# Arrangements and Duality

## Motivation: Ray-Tracing

6.838 Fall 2001, Lecture 9

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