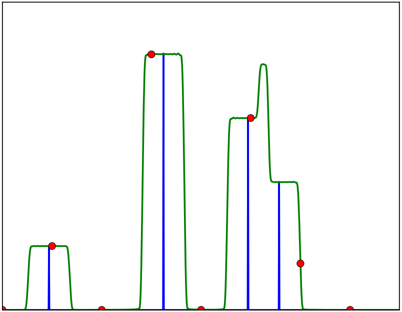
Nearly Optimal Sparse Fourier Transform

Haitham Hassanieh Piotr Indyk Dina Katabi **Eric Price**

MIT

2012-04-27



Outline

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2. Special case: exactly sparse signals

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2. Special case: exactly sparse signals
3. General case: approximately sparse signals
4. Experiments

The Dicrete Fourier Transform

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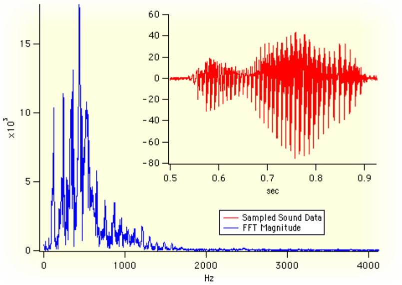

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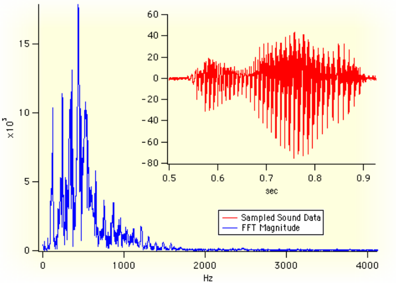

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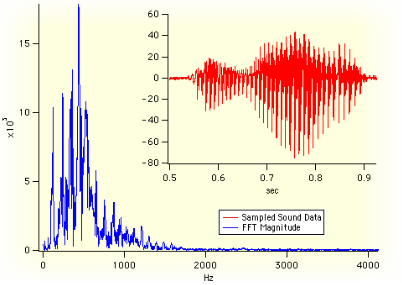
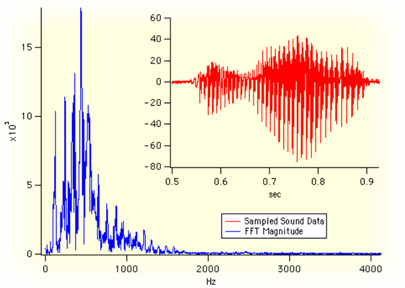
- Fundamental tool
  - Compression (audio, image, video)
  - Signal processing
  - Data analysis
  - ...
- FFT: $O(n \log n)$ time.

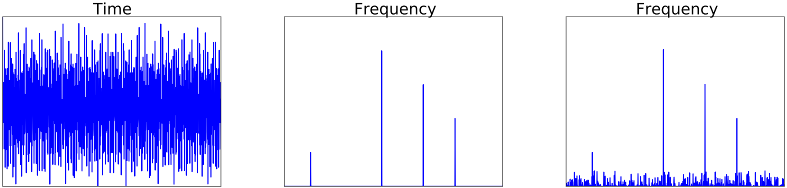


Sparse Fourier Transform



- Often the Fourier transform is dominated by a small number of "peaks"
  - Precisely the reason to use for compression.
- If most of mass in k locations, can we compute FFT faster?

Sparse Fourier Transform



- If at most k non-zero coefficients, then "exactly k-sparse."
- More often well approximated by k largest coefficients: "approximately k-sparse."

Previous work

- Boolean cube: [KM92], [GL89]. $\mathbb{C}^n$: [Mansour92] $k^c \log^c n$.
- Long line of additional work [GGIMS02, AGS03, Iwen10, Aka10]
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  - All have poor constants, many logs.
  - Need $n/k > 40{,}000$ or $\omega(\log^3 n)$ to beat FFTW.
  - Our goal: faster, beat FFTW for smaller n/k in theory and practice.

Our results

- $O(k \log(n/k) \log n)$ time.
- $O(k \log n)$ for special case: exactly k-sparse.
- Faster than FFT when $n/k = \omega(1)$.
- Lower bounds:
  - $\Omega(k \log k)$ for special case assuming FFT is optimal.
  - For general case, $\Omega(k \log(n/k)/ \log\log(n/k))$ samples even with *adaptive* sampling.

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- If $\widehat{x}$ is sparse, recover it exactly.
  - In $O(k \log n)$ time.
- Caveats:
  - Additional $\|x\|_2^2 / n^{\Theta(1)}$ error. Alternatively, $\widehat{x}$ has poly(n) precision.
  - n must be a power of 2.

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Algorithm

Suppose $\widehat{x}$ is k-sparse, with integer coefficients in $\{-n^{\Theta(1)}, \ldots, n^{\Theta(1)}\}$.

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We can recover $\widehat{x}$ in $O(k \log n)$ time with $3/4$ probability.

