Discrete Inverse Problems and Fourier Sampling

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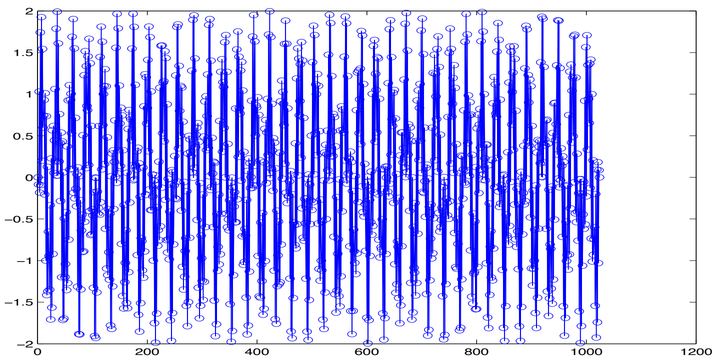
sFFT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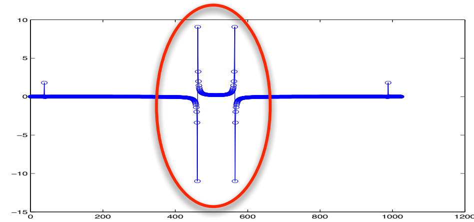

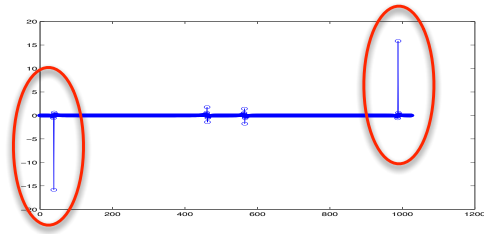

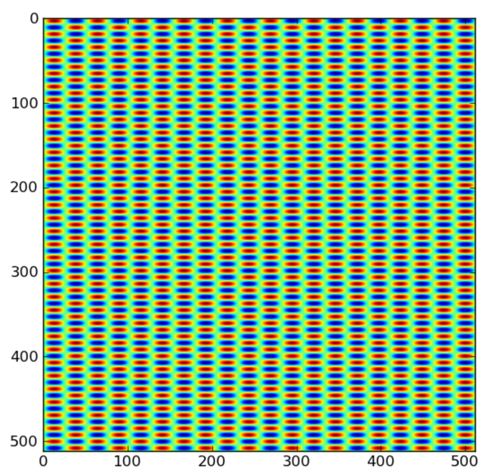
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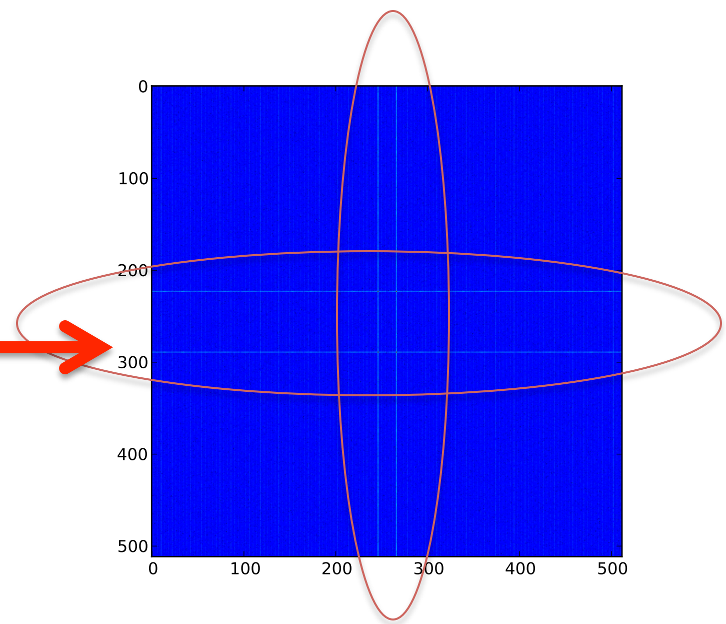
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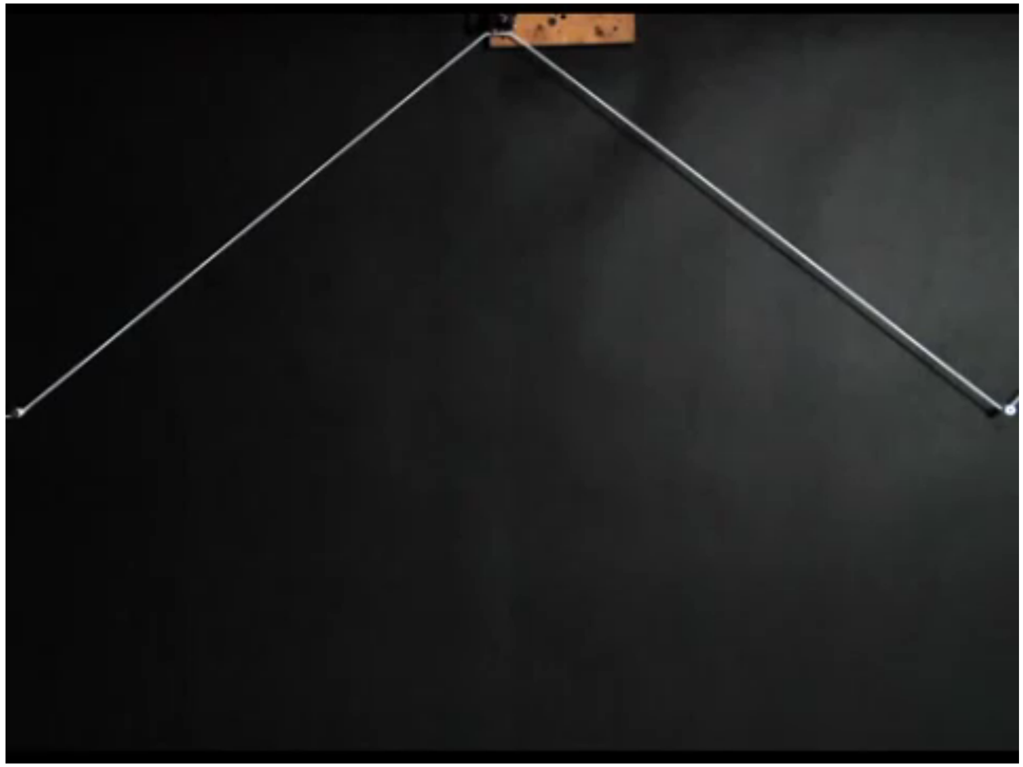


