{bmatrix} 1 & x_0^\top \\ x_0 & W_{i(0)} \end{bmatrix} \geq 0 \tag{22}$$

This still depends on the initial state. We summarize the variants of the objective below.

1) *SDP1*
   SDP (18).
2) *SDP2*

$$\begin{aligned} &\text{maximize trace}(\sum_{i=1}^{s} W_i) \\ &\text{subject to } W_i > 0, \text{ and (17)}. \end{aligned} \tag{23}$$

3) *SDP3*

$$\begin{aligned} & \text{minimize } 0 \\ & \text{subject to } W_i > 0, \text{ and } (17). \end{aligned} \tag{24}$$

4) *SDP4*

$$\begin{aligned} & \text{minimize } \gamma \\ & \text{subject to } \begin{bmatrix} W_i & * & * & * & * \\ 0 & 0 & * & * & * \\ A_iW_i + B_iY_i & \gamma a_i & W_j & * & * \\ Q_i^{1/2}W_i & 0 & 0 & \gamma I & * \\ R_i^{1/2}Y_i & 0 & 0 & 0 & \gamma I \end{bmatrix} \geq 0, \\ & \gamma > 0, \forall (i,j) \in \mathcal{S}, \\ & \text{and } (22). \end{aligned} \tag{25}$$

In practice, we use SDP2. It is experimentally verified with a large number of test cases that when SDP2 is applied to the linear system ($s = 1$), a special case of the PWA system, we always get back exactly the normal LQR controller for the linear system. This suggests that our choice of the objective is more natural. However, we have not yet been able to prove it, so we leave it as a conjecture.

Conjecture 1. Let $x_{k+1} = Ax_k + Bu_k$ be a linear system. Let $Q \geq 0, R > 0$ be cost matrices. Assume that (A, B) is a stabilizable pair, and that (A, C) is a detectable pair, where C is a full-rank factorization of Q (i.e., $C^\top C = Q$ and rank(C) = rank(Q)).

Suppose W and Y is a pair of solutions to the SDP (23). Let $P_1 = W^{-1}$ and $K_1 = YW^{-1}$.

Suppose P_2 is the unique stabilizing solution to the Discrete Algebraic Riccati Equation

$$X = A^\top XA - (A^\top XB)(R + B^\top XB)^{-1}(B^\top XA) + Q,$$

and the optimal gain matrix is

$$K_2 = -(R + B^\top P_2 B)^{-1} B^\top P_2 A.$$

Then

$$P_1 = P_2, K_1 = K_2.$$

□

There are also ellipsoid approximation counterparts for these SDP's, which we do not explicitly write down here. We denote the ellipsoid approximation counterparts of the SDP's by SDP1*, SDP2*, SDP3*, and SDP4*, correspondingly.

IV. PWA Approximation of PWAQR Controller

So far, we have considered PL controllers in the approximation of the PWAQR controller. In this section, we generalize the previous results to the synthesis of a PWA controller

$$u_k = K_i x_k + b_i, \forall x_k \in \mathcal{X}_i, \tag{26}$$

for the PWA system (1) that stabilizes the origin by means of a PWQ Lyapunov function of the full form

$$V(x) = x^\top P_i x + 2q_i^\top x + r_i, \forall x_k \in \mathcal{X}_i. \tag{27}$$

The analogue of (13) is

$$\begin{aligned} & \Delta V(x_{k+1}, x_k) + x_k^\top Q_{i(k)} x_k \\ & + (K_{i(k)}x_k + b_{i(k)})^\top R_{i(k)}(K_{i(k)}x_k + b_{i(k)}) \leq 0, \forall x_k. \end{aligned} \tag{28}$$

Denote $a_{cl,i} = B_i b_i + a_i$. Then $x_{k+1} = A_i x_k + B_i u_k + a_i = A_{cl,i}x_k + a_{cl,i}$.

