

## Basic Fourier Analysis

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### AGENDA

#### I DFT, basic def =

orthonormality / unitary transform

FFT algorithm

linear approx<sup>n</sup>

non-linear (or sparse) approx<sup>n</sup>

basic exponential sums, definite integrals

e.g., sinc funct<sup>n</sup>, Dirichlet kernel

#### II FT on other domains

DFT  $Z_n \leftrightarrow Z_n$

FS  $Z \leftrightarrow S^1$

Cts FT  $\mathbb{R} \leftrightarrow \mathbb{R}$

#### III Properties/Invariants of FT

convolution/multiplication } algebra

dilation

translation

differentiation

} calculus/analysis

eigenfunctions of Laplacian } (sometimes graphs)

#### IV Engineers' fallacies/Mathematicians' pedantries

band-limited

time-limited

BOTH?!

}  $\rightarrow$  prolates

define FT on  $L^2(\mathbb{R})$  properly

convergence of FS

#### V Shannon sampling theorem (Poisson summation formula)

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C adjoint!

## ① Discrete Fourier transform (DFT)

- begin with vector  $x \in \mathbb{C}^n$ , length n, dot prod  $\langle x, y \rangle = x^* y$
  - let  $F_{j,k}$  be DFT matrix :  $j, k = 0, 1, \dots, n-1$

$$f_{j,k} = \frac{1}{\sqrt{n}} e^{\frac{2\pi i j k}{n}} = \frac{1}{\sqrt{n}} \left( e^{\frac{2\pi i j}{n}} \right)^k \cdot \text{so } f \text{ is a vandermonde matrix; it's invertible.}$$

Note: this means that we must treat  $x$  as a periodic function; func<sup>n</sup> defined on  $\mathbb{Z}^n$ .

• also  $e^{\frac{2\pi i j}{n}}$  is  
 with period  $n$ , so  
 $j, k \in \mathbb{Z}_n$ .

- define  $\hat{x} = Fx$  discrete Fourier transform

$$\hat{x}(j) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi j k / n} x(k)$$

$$j \quad F \quad x = \hat{x}$$

so  $\hat{x}$  is periodic, too.  $\hat{x} \in \mathbb{C}^n$  too.

- $F$  is a unitary matrix (you check that dot prods between cols are zero and cols have unit  $\ell_2$  norm)

you verify that  $F^{-1} = F^*$  note: adjoint,  $\mathbb{C}$ -valued!

- to compute the DFT, you could just do the naive MVM.

$$\hat{x} = f x$$

which for  $x$  of length  $n$ , takes  $O(N^2)$  flops.

BUT,  $\mathcal{F}$  is a very special matrix; it's got a recursive factorizat<sup>n</sup>.

(if  $n$  is a power of 2,

(if not, slightly more complicated)

If you haven't seen the derivation of this, you should look it up! Piotr/Mark cover?

$$= \sum_{k=0}^{n/2-1} x(2k) e^{\frac{2\pi i j(2k)}{n}} + e^{\frac{2\pi i j}{n}} \sum_{k=0}^{n/2-1} x(2k+1) e^{\frac{2\pi i j(2k+1)}{n}}$$

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$$= \hat{x}(\text{even})[j]$$

$$= \underbrace{\hat{x}_{\text{even}}(j)}_{\substack{\text{DFT of} \\ \text{size } n/2 \\ (\text{and } n/2 \\ \text{periodic})}} + e^{\frac{2\pi i j}{n}} \underbrace{\hat{x}_{\text{odd}}(j)}_{\substack{\text{DFT of size } n/2 \\ (\text{and } n/2 \\ \text{periodic})}} \quad \text{for } j < n/2$$

for even elts  
of  $x$

for odd elts of  $x$ .

note that for  $j \geq n/2$

$$\hat{x}(j) = \hat{x}(j-n/2)$$

leads to  
butterfly algo.  
with twiddle  
factors on  
edges...

