Tutorial on Sparse Fourier Transforms

Eric Price

UT Austin

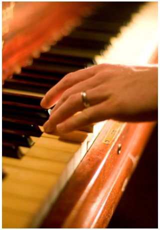
The Fourier Transform

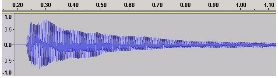
Conversion between time and frequency domains

Time Domain

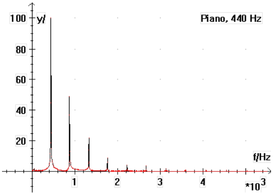
Frequency Domain

Fourier Transform ⟷





Displacement of Air

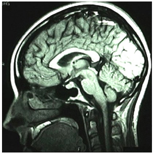


Concert A

The Fourier Transform is Ubiquitous

Audio

Video



Medical Imaging

Radar

GPS

Oil Exploration

Computing the Discrete Fourier Transform

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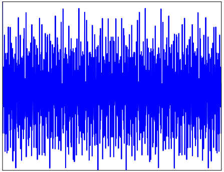
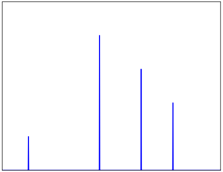
When can we compute the Fourier Transform in *sublinear* time?

Idea: Leverage *Sparsity*

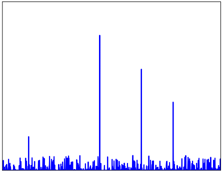
Often the Fourier transform is dominated by a small number of peaks:



Time Signal



Frequency
(Exactly sparse)



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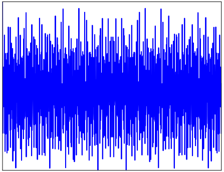

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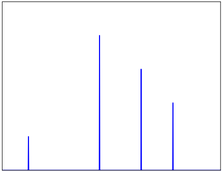

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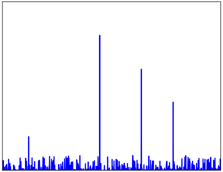

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Sparsity is common:

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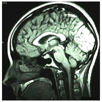

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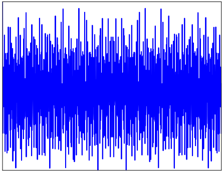

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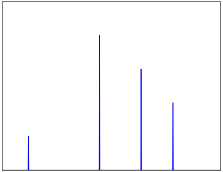

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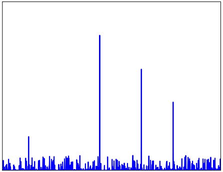

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Sparsity is common:

Goal of this workshop: *sparse* Fourier transforms

Faster Fourier Transform on sparse data.

Classes of sparse Fourier transform algorithms

For recovering a k-sparse signal in n dimensions.

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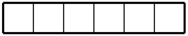

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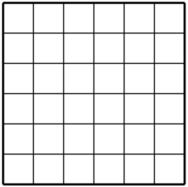

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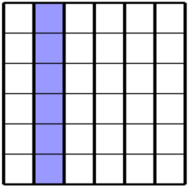

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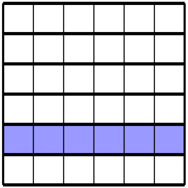

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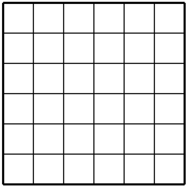

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- Hadamard transform: $x \in \mathbb{C}^{2 \times 2 \times \cdots \times 2}$:

$$\widehat{x}_i = \sum_{j}^{n} (-1)^{\langle i,j \rangle} x_j$$

Generic Algorithm Outline

- Goal: given access to x, compute $\overline{x} \approx \widehat{x}$
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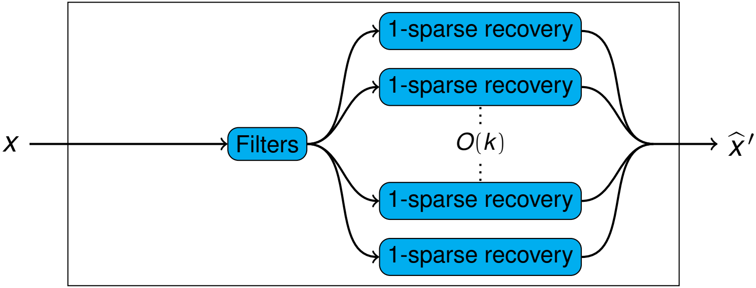

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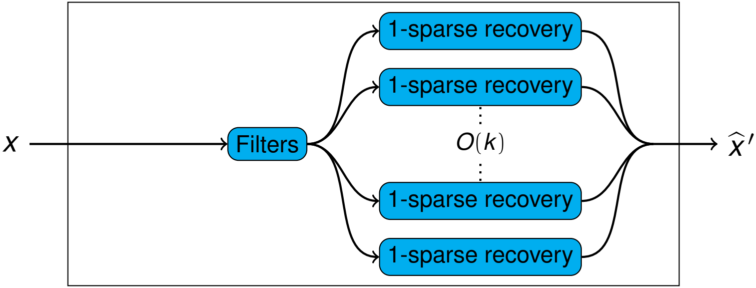

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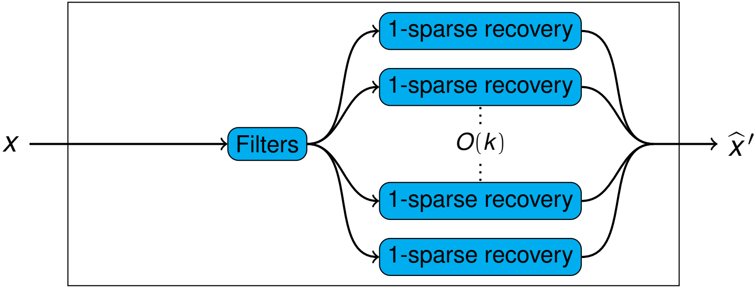