We obtain BMIs in the variables (W, q, r, Y, b) for controller synthesis

$$\begin{aligned} & \begin{bmatrix} W_i & * & * & * & * \\ -q_j^\top A_{cl,i}W_i + q_i^\top W_i & -2q_j^\top a_{cl,i} + r_i - r_j & * & * & * \\ A_{cl,i}W_i & a_{cl,i} & W_j & * & * \\ Q^{\frac{1}{2}}W_i & 0 & 0 & I & * \\ R^{\frac{1}{2}}Y_i & R^{\frac{1}{2}}b_i & 0 & 0 & I \end{bmatrix} \\ & \geq 0, (i,j) \in \mathcal{S}, (\text{notice that } A_{cl,i}W_i = A_iW_i + B_iY_i) \\ & W_i > 0, i = 1, \ldots, s, \\ & (E_iW_iq_i + e_i)_h > 0, i \in \mathcal{I}_1, \text{for some } h \text{ depending on } i, \\ & q_i = 0, r_i = 0, i \in \mathcal{I}_0. \end{aligned} \tag{29}$$

The first inequality in (29) ensures that V decreases along all state trajectories. In the third inequality, $(\cdot)_h$ represents the h-th component of a vector. The local minima of the PWQ Lyapunov function candidate V are in the set

$$\mathcal{Q} = \{-W_1q_1, \ldots, -W_sq_s\}.$$

In order to stabilize the system to the origin, we require $-W_iq_i, i \in \mathcal{I}_1$ to be outside the region $\mathcal{X}_i$, i.e., $-W_iq_i$ is not in the set $\mathcal{X}_i = \{x \mid E_ix_i \geq e_i\}$. So for each $i \in \mathcal{I}_1$, at least one of the inequalities in $E_iW_iq_i + e_i \leq 0$, viewed as component-wise inequalities for a vector, should be violated, and that is exactly the third inequality in (29). The fourth inequality in (29) guarantees $-W_iq_i, i \in \mathcal{I}_0$ to be the origin. Together with the second inequality, they ensure that V is positive.

This is a system of BMIs, and we need to enumerate the indices $h(i)$ for each $i \in \mathcal{I}_1$ until a solution is found. Solving the BMIs is NP-hard [15]. Nevertheless, there exist fast heuristic methods for solving them, for example, based on SDP [16]. However, the approach in [16] does not guarantee to find the global minimum.

V. Experiment

A. Cart-Pole Balance Control

We consider the problem of balancing the cart-pole system as shown in Figure 2. We want to show that SDP2 produces the LQR controller for the linearized system around the fixed point. Let $q = [x, \theta]^\top$, $\mathbf{x} = [q^\top, \dot{q}^\top]^\top$ and $\mathbf{u} = f$. We are interested in balancing the simple pendulum around its unstable fixed point $\mathbf{x}^* = [0, \pi, 0, 0]^\top$ using only horizontal force on the cart [17]. We assume there is no friction or air resistance.

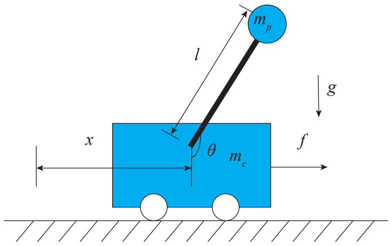

Fig. 2. The cart-pole system.

The manipulator equation is

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B\mathbf{u}$$

where

$$H(q) = \begin{bmatrix} m_c + m_p & m_p l\cos\theta \\ m_p l\cos\theta & m_p l^2 \end{bmatrix}, G(q) = \begin{bmatrix} 0 \\ m_p g l\sin\theta \end{bmatrix},$$
$$C(q,\ddot{q}) = \begin{bmatrix} 0 & -m_p l\dot{\theta}\sin\theta \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

TABLE I
PHYSICAL PARAMETERS OF THE CART-POLE SYSTEM

	Explanation	Value
m_c	cart mass	10
m_p	pole mass	1
l	pole length	0.5
g	gravitational acceleration	9.81
Δt	discretization time interval	0.05