$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 f(x,t)}{\partial t^2}$$

Wave Equation

Solve PDE for string position

1. Posit: separation of variables

$$f(x,t) = u(x)v(t)$$

2. Plug into PDE

$$v(t)\frac{d^2u(x)}{dx^2} = c^2u(x)\frac{d^2v(t)}{dt^2}$$

Depends on *x* only

$$\frac{1}{u(x)}\frac{d^2u(x)}{dx^2} = \frac{c^2}{v(t)}\frac{d^2v(t)}{dt^2} = \lambda$$

Depends on *t* only

3. Solve eigenfunction problem

$$\frac{d^2u(x)}{dx^2} = \lambda u(x)$$

$$u(0) = u(1) = 0$$

$u_k(x) = \sin(k$

Solutions are eigenfunctions of diff. operator with eigenvalue $\lambda=-k^2\pi^2$

General solution

$$f(x,t) = \sum_k \sin(k\pi x)\left(\alpha_k \cos(ck\pi t) + \beta_k \sin(ck\pi t)\right)$$

Laplacian: self-adjoint operator with nice spectrum and eigenfunctions

$$Lu = \lambda u$$

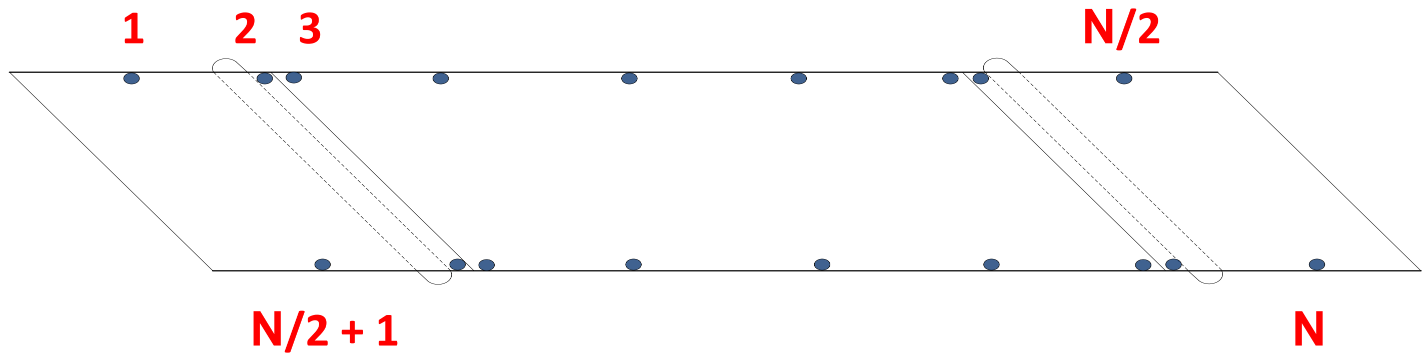
$$L = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$ (physical, Fourier analysis)