$$\hat{x}_e(j) = \hat{x}_e(j-n/2)$$

$$\hat{x}_o(j) = \hat{x}_o(j-n/2)$$

and twiddle factor =  $e^{\frac{2\pi i (j-n/2)}{n}}$

### - Def<sup>n</sup> LINEAR APPROX<sup>n</sup>

(linear)

idea is to use a ~~fixed~~ fixed subspace to approximate  $x$

e.g. take first  $K$  (discrete) Fourier basis vectors

Define  $S_K = \text{span of } \dots$

$$\text{Then } x|_{S_K}^{(k)} = \underset{\substack{\text{ortho} \\ \wedge \\ K-1}}{\text{proj}} \text{ of } x \text{ onto } S_K$$

$$= \frac{1}{\sqrt{n}} \sum_{j=0}^{K-1} \hat{x}(j) e^{-\frac{2\pi i j k}{n}}$$

i.e., just keep FIRST  $K$   
coeffs and do IDFT.

it's linear because  $S_K$  is a subspace

$$(x+y)|_{S_K} = x|_{S_K} + y|_{S_K}$$

frequently used in numerical analysis ("first  $K$  Fourier modes"  
old-school compression?)

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- Def<sup>n</sup> NON LINEAR APPROX: (this is sparse approx)

idea is to pick best K Fourier modes for a given  $x$

↳ so as to minimize  $\ell_2$  error

$$x_K = \arg \min_{\substack{\hat{z} \\ \hat{z} \text{ has } K \\ \text{non-zero FCs}}} \|x - \hat{z}\|_2$$

$$\text{Planchet/Parseval: } \|x\|_2^2 = \sum_{k=0}^{n-1} |x(k)|^2 = \sum_{j=0}^{n-1} |\hat{x}(j)|^2 = \|\hat{x}\|_2^2$$

$$\Rightarrow \|x - \hat{z}\|_2^2 = \|\widehat{(x - z)}\|_2^2 = \|\hat{x} - \hat{z}\|_2^2$$

$$= \sum_{j=0}^{n-1} |\hat{x}(j) - \hat{z}(j)|^2 \quad \begin{matrix} \hat{z} \text{ has } K \text{ non-zeros} \\ \text{in } \Delta_K \end{matrix}$$

$$= \sum_{j \in \Delta_K} |\hat{x}(j) - \hat{z}(j)|^2 + \sum_{j \notin \Delta_K} |\hat{x}(j)|^2$$

whole expression minimized

when  $\Delta_K$  = set of K largest (in abs. val.)

FCoeffs of  $x$

$$\therefore x_K^{(k)} = \frac{1}{\sqrt{n}} \sum_{j \in \Delta_K} \hat{x}(j) e^{-2\pi i j k / n}$$

$\Delta_K$  = K biggest FCoeffs of  $x$

It's NONLINEAR because  $(x+y)_K \neq x_K + y_K$ . duh.

- Helpful exponential sums and definite integrals

$$\textcircled{1} \quad \sum_{k=0}^{n-1} e^{ikx} = \sum_{k=0}^{n-1} (e^{ix})^k = \frac{1 - (e^{ix})^n}{1 - e^{ix}}$$

$$= \frac{1 - e^{inx}}{1 - e^{ix}} = \left[ \frac{\sin(\frac{nx}{2})}{\sin(\frac{x}{2})} e^{ix(n-1)/2} \right]$$

• if  $e^{ix} = e^{2\pi i j / n}$  for  $e^{2\pi i / n}$  a primitive root of unity,  
you'll easily verify that  $\sum_{k=0}^{n-1} e^{2\pi i j k / n} = 0$  for any  $j \in \mathbb{Z}_n$ .

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- also note Dirichlet kernel def<sup>n</sup> follows from this

$$\sum_{k=-n}^n e^{ikx} = \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})} = D_{2n}(x)$$

$$\textcircled{2} \quad \int_a^b e^{iwx} dx = \left. \frac{e^{iwx}}{iw} \right|_a^b = \frac{e^{iwb} - e^{iwa}}{iw}$$

- let  $a = -L, b = +L$  and observe def<sup>n</sup> of sinc funct<sup>n</sup> follows from this

$$\begin{aligned} & \int_{-L}^L e^{ix} dx \\ &= \left. \frac{e^{ix}}{ix} \right|_{-L}^L = \frac{e^{iLx} - e^{-iLx}}{2Lix} \\ &= \frac{\sin(Lx)}{Lx} = \text{sinc}(Lx) \end{aligned}$$