Linearizing around the fixed point $(\mathbf{x}^*, \mathbf{u}^*) = ([0, \pi, 0, 0]^\top, 0)$ using Taylor expansion, and choosing the physical parameters of the cart-pole system as described in the Table I, we get the discrete-time system dynamics $\mathbf{x}_{t+1} = A\mathbf{x}_t + B\mathbf{u}_t$, where

$$A = \begin{bmatrix} 1 & 0 & 0.05 & 0 \\ 0 & 1 & 0 & 0.05 \\ 0 & 0.0491 & 1 & 0 \\ 0 & 1.0791 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0.005 \\ 0.01 \end{bmatrix}.$$

Choose the cost matrices $Q = I_4, R = I_1$. Our SDP2 produces exactly the LQR optimal state feedback gain

$$K = \begin{bmatrix} 0.8027 & -214.6740 & 4.2763 & -46.4030 \end{bmatrix}.$$

This again strengthens our belief that the Conjecture 1 is true.

B. A 4-cell PWA system

We next consider the 4-cell PWA system as described in [18]. We use this example to evaluate the controllers returned by SDP1, . . ., SDP4. The dynamics of the system is

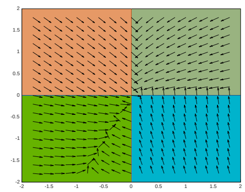

Fig. 3. x-plane: $x_{k+1} - Ax_k$ normalized vector field.

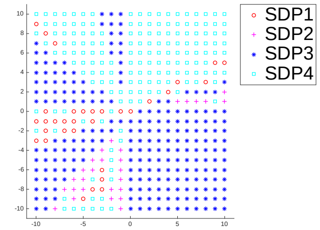

Fig. 4. The minimum cost controller varies with the initial states. SDP1 and SDP4 are solved separately for every initial state.

$$x_{k+1} = \begin{cases} \begin{pmatrix} -0.04 & -0.461 \\ -0.139 & 0.341 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k, & E_1 x_k \geq 0 \\ \begin{pmatrix} 0.936 & 0.323 \\ 0.788 & -0.049 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k, & E_2 x_k \geq 0 \\ \begin{pmatrix} -0.857 & 0.815 \\ 0.491 & 0.62 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k, & E_3 x_k \geq 0 \\ \begin{pmatrix} -0.022 & 0.644 \\ 0.758 & 0.271 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k, & E_4 x_k \geq 0 \end{cases}$$

where the partitioning corresponds to the four quadrants of the two dimensional x plane, i.e.,

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, E_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, E_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The vector field of the system in the two-dimensional x plane is plotted in Figure 3. (Note that the vector field plot in [18] is incorrect.)

We choose the cost matrices to be $Q = I_2, R = 10I_1$. We simply let $\mathcal{S} = \mathcal{I} \times \mathcal{I}$ and compute controllers using SDP1, . . ., SDP4. Since all cells contain the origin, ellipsoid approximation is not useful here. We then compute the costs of the trajectories generated by these controllers starting from various different initial states, and mark the minimum cost

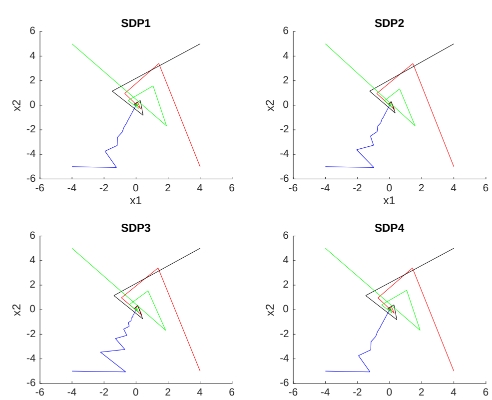

Fig. 5. Trajectories generated by four controllers starting at the initial states $x_0 = [\pm 4, \pm 5]^\top$.