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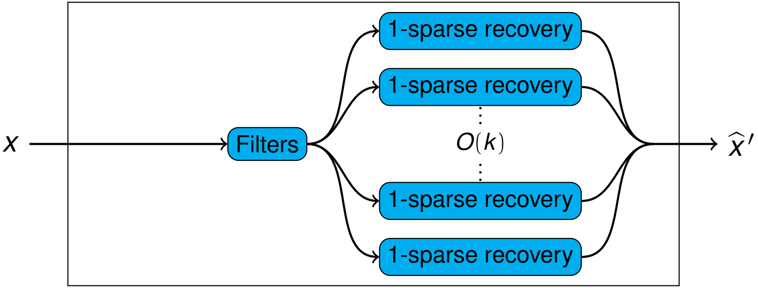

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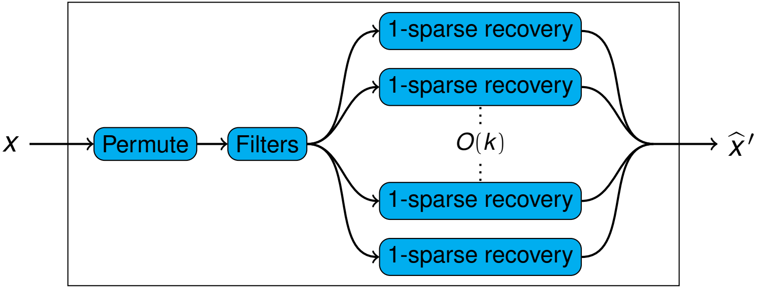

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3. Finds "most" of signal; repeat on residual

Talk Outline

1. Algorithm for $k = 1$

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Algorithm for $k = 1$: one dimension, exact case

Lemma

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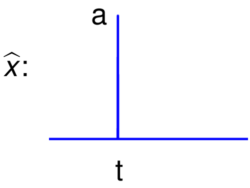

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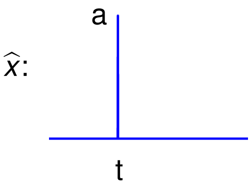

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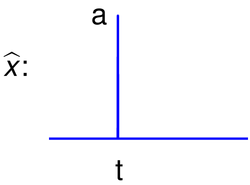

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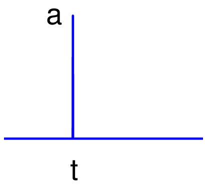
$\widehat{x}$: 

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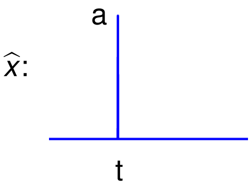

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- Then $x = (a, a\omega^t, a\omega^{2t}, a\omega^{3t}, \ldots, a\omega^{(n-1)t})$.

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- $x_1 / x_0 = \omega^t \implies t$.

Algorithm for $k = 1$: one dimension, exact case

Lemma

We can compute a 1-sparse $\widehat{x}$ in $O(1)$ time.

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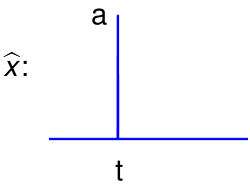

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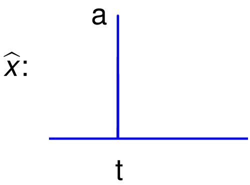

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- $x_1/x_0 = \omega^t \implies t$. ∎
- (Related to OFDM, Prony's method, matrix pencil.)

Algorithm for $k = 1$: one dimension, approximate case

Lemma

Suppose $\widehat{x}$ is approximately 1-sparse:

$$|\widehat{x}_t| / \|\widehat{x}\|_2 \geqslant 90\%.$$

Then we can recover it with $O(\log n)$ samples and $O(\log^2 n)$ time.

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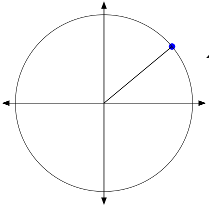

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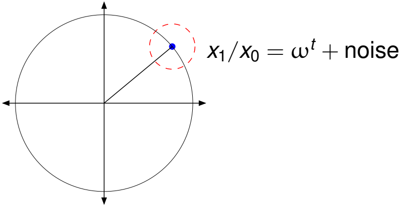

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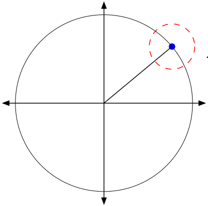

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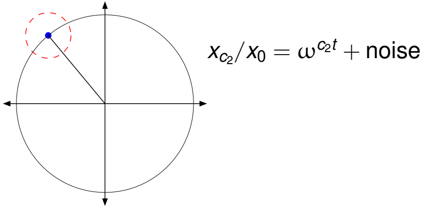

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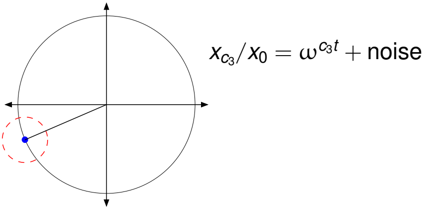

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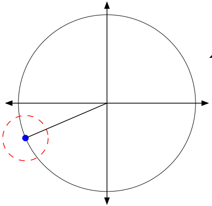

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- Error correcting code with efficient recovery $\implies$ lemma. ∎

Algorithm for $k = 1$: Hadamard setting

Levin '93, improving upon Goldreich-Levin '89

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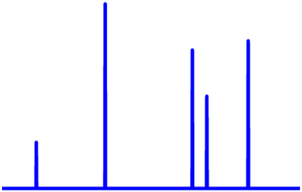
- Choose i to be the $O(\log n)$ rows of generator matrix for constant rate and distance binary code.

