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Chaotic behaviour in the Solar System

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There are several physical situations in the Solar System where chaotic behaviour plays an important role. Saturn's satellite Hyperion is currently tumbling chaotically. Many of the other irregularly shaped satellites in the Solar System had chaotic rotations in the past. There are also examples of chaotic orbital evolution. Meteorites are most probably transported to Earth from the asteroid belt by way of a chaotic zone. Chaotic behaviour also seems to be an essential ingredient in the explanation of certain non-uniformities in the distribution of asteroids. The long-term motion of Pluto is suspiciously complicated, but objective criteria have not yet indicated that the motion is chaotic.

1. Introduction

The Solar System is generally perceived as evolving with clockwork regularity. Indeed, it was a search for the principles that underlie the perceived regularities in the motions of the planets that culminated in Newton's formulation of the laws of mechanics and universal gravitation 300 years ago. Recently it has been widely recognized that dynamical systems possess irregular as well as regular solutions. Irregular solutions of deterministic equations of motion are termed 'chaotic'. The Solar System is just another dynamical system; the study of this preeminent dynamical system is not untouched by the discoveries in nonlinear dynamics. Solar System dynamics encompasses the orbital and rotational dynamics of the planets and their natural satellites, the coupling between them, and the slow evolution of the orbits and spins due to tidal friction. It is primarily the dynamics of resonances and resonances are almost always associated with chaotic zones. Chaotic behaviour must be considered a possibility in almost any dynamical situation in the Solar System. In this paper a number of physical applications of modern dynamics to the Solar System will be reviewed. Applications to rotational dynamics will be considered first, followed by applications to orbital dynamics.

2. Tumbling of Hyperion

The chaotic tumbling of Hyperion, one of Saturn's more distant satellites with an orbit period of 21 days, offers one of the most dramatic physical examples of chaotic behaviour (Wisdom *et al.* 1984). The rotation rate and spin-axis orientation are predicted to undergo significant changes in only a few orbit periods. The chaotic tumbling of Hyperion is primarily a consequence of Hyperion's highly aspherical shape, which was determined from *Voyager 2* images to have radii of

190 km × 145 km × 114 km ± 15 km (Smith *et al.* 1982), and to a lesser extent a consequence of the large eccentricity of Hyperion's orbit ($e \approx 0.1$). Weak tidal friction acting over the age of the Solar System is responsible for bringing Hyperion to this chaotic state.

An out-of-round satellite in a non-uniform gravitational field is subject to a torque. The torque arises because the attractive force on the side of the satellite nearest the planet is stronger than the attractive force on the far side of the satellite. For an out-of-round body the torques arising from these forces do not balance, and give rise to a net torque, a 'gravity gradient torque'. Hyperion is subject to especially large torques because of its highly aspherical shape. In addition, these torques have a strong time dependence because of the large eccentricity of Hyperion's orbit.

Earth's Moon very nearly always points the same face toward Earth; the equality of the rotation period and the orbit period of the Moon is a natural consequence of the action of tidal friction. Tidal friction tends to bring the spin axis into coincidence with the axis of largest moment of inertia, and over longer times brings the spin axis perpendicular to the orbit plane as the rotation rate is slowed until the rotation period equals the orbital period (see Goldreich & Peale 1966; Peale 1977). All satellites in the Solar System which are sufficiently close to their host planet for the tidal torques to have been strong enough to significantly affect the rotation rate over the age of the Solar System are observed to be in this state where the spin period is locked to the orbit period. The timescale for the spin of Hyperion to be slowed by tidal friction to synchronous rotation is on the order of the age of the Solar System. Thus, the magnitude of Hyperion's rate of rotation is near that which Hyperion would need to always point one face toward Saturn. Hyperion's rotation would not be chaotic if it were not tidally evolved. At the same time, if the timescale for tidal despinning were much shorter than the age of the Solar System, Hyperion might have already found its way into a stable commensurate rotation state.

The chaotic rotation of Hyperion is best illustrated in a simplified model. In this model the orbit of Hyperion is taken to be a fixed ellipse; the timescale for chaotic variations in the spin rate is much shorter than the timescale for significant variations in Hyperion's orbit, which are, in any case, not large. Furthermore, the spin axis is taken to be perpendicular to the orbit plane and aligned with the axis of largest moment of inertia; this is the usual outcome of tidal evolution. In this simplified problem the equation of motion for the orientation is quite simple:

$$C\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -n^2(B-A)\,\frac{3}{2}\left(\frac{a}{r}\right)^3 \sin 2(\theta - f).$$

The orientation of the satellite is specified by a single angle, θ, which is taken to be the angle between the axis of smallest principal moment of inertia (the longest axis of a triaxial ellipsoid) and the inertially fixed line of periapse (the line joining the planet and the point in the orbit closest to the planet). The angular position of the satellite in its orbit is also measured from the periapse of the orbit. This angle is the true anomaly, denoted here by the symbol f. The principal moments of inertia are $A < B < C$; C is the moment of inertia about the spin axis. The mean angular motion of the satellite in its orbit is n, the instantaneous distance from the planet to the satellite is r, and the semimajor axis of the orbit is a. The equation of motion equates the rate of change of the angular momentum, or equivalently, the product of the moment of inertia about the spin axis and the acceleration of the orientation, to the external torque. The inverse cube dependence of the torque on the distance from the planet reflects the origin of the torque as a gravity gradient. There would be no net torque if the body were axisymmetric about the spin axis; the asymmetry of the body enters the equation of motion through the difference of the principal moments of inertia in the plane of the orbit, $B-A$. Instantaneously, the torque always tends to try to align the long axis of the satellite with the line between the satellite and the planet; the angle between the long axis and the planet–satellite line is $\theta - f$. This equation keeps only the lowest moments of the mass distribution in the orientation dependent part of the potential energy. The contributions that are ignored are of one higher order in the small ratio of the radius of the satellite to the orbital radius. In this approximation all bodies have a symmetry under which a rotation by 180° gives a dynamically equivalent configuration. The factor of two multiplying the difference of angles reflects this symmetry.

This equation of motion has only a single degree of freedom, the orientation angle θ, but depends explicitly on the time through the distance to the planet, r, and the non-uniform keplerian motion of the true anomaly, f. It is worth emphasizing that it is the non-zero eccentricity of the orbit that spoils the integrability of this simplified problem. If the eccentricity is set to zero then the planet to satellite distance remains equal to the semimajor axis, and the true anomaly becomes simply the mean motion times the time. The equation of motion for the angle $\theta' = \theta - nt$ is

$$C\frac{\mathrm{d}^2\theta'}{\mathrm{d}t^2} = -\frac{3n^2(B-A)}{2}\sin 2\theta'.$$

Except for the factor of two, which could easily be removed by a further change of variables, this is the equation of motion for a pendulum, which of course can be explicitly integrated. An important feature is that the problem now has an integral

$$E = \tfrac{1}{2}C\left(\frac{\mathrm{d}\theta'}{\mathrm{d}t}\right)^2 - \frac{3n^2(B-A)}{4}\cos 2\theta'.$$

