

Sparse FFT via matrix pencil

Laurent Demanet

Department of Mathematics, MIT

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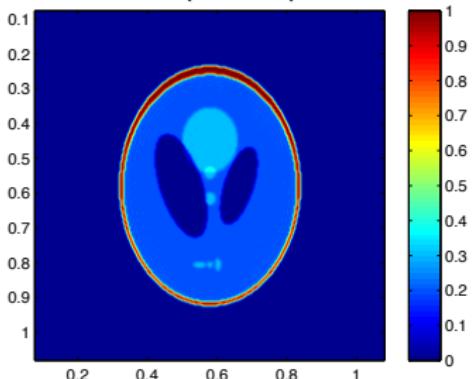
Projects in the Imaging and Computing Group

Interferometric Waveform Inversion

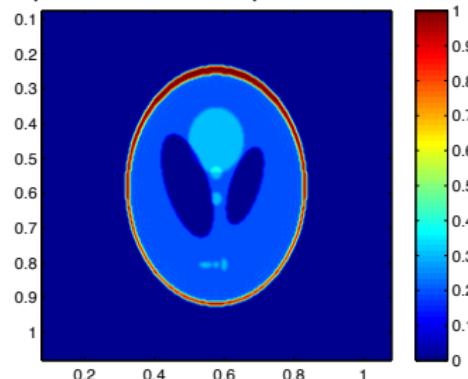
$$\min_x \|b - Ax\|$$

$$\min_{X \succeq 0} \|bb^T - AXA^T\|$$

Classic least squares optimization



Optimization over quadratic data



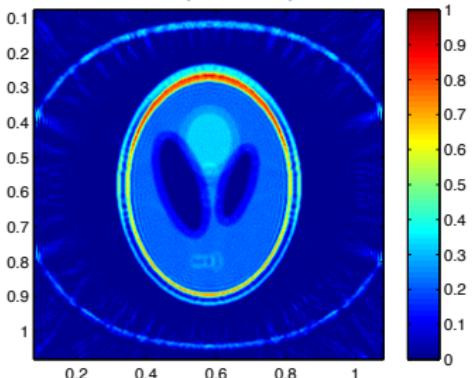
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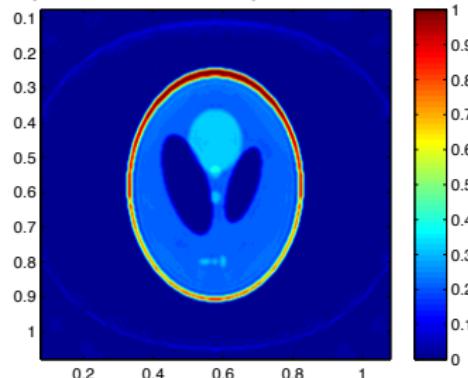
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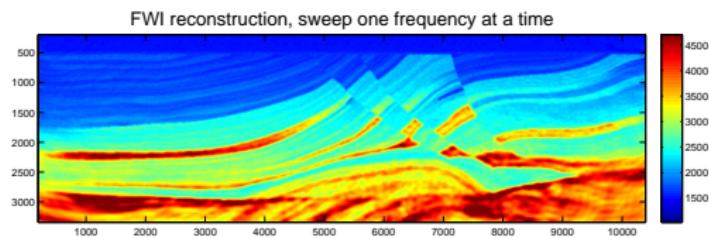
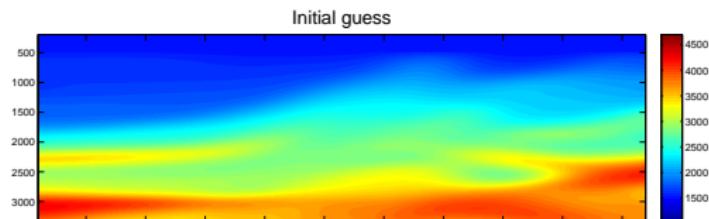
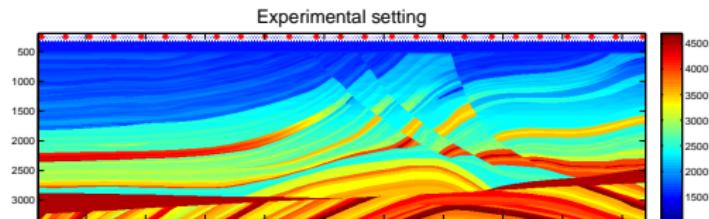


Optimization over quadratic data



Projects in the Imaging and Computing Group

Model velocity estimation: $\min_x \|b - \mathcal{A}(x)\|$

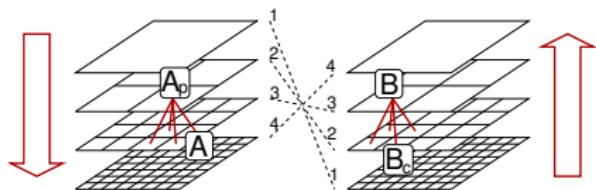


Projects in the Imaging and Computing Group

Fast algorithms for synthetic aperture radar (SAR)