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- Then: repeat on $\widehat{x} - \widehat{x'}$, with $k \to k/2$ and decreasing the error probability. [Eppstein-Goodrich '07]

Inspiration: arbitrary linear measurements

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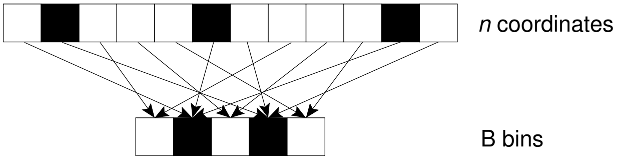

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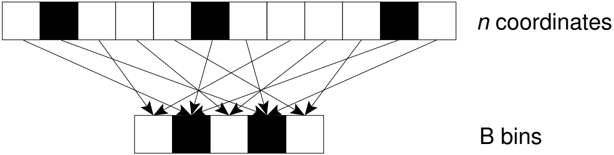

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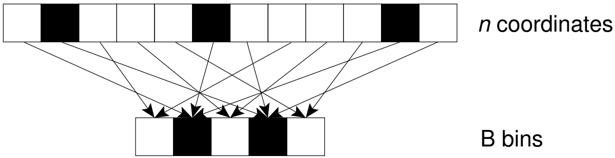

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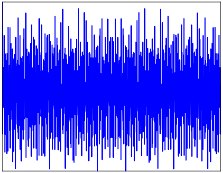
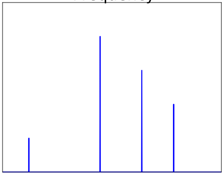
  - Will be able to do this in $O(B \log n)$ time.

What can you do with Fourier measurements?





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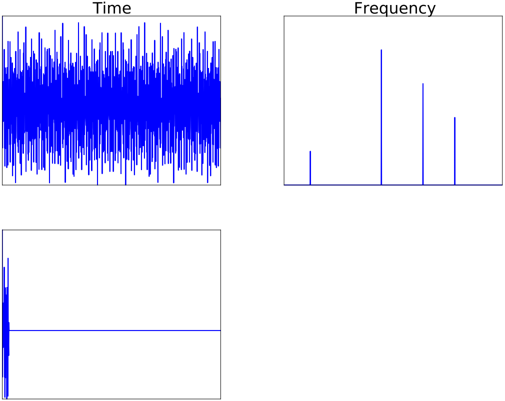

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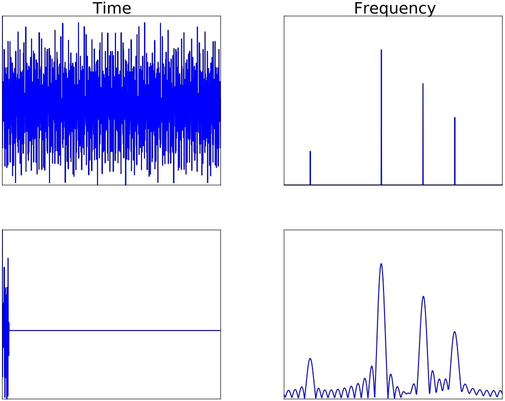

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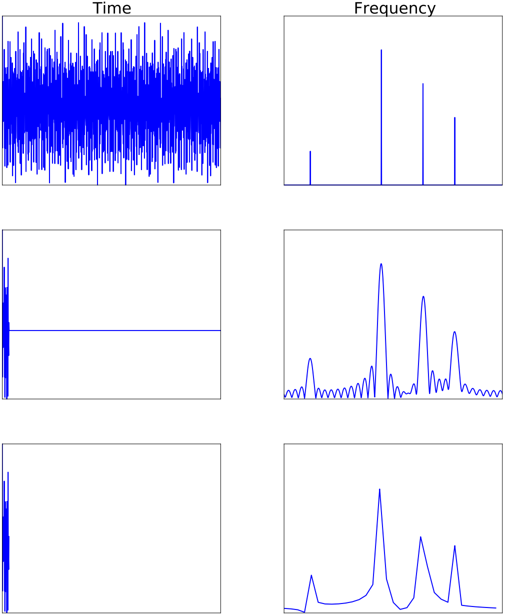

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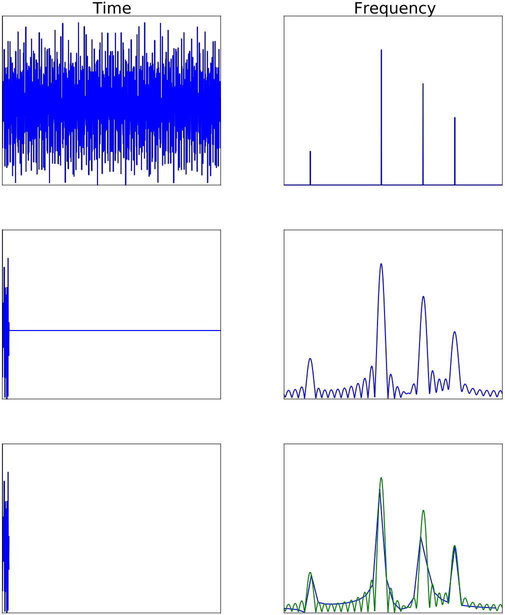

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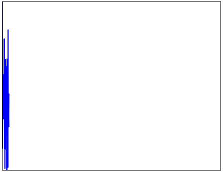

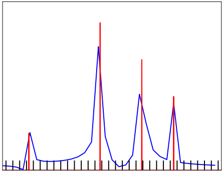

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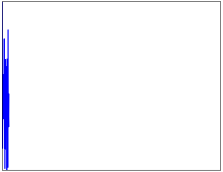

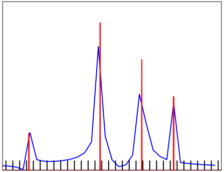

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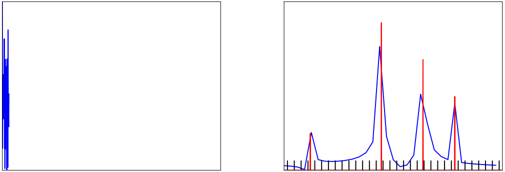

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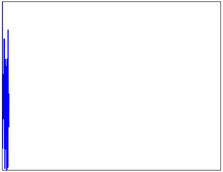