## II FT on other domains.

So far we have discussed periodic vectors of length  $n$   
OR another way to say this, we have discussed the FT  
for funct<sup>n</sup>s defined on  $\mathbb{Z}_n$  (for periodic functions defined  
on a finite, periodic domain). The FTs of such functions  
are themselves functions defined on a finite, periodic (discrete)  
domain  $\mathbb{Z}_n$ . That is,

$$\begin{aligned} x &= \text{funct<sup>n</sup> on } \mathbb{Z}_n & \hat{x} &= \text{function on } \mathbb{Z}_n \\ \text{(Fourier)} \\ \text{or, the dual of } \mathbb{Z}_n & \text{ is } \mathbb{Z}_n & \text{ or } & \text{ } \\ & \text{ " } & \text{ " } & \text{ } \\ & g & \hat{g} & (\text{notat<sup>n</sup>}) \end{aligned}$$

There's a lot more going on here that we  
could discuss.

$\rho: G \times C^* \rightarrow C^*$ , i.e.  $\rho(g) \cdot \cdot$  is

$$\chi(g) = \text{Tr}(\rho(g))$$

$$\chi: G \rightarrow \mathbb{C}^\times \quad (\text{if } \rho(g) \in \mathbb{C}^\times)$$

unitary operator, i.e.,

$\rho(g)$  = unitary matrix it's in  $T$ .

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let  $G$  be a (locally) compact abelian group.

A character  $\chi$  of  $G$  is a (continuous) group

homomorphism from  $G \rightarrow T$  (unit circle in  $C$ )  
or torus

there  
may be  
a number  
of characters.

$$\chi(g) \in T; \text{ i.e., } \chi(g) = e^{i\theta g} \text{ for some } \theta \in [0, 2\pi)$$

a homomorphism means

$$\chi(e) = 1 \in T \quad \text{unit} \mapsto \text{unit}$$

$$\chi(g_1 g_2) = \chi(g_1) \chi(g_2) \quad \text{mult.} \mapsto \text{mult.}$$

ex:  $(\mathbb{Z}_n, +)$   $\mathbb{Z}_n$  ints. modulo  $n$  with  $+$  as group op.

$\Rightarrow 0$  is the unit for  $+$ . let's let  $\chi_1$  be one char. for  $\mathbb{Z}_n$

$$\chi_1(0) = e^{i0} \exp(2\pi i 0/n) = 1$$

$$\chi_1(1) = \exp(2\pi i 1/n)$$

$$\begin{aligned} \chi_1(1+1) &= \exp(2\pi i 1/n) \cdot \exp(2\pi i 1/n) \\ &= e^{2\pi i / n} \cdot e^{2\pi i / n} = e^{2\pi i 2/n} \end{aligned}$$

etc.

There are other characters for  $\mathbb{Z}_n$

~~$$\chi_2(0) = e^{2\pi i 1/n}$$~~

~~$$\chi_2(1) = e^{2\pi i 2/n}$$~~

~~$$\chi_2(2) = e^{2\pi i 0/n}$$~~

~~$$\chi_2(3) = e^{2\pi i 1/n}$$~~

let's take  $\chi_j: \mathbb{Z}_n \rightarrow T$  for  $j \in \mathbb{Z}_n$ . then

and define  $\chi_j(l) = e^{2\pi i j l / n} \in T$ .

$$\text{Then } \chi_j(0) = e^{2\pi i j 0 / n} = 1$$

$$\chi_j(2) = \chi_j(1+1) = \chi_j(1) \chi_j(1)$$

$$= e^{2\pi i j 2 / n} = e^{2\pi i j n} e^{2\pi i j / n}$$

yes, it's a homomorphism

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How many characters are there?

How do we know ~~too many~~ we have them all? How to char. chars?The set of all characters on  $G$  is itself a (locally)compact abelian group, called the dual group  $\widehat{G} \approx \text{Hom}(G, \mathbb{T})$ (a la linear algebra  $V^* \approx \text{Hom}(V, \mathbb{F})$ .)Pontryagin duality says  $(\widehat{G})^* \approx G$ and we use the dual group as the underlying space for the  
FT defined of funct's  $f \in L^1(G)$  [later, extend to  $L^2(G)$ ].

So, what do you need to know, really?! groups and their duals!