Hamiltonian systems with more than one degree of freedom almost always exhibit a divided phase space: for some initial conditions the trajectory is chaotic, and for others the trajectory is regular (Hénon & Heiles 1964). The explicit time-dependence cannot be eliminated from the equations of motion for the simplified spin-orbit problem when the eccentricity is non-zero. Thus the spin-orbit problem may be expected to display the generic mixed phase space. The structure of the phase space is most easily understood by computing surfaces of section. For the simplified spin-orbit problem surfaces of section are generated by looking at the rotation state stroboscopically, once per orbit. The equation of motion is numerically integrated, and every time the satellite goes through periapse the rate of change of the orientation, $\mathrm{d}\theta/\mathrm{d}t$, is plotted against the orientation, θ. The surface of section for Hyperion is shown in figure 1. A number of different trajectories have been used to illustrate the principal types of motion that are possible. Recall that if the motion is quasiperiodic the points will fill a one-dimensional curve; if the points seem to fill an area the motion is chaotic. All of the scattered points in

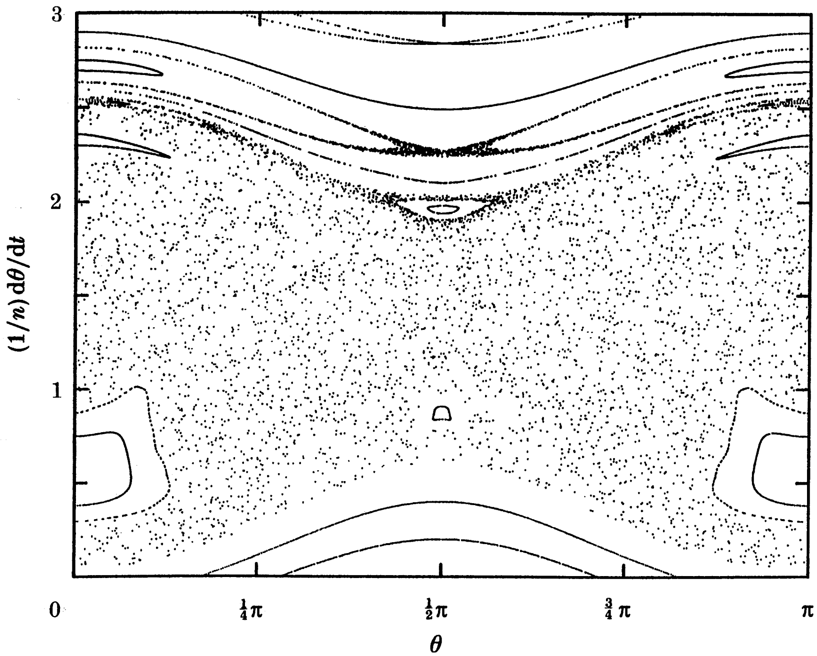

FIGURE 1. Surface of section for Hyperion (with $\alpha = \sqrt{[3(B-A)/C]} = 0.89$ and $e = 0.1$). The rate of change of the orientation is plotted against the orientation at every periapse passage. The spin axis is fixed perpendicular to the orbit plane.

the centre of the section belong to the same trajectory. The two trajectories that generate an X in the upper centre part of the section are also chaotic. The other trajectories appear, without close examination, to be quasiperiodic, and certainly identify the main regions of quasiperiodic motion. The islands in the chaotic sea correspond to various states where the rotation period is commensurate with the orbit period. The island in the lower part of the section near $\theta = 0$ is the synchronous island, where Hyperion would on the average always point one face toward Saturn (i.e. never make a complete relative rotation). The island in the upper part of the large chaotic zone is the 2-state, where Hyperion would on the average rotate twice every orbit period. A number of other islands are shown. The curves in the bottom of the section near $\theta = \frac{1}{2}\pi$ represent a non-commensurate quasiperiodic rotation. If the range of the ordinate were greater it would be seen that they stretch all the way across the figure, as do other non-commensurate quasiperiodic curves near the top of the section. Only the portion of the section between 0 and π is shown because the addition of π to θ gives a dynamically equivalent state. Note, however, that the synchronous states on the section at $\theta = 0$ and $\theta = \pi$ differ in that they present opposite faces to Saturn.

This simplified model was motivated by the standard picture of the tidal evolution of rotations in which the spin axis is driven to the orbit normal as the spin is slowed to synchronous rotation. Elements of the standard picture must now be reexamined. In particular, it is necessary to reexamine the stability of the spin axis orientation perpendicular to the orbit plane. Without giving the details of the methods used, it turns out that the chaotic zone is attitude unstable. This means that if Hyperion were placed in the chaotic zone with the slightest deviation of its spin axis from the orbit normal this deviation would grow exponentially. The timescale is just a few orbit periods. This is also true of the synchronous state; that state in which all other tidally evolved satellites in the Solar System are found is attitude unstable for Hyperion! The attitude stability of the other commensurate islands is mixed, some are stable while others are unstable. The equations that govern the three-dimensional tumbling motion are Euler's equations with the full three-dimensional gravity gradient torque. These equations have three degrees of freedom, through, say, the three Euler angles, plus the explicit time-dependence from the non-uniform keplerian motion in an orbit with non-zero eccentricity. It is no longer possible to plot a surface of section for a problem with so many degrees of freedom. However, another property of chaotic trajectories is that neighbouring trajectories separate exponentially from one another. The rates of exponential separation are quantified by the Lyapunov characteristic exponents. The three-dimensional tumbling state which is entered as the spin axis falls away from the orbit normal is a fully chaotic state. There are no hidden integrals of the motion; the chaotic tumbling motion has three positive Lyapunov exponents.

When the evolution due to tidal friction is included the problem is no longer strictly hamiltonian. However, there is a tremendous disparity between the dynamical timescale and the timescale over which the tides are important. The tidal evolution is consequently viewed as a slow evolution through the phase space of the hamiltonian system. Most likely Hyperion at one time had a rotation period much shorter than its orbital period and began its evolution high above the top of the section in figure 1. Over the age of the Solar System its spin gradually slowed, while the obliquity damped nearly to zero. As it damped to zero the assumptions made in computing figure 1 came closer to being realized. By the time Hyperion reached the large chaotic zone its spin axis was nearly normal to the orbit plane. Once the large chaotic zone was entered, however, the work of the tides over aeons was undone in a matter of days. Because the large chaotic zone is attitude unstable, Hyperion quickly began to tumble through all orientations. Ultimately, Hyperion may be captured by one of the small attitude stable islands. It can never be captured by the synchronous island because the synchronous island is attitude unstable.

Observations of Hyperion are not yet adequate to fully confirm the chaotic tumbling, though they are all consistent with it. The most convincing evidence for chaotic tumbling comes from the *Voyager* pictures themselves, which show that the long axis of Hyperion is out of the orbit plane and the spin axis is near the plane. This is consistent with chaotic tumbling, but inconsistent with other known regular rotation states. Further observations of Hyperion will be needed to unambiguously determine whether its rotation is chaotic. If the observations are complete enough it might be possible to invert the light curve for the initial conditions and moments. Numerical simulations indicate that the moments may be determined with accuracy which increases exponentially with the time interval over which the observations are made.

3. Irregularly shaped satellites

Are there other examples of chaotic tumbling in the Solar System? Hyperion appears to be alone in its chaotic dance, the result of a unique combination of factors which are nowhere else realized in the Solar System. It turns out though that many other satellites tumbled chaotically in the past. In fact, *all* irregularly shaped satellites in the Solar System must tumble chaotically just at the point where the spin is about to be captured into synchronous rotation (Wisdom 1987*a*).