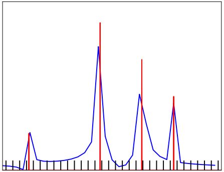

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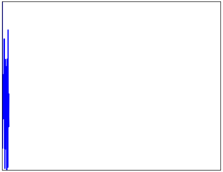

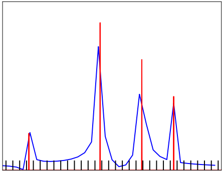

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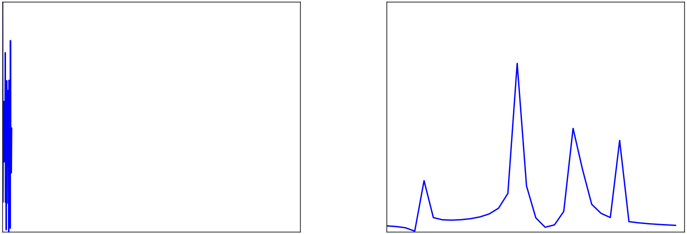

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- Observe

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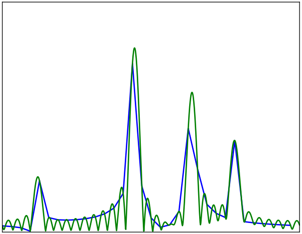

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$$\mathrm{DFT}(F \cdot x, B) = \mathrm{subsample}(\mathrm{DFT}(F \cdot x, n), B)$$

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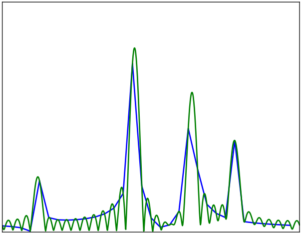

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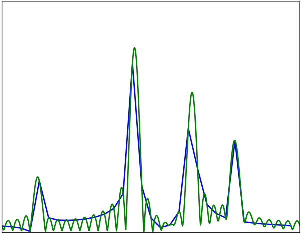
$$\mathrm{DFT}(F \cdot x, B) = \mathrm{subsample}(\mathrm{DFT}(F \cdot x, n), B) = \mathrm{subsample}(\widehat{F} * \widehat{x}, B).$$

Leakage



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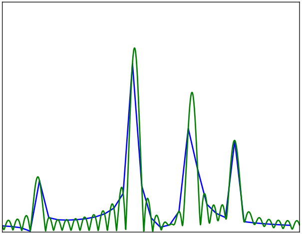

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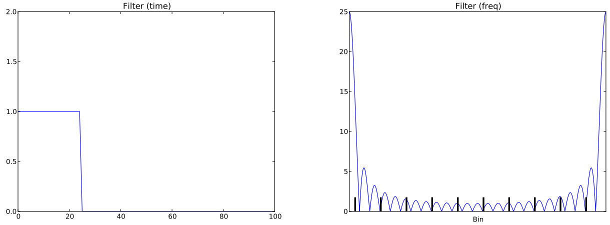
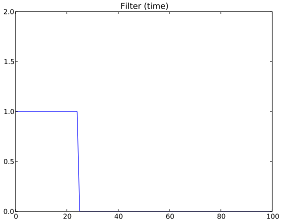
- DFT $\widehat{F}$ of boxcar filter is sinc, decays as $1/i$.
- Need a better filter F!

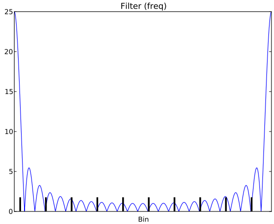
Filters



- Given $|\mathrm{supp}(F)| = B$, concentrate $\widehat{F}$.

Filters





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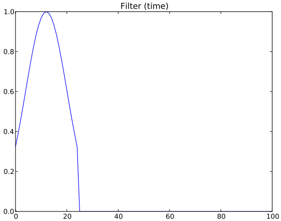
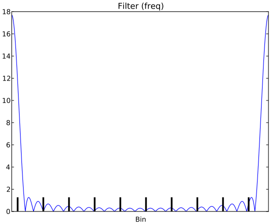
- Boxcar filter: decays perfectly in time, $1/t$ in frequency.
  - Non-trivial leakage everywhere.

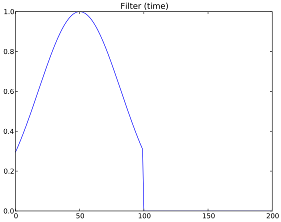
Filters

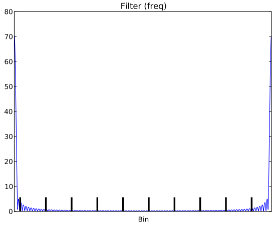




- Given $|\text{supp}(F)| = B$, concentrate $\widehat{F}$.
- Boxcar filter: decays perfectly in time, $1/t$ in frequency.
  - Non-trivial leakage everywhere.
- Gaussians: decay as e^{-t^2} in time and frequency.
  - Non-trivial leakage to $O(\sqrt{\log n} \cdot \sqrt{\log n}) = O(\log n)$ buckets.

Filters





- Given $|\text{supp}(F)| = B \log n$, concentrate $\widehat{F}$.
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- Gaussians: decay as e^{-t^2} in time and frequency.
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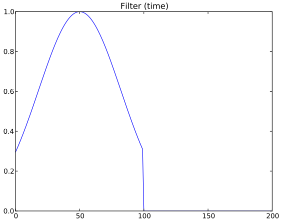
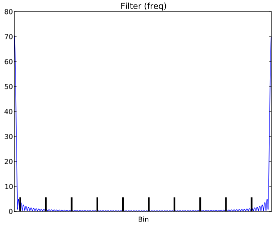
- Still $O(B \log n)$ time when $|\text{supp}(\widehat{F})| = B \log n$.
  - Non-trivial leakage to 0 buckets.