$$L = D - A$$ (combinatorial, spectral graph theory)

Joint work with
Michael Wakin (Mines)
Jae Young Park (Umich)

Structural Dynamics

N sensors on structure



- Each sensor observes displacement data $x_l(t)$
- Concatenate to get: $[x(t)] = [x_1(t), x_2(t), \ldots, x_N(t)]^T$
- An N-degree-of-freedom structure with ***no damping*** can be modeled by:

$$[M]\left[\frac{d^2x(t)}{dt^2}\right] + [K][x(t)] = [0(t)] \qquad [M], [K] : \text{unknown}$$

$N \times N$ mass matrix

$N \times N$ stiffness matrix

free decay

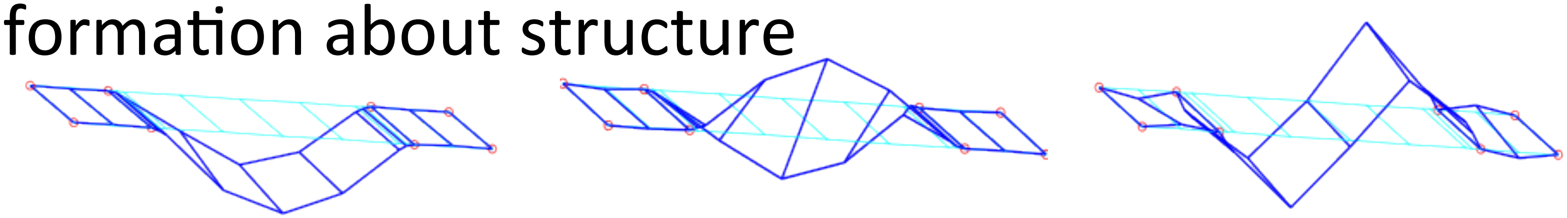
Structural Dynamics

- Homogeneous solution:

$$[x(t)] = \sum_{n=1}^{N} \rho_n \sin(\underbrace{\omega_n t}_{\text{modal frequency}} + \theta_n)[\underbrace{\psi_n}_{N \times 1 \text{ mode shape}}]$$

Generalized eigenvectors
$([K] - \lambda^2[M])\psi = 0$

- $[\psi_n]$ are orthonormal, independent of time, physical information about structure



2.44 Hz   2.83 Hz   10.25 Hz

- **Modal analysis:**
  - Extract modal frequencies, mode shapes, e
  - Populate/update FE models, detect changes o

Discrete Inverse Problems

Data Collection

- Recall:

$$[x(t)] = \sum_{n=1}^{N} \rho_n \sin(\underbrace{\omega_n t}_{\text{modal frequency}} + \theta_n)[\underbrace{\psi_n}_{N \times 1 \text{ mode shape}}]$$

- Consider analytic signal:

$$[x(t)] = \sum_{n=1}^{N} A_n \mathrm{e}^{\mathrm{i}\omega_\mathrm{n}\mathrm{t}}[\psi_\mathrm{n}]$$

- Sample $[x(t)]$ at times $t_1, t_2, \ldots, t_M$
- Stack samples into $M \times N$ matrix $[X]$.

$$[X] = \begin{bmatrix} x_1(t_1) & x_2(t_1) & \cdots & x_N(t_1) \\ x_1(t_2) & x_2(t_2) & \cdots & x_N(t_2) \\ \vdots & & & \vdots \\ x_1(t_M) & x_2(t_M) & \cdots & x_N(t_M) \end{bmatrix}$$