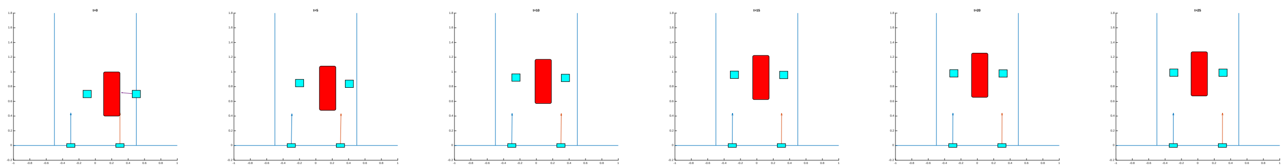

Fig. 6. Stabilizing box Valkyrie. From left to right, the time stamps are $0, 5, 10, 15, 20, 25$.

controller for the various initial states in Figure 4. The initial-state-dependent controllers (obtained from SDP1 and SDP4) are solved separately for every initial state.

For example, starting from the initial state $x_0 = [-4, -5]^\top$, the costs of the four trajectories generated by the four controllers are $145.3563, 143.5464, 144.4355$, and 145.3563, respectively. So the controller returned by SDP2 is the minimum cost controller for the initial state $x_0 = [-4, -5]^\top$ (minimum among the four controllers at hand). The upper bound γ computed in SDP1 and SDP4 is in both cases 151.7155. The trajectories are shown in Figure 5. Notice that although the trajectories corresponding to SDP1 and SDP4 are similar, the PL controllers are not exactly the same.

Figure 4 indicates that there is no “best controller” among the four controllers. Every controller has the chance to be the minimum cost controller starting from some initial state. It is amazing that even the feasibility SDP, SDP3, can produce the minimum cost controller at many initial states. We cannot conclude that SDP3 is the best simply based on the observation that it covers the most number of initial states in Figure 4 – it really depends on the dynamics of the system and the choice of the cost matrices.

C. *Simplified Humanoid Model*

Finally, we consider the “box Valkyrie model” (Figure 7), a simplified 2-dimensional model for the Valkyrie bipedal robot. It has four massless, velocity-controlled limbs, depicted by the four black dots inside the blue rectangles, and a center of mass, depicted by the black dot inside the red rectangle. The rectangles surrounding the limb dots and the center of mass are just for better visual effects. It was called ”box Valkyrie” because it was situated in a box, not because the limbs needed to be visualized as boxes. Two feet are on the floor. The arrows pointing upwards are the normal forces exerted by the floor. There are two walls at $x = -0.5$ and $x = 0.5$, respectively.

The goal is to keep the center of mass at the origin. The center of mass is controlled by the contact forces, which are exerted upon those limbs that are in contact with the environment. Different from previous work on humanoid push recovery, we consider the recovery strategies where the robot can reach out with its hand and push on the surrounding environment.

We use the following contact model. When a limb p is in contact with the wall or the floor, we model it using two points: a non-penetrating point p_1 staying on the contact surface, and a penetrating point p_2 penetrating the contact surface, as shown in Figure 8. Both p_1 and p_2 are velocity-controlled. The normal force F_N is proportional to the vertical displacement between p_1 and p_2, and the frictional force F_f is proportional to the horizontal displacement between p_1 and p_2, i.e., $F_N = -k\Delta z$ and $F_f = -k\Delta x$. The blue dashed lines in the Figure 8 are the boundary lines of the friction cone. We keep p_2 inside the “reflected friction cone” so that the frictional force lies in the friction cone. Once p_1 or p_2 goes outside the contact surface, we recombine them into one point, p. In Figure 7, the two feet are in contact with the floor, so there are two black dots at each foot. The two hands are not in contact with the wall or the floor, so there is only one dot at each hand.

The box Valkyrie model has 20 states and 8 control inputs.