Filters





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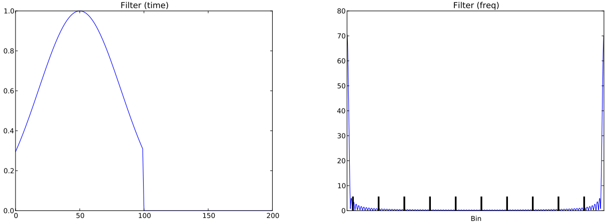
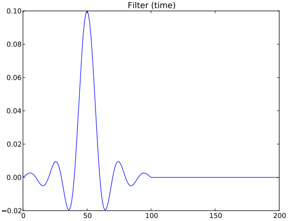
- Still $O(B \log n)$ time when $|\mathrm{supp}(\widehat{F})| = B \log n$.
  - Non-trivial leakage to 0 buckets.
  - Trivial contribution to correct bucket.

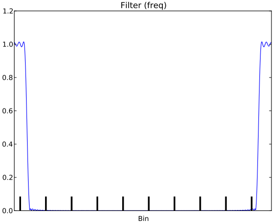
Filters



- Let G be Gaussian with $\sigma = B\sqrt{\log n}$
- H be box-car filter of length n/B.

Filters





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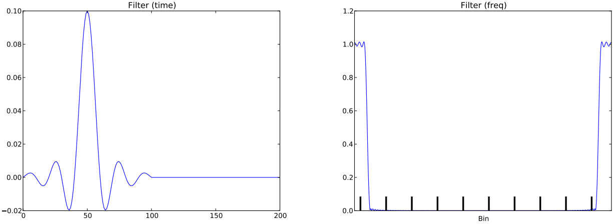
- Use $\widehat{F} = \widehat{G} * H$.

Filters



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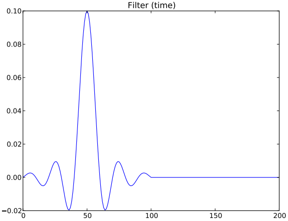

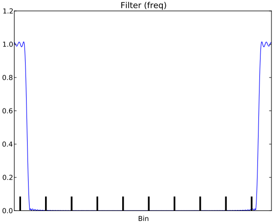

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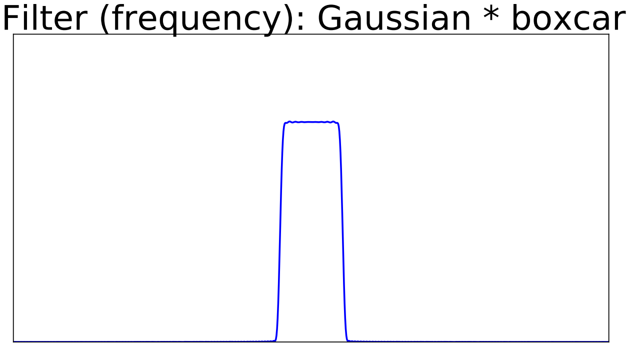
Properties of filter



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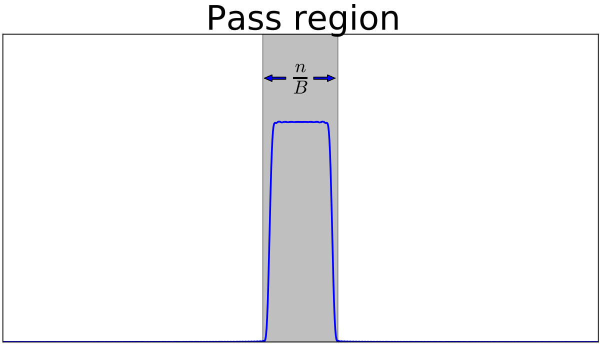
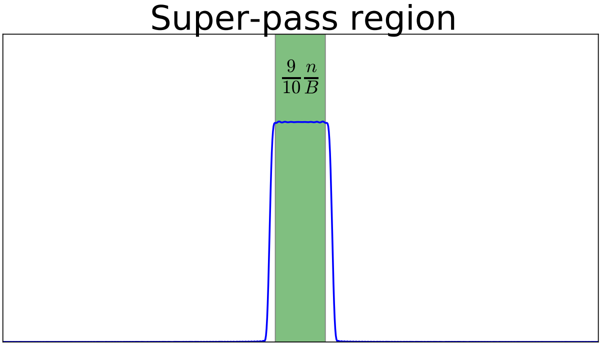


- "Pass region" of size n/B, outside which is negligible δ.