SVD for Modal Analysis

$$[X] = \begin{bmatrix} x_1(t_1) & x_2(t_1) & \cdots & x_N(t_1) \\ x_1(t_2) & x_2(t_2) & \cdots & x_N(t_2) \\ \vdots & & & \vdots \\ x_1(t_M) & x_2(t_M) & \cdots & x_N(t_M) \end{bmatrix}$$

recall

$$[x(t)] = \sum_{n=1}^{N} A_n e^{i\omega_n t}[\psi_n]$$

$$\begin{bmatrix} e^{i\omega_1 t_1} & e^{i\omega_2 t_1} & \cdots & e^{i\omega_N t_1} \\ e^{i\omega_1 t_2} & e^{i\omega_2 t_2} & \cdots & e^{i\omega_N t_2} \\ \vdots & & & \vdots \\ e^{i\omega_1 t_M} & e^{i\omega_2 t_M} & \cdots & e^{i\omega_N t_M} \end{bmatrix} \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_N \end{bmatrix} \begin{bmatrix} [\psi_1] & [\psi_2] & \cdots & [\psi_N] \end{bmatrix}$$

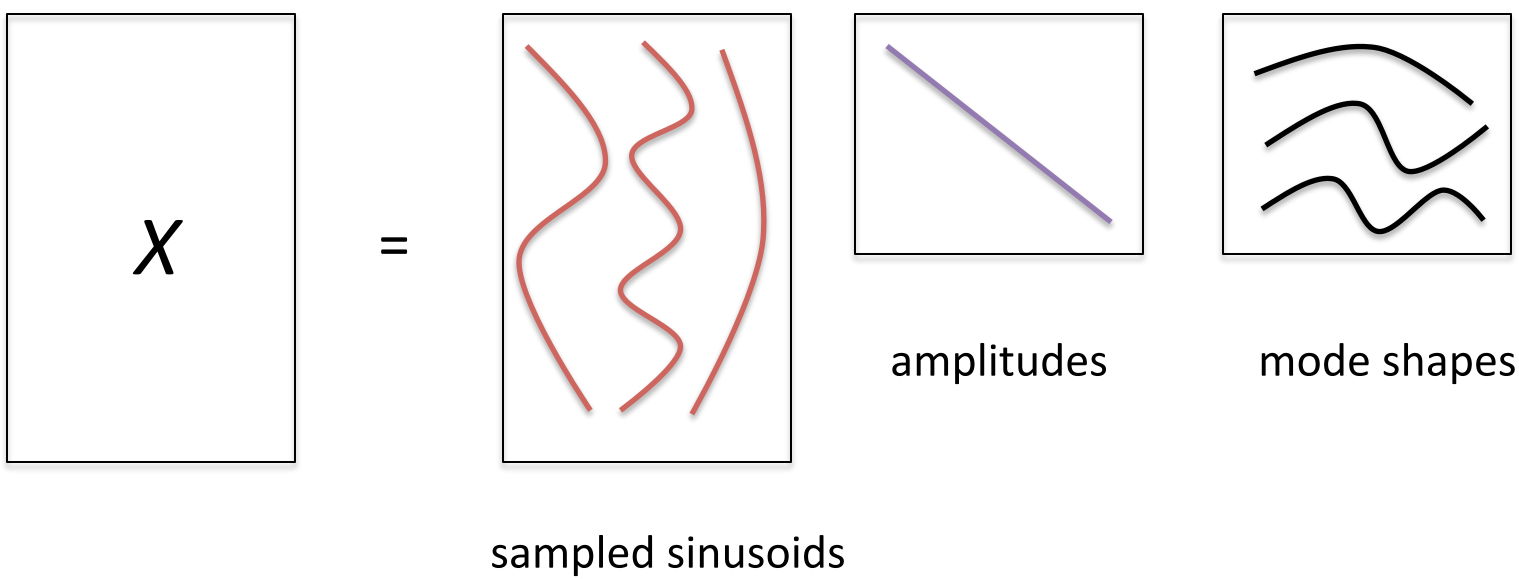
SΣΨ*

sampled sinusoids
can make *nearly* orthogonal

diagonal
amplitudes

unitary
mode shapes

SVD for Modal Analysis



SVD for Modal Analysis

- **Key idea:**
  right singular vectors of $[X] \approx$ true mode shapes
- **Accuracy depends on:**
  - strategy for choosing sample times $t_1, t_2, \ldots, t_M$
  - number of samples *M*
  - total sampling duration *T*
  - number of active modes *K* < *M*
  - minimum separation between modal frequencies
    $$\delta_{\min} = \min_{l \neq n} |\omega_l - \omega_n|$$
  - maximum separation between modal frequencies
    $$\delta_{\max} = \max_{l \neq n} |\omega_l - \omega_n|$$
- **Our contributions:**
  - three sampling strategies, including compressive sampling
  - non-asymptotic bounds