	$G$	$\widehat{G}$	
"discrete Fourier transform"	$\mathbb{Z}_n$	$\mathbb{Z}_n$	the implications of Pontryagin duality is that it doesn't matter which group $G$ or $\widehat{G}$ we use for the funct's, its FT is defined on the dual...
"Fourier series expansion of a periodic funct."	$S^1$	$\mathbb{Z}$	
"continuous Fourier transform"	$\mathbb{R}$	$\mathbb{R}$	

We've all seen the FT defined on these different domains but perhaps never in such a clean unified fashion. This is how I tend to think about FT (as characters of reps on  $\mathbb{R}, \mathbb{Z}_n, \mathbb{Z}$ , etc.) and why I will go through a bunch of the invariants and properties of FT in a rather sloppy fashion, mostly "proving" all of the invariants for FT defined on  $\mathbb{R}$ , ignoring all questions of convergence, integrals well-defined, etc.

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Shoopy FT defns. sort of diff sheets etc.

•  $(S^1, \mathbb{Z})$

Let  $f \in L^1(S^1)$ ; i.e.,  $x \in [0, 2\pi]$ .

$$f(x) = f(x + 2\pi) \text{ and } \int_0^{2\pi} |f(x)| dx < \infty$$

$$\hat{f}(l) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ilx} dx, l \in \mathbb{Z}$$

$$f(x) = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \hat{f}(l) e^{ixl}$$

$$j - n = m \text{ or } j = m + D_{2n}(x)$$

can see  $2\pi$ -periodicity here

!! Recall Dirichlet kernel. It's a  $2\pi$ -periodic functy and its FT is 1 from  $-n, \dots, n$  and 0 elsewhere

•  $(\mathbb{R}, \mathbb{R})$

Let  $f \in L^1(\mathbb{R})$ ; i.e.,  $x \in \mathbb{R}$  and  $\int |f(x)| dx < \infty$

$$\hat{f}(s) = \int f(x) e^{ixs} dx, s \in \mathbb{R} =$$

$$f(x) = \int \hat{f}(s) e^{-ixs} ds, x \in \mathbb{R}$$

III

### Properties / Invariants of FT.

① convolution/multiplication: "convol" on one side of FT  
is mult. by FT on the other side..."

defn  $f * g$  "f convolved with g"

$$(f * g)(y) = \int f(x) g(y-x) dx$$

I'll do everything on  $\mathbb{R}$

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$$\begin{aligned}
 (\hat{f} * g)(\xi) &= \int (f * g)(y) e^{iy\xi} dy \\
 &= \int \left( \int f(x)g(y-x) dx \right) e^{iy\xi} dy \quad \text{change order of integration, desktch} \\
 &= \int f(x) \left( \int g(y-x) e^{iy\xi} dy \right) dx \quad \text{you tell me FUBINI} \\
 &\quad \text{change vars } z = y-x \quad \text{you check bds. on integral, we're on } \mathbb{R} \\
 &\quad \Rightarrow y = z+x \\
 &= \int f(x) \left( \int g(z) e^{iz(z+x)} dz \right) dx \\
 &= \int f(x) e^{ix\xi} \left( \int g(z) e^{iz\xi} dz \right) dx \\
 &= \hat{f}(\xi) \hat{g}(\xi).
 \end{aligned}$$

implicatn: if I filter  $f$  on one side of FT with  $g$ ,  
 that's equiv. to multiplying their FTs on the  
 other side. convolving

convolution  $\leftrightarrow$  blurring by a func (think kernel...)  
 filtering  $\leftrightarrow$  multiplying by a func

(2) dilation

$$\begin{aligned}
 \text{set } D_a f(x) &= f(ax) \Rightarrow \hat{f} \\
 &\quad \text{dilated domain by } \frac{1}{a}; D_a f(\frac{x}{a}) = \frac{1}{a} f(x) \\
 (D_a f)^{\wedge}(\xi) &= \int (D_a f)(x) e^{ix\xi} dx = \int a f(ax) e^{ix\xi} dx \\
 &\quad \text{change vars } \frac{1}{a} dz = dx \\
 &= \int \frac{1}{a} f(z) e^{i\frac{z}{a}\xi} dz \quad x = \frac{1}{a} z \\
 &= \int \frac{1}{a} f(z) e^{iz\xi/a} dz = \frac{1}{a} \hat{f}\left(\frac{\xi}{a}\right) = D_{1/a} \hat{f}(\xi)
 \end{aligned}$$

to check this in the discrete setting  
(easier analysis but annoying arithmetic!)