Butterfly algorithm



RMS 3e-2, speedup 400

Agenda

Sparse FFT via matrix pencil (MPFFT) w/ Jiawei Chiu

- sFFT identifies modes by **phase extraction** from bin coeff.
- Old idea from signal processing: frequency identification by comparing a signal to its translates. Known as **matrix pencil** / Prony / FRI / HSVD / Pisarenko / AAK / ...
- Judicious use of the matrix pencil **speeds up sFFT**: complexity $O(S \log N (\log \frac{N}{S})^2)$, robust to noise, benchmarkable.

Binning for sFFT (GGIM'02, GMS'05, IGS'07, HIKP'12,...)

Classical mode randomization.

Sampling in t : $x(t)$, $t = 0, \dots, B - 1$, and chunks translated by τ .

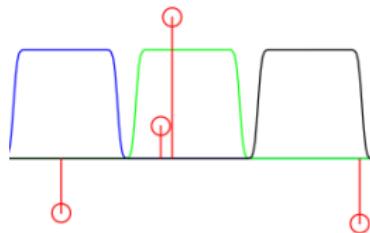
• 0 1 2 • 4 • 8 • •

• 16 • •

• 32 • •

• 64 • •

Sampling in k :



Expect $B \sim S$.

B “basic” bin coefficients:

$$y_b^0 = \text{FFT}_b(w x), \quad b = 0, \dots, B - 1$$

B bin coefficients from each τ :

$$y_b^\tau = \text{FFT}_b(w_\tau x), \quad b = 0, \dots, B - 1$$

Phase extraction

Bin coefficient = mode superposition:

$$y_b^0 = \sum_k \hat{w} \left(k - b \frac{N}{B} \right) \hat{x}(k)$$

Bin coefficient for **translates** = **modulated** mode superposition:

$$y_b^{\tau} = \sum_k \hat{w} \left(k - b \frac{N}{B} \right) \hat{x}(k) e^{2\pi i \tau k / N}$$

No collision / isolated mode at $k = k_0$:

$$\frac{y_b^1}{y_b^0} = e^{2\pi i \tau k_0 / N} \quad \Rightarrow \quad k_0$$

Then $\hat{x}(k_0) = y_b^0 / \hat{w} \left(k_0 - b \frac{N}{B} \right)$.

Phase extraction via a Hankel matrix

$$\begin{pmatrix} y^0 & y^1 \end{pmatrix} = y^0 \begin{pmatrix} 1 & e^{i\theta} \end{pmatrix}.$$

- “**OFDM trick**”: $y^1/y^0 = e^{i\theta}$ when the mode is isolated
- **Hankel matrix**: pick the next y^2 , form

$$Y = \begin{pmatrix} y^0 & y^1 \\ y^1 & y^2 \end{pmatrix} = y^0 \begin{pmatrix} 1 & e^{i\theta} \\ e^{i\theta} & e^{2i\theta} \end{pmatrix}.$$

- $\text{rank}(Y) = 1$ when the mode is isolated
- Find θ from

$$\begin{pmatrix} y^1 & y^2 \end{pmatrix} = e^{i\theta} \begin{pmatrix} y^0 & y^1 \end{pmatrix}$$

Matrix pencil method (Hua, Sarkar, 1990)

Generalization: $y^j = \sum_{\ell=1}^L c_\ell e^{ij\theta_\ell}$, $j = 0, \dots, 2J - 2$

$$Y = \begin{pmatrix} y^0 & y^1 & \cdots & y^{J-1} \\ y^1 & y^2 & \cdots & y^J \\ \vdots & \vdots & & \vdots \\ y^{J-1} & y^J & \cdots & y^{2J-2} \end{pmatrix}.$$

- $\text{rank}(Y) = \text{number of modes}$ (when $J > L$)
- Find $z_\ell = e^{i\theta_\ell}$ the **rank-reducing** numbers of the pencil

$$\overline{Y} - z\underline{Y}$$

(\overline{Y} : knock off top row; \underline{Y} : knock off bottom row)

Prony's method

Return to

$$Y = \begin{pmatrix} y^0 & y^1 \\ y^1 & y^2 \end{pmatrix} = y^0 \begin{pmatrix} 1 & e^{i\theta} \\ e^{i\theta} & e^{2i\theta} \end{pmatrix}.$$

Alternatively, $Yc = 0$ with

$$c = (e^{i\theta} \quad -1)^t$$

- Form $p(x) = \sum c_n x^n = e^{i\theta} - x$.
- Then $e^{i\theta}$ is a root of $p(x)$.