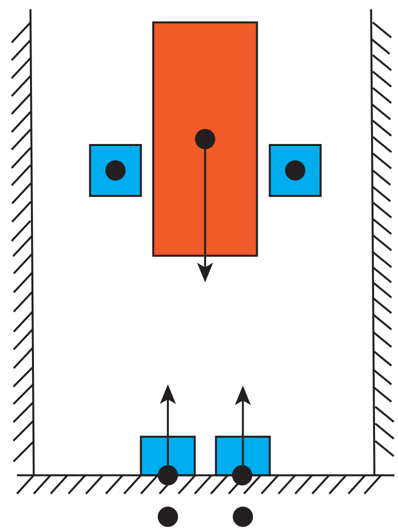

Fig. 7. The “box Valkyrie model”, a simplified 2-dimensional model for the Valkyrie bipedal robot

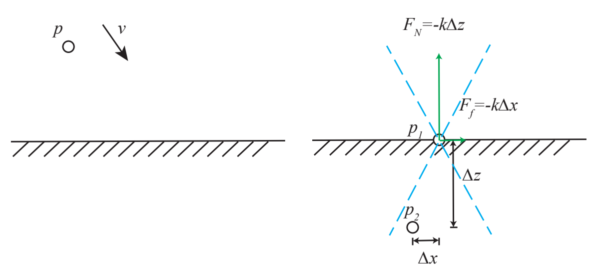

Fig. 8. Contact model. Left: Without contact, a limb is modelled as a point p. Right: In contact, p splits into two points, the non-penetrating point p_1 and the penetrating point p_2.

Since the controller is linear in any specific mode, we cannot expect the controller to lift a foot and put it down somewhere else. So we simply put the feet at their equilibrium points in the initial state. The left hand may touch the left wall, and the right hand may touch the right wall. So the model is a piecewise affine system with four modes: no hands in contact with the walls, the left hand in contact with the left wall, the right hand in contact with the right wall, or two hands both in contact with the corresponding walls. The state-of-the-art MIQP approach would have two binary variables for each time step, one indicating if the right hand is touching the wall, and the other indicating if the left hand is touching the wall. It would have 4 possible modes at every time step, and the number of mode switch sequences grows exponentially with the number of time steps. Using our approach, we only need to think about the set $\mathcal{S}$, which is polynomial (quadratic) in the number of modes. We use SDP2 or SDP2* to solve for the controller with $\mathcal{S} = \mathcal{I} \times \mathcal{I}$. The sequence of poses in Figure 6 shows that when the robot body is pushed to the right and one hand is in contact with the wall, the controller stabilizes the center of mass in approximately 30 time steps.

VI. CONCLUSION AND FUTURE WORK

We have derived a procedure for synthesizing controllers that approximately solve the PWAQR problem. We applied this method to a PWA approximation of a nonlinear hybrid system representing a humanoid robot's centroidal dynamics in the plane. As opposed to the MPC and the MIQP approaches, our method does not require enumerating the mode switch sequences, and hence scales better.

There are some limitations. (i) For the humanoid model, we do not take the centroidal angular momentum into account. (ii) We cannot incorporate the PWQ Lyapunov function of the full form (27) into the LMIs for the controller synthesis. So we have to express the Lyapunov stability conditions as BMIs, solving which is NP-hard. Also, the domain information, either the polyhedron or the ellipsoid approximation, cannot be incorporated into the BMIs.

Since the PWAQR naturally does not have any constraints on the control input, we impose the constraints on the control inputs by limiting the state of the system. For example, in the box Valkyrie model, we keep the position of the penetrating point p_2 inside the "reflected friction cone" so that the frictional force always lies inside the friction cone. In the future, another possible way to try is to incorporate the force constraints into the system dynamics as in [13].

Another limitation of the current approach is that the control law does not switch inside a polyhedron region $\mathcal{X}_i$, while in practice it is common for the control to switch in the same polyhedron region. The search for the control switching surface might be done together with the search for the control law in an alternative fashion. This will be a subject of further investigations.