Almost all resonances are surrounded by chaotic zones, though in some cases these chaotic zones may be very narrow. The commensurate spin-orbit states are examples of resonances. Resonances appear as islands on a surface of section. Two moderately narrow chaotic zones were illustrated in the upper part of figure 1. There exist approximate methods of estimating the size of these chaotic zones (see Chirikov 1979). The width of the chaotic zone surrounding the synchronous island may be specified in terms of the magnitude of the chaotic variations of the integral E of the zero eccentricity problem:

$$\frac{\Delta E}{E} \approx \frac{14\pi e}{\alpha^3}\, \mathrm{e}^{-\pi/2\alpha},$$

where α, the asphericity parameter, is $\sqrt{[3(B-A)/C]}$. Note that this estimate has the correct limit for zero orbital eccentricity, where the simplified problem is integrable. Although the width of the chaotic zone depends exponentially on the asphericity parameter, it only depends linearly on the orbital eccentricity. Thus satellites with large deviations from spherical symmetry, but small eccentricities may still have significant chaotic zones.

Phobos, a satellite of Mars, is almost as out-of-round as Hyperion, but its orbital eccentricity is only 0.015. A surface of section for Phobos is shown in figure 2. The chaotic zone is a significant feature on the section. Even for Deimos, the other satellite of Mars, where the orbital eccentricity is considered to be anomalously small ($e \approx 0.0005$), the chaotic zone is not microscopic (see figure 3.) Surfaces of section for several other irregularly shaped satellites with α near unity confirm the existence of significant chaotic zones surrounding the synchronous island.

Stability analysis shows that the chaotic zones of these irregular satellites is in every case attitude unstable, just as it is for Hyperion. A slight displacement of the spin axis from the orbit normal grows exponentially, leading to chaotic tumbling. The surprising result is the strength of this attitude instability. In every case the timescale for the exponential growth of obliquity is only a few orbit

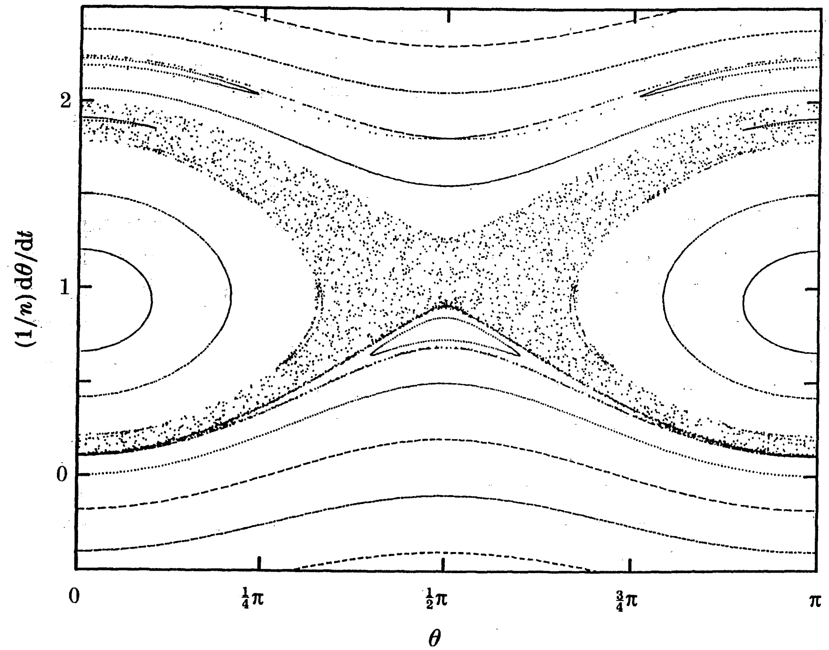

FIGURE 2. Surface of section for Phobos (with $\alpha = 0.83$ and $e = 0.015$). The chaotic zone is a significant feature on the section.

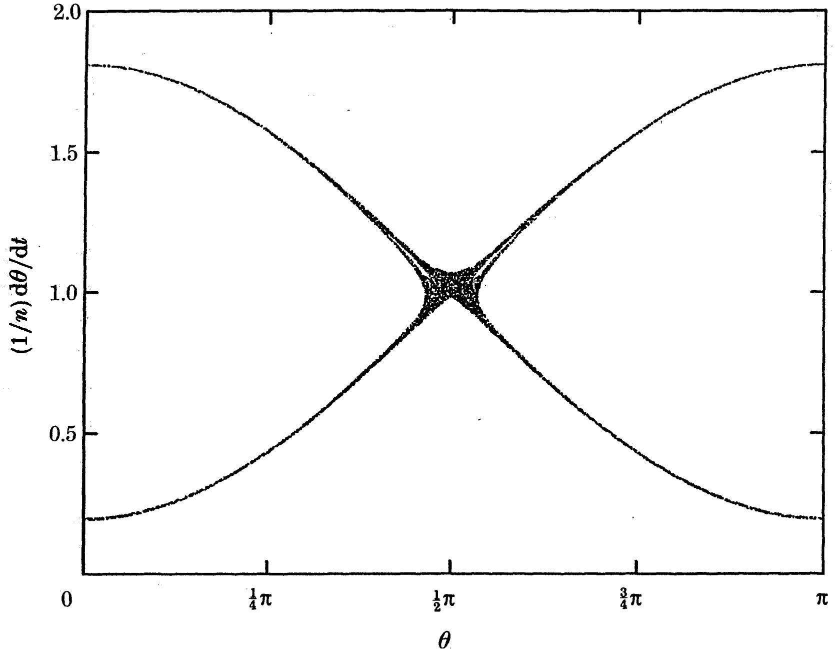

FIGURE 3. Chaotic separatrix for Deimos (with $\alpha = 0.81$ and $e = 0.0005$). The chaotic zone is sizable considering the very low orbital eccentricity.

periods. This is true even for Deimos, with its low orbital eccentricity, and narrow chaotic zone. The orbital eccentricity does not play a crucial role in this attitude instability. It turns out that even for zero eccentricity the synchronous separatrix is attitude unstable, and leads to chaotic three-dimensional tumbling, even though the simplified problem with the spin axis perpendicular to the orbit plane is integrable.

It is not possible to tidally evolve into the synchronous state without passing through a region that is attitude unstable. The resulting tumbling motion is always chaotic. All synchronously rotating satellites with significantly irregular shapes must have spent a period of time tumbling chaotically. The length of time spent in this state must be comparable with, and probably somewhat greater than, the despinning timescale; it is not yet possible to make a rigorous estimate. Thus Deimos probably spent of the order of 100 Ma tumbling chaotically, and Phobos spent on the order of 10 Ma in this tumbling state.

Enhanced dissipation of energy during the chaotic tumbling phase may help explain the anomalously low eccentricity of Deimos, and certainly must be taken into account in future studies of the orbital histories of the irregularly shaped satellites. This new episode in the adolescence of the irregularly shaped satellites is, however, fascinating in itself. The World, and newtonian mechanics in particular, works in a surprising way.