Properties of filter



- "Pass region" of size n/B, outside which is negligible δ.
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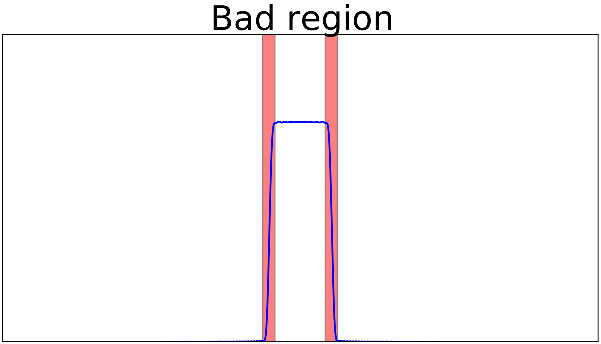

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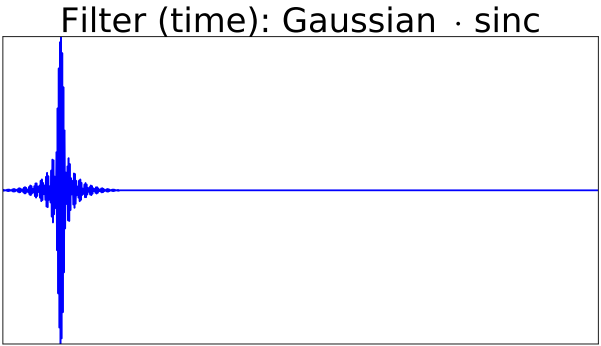
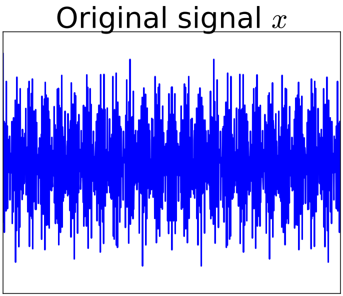
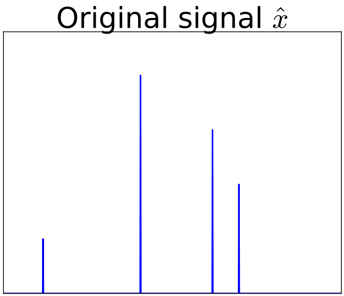


- "Pass region" of size n/B, outside which is negligible δ.
- "Super-pass region", where ≈ 1.
- Small fraction (say 10%) is "bad region" with intermediate value.
- Time domain has support size $O(B \log n)$.

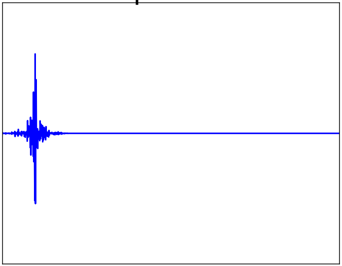
Algorithm for *exactly sparse* signals



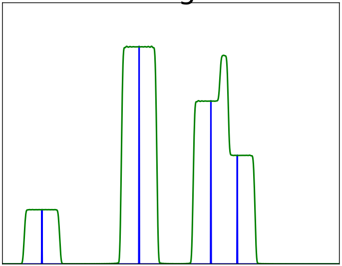


Algorithm for *exactly sparse* signals

Computed $F \cdot x$



Filtered signal $\widehat{F} * \widehat{x}$



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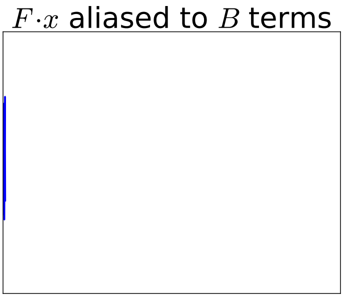

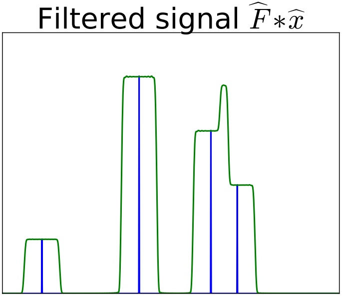

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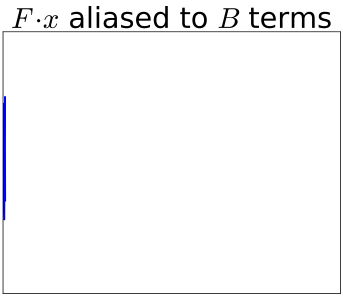

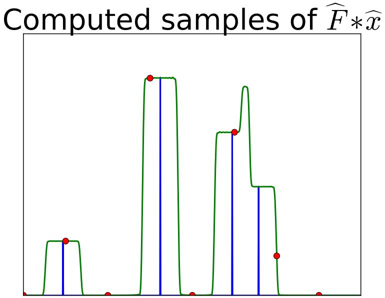

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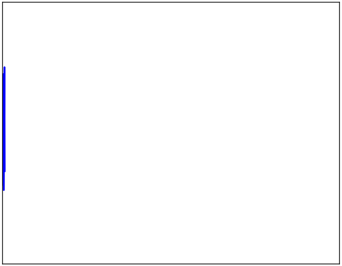
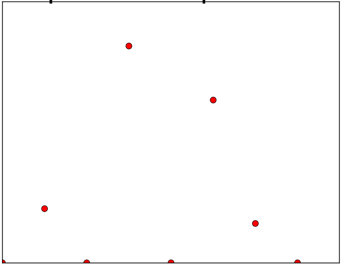
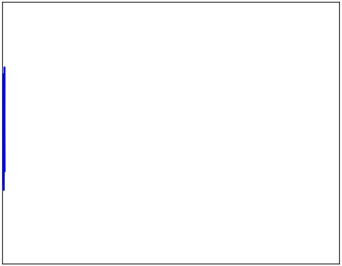