Related work

- **Modal Analysis**
  - Ibrahim Time Domain (ITD), Frequency Domain Decomposition (FDD), Eigensystem Realization Algorithm (ERA)
- **Proper Orthogonal Decomposition (POD)**
  - Proper Orthogonal Modes (POM) converge asymptotically to true mode shapes [Feeny and Kappagantu, 1998], [Kerschen and Golinval, 2002]
- **CS in SHM**
  - Use CS to collect data, reconstruct signal, modal analysis [O'Connor, Lynch, Gilbert, 2014], [Bao, et al., 2010, 2013]
- **Randomized Numerical Linear Algebra (randNLA)**
  - Compressive PCA [Fowler, 2009], [Qi and Hughes, 2012]
  - Subspace approximation [Halko and Tropp, 2010] and many others

Uniform Sampling

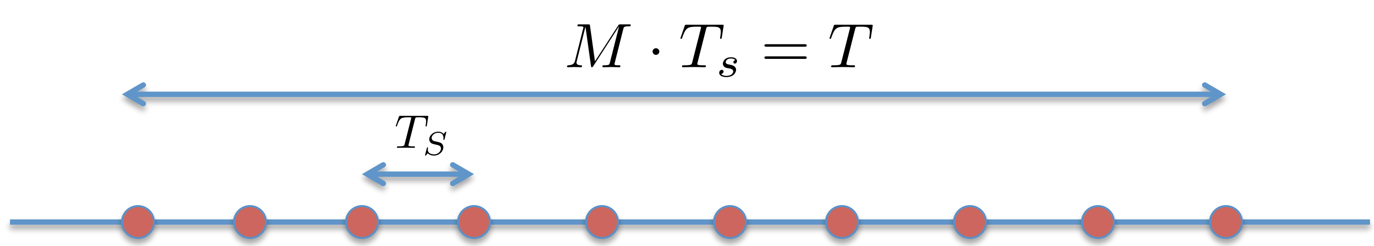
- **Theorem 1:**

  Suppose $t_1, t_2, \ldots, t_M$ are uniformly spaced over $[0, T]$ with sampling interval T_s, where

$$T \sim \frac{\log K}{\epsilon \cdot \delta_{\min}} \quad \text{and} \quad T_s = \frac{\pi}{\delta_{\max}}.$$

  Then

$$|\langle \{\psi_n\}, \{\widehat{\psi}_n\} \rangle|^2 \geq 1 - \frac{\epsilon^2(1+\epsilon)}{1-\epsilon} \cdot \max_{l \neq n} \frac{|A_l|^2 |A_n|^2}{\min_{c \in [-1,1]} ||A_l|^2 - |A_n|^2(1+c\epsilon)|^2}$$



Random Sampling

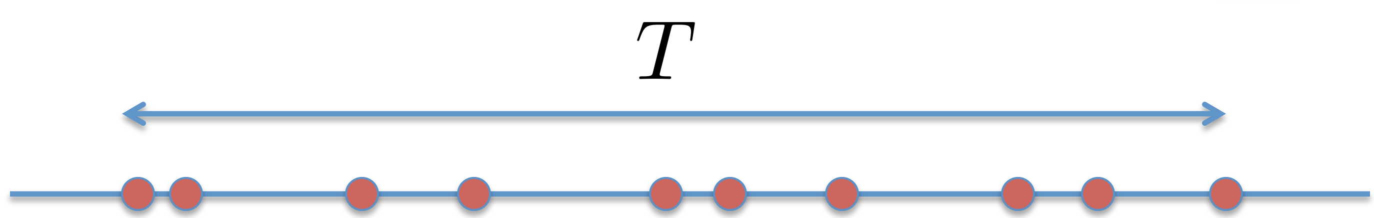
- **Theorem 2:**

  Suppose $t_1, t_2, \ldots, t_M$ are chosen uniformly at random over $[0, T]$ with

$$T \sim \frac{\log N}{\epsilon \cdot \delta_{\min}} \qquad \text{and} \qquad M \sim \frac{K \log K}{\epsilon^2}.$$