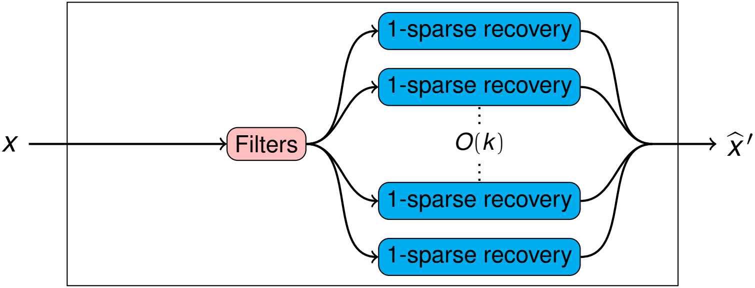
Talk Outline

1. Algorithm for $k = 1$
2. Reducing k to 1
3. Putting it together

Algorithm for general k

- Reduce general k to $k = 1$.





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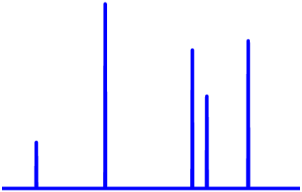

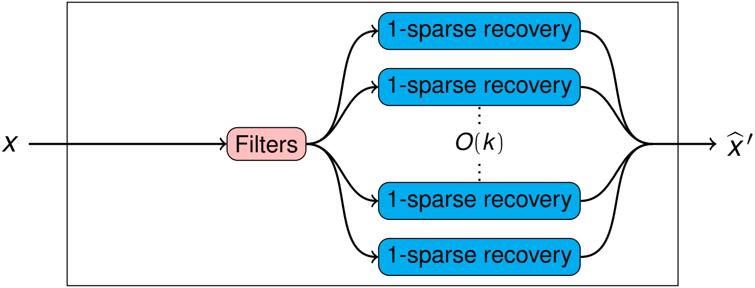

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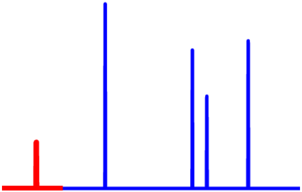

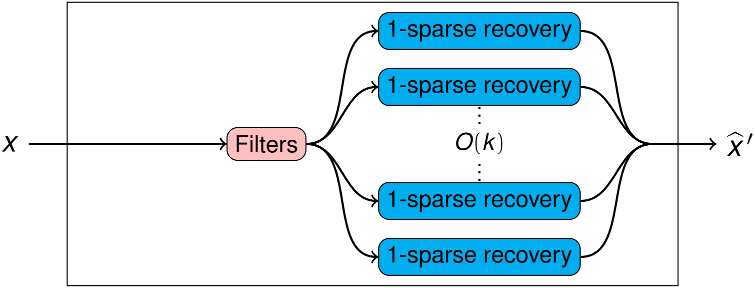

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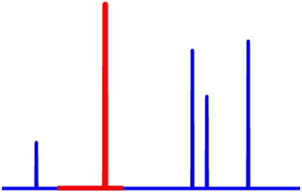

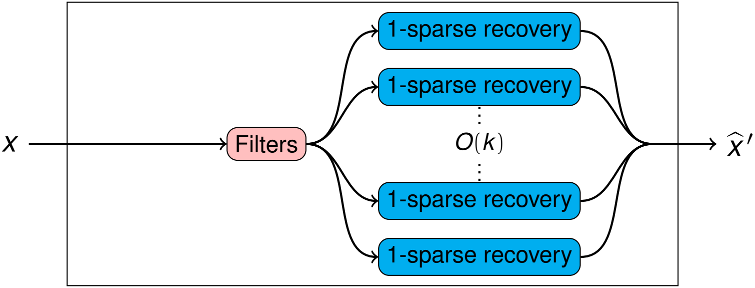

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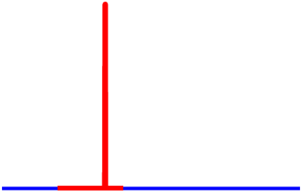

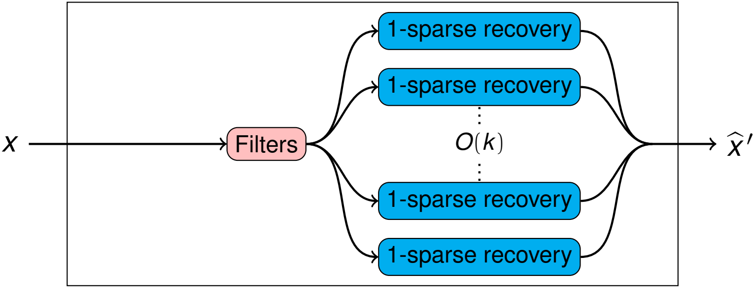

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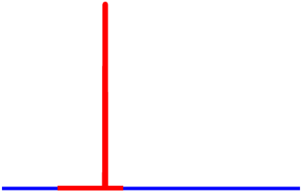

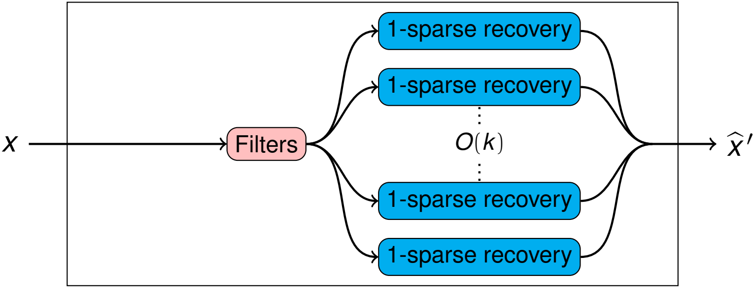

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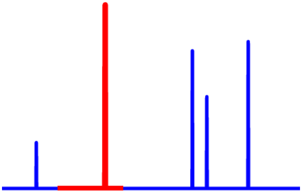