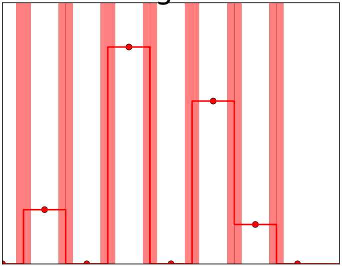


Algorithm for *exactly sparse* signals

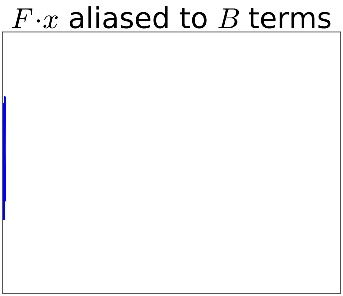
$F \cdot x$ aliased to B terms

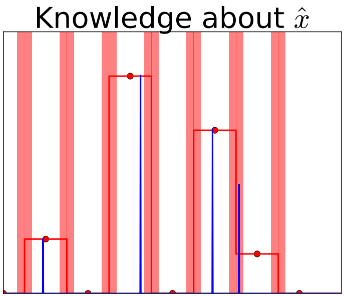


Knowledge about $\hat{x}$



Algorithm for *exactly sparse* signals





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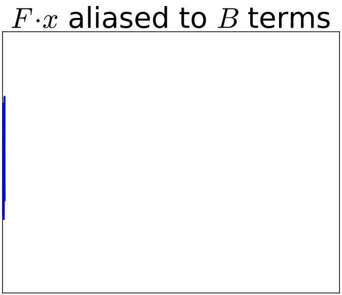
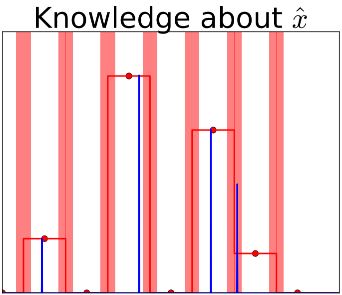




Lemma

If i is alone in its bucket and in the "super-pass" region,

$$u_{h(i)} = \widehat{x}_i.$$

Computing u takes $O(B \log n)$ time.

Algorithm for *perfectly sparse* signals

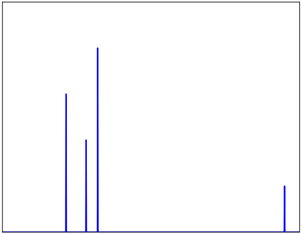
Lemma

If i is alone in its bucket and in the "super-pass" region,

$$u_{h(i)} = \widehat{x}_i.$$

- Time-shift x by one and repeat: $u'_{h(i)} = \widehat{x}_i \omega^i$.
- Divide to find i.

Permutation in time and frequency



- Can recover coordinates that are alone in their bucket and in the super-pass region.
- What if coordinates are near each other?

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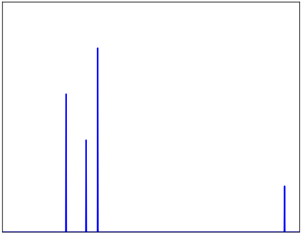

- Can recover coordinates that are alone in their bucket and in the super-pass region.
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- Define the "permutation"

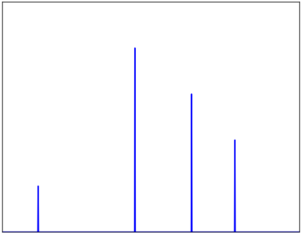
$$(P_{a,b}x)_i = x_{ai}\omega^{-ib}.$$

Then

$$\widehat{(P_{a,b}x)}_{ai+b} = \widehat{x}_i.$$

- For random a and b, each i is probably "well-hashed."

Permutation in time and frequency



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Overall algorithm

- Weak sparse recovery:
  - Permute with random a, b.
  - Hash to u
  - Time shift by one, hash to u'.
  - For $j \in [B]$
    - Choose i^* by $u'_j / u_j = \omega^{i^*}$.
    - Set $\widehat{x'}_{i^*} = u_j$.

Overall algorithm

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    - Set $\widehat{x'}_{i^*} = u_j$.
- Full sparse recovery:
  - $\widehat{x'} \leftarrow \texttt{WeakRecovery}(x, k)$
  - $k \rightarrow k/2$, $x \rightarrow (x - x')$, repeat.

Overall algorithm

- Weak sparse recovery:
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  - Time shift by one, hash to u'.
  - For $j \in [B]$
    - ★ Choose i^* by $u'_j / u_j = \omega^{i^*}$.
    - ★ Set $\widehat{x'}_{i*} = u_j$.
- Full sparse recovery:
  - $\widehat{x'} \leftarrow \texttt{WeakRecovery}(x, k)$
  - $k \rightarrow k/2$, $x \rightarrow (x - x')$, repeat.
- Time dominated by hash to $B_r = k/2^r$ buckets in round r:
  - $B_r \log n$ to hash x.
  - Hashing $\widehat{x'}$ takes $O(|\text{supp}(\widehat{x'})|) = O(k)$.

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- Time $\sum(\frac{k}{2^r} \log n + k) = O(k \log n)$.

Outline

1. Introduction
2. Special case: exactly sparse signals
3. General case: approximately sparse signals
4. Experiments

Nearly sparse signals

- What happens if only 90% of the mass lies in top k coordinates, not 100%?
- Want to find most "heavy" coordinates i with $|\widehat{x}_i|^2 > \|x_{\text{tail}}\|_2^2/k$.