$$(f * g)(l) = \sum_{j=0}^{n-1} f(j)g(l-j)$$

$$\begin{aligned} (\hat{f * g})(k) &= \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} (f * g)(l) e^{-2\pi i k l / n} = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \sum_{j=0}^{n-1} f(j)g(l-j) e^{-2\pi i k l / n} \\ &= \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f(j) \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} g(l-j) e^{-2\pi i k l / n} \end{aligned}$$

changing order  
of summation  
EASY!

change vars

$$m=j \Rightarrow m=l-j \Rightarrow l=m+j$$

$$\text{bds: } m=-j \text{ to } m=n-1-j$$

use periodicity to straighten

enclosed a sum  $n-(-n)$  out sum.

$$= \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} g(m) e^{2\pi i k(m+j)/n}$$

$$= e^{2\pi i k j} \hat{g}(k)$$

$$= \hat{g}(k) \sum_{j=0}^{n-1} f(j) e^{2\pi i k j} \quad (\text{need } \sqrt{n} \text{ factor?})$$

why we do calculus instead!

TF for this are the "discrete Fourier transform"

...this stuff isn't in the book at

Sk is pointing out of it

$$\left. \begin{array}{l} \text{Fourier transform} \\ \text{discrete Fourier transform} \end{array} \right\} = \{ \hat{f}, \hat{g}, \hat{h} \}$$

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2013implication:

If we dilate domain by  $a$  on one side of FT, that's equivalent to dilating <sup>domain</sup><sub>on</sub> other side by  $1/a$ .

## (3) translation

set  $(T_b f)(x) = f(x-b) \Rightarrow$  translate domain by  $b$

$$(\hat{T}_b f)(\xi) = \int (T_b f)(x) e^{ix\xi} dx$$

$$= \int f(x-b) e^{ix\xi} dx$$

$$\text{change vars } z = x-b$$

$$dz = dx$$

$$x = z + b$$

$$= \int f(z) e^{i\xi(z+b)} dz = e^{ib\xi} \int f(z) e^{iz\xi} dz$$

$$= e^{ib\xi} \hat{f}(\xi)$$

implicat<sup>n</sup>: translating domain on one side of FT

$\Leftrightarrow$  modulat<sup>n</sup> ~~by~~ on other side

(this is very handy for interpolat<sup>n</sup>!)

## (4) differentiation

Assume  $f, \frac{df}{dx} \in L^1(\mathbb{R})$ , then

| let's be somewhat rigorous.

$$(\hat{\frac{df}{dx}})(\xi) = \int \frac{df}{dx}(x) e^{ix\xi} dx \quad \text{integration by parts}$$

$$= - \int f(x) \frac{d}{dx}(e^{ix\xi}) dx \quad \text{careful argument about bdry terms as } b \rightarrow \pm\infty.$$

$$= -i\xi \int f(x) e^{ix\xi} dx$$

$$= -i\xi \hat{f}(\xi)$$

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## ⑤ eigenfunctions of Laplacian

solve following ODE (in higher dim's, it's a PDE...)

$$\Delta f + \lambda f = 0$$

Helmholtz eqn.

$$\frac{\partial f}{\partial t} = \Delta f \text{ heat eqn}$$

$$+\Delta f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} \text{ wave eqn.}$$

all of the fundamental PDEs have Laplacian op. in spatial coords so it's vital to know about eigenfuns of Laplacian on various domains.

$$\left. \begin{array}{l} \frac{d^2 f}{dx^2} + \lambda f = 0 \quad \text{or} \quad \frac{d^2 f}{dx^2} = -\lambda f \\ f(0) = f(2\pi) \\ x \in [0, 2\pi] \end{array} \right\} \quad \boxed{\text{Laplacians defined on graphs, too!}}$$

i.e., find eigenfuns of Laplacian on  $S^1$  (or a compact domain  $\Omega$ , in general)note  $f_n(x) = e^{inx}$  for  $n \in \mathbb{Z}$  is eigenfun with eigenvalue  $\lambda = n^2$ 

$$f_n(0) = 1 \Rightarrow f_n(2\pi) = e^{in \cdot 0} = e^{in \cdot 2\pi} \quad \checkmark$$

$$\frac{d^2 f_n}{dx^2} = -n^2 e^{inx}$$

since  $\Omega = S^1$  is compact,  $\{f_n\}_{n \in \mathbb{Z}}$  are an ONB for  $L^2(\Omega)$ .  
spectral decompos'n for  $\Delta$ .

## IV

## Fallacies / Pedantries

Engineers talk about time or band-limited (or, god forbid, both band & time limited!!) functions rather flippantly. There are times & places for playing fast and loose with proper analysis and times to pay attention to those pedantic mathematicians.

- band-limited — frequently engineers model "real" signals

Think about FM radio signals: the signal is in a freq. of the stat' band about the

as band-limited, which means functions in  $L^2(\mathbb{R})$  with  $\text{supp } \hat{f} \subset [-\Omega/2, +\Omega/2]$  (for bandwidth  $\Omega$ ). The idea is that real signals are transmitted over some medium that can't support freqs. higher than  $\Omega/2$ .

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- time-limited - just as "real" signals are modeled as being having FTs confined to some freq. band, engineers model signals as being confined to some time interval as well:

$$\mathcal{L}^2(\Omega) \stackrel{\text{def}}{=} \left\{ f \in L^2(\mathbb{R}) \text{ s.t. } \text{supp } f \subset [-T/2, +T/2] \right\}$$

/ after all, we're only going to observe them for some period of time....

this is a closed subspace of  $L^2(\mathbb{R})$ . It's actually a RKHS....

BUT, these are really only models and perhaps not all that realistic physically. A radio signal, for instance, may have a spectrum that with non-zero energy (math'lly) outside its supposed band but that energy isn't discernible by any physically realizable receiver. or, by imposing such mathematical constraints, we force strange physically unrealistic consequences in other ways. E.g.,

a truly band-limited function will NOT be time-limited (the radio station transmits over all of time...). so, it's important to understand

1. what's physically realistic (and what's math'lly realistic)

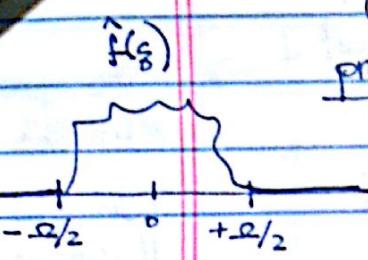
2. what mathematical implications

there are for what might be construed as good physical models

3. when/how to sweep everything under the rug.

- The reason we are so concerned about these issues is because a function cannot be both time- AND freq'n band-limited simultaneously; a function that is compactly supported in both time and frequency ( $\text{time} = \mathbb{R}$ ) must be identically equal to zero.

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(sketch)

proof: if  $\text{supp } \hat{f} \subset [-z/2, z/2]$ , then the extension of  $f$  to entire C-plane is an entire funct<sup>n</sup> of exponential type

$$f(z) = \int_{-z/2}^{z/2} \hat{f}(s) e^{zs} ds \quad z \in \mathbb{C}$$

$$\text{i.e., } |f(z)| \leq C e^{\frac{|z|}{\text{Im } z}}$$

it's a C-differentiable funct<sup>n</sup> whose growth is bdd. by an expon. funct<sup>n</sup>.

But

~~some restrictions for f~~ since  $f$  is compactly supp'd, too. The only entire funct<sup>n</sup>s that are compactly supp'd is  $\equiv 0$  (as analytic funct<sup>n</sup>s have isolated zeroes).

- There are lovely ways to deal with this problem —

prolate spheroidal wave functions (and their discrete

versions, discrete prolate spheroidal sequences, DPSS),

$P_\alpha$  = band limiting operator

$Q_T$  = time limiting operator  $Q_T P_\alpha Q_T$

these are eigenfunct<sup>n</sup>s of  $\hat{f}$ ; i.e,

$$\hat{f}_n(Q_T P_\alpha Q_T f) = \lambda_n f$$

### Pedancies

- All of our defns of FT on different groups

(other than  $\mathbb{Z}_n$ ) were well-defined for funct<sup>n</sup>s that are integrable,

$f \in L_1^{(1)}(G)$ , NOT for Hilbert space  $L^2(G)$ .

NOT 2 Even more functions that you think of such

~~especially annoying, for FT as~~

$$e^{ixs} = f(x)$$

are NOT square integrable. ( $\int |e^{ixs}|^2 dx > \int 1 dx / +\infty$ )

define  $1$  on  $L^1(\mathbb{R})$ ,  
extend to  $L^2(\mathbb{R})$  by a  
limiting argument

so, there's mathematical work to do to define  
FT for  $L^2(\mathbb{R})$ . This is not super-critical.

$L^1$  dense in  $L^2$ . What is more important is the right way to deal with

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tempered  
distrib<sup>ns</sup>.

let's NOT  
do this!!

← funct<sup>ns</sup> like  $f(x) = e^{ix\zeta_0}$  and what you  
think its FT  $\hat{f}(\xi) = \delta_{\xi_0}(\xi)$  should be  
(a point mass at  $\xi_0$ ).

→ instead, let's model all of our funct<sup>ns</sup>  
defined on  $\mathbb{R}$  as beautifully smooth,  
decaying funct<sup>ns</sup>

All of this is a pedantic way of saying we have  
to be careful in doing Fourier analysis on any other  
setting besides  $\mathbb{Z}_n$  and our models for a finite #  
of samples from a real-world signal might not  
be as straight forward as we'd like; i.e., a compressible  
or sparse discrete spectrum vector is a model problem.

## IV

### Shannon sampling theorem

Let's put all of this together and state precisely and  
prove (with hand-waving at rigorous points) the  
Shannon sampling theorem — so that we all have  
a mathematician's idea of what "Nyquist rate" means.

Thm: Poisson summation formula

If  $f$  is suffly smooth and has rapid enough decay, then

$$\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{2\pi i mx}$$

" $f$  periodized  $\hat{f}$  has as its FS,  $\hat{f}$  sampled  
at the integers."

(15)

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We'll use the dual form to "prove" the Shannon sampling theorem:

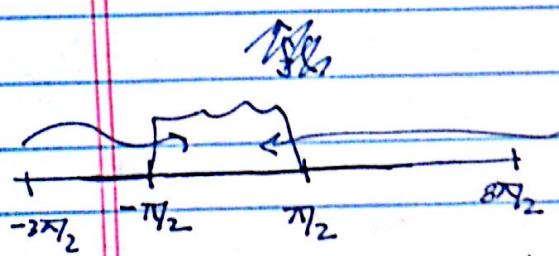
" $\hat{f}$  periodized has as its FS,  $f$  sampled at the integers"

i.e., by sampling  $f$  at the integers, one can linearly reconstruct  $\hat{f}$ , using sinc-functns.

proof:

$$\hat{f}(R) = \sum_{k=-\infty}^{\infty} f(k) e^{-ikR}$$

set  $R = 2\pi$



$$\text{set } R = 2\pi$$

$$\begin{aligned} \text{use } & \int_{-\infty}^{\infty} e^{-izs} ds \\ & = \text{sinc}(z). \end{aligned}$$

define  $\hat{f}_p(s) = \sum_k \hat{f}(s + 2\pi k)$  i.e., periodize or fold in spectrum  $\hat{f}$

then  $\hat{f}_p$  is a  $2\pi$  periodic functn and so it has a FS.

$$\hat{f}_p(l) = \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} \hat{f}_p(s) e^{ils} ds \quad l \in \mathbb{Z}$$

$$\text{and } \hat{f}_p(s) = \sum_{l \in \mathbb{Z}} \hat{f}_p(l) e^{-ils}$$

$$\begin{aligned} &= \int_{-\pi}^{\pi} \hat{f}_p(s) e^{-ils} ds \\ &= \int_{-\pi}^{\pi} \sum_{l \in \mathbb{Z}} \hat{f}_p(l) e^{-ils} e^{-ils} ds \\ &= \sum_{l \in \mathbb{Z}} \hat{f}_p(l) \int_{-\pi}^{\pi} e^{-il(2\pi s)} ds \end{aligned}$$

$$\text{know } f(x) = \int_{-\infty}^{\infty} \hat{f}(s) e^{-ixs} ds = \left( \sum_{l \in \mathbb{Z}} \int_{-\pi}^{\pi} \hat{f}(s + 2\pi l) e^{-ixs} ds \right)$$