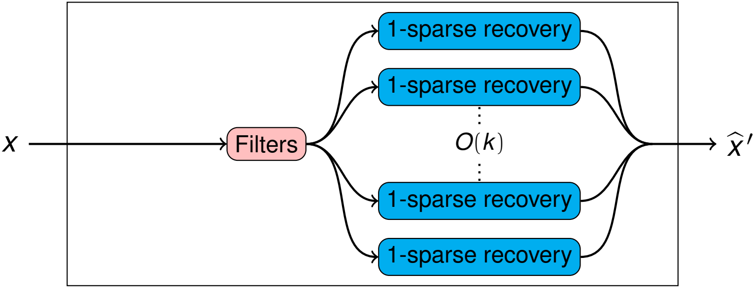

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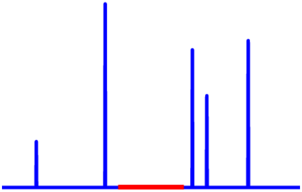

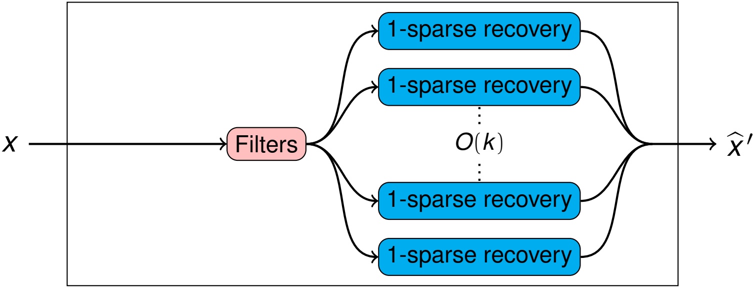

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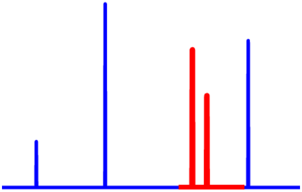

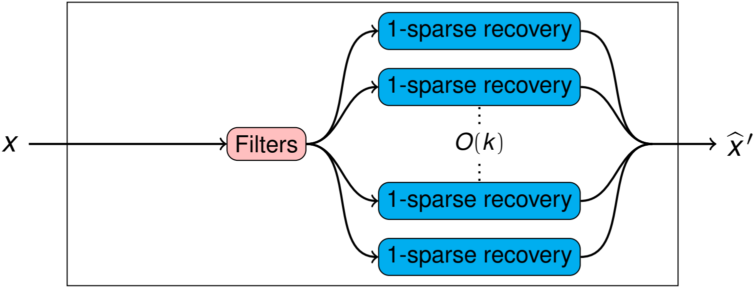

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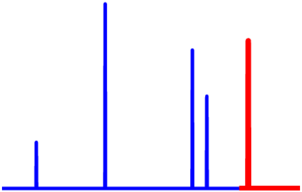

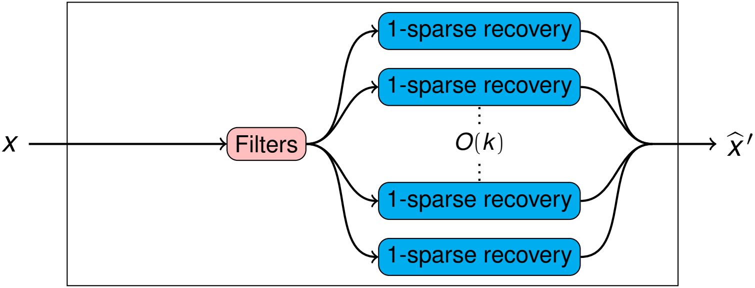

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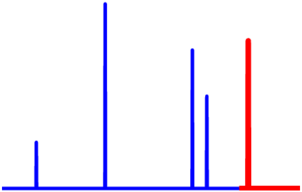

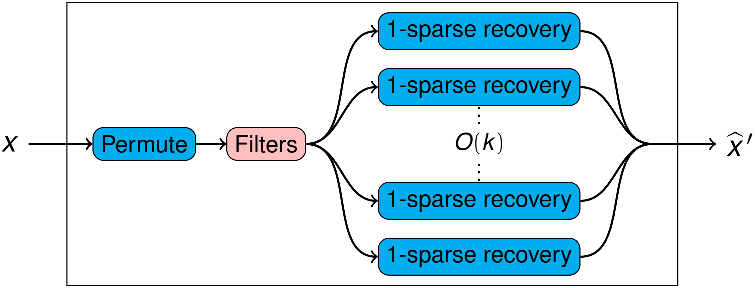

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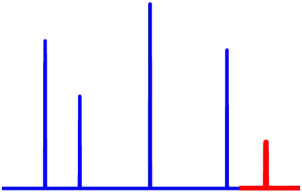

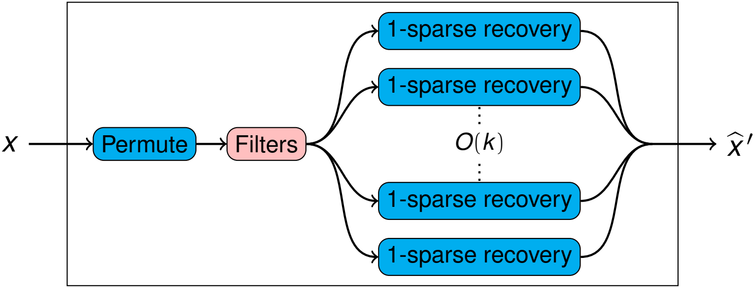

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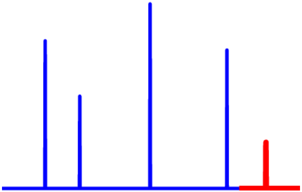