4. 3/1 Kirkwood gap

The distribution of the semimajor axes of the asteroids is not uniform; it shows several gaps as well as several enhancements. The origin of these gaps has been the object of a great deal of speculation. One major clue to the cause of these non-uniformities, which was noted at the time of their discovery by D. Kirkwood, is that they occur near mean-motion commensurabilities with Jupiter. That is, a small integer times the mean motion of an asteroid in a gap will nearly equal the product of another small integer times the mean motion of Jupiter. However, the mere association of a gap with a resonance does not in itself explain the formation of the gap. Nature herself provides the counterexamples: there are gaps at some resonances and enhancements at others. The basic difficulty in understanding the formation of the gaps was that the motion near complex resonances was not well understood analytically, and numerical simulations could not alleviate the problem because of the great amount of computer time required. Integrations over 10 ka did not uncover a mechanism for the formation of gaps. Integrations over significantly longer times were prohibitively expensive and did not seem warranted.

Longer integrations were made possible by the introduction of a new method for following the trajectories of asteroids (Wisdom 1982). Following the ideas of Chirikov (1979), an algebraic mapping of the phase onto itself was derived which approximates motion near the 3/1 commensurability. The map is an approximation to the stroboscopic section that would be obtained by looking at the coordinates of the asteroid once each Jupiter period (which is about 12 years); evolution of an asteroid is followed by successively iterating this map. The derivation of the map relies on the averaging principle, as have most other studies of the long-term evolution of asteroid orbits. The terms with highest frequency, the orbital frequency, are first removed by averaging, leaving the resonant terms and the secular terms. New high-frequency terms are added in such a way that delta functions are formed. The new equations can be integrated across the delta functions and between them, giving a map of the phase space onto itself. The map is significantly faster than more conventional methods; it is more than a thousand times faster than a full integration and several hundred times faster than the methods that rely on numerical averaging to increase the basic step size. This great increase in computation speed made integrations over much longer intervals possible.

Integration over longer times was justified. Fig. 4 shows the orbital eccentricity as a function of time for a chaotic trajectory near the 3/1 commensurability

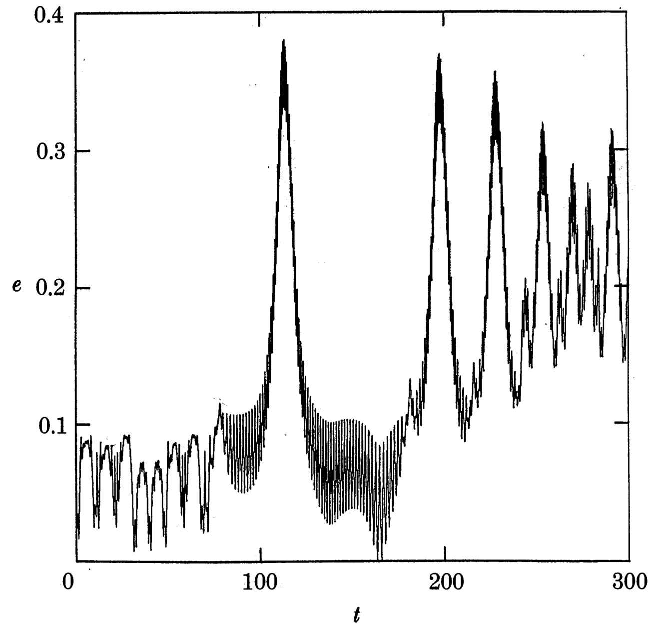

FIGURE 4. Eccentricity against time for a chaotic trajectory near the 3/1 commensurability. Time is measured in millennia. A short (10 ka) integration would give a very poor idea of the nature of this trajectory.

computed with the planar elliptic map. Although excursions in eccentricity of this magnitude were previously known (Scholl & Froeschlé 1974), the possibility that an orbit could spend a hundred thousand years or longer at low eccentricity and then 'suddenly' take large excursions was quite unexpected. Subsequent numerical integrations of the full, unaveraged, differential equations have verified that the behaviour is not an artifact of the method (Wisdom 1983; Murray & Fox 1984).

It is only when the trajectory is computed over millions of years that one begins to feel as though the true nature of the motion is represented. Figure 5 shows the typical behaviour of the eccentricity of a chaotic trajectory near the 3/1 resonance in the planar elliptic problem. There are bursts of irregular high-eccentricity behaviour interspersed with intervals of irregular low-eccentricity behaviour, with

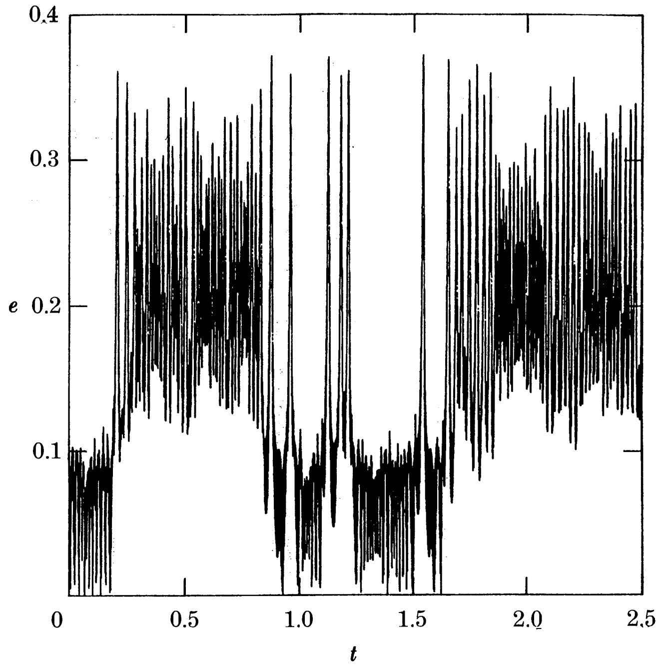

FIGURE 5. Eccentricity of a typical chaotic trajectory over a longer time interval. The time is now measured in millions of years. Bursts of high-eccentricity behaviour are interspersed with intervals of irregular low-eccentricity behaviour, broken by occasional spikes.

an occasional eccentricity spike. Figure 6 shows a very interesting, though relatively rare behaviour. The eccentricity jumps shown in figure 6 all reach the same eccentricity, but seem to occur at irregular intervals. A most surprising result is that if the plot is expanded near two different jumps and then superimposed the eccentricity jumps are practically identical. Both of these trajectories were computed with the map. For discussions of the growth of numerical error see Wisdom (1983, 1987*b*).

The unexpected behaviour of the eccentricity can be understood by putting the trajectory in context on a surface of section (Wisdom 1985*a*). At first sight this is not possible because the planar elliptic problem has two and a half degrees of freedom (through, say, the x and y coordinates and the explicit time dependence resulting from the keplerian motion of Jupiter in its elliptical orbit), but the problem may be reduced to two degrees of freedom by averaging over the orbital period. A surface of section corresponding to figure 5 is shown in figure 7. Here the variables $x = e\cos(\varpi - \varpi_J)$ and $y = e\sin(\varpi - \varpi_J)$ are plotted each time a particular combination of the mean longitudes goes through zero. The distance from the

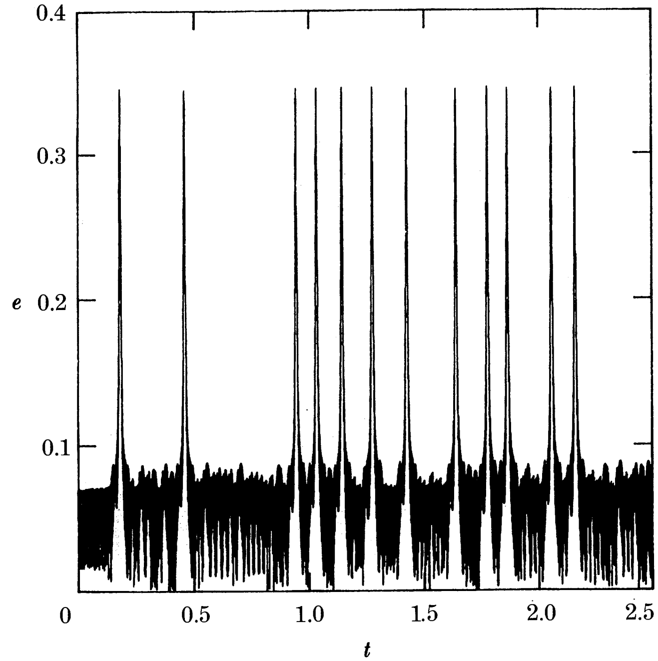

FIGURE 6. Eccentricity of a relatively rare, but quite interesting chaotic trajectory. The time is measured in millions of years.