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Lemma

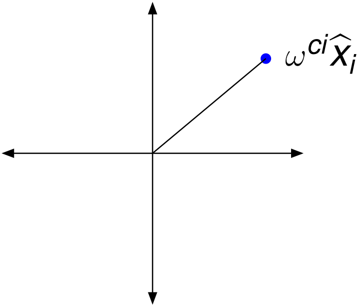
Each i is "well-hashed" with large constant probability over the permutation (a, b). If i is well-hashed, then with time shift c we have

$$u_{h(i)} = \widehat{x}_i \omega^{ci} + \eta$$

so that for random c, the noise η is bounded by

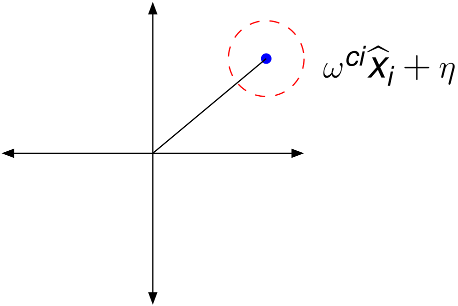
$$E[|\eta|^2] \lesssim \|x_{tail}\|_2^2/B$$

Recovering well-hashed i



- With good probability over c, get $u_{h(i)} = \widehat{x}_i \omega^{c\pi(i)} + \eta$ with $|\eta| < |\widehat{x}_i|/10$.

Recovering well-hashed i



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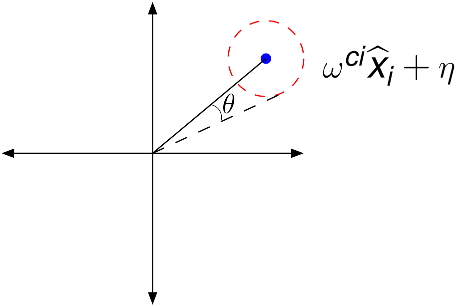

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- Phase error $|\theta| \leq \sin^{-1}(\frac{|\eta|}{|\widehat{x}_i|}) < 0.11$.

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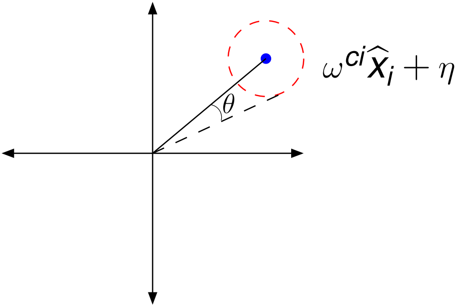
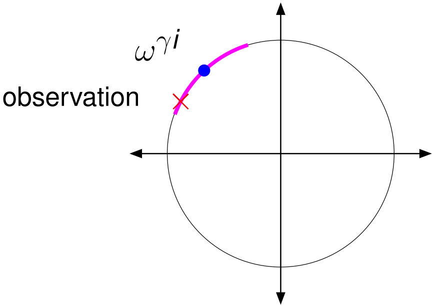


- With good probability over c, get $u_{h(i)} = \widehat{x}_i \omega^{c\pi(i)} + \eta$ with $|\eta| < |\widehat{x}_i|/10$.
- Phase error $|\theta| \leq \sin^{-1}(\frac{|\eta|}{|\widehat{x}_i|}) < 0.11$.
- True for random c. For a fixed γ, run on c and $c + \gamma$ to observe

$$\omega^{\gamma\pi(i)}$$

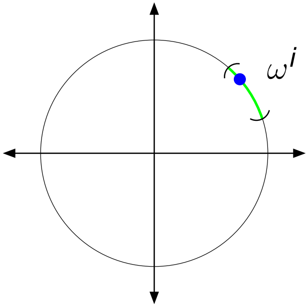
to within 0.22.

Recovering well-hashed i



- Find i from n/k possibilities in bucket.
- Choose any γ, then observe $\omega^{\gamma i}$ to within ± 0.1 radians.
- Constant number of bits, so hope for $\Theta(\log(n/k))$ observations.

Recovering well-hashed i



- We know i to within R.
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- Restrict and repeat, $\log(n/k)$ times.

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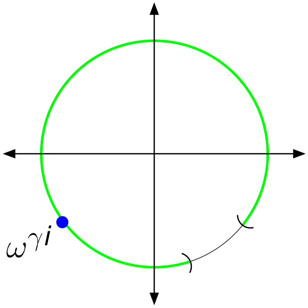

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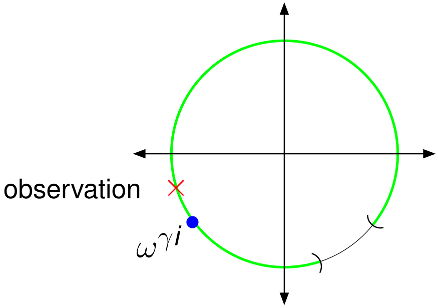

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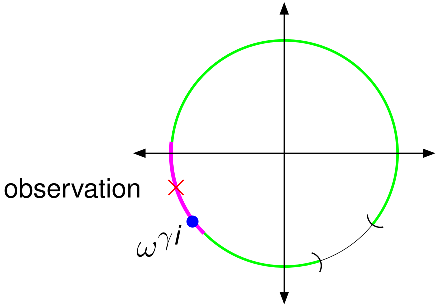

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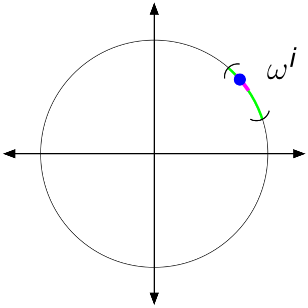

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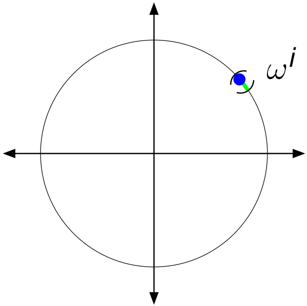

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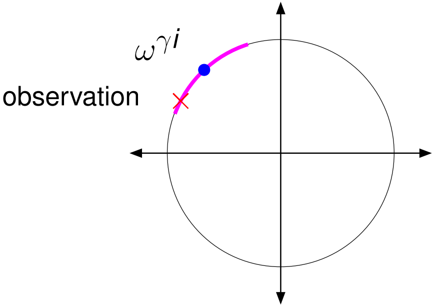