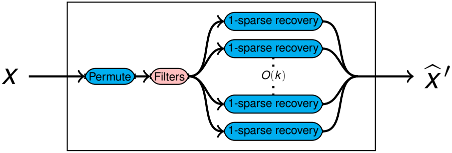

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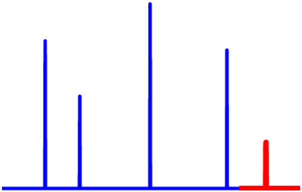
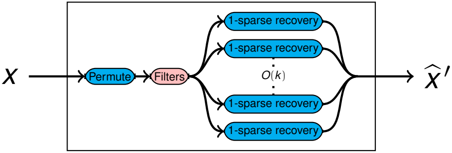


Recovers *most* of $\widehat{x}$:

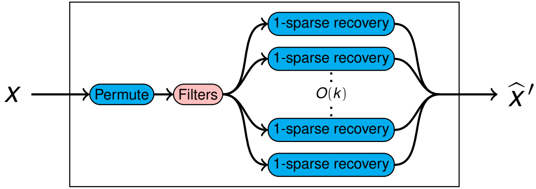


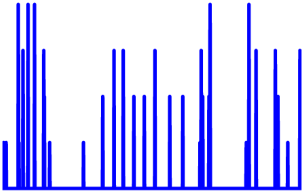
Lemma (Partial sparse recovery)

In $O(k \log n)$ expected time, we can compute an estimate $\widehat{x}'$ such that $\widehat{x} - \widehat{x}'$ is $k/2$-sparse.

Going from finding most coordinates to finding all $\widehat{x}$

Partial k-sparse recovery





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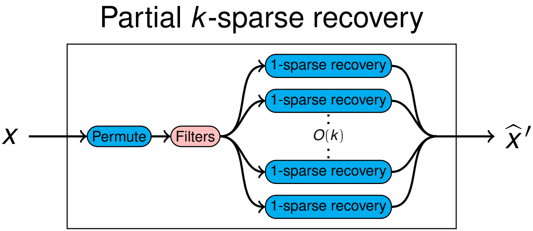

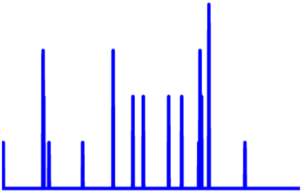

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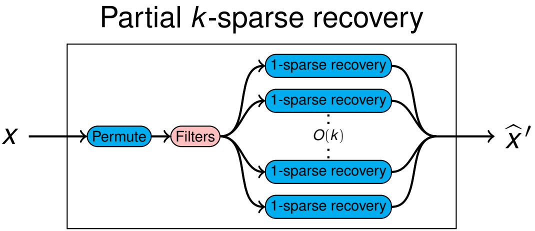

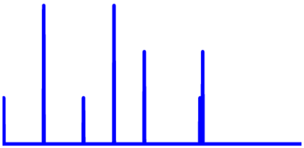

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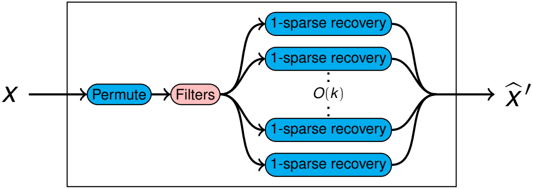

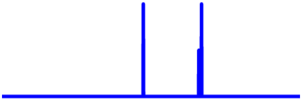

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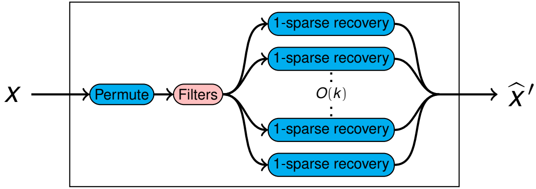

Lemma (Partial sparse recovery)

In $O(k \log n)$ expected time, we can compute an estimate $\widehat{x}'$ such that $\widehat{x} - \widehat{x}'$ is $k/2$-sparse.

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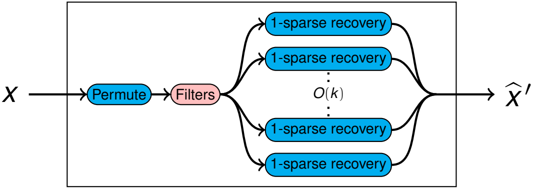

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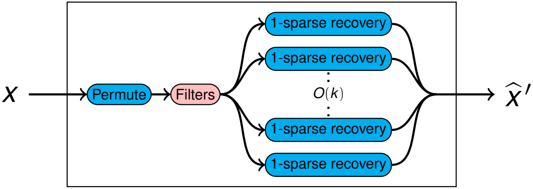

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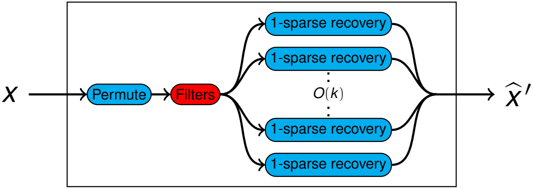

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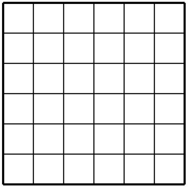

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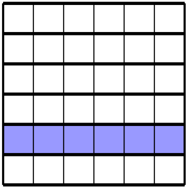

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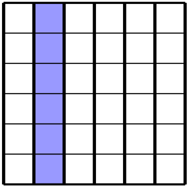

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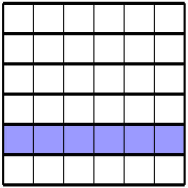

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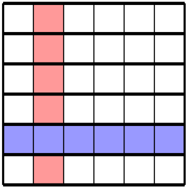

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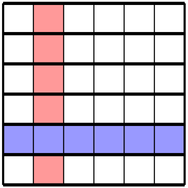

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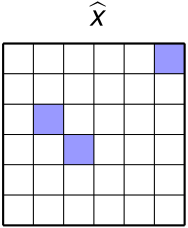

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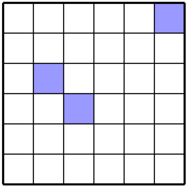

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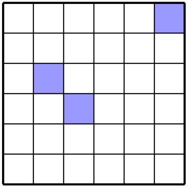

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- If $k = \sqrt{n}$ random nonzeros, expect to recover most of them.