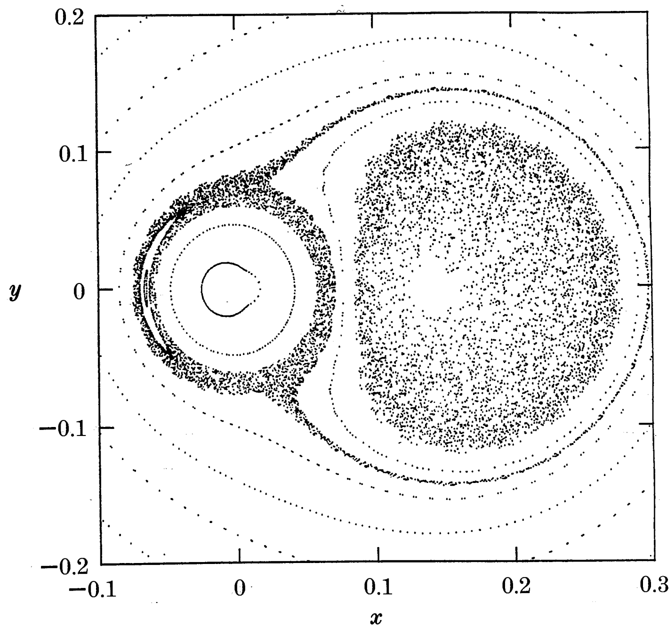

FIGURE 7. Surface of section corresponding to the trajectory of figure 5. The coordinates are $x = e\cos(\varpi - \varpi_J)$, and $y = e\sin(\varpi - \varpi_J)$. The orbital eccentricity is the radius from the origin. The trajectory is free to explore a rather large chaotic zone, but sometimes spends a period of time near the islands close to the origin.

origin on this section is the orbital eccentricity, e, and the ϖs are the longitudes of perihelia for the asteroid and Jupiter. During the intervals of low-eccentricity behaviour the trajectory stays in that part of the chaotic zone near the origin, encircling one of the islands; during the high eccentricity intervals the trajectory is moving in the extended chaotic zone to the right of the figure. The origin of the peculiar behaviour of the eccentricity shown in figure 6 is apparent on the surface of section shown in figure 8. The chaotic zone that surrounds the origin has a very

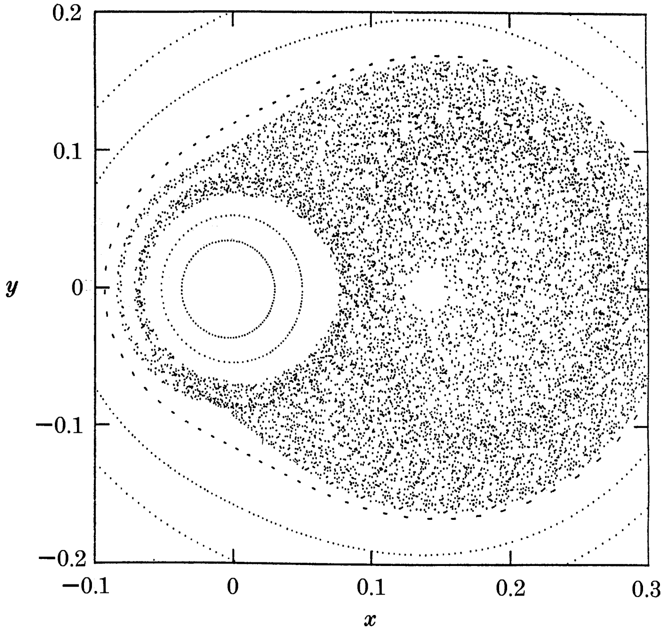

FIGURE 8. Surface of section for a trajectory similar to the one used to generate figure 7. The narrow branch of the chaotic zone explains the irregularly appearing, but nearly identical jumps in eccentricity.

narrow branch that extends to high eccentricity. The similarity of the different jumps is explained by the narrowness of the chaotic zone. The fact that the jumps occur at irregular intervals simply reflects the irregular nature of the motion in the chaotic zone. Thus the peculiar behaviour of the eccentricity of chaotic trajectories near the 3/1 commensurability can be understood as a simple manifestation of chaotic behaviour in a problem with two degrees of freedom.

The long-period motion can also be understood semianalytically (Wisdom 1985*b*). Over much of the phase space the resonance timescale and the secular timescale are well separated. An analytic average over the resonance timescale gives a long-period hamiltonian with one degree of freedom. This approximation, for instance, recovers the large jumps in eccentricity. It also gives a rather interesting picture of the evolution in the chaotic zone, where the chaotic trajectories are for the most part predictable, but occasionally enter a region where the motion is essentially four dimensional.

Now that the nature of trajectories near the 3/1 commensurability is better understood, can the formation of the 3/1 Kirkwood gap be explained? The large eccentricity increases are important for the formation of the gap because at the location of the gap eccentricities above 0.3 are Mars-crossing. It turns out that all of the chaotic trajectories cross the orbit of Mars. Orbits that previously appeared to be limited to low eccentricity are now understood to have large excursions in eccentricity on longer timescales. The quasiperiodic resonance librators also generally have sufficient variation in eccentricity to cross the orbit of Mars. Thus asteroids with both resonant quasiperiodic trajectories and chaotic trajectories near the 3/1 commensurability can be removed by close encounters or collisions with Mars. Comparison of the outer boundary of the chaotic zone with the actual distribution of asteroids shows remarkably good agreement (figure 9). This figure gives strong evidence that chaotic behaviour has indeed played a role in the formation of this gap.