ACKNOWLEDGMENT

This work was supported by NASA Award NNX16AC49A. The views, opinions and positions expressed by the authors are theirs alone, and do not necessarily reflect the views, opinions or positions of NASA. The authors also thank Twan Koolen for many helpful comments.

REFERENCES

[1] S. Kuindersma, R. Deits, M. Fallon, A. Valenzuela, H. Dai, F. Permenter, T. Koolen, P. Marion, and R. Tedrake. Optimization-based locomotion planning, estimation, and control design for the Atlas humanoid robot. *Autonomous Robots*, 40(3):429-455, 2016.

[2] M. Vukobratovic, and D. Juricic, Contribution to the synthesis of biped gait, *Biomedical Engineering, IEEE Transactions on*, (1):16, 1969.

[3] M. Vukobratovic, A. A. Frank, and D. Juricic, On the stability of biped locomotion, *IEEE Transactions on Biomedical Engineering*, pp. 2536, January 1970.

[4] B. Stephens, Humanoid push recovery, *Humanoid Robots, 2007 7th IEEE-RAS International Conference on*, 2007.

[5] T. Koolen, T. de Boer, J. Rebula, A. Goswami, and J. Pratt Capturability-based analysis and control of legged locomotion. Part 1: theory and application to three simple gait models. *International Journal of Robotics Research* 31(9): 10941113, 2012.

[6] Piecewise Affine Systems. In: *Optimal Control of Constrained Piecewise Affine Systems*. Lecture Notes in Control and Information Sciences, vol 359. Springer, Berlin, Heidelberg, 2007.

[7] E. Sontag, Nonlinear regulation: The piecewise linear approach, *IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 346-358, 1981.

[8] A. K. Valenzuela. *Mixed-Integer Convex Optimization for Planning Aggressive Motions of Legged Robots Over Rough Terrain*. PhD thesis, Massachusetts Institute of Technology, Feb 2016.

[9] A. Alessio and A. Bemporad, A Survey on Explicit Model Predictive Control, *Nonlinear Model Predictive Control*, LNCIS 384, pp. 345369.

[10] A. Hassibi, S. Boyd, Quadratic Stabilization and Control of Piecewise-Linear Systems, in the *Proceedings of the American Control Conference*, Philadelphia, Pennsylvania, June 1998.

[11] L. Rodrigues, *Dynamic output feedback controller synthesis for piecewise-affine systems*, Ph.D. dissertation, Stanford University, CA, 2002.

[12] L. Özkan, M. V. Kothare, C. Georgakis, Model predictive control of nonlinear systems using piecewise linear models, *Computers and Chemical Engineering*, Volume 24, Issues 27, 15 July 2000, Pages 793-799.

[13] M. Posa, M. Tobenkin, and R. Tedrake. Lyapunov analysis of rigid body systems with impacts and friction via sums-of-squares. In *Proceedings of the 16th International Conference on Hybrid Systems: Computation and Control (HSCC)*, Philadelphia, PA, 2013.

[14] D. Stewart and J. Trinkle, An implicit time-stepping scheme for rigid body dynamics with Coulomb friction, in the *Proceedings of International Conference on Robotics and Automation*, San Francisco, CA, May 2000, pp. 162-169.

[15] J. G. VanAntwerp, R. D. Braatz, A tutorial on linear and bilinear matrix inequalities, *Journal of Process Control* 10 (2000) 363-385.

[16] S. Ibaraki, M. Tomizuka, Rank Minimization Approach for Solving BMI Problems with Random Search, *Proceedings of the American Control Conference*, Arlington, VA, June 25-27, 2001.

[17] R. Tedrake. *Underactuated Robotics: Algorithms for Walking, Running, Swimming, Flying, and Manipulation (Course Notes for MIT 6.832)*. Downloaded on July 11th, 2017 from http://underactuated.mit.edu/

[18] D. Mignone, G. Ferrari-Trecate, M. Morari, Stability and Stabilization of Piecewise Affine and Hybrid Systems: An LMI Approach, in *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, December 2000.