Filters more generally

- Fourier transform switches multiplication and convolution.

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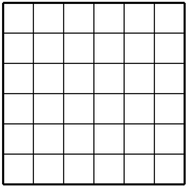

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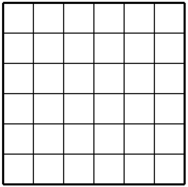

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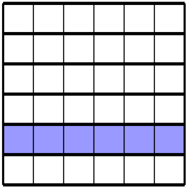

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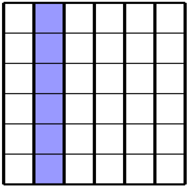

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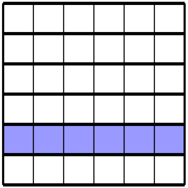

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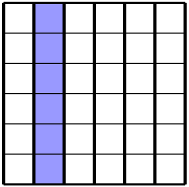

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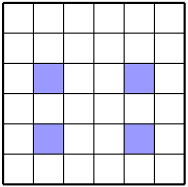

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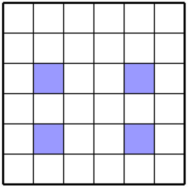

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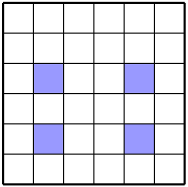

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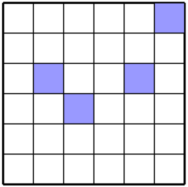

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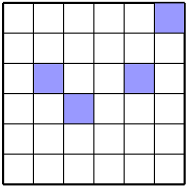

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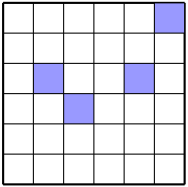

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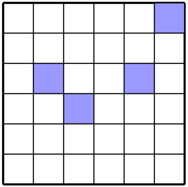

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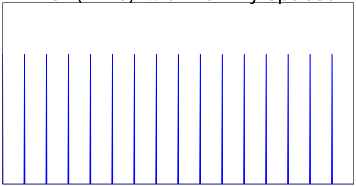
- For worst-case inputs, need other filters

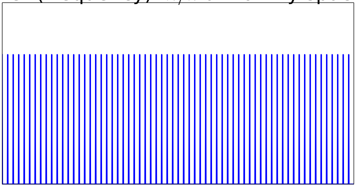
A different style of filter

GMS05, HIKP12, IKP14, IK14

Filter (time): k uniformly spaced



Filter (frequency): n/k uniformly spaced



- Previous slides used *comb filter*

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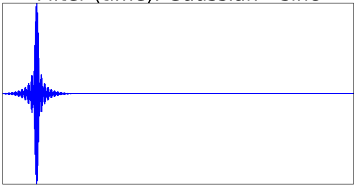

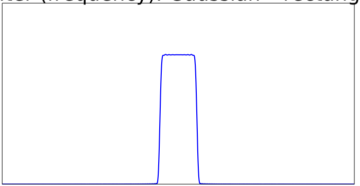

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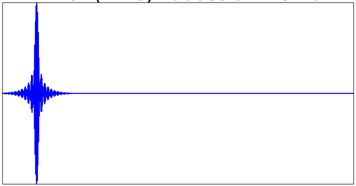
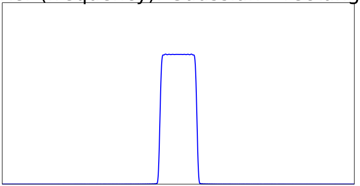
Filter (time): Gaussian · sinc



Filter (frequency): Gaussian * rectangle



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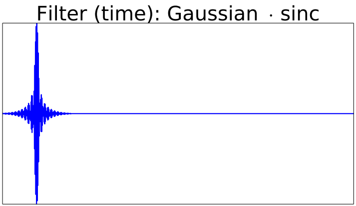
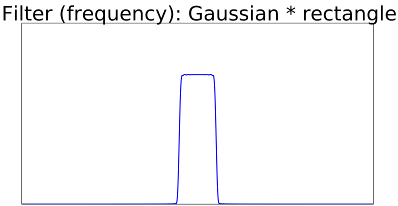
- We can permute the frequencies:

$$x'_i = x_{\sigma i} \implies \widehat{x}_i = \widehat{x}_{\sigma^{-1} i}$$

A different style of filter

GMS05, HIKP12, IKP14, IK14





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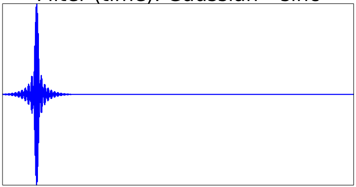

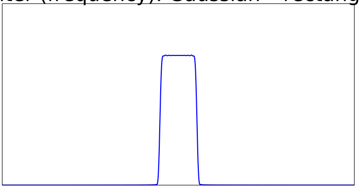

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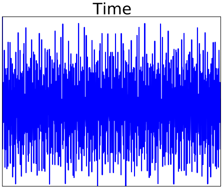
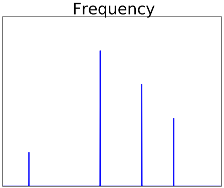
- Allows us to convert worst case to random case.