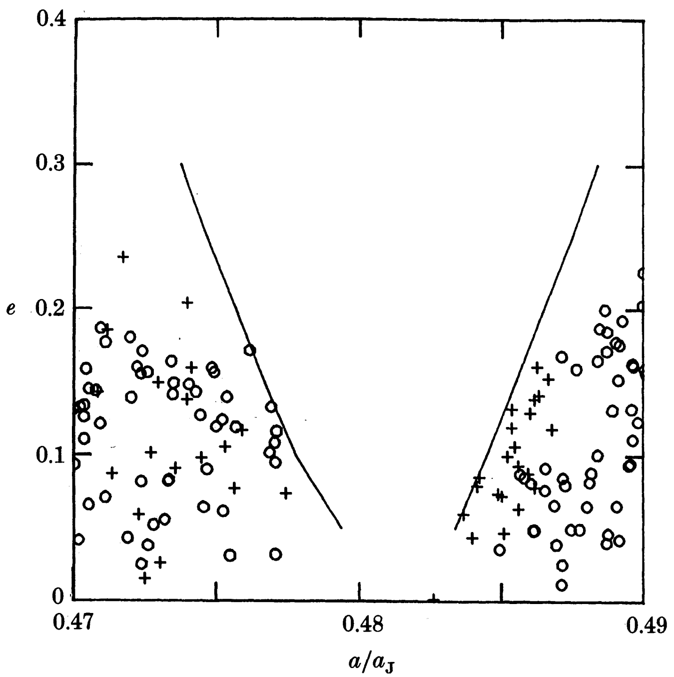

FIGURE 9. Comparison of the actual distribution of asteroids with the outer boundaries of the chaotic zone. There is both a chaotic region and quasiperiodic region in the gap, but trajectories of both types are planet-crossing.

5. Transport of meteorites

It is rather surprising that the origin of the meteorites, those stones which contain so many clues concerning the formation of the Solar System, is still not definitively known. It is widely believed that meteorites originate in the asteroid belt, yet until recently a dynamical mechanism for transporting them to Earth which was consistent with the meteorite date eluded discovery. Monte-Carlo simulations of Wetherill (1968) ruled out the previously suggested sources, and seemed to indicate that there was another source of high-eccentricity orbits.

When the orbit of Jupiter is assumed to be fixed, and the motion of the asteroid is limited to Jupiter's orbit plane, chaotic trajectories near the 3/1 commensurability are limited to eccentricities below about 0.4. As the integration is made more realistic, though, by allowing three-dimensional motion and including the variations of Jupiter's orbit which result from the perturbations of the other planets, the variations in eccentricity become more extreme. Chaotic trajectories which begin at normal asteroidal eccentricities ($e \approx 0.15$) reach eccentricities above 0.6, which is large enough for the orbit to cross the orbit of Earth. Figure 10 shows an example of such behaviour. Besides giving a stronger mechanism for clearing the 3/1 Kirkwood gap, these chaotic trajectories provide a new dynamical mechanism for bringing debris from asteroidal collisions near the 3/1 resonance directly to Earth (Wisdom 1985*b*). Wetherill (1985) has shown that this new source is consistent with the meteorite data, and that the larger fragments from asteroid collisions partly account for the observed population of Earth crossing asteroids. This discovery of a dynamical route from the asteroid belt to Earth is an important scientific application of chaotic behaviour.

6. 2/1 Kirkwood gap and the Hilda asteroids

The fact that there is a gap in the distribution of asteroid semimajor axes near the 2/1 commensurability and an enhancement in the distribution near the 3/2 commensurability needs an explanation. Unfortunately, the dynamics of the 2/1 and 3/2 resonances are considerably more complicated that the dynamics of the 3/1 resonance. These resonances are not well represented by low-order truncations of the disturbing potential. This makes analytic investigations difficult if not impossible.

The only alternative appears to be direct numerical integrations. Numerical integration of problems in celestial mechanics is particularly time consuming because of the great range of timescales involved. The orbital dynamics of the Solar System only begins to be interesting when studied over timescales of millions of years. The single trajectory in figure 10 used the equivalent of about 200 VAX hours. The Digital Orrery (Applegate *et al.* 1985) is a special purpose computer specifically designed to study problems in celestial mechanics. The design and construction of the Orrery were led by Gerald J. Sussman, from the Artificial Intelligence Laboratory and the Department of Electrical Engineering at MIT. The construction of the Orrery is an extremely important advance for planetary dynamics, which evidently had a need for a dedicated supercomputer. The Orrery runs at about 60 times the speed of a VAX for celestial mechanics problems, or about a third the speed of a Cray.

Chaotic behaviour near the 2/1 commensurability was first discovered by Giffen (1973), though short integrations by Froeschlé & Scholl (1976, 1981) indicated that chaotic behaviour was not very widespread or catastrophic. A new survey of the structure near the 2/1 and 3/2 resonances is currently underway

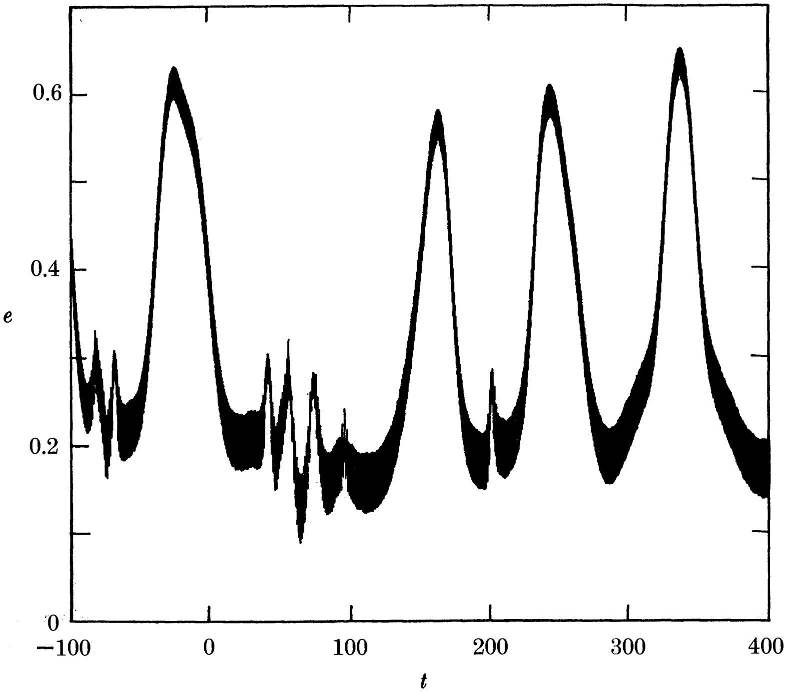

FIGURE 10. Eccentricity against time in millennia for a test particle perturbed by the four jovian planets. At times the eccentricity is typical of asteroids, at other times it is large enough for the orbit to cross the orbit of Earth. This integration shows meteoritic material may be directly transported to Earth from the asteroid belt by way of the 3/1 chaotic zone.

with the Digital Orrery. The preliminary results of this survey are quite interesting. Figure 11 shows the chaotic zone near the 2/1 commensurability. In this figure a cross indicates that a trajectory of the planar elliptic problem with this initial eccentricity and semimajor axis is chaotic. Though the exploration is not yet complete, the outline of the chaotic region is probably well represented. There appears to be a sizable chaotic zone near the 2/1 commensurability. On the other hand, the corresponding plot for the region near the 3/2 commensurability shows that the resonance region is basically devoid of chaotic behaviour. There is thus a *qualitative* difference in the structure of the phase space near the 2/1 and 3/2 commensurabilities which corresponds to the qualitative difference in the observed distribution of asteroids. There is a large chaotic zone at the 2/1 resonance and there is also a Kirkwood gap at that resonance. The Hildas are located near the 3/2 resonance and this region is devoid of chaotic behaviour. However, detailed comparison with the actual distribution of numbered asteroids near the 2/1 resonance, figure 12, does not show perfect agreement. This discrepancy is most likely a result of the use of the planar-elliptic approximation in the survey. The integration of a test-particle with initial conditions in the discrepant region perturbed by the four jovian planets showed it to be chaotic.