Generalizes to $J > 1$: **Prony's method**.

Special case of matrix pencil. Numerically inferior.

Back to sFFT (GGIM'02, GMS'05, IGS'07, HIKP'12,...)

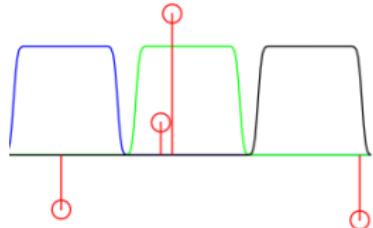
Sampling in t : $x(t)$, $t = 0, \dots, B - 1$, and chunks translated by τ .

• 0 1 2 4 • 8 • • 16 •

• 32 •

• 64 •

Sampling in k :



- Compute bin coefficients y_b^τ .
 - $\forall b$, build Y_b with $J = 2$ or 3
 - If $\text{rank}(Y_b) > 1$, discard bin
 - MP estimation of k_0
 - Estimate coefficients directly from (the reliable) y_b^τ
- ⇒ “no junk” from leaking modes

Complexity

Robustness to noise: **Binary mode index expansion.**

Use MP to get individual bits of k . Requires $\tau = 2^n$.

Complexity: $O(S \log N (\log \frac{N}{S})^2)$.

- $O(S \log N)$ for the bin coefficients at each scale
- $O(\log N/S)$: number of scales
- $O(\log N/S)$: because coeff. estimation is within freq. id step.

Provided collision detector works. Constant proba. success.

Additive error estimate.

Complexity

Theoretical difficulties with matrix pencil:

Theorem (Chiu, D., 2013)

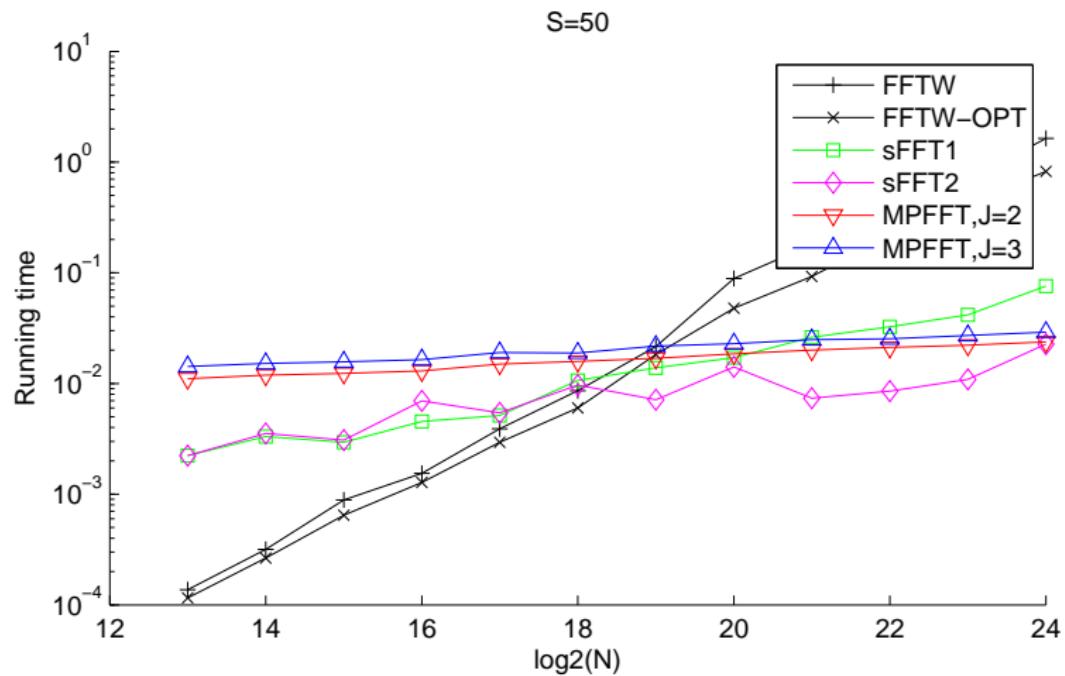
Let $\theta_\ell = \ell/N$ be drawn uniformly at random from $\ell = 0, \dots, N-1$, and