Talk Outline

1. Algorithm for $k = 1$
2. Reducing k to 1
3. Putting it together

How can you hope for sublinear time?





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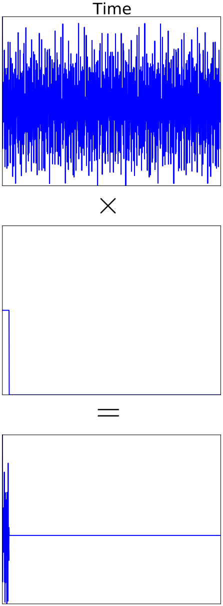

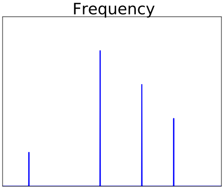

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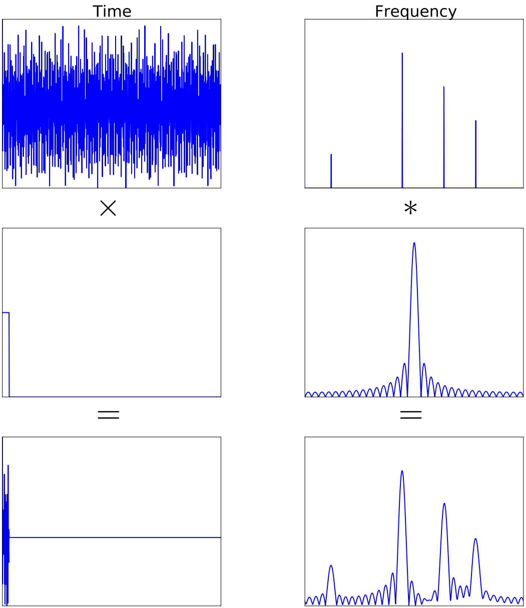

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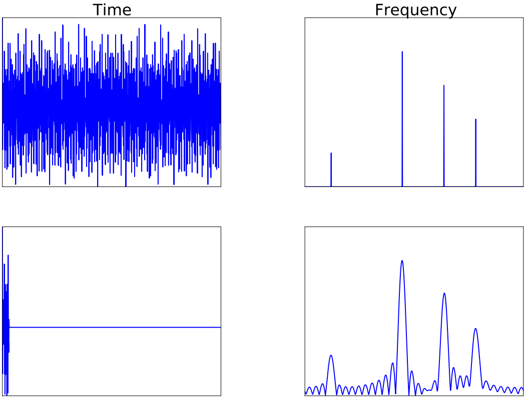

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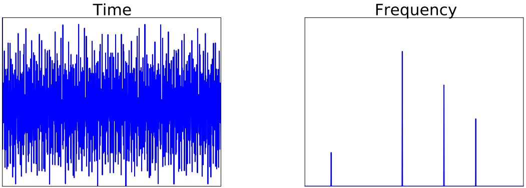

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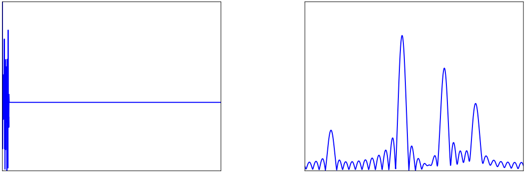

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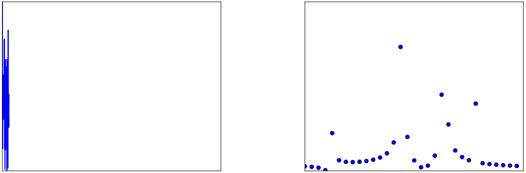

k-dimensional DFT of first *k* terms: $O(B \log B)$
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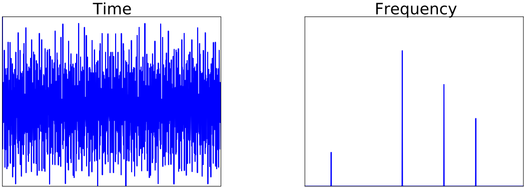

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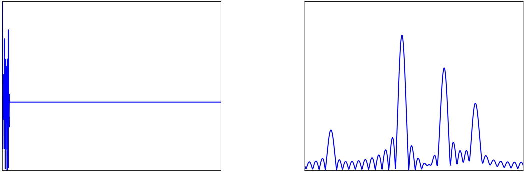

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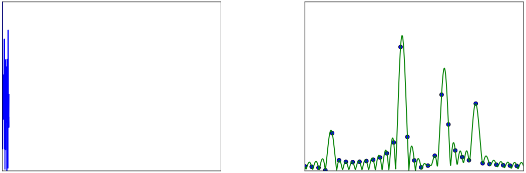

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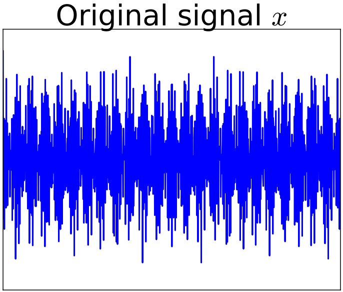
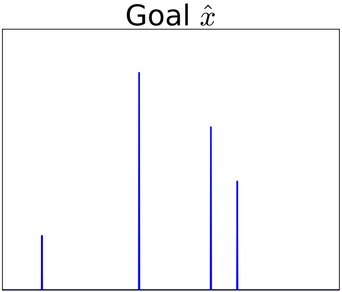
Algorithm for *exactly sparse* signals





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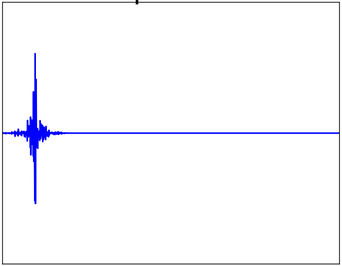