The association of chaotic behaviour with a gap in the distribution of asteroids does not by itself explain the formation of the gap. For the 3/1 resonance an

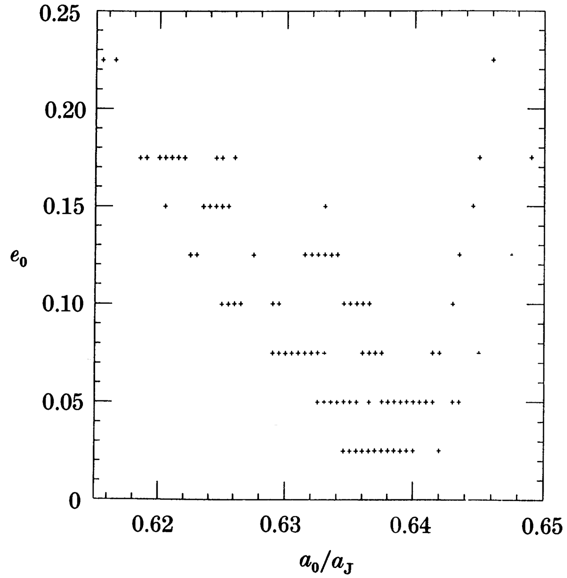

FIGURE 11. Chaotic trajectories near the 2/1 commensurability in the planar elliptic approximation. A cross marks those initial conditions (eccentricity e, and semimajor axis a, referred to Jupiter's semimajor axis) which lead to chaotic behaviour. The survey is not complete, but the basic extent of the chaotic zone is apparent. There is a significant chaotic zone.

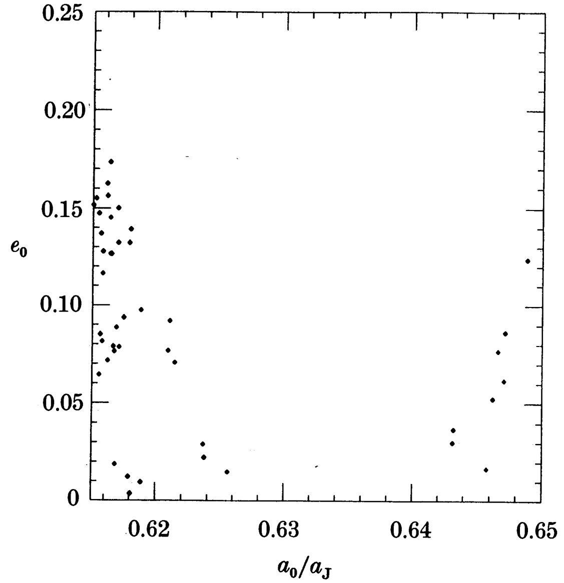

FIGURE 12. Actual distribution of asteroids near the 2/1 commensurability, each evolved to the same longitudes used in the survey. There is a good qualitative agreement between the gap and the region of chaotic behaviour shown in figure 14. The boundaries on the high semimajor axis side are in excellent agreement, whereas there seems to be a discrepancy on the low semimajor axis side.

essential ingredient was the fact that chaotic trajectories, as well as resonant quasiperiodic trajectories, crossed the orbits of Mars and Earth. The sweeping action of these planets can clear the 3/1 gap. It seems *a priori* unlikely that a similar mechanism can account for the formation of those gaps that are significantly more distant from Mars and Earth. Froeschlé & Scholl (1981) attempted to answer this question. They integrated Giffen's chaotic trajectory in the planar elliptic approximation for 100 ka, but found that it seemed to be limited to eccentricities below 0.15, which is no larger than the eccentricity of a typical asteroid. As before, the planar elliptic problem is not an adequate representation of the problem. The integration of test particles perturbed by the jovian planets shows more dramatic increases in eccentricity. For these trajectories the eccentricity and the inclination show the remarkable correlation exemplified in figure 13. Initially the inclination is low and the eccentricity seems to be limited to values below about 0.25. Over the span of the integration though the trajectory seems to trace out a pathway to high eccentricity which temporarily takes it through inclinations as high as 0.44 rad. Thus the three-dimensional nature of the motion is crucial. At the peak in eccentricity this trajectory is marginally

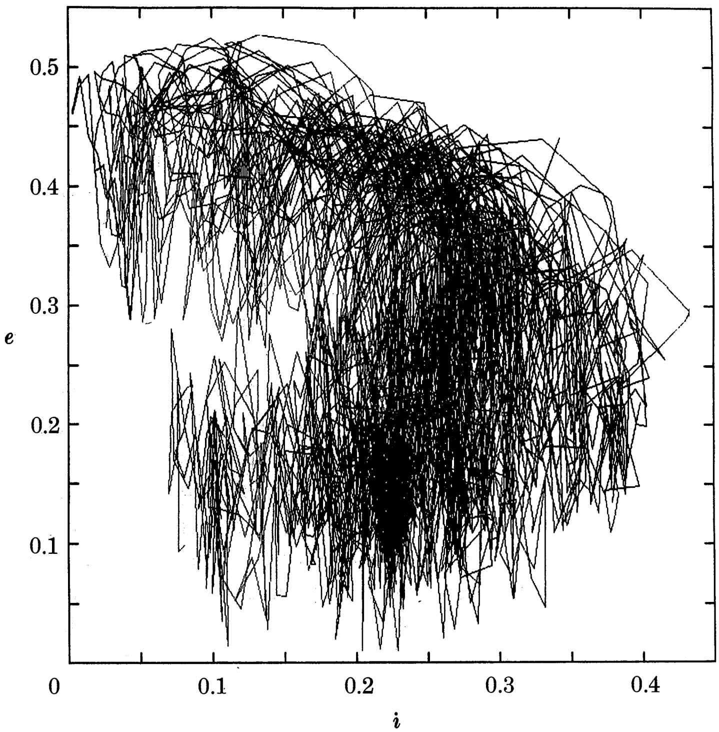

FIGURE 13. Including perturbations of the jovian planets, chaotic trajectories near the 2/1 commensurability show this remarkable correlation between eccentricity and inclination. There is a path in the phase space that takes the trajectory from low eccentricity to high eccentricity which requires that it temporarily take moderate inclination. The three-dimensional nature of the motion is crucial.

Mars-crossing. This does provide a mechanism for the removal of the asteroids on these chaotic trajectories, but I am not convinced that this is the final answer.

A few words must be said to clear up some misunderstandings. It is often said that the Kirkwood gaps are a 'simple consequence of the breakdown of K.A.M. tori near resonances'. This is not so. Resonances occur in the circular restricted problem, but the microscopic chaotic zones associated with them could never account for the creation of gaps. Such a global statement about the stability of resonances can also not account for the contrasting stability of the Hilda asteroids and the Kirkwood gaps. Only a detailed examination of successively more realistic representations of the dynamics begins to account for the distribution of asteroids. Chaotic behaviour near resonances also has nothing to do with the formation of gaps in Saturn's rings. The gaps have an entirely different origin in the collective response to resonant perturbation of a large number of particles which collide frequently, on the order of 20 times per orbit period. The qualitative character of the long-period evolution of individual trajectories is irrelevant.