$$y^j = \sum_{\ell=1}^L c_\ell e^{ij\theta_\ell}.$$

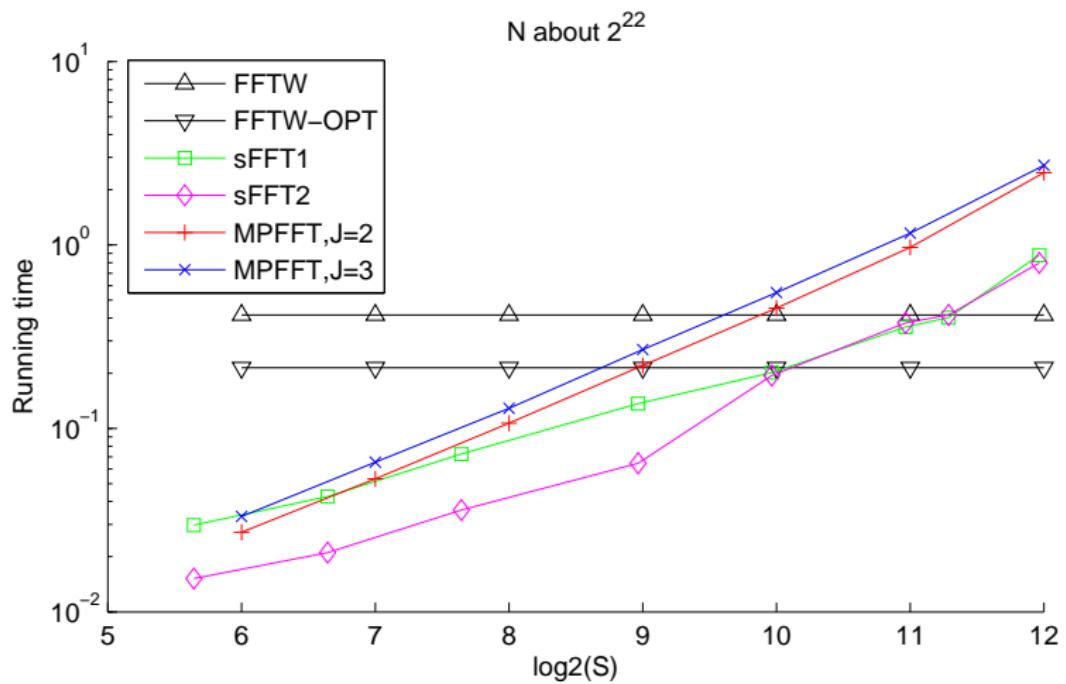
Assume $|\sum c_j| \asymp \sum |c_j|$. Let $Y = \text{Hankel}_J(y)$. Then

$$\sigma_2(Y) \gtrsim \frac{1}{J} \sum_{j \geq 2} |c_j|^2.$$

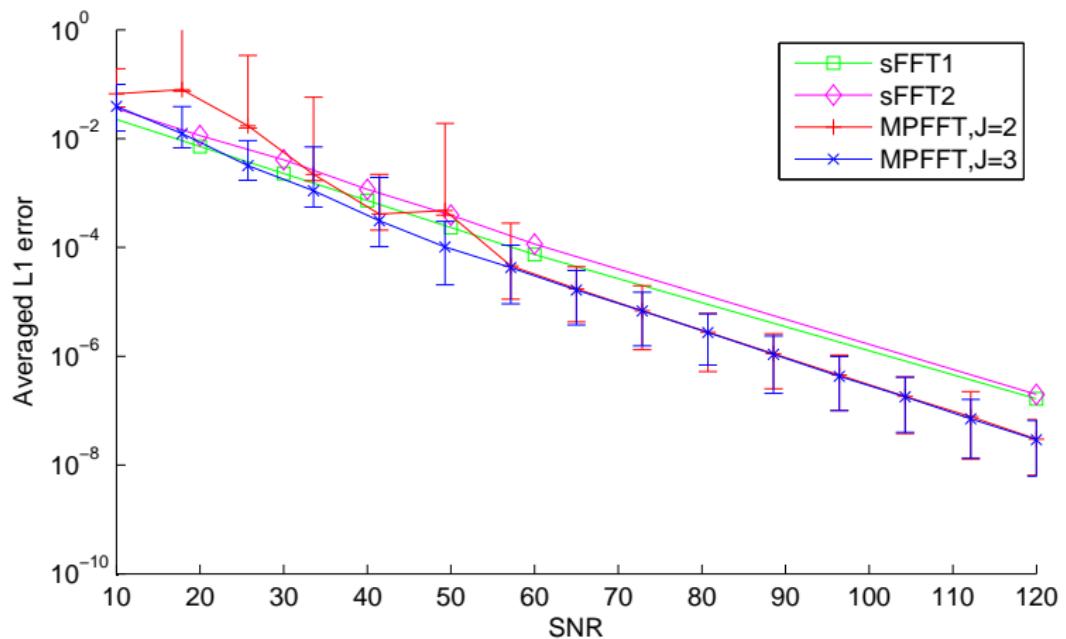
Numerical results



Numerical results



Numerical results



Conclusions

Matrix pencil (MP) for sFFT:

- MP by itself is not a sFFT algorithm
- MP is a great **collision detector**
- MP cannot (robustly) handle mode collisions (yet)