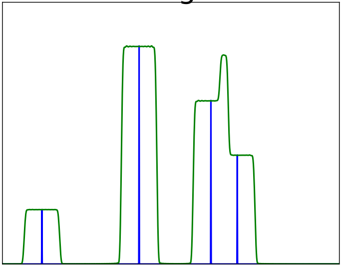

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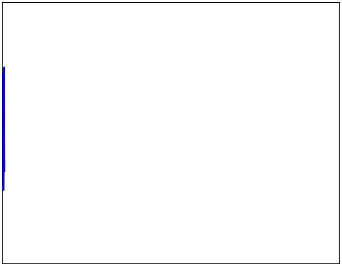
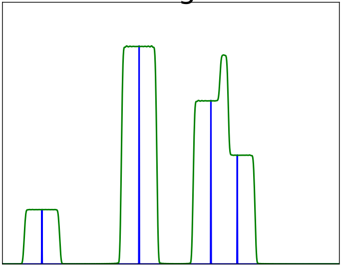
$F \cdot x$ aliased to k terms



Filtered signal $\widehat{F} * \widehat{x}$



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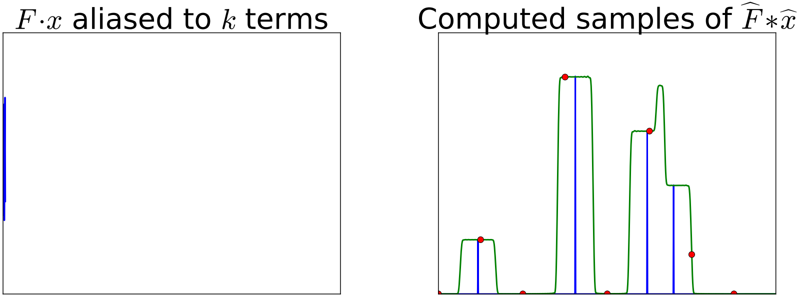

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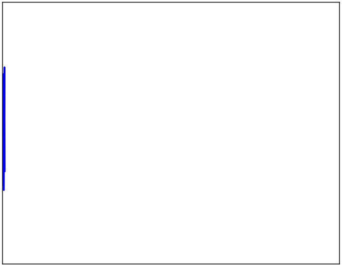
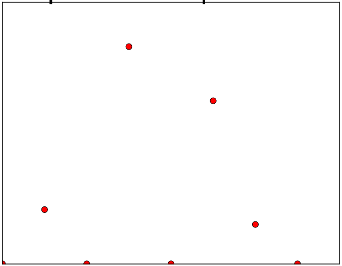


Computed samples of $\widehat{F} * \widehat{x}$



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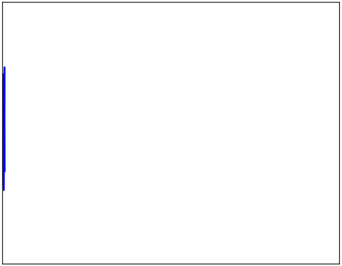
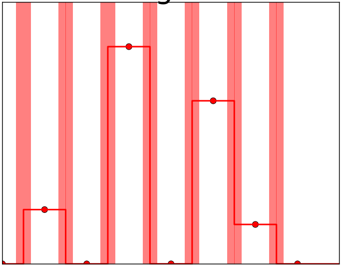


Knowledge about $\hat{x}$



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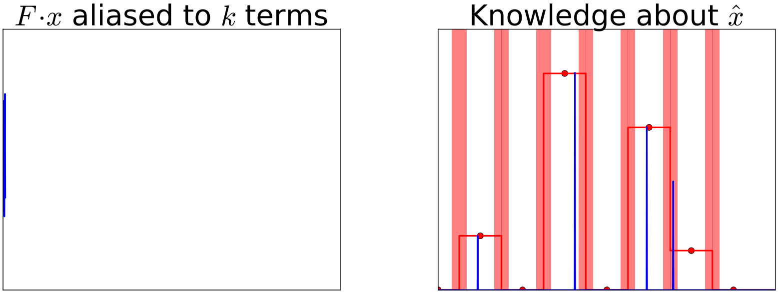

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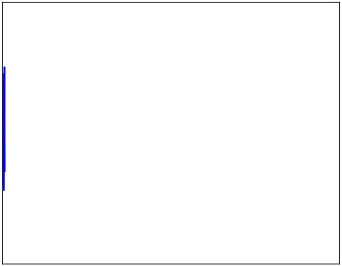

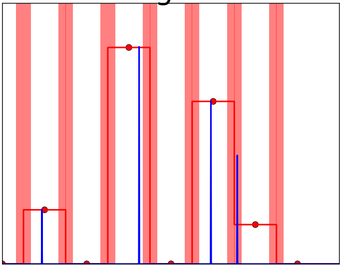

Lemma

If t is isolated in its bucket and in the "super-pass" region, the value b we compute for its bucket satisfies

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Computing the b for all $O(k)$ buckets takes $O(k \log n)$ time.

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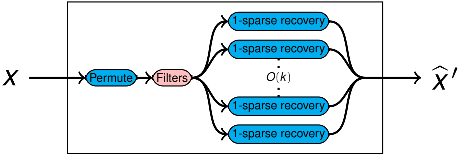

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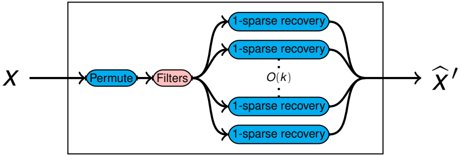

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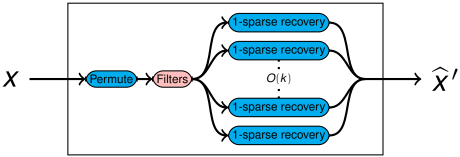

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Thank You