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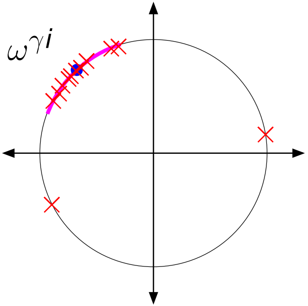

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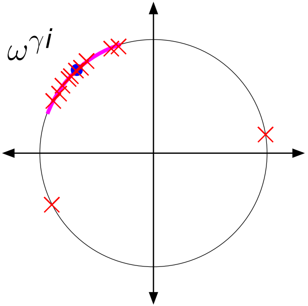

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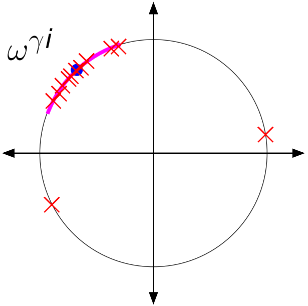

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- Two options:
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  - Can avoid the loss: learn $\log\log(n/k)$ bits at a time.

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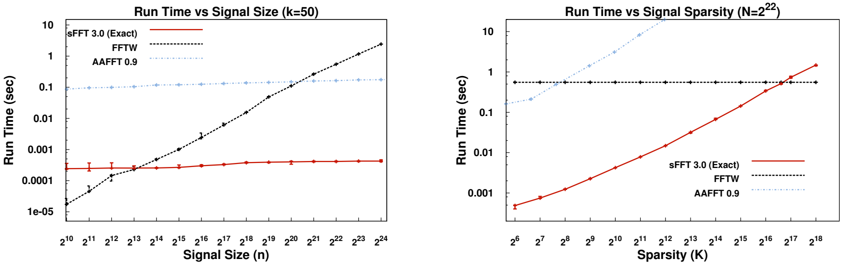
  - Instead: $B_i \asymp k/i^{\Theta(1)}$, $k_i \asymp k/i!$ gives

$$k \log n \log(n/k)$$

Outline

Empirical performance of exact sparse algorithm

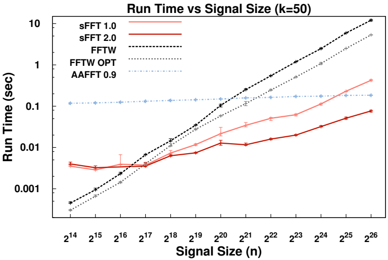


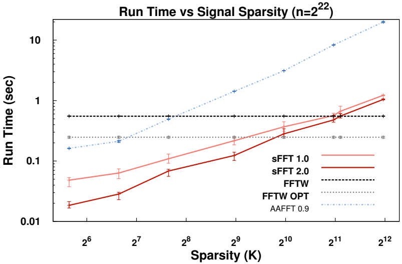
- Compare to FFTW, previous best sublinear algorithm (AAFFT).
- Faster than FFTW for $k/n < 3\%$.

Conclusions and Future Work

- $O(k \log n)$ for exactly sparse $\widehat{x}$
- $O(k \log \frac{n}{k} \log n)$ for approximation.
- Beats FFTW for $k/n < 3\%$ (in the exact case).
- Open problems:
  - Can we get $k \log n$ for approximate recovery?
  - Hadamard matrix / FFT over finite fields?
  - n not a power of 2?
  - Higher probability of success without $\log(1/\delta)$ slowdown?
  - Stronger approximation guarantee, like ℓ_∞/ℓ_2?
  - Better recovery of off-grid frequencies?

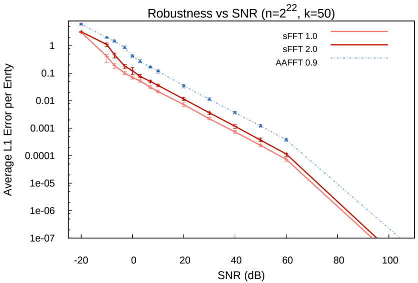
SODA empirical Performance: runtime





- Compare to FFTW, previous best sublinear algorithm (AAFFT).
- Offer a heuristic that improves time to $\widetilde{O}(n^{1/3}k^{2/3})$.
  - Filter from [Mansour '92].
  - Can't rerandomize, might miss elements.
- Faster than FFTW for $n/k > 2{,}000$.
- Faster than AAFFT for $n/k < 1{,}000{,}000$.

SODA empirical Performance: noise



- Just like in Count-Sketch, algorithm is noise tolerant.

Saving a $\log\log(n/k)$ factor

- Could use only $\log(n/k)$ samples by taking random γ:
  - For $\tau' \neq \tau$, $\omega^{\gamma(\tau'-\tau)}$ uniform over circle.
  - Hence $\omega^{\gamma\tau'}$ probably far from the observations.
  - Distinguish among n/k possibilities with $\log(n/k)$ samples.
- Takes $n/k \log(n/k)$ time to test all possibilities.
- Idea: mix the two approaches.
  - Split region into $\log(n/k)$ subregions of size w.
  - Choose random $\gamma \in [\frac{n}{8w}, \frac{n}{4w}]$.
  - Small enough that subregions remain local.
  - Large enough that far subregions roughly uniform.
  - Identify subregion exhaustively: $\log\log(n/k)$ measurements and $\log(n/k)\log\log(n/k)$ time.
  - Repeat $\log_{\log(n/k)}(n/k)$ times to identify τ.
  - Total $\log(n/k)$ measurements, $\log^2(n/k)$ time.