7. Outer planets and Pluto

The determination of the stability of the Solar System is one of the oldest problems in dynamical astronomy. While Arnol'd's proof of the stability of a large measure of solar systems with sufficiently small planetary masses, eccentricities, and inclinations marks tremendous progress towards a rigorous answer to this question (Arnol'd 1961), the stability of the actual Solar System remains unknown. Certainly, the great age of the Solar System demands a high level of stability, but weak instabilities may still be present. Experience with the motion of asteroids has demonstrated that weak instabilities may sometimes even lead to sudden, dramatic changes in orbits. The stability of the Solar System should thus not be taken for granted.

The first application of the Digital Orrery was to the long-term evolution of the outer planets. For many years the million-year integration of Cohen *et al.* (1973) held the title of the longest integration of the Solar System. With the Orrery, the interval of integration has been extended to 210 Ma (Applegate *et al.* 1986).

The integrations showed that the best analytic approximations of the motion of the outer planets were in serious need of improvement. Bretagnon (1974) lists over 200 corrections to the Lagrange solutions. It turns out that there are contributions to the motion of the jovian planets which are larger than all but seven of those corrections. Higher-order terms were more important than the terms taken into account by Bretagnon. More recent work, particularly that of Laskar (1986, and unpublished work) is in better agreement, and provides independent confirmation of the results of the numerical integration.

The motion of Pluto is extraordinarily complicated. Pluto's orbit is unique among the planets. It is both eccentric ($e \approx 0.25$) and inclined ($i \approx 16°$). The orbits of Pluto and Neptune cross one another, a condition which is only permitted by the libration of a resonant argument associated with the 3/2 mean motion commensurability. This resonance assures that Pluto is at aphelion when Pluto and Neptune are in conjunction and thus prevents close encounters. The next level of complexity is that the argument of perihelion of Pluto librates about $\frac{1}{2}\pi$ with a period near 3.8 million years (Williams & Benson 1971). The Orrery integrations confirmed this libration, but found that the picture was not yet complete; there are significant contributions to the motion of Pluto with much longer periods. The amplitude of libration of the argument of perihelion has a strong modulation with a period of 34 million years. In fact, the frequency of the second largest contribution to the eccentricity of Pluto corresponds to a period of 137 Ma. This long period results from a near commensurability between the frequency of circulation of the longitude of the ascending node of Pluto and one of the fundamental frequencies in the motion of the jovian planets. Figure 14 shows the inclination of Pluto over 214 Ma. There seems to be an even longer period present (or perhaps even a secular drift)! The motion of Pluto is suspiciously complicated. However, the computation of the Lyapunov characteristic exponent for Pluto does not yet show any objective evidence for chaotic behaviour. Much longer integrations seem to be required to determine the true nature of Pluto's motion.

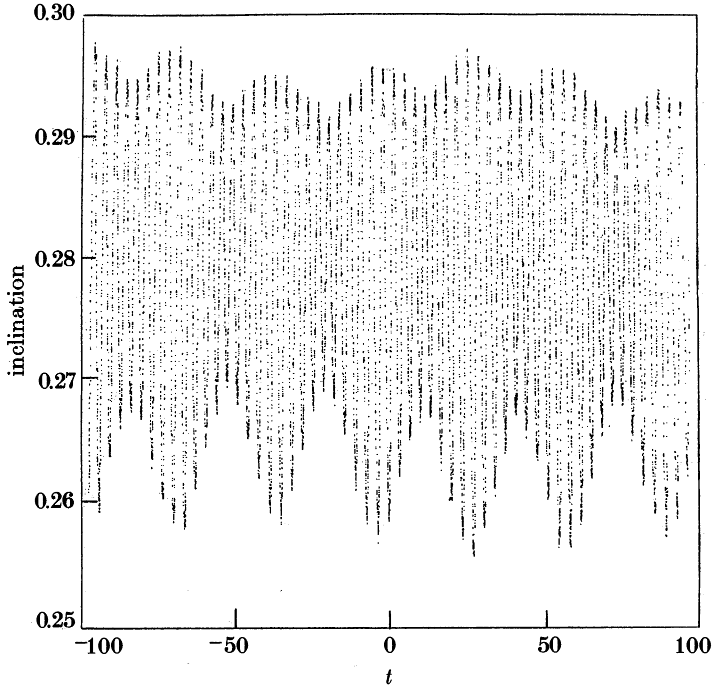

FIGURE 14. The inclination of Pluto for 214 Ma. Time is given in millions of years. Besides the 34 Ma modulation of the 3.80 Ma oscillation, there is evidence of much longer period variations (or perhaps even a secular drift!).

CONCLUSIONS

Several physical examples of chaotic behaviour in the Solar System have been presented. Hyperion tumbles irregularly as a consequence of its out-of-round shape, large orbital eccentricity, and tidally evolved rotation. Hyperion is currently the only example of this chaotic tumbling in the Solar System. However, *all* of the tidally evolved, irregularly shaped satellites in the Solar System tumbled chaotically in the past, just at the point of entry into the synchronous rotation state.

Physical examples of chaotic orbital behaviour have also been presented. The distribution of asteroids seems to be, in several instances, a reflection of the character of the trajectories in the underlying phase space. This is clearly the case for the 3/1 Kirkwood gap. There is a sizable chaotic zone, and the phase space boundary of the distribution of asteroids corresponds quite well with the outer boundary of the chaotic zone. In this case, the fact that both chaotic and quasi-periodic trajectories cross planetary orbits explains the removal of any asteroids originally in the gap.

Allowing three-dimensional motion and taking into account the perturbations of the outer planets, trajectories in the 3/1 chaotic zone reach Earth-crossing eccentricities. These trajectories seem to provide the long-sought dynamical route for the transport of meteoritic material from the asteroid belt to Earth. Studies by Wetherill have shown this source to be consistent with the ordinary chondrite data.

The 2/1 Kirkwood gap and the Hilda group have long presented a paradox to classical dynamical astronomy. A new survey with the Digital Orrery indicates that the qualitative difference in the distribution of asteroids at these two resonances is reflected in a qualitative difference in the underlying dynamics. Discrepancies in the detailed comparison probably result from the use of the planar elliptic approximation in the survey. When perturbations of the jovian planets are taken into account and three-dimensional motion is allowed, chaotic trajectories at the 2/1 resonance reach very large eccentricities at low inclinations, by way of a path that temporarily takes them to high inclinations. The three-dimensional aspect of the problem is essential. The eccentricities become large enough that the chaotic trajectories cross the orbit of Mars, but there may yet be other mechanisms for clearing the distant gaps.

The stability of the Solar System itself has been examined through a 210 Ma integration of the outer planets. The motion of the jovian planets themselves seems to be regular, though perhaps a bit more complicated than might have been expected. On the other hand, the motion of Pluto is extraordinarily complicated. Besides the well-understood mean-motion resonance which prevents the close approach of Pluto and Neptune even though their orbits cross, Pluto participates in at least two other resonances. First, it has been known for some time that the argument of perihelion librates about $\frac{1}{2}\pi$. Then, the frequency of the circulation of Pluto's ascending node is nearly commensurate with one of the fundamental frequencies in the motion of the jovian planets. This near commensurability gives rise to strong variations in the eccentricity with a period of 137 Ma. There is also evidence of much longer periods in the inclination, which appears to be secularly declining over the 210 Ma integration. Although the abundance of resonances raises suspicions about the stability of Pluto, there is not yet any objective evidence that the motion of Pluto is chaotic.

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