  Then with exponentially small failure probability,

$$|\langle \{\psi_n\}, \{\widehat{\psi}_n\} \rangle|^2 \geq 1 - \frac{\epsilon^2(1+\epsilon)}{1-\epsilon} \cdot \max_{l \neq n} \frac{|A_l|^2 |A_n|^2}{\min_{c \in [-1,1]} \left| |A_l|^2 - |A_n|^2 (1 + c\epsilon) \right|^2}$$



Lessons

- Uniform sampling = "Nyquist-like" bound
  - sampling interval inversely proportional to $\delta_{\max}$
  - time span T inversely proportional to $\delta_{\min}$
- Random sampling requires
  - same time span T (up to a constant)
  - fewer total measurements M when $\delta_{\max}/\delta_{\min}$ large
- Similar results for mode shapes with closely spaced amplitudes

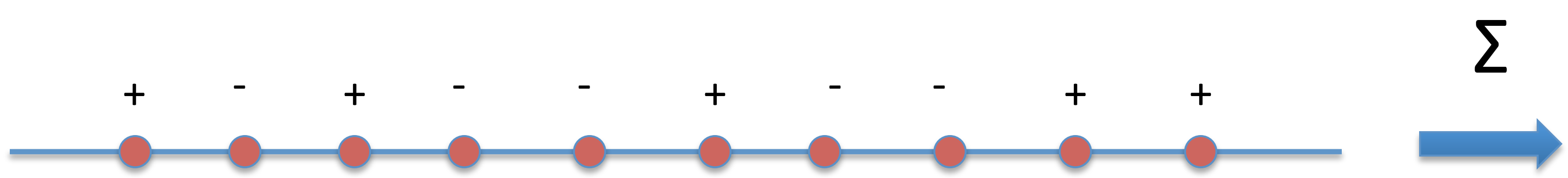
$$M \sim \max\left(\frac{\log K}{\epsilon} \cdot \frac{\delta_{\max}}{\delta_{\min}}, \; K\right) \quad \text{vs.} \quad M \sim \frac{K \log K}{\epsilon^2}$$

Practical sampling in HW

- **Goal:** reduce transmission, save batteries/use solar power

- Uniform samples possible, generates too much data
- Uniformly random in time too hard to implement

- Uniform samples but randomly "reduced" or sketched



Algorithm: Compressive SVD

- Collect data

$$Y = \Phi X$$

- Compute SVD[Y]

$$Y = \hat{S}\hat{\Sigma}\hat{\Psi}^*$$

- Return mode shapes

$$\hat{\psi}_n \approx \psi_n$$

Uniform Sampling with Random Matrix Multiplication

- **Theorem 3:** Suppose $t_1, t_2, \ldots, t_M$ are uniformly spaced with sampling interval $T_s = \frac{\pi}{\delta_{\max}}$ and

$$M \sim \max\left(\frac{\log N}{\epsilon} \cdot \frac{\delta_{\max}}{\delta_{\min}},\ N\right).$$

Let $[Y] = [\Phi][X]$ with $[\Phi]$ random JLT with $m \sim \frac{\text{rank}[X]}{\epsilon'^2}$ rows.

For the right singular vectors of $[Y]$, with high probability,

$$\|\{\psi_n\} - \{\tilde{\psi}_n\}\|_2 \leq C \cdot \epsilon \cdot \max_{l \neq n} \frac{|A_l||A_n|}{\min\limits_{c \in [-1,1]} \{||A_l|^2 - |A_n|^2(1 + c\epsilon)|\}}$$

$$+ C \cdot \epsilon' \cdot \max_{l \neq n} \frac{\sigma_l \sigma_n}{\min\limits_{c \in [-1,1]} \{|\sigma_l^2 - \sigma_n^2(1 + c\epsilon')|\}}$$

Spectral gap in [X]

Conclusions

- Simple analysis of compressed data via SVD
  - many other data-centric applications need spectral estimates
  - promising non-asymptotic results in modal analysis
  - simple hardware implementation

- Future work
  - modal analysis of systems with damping
  - estimation bounds for modal frequencies
  - more sophisticated estimation strategies
  - robustness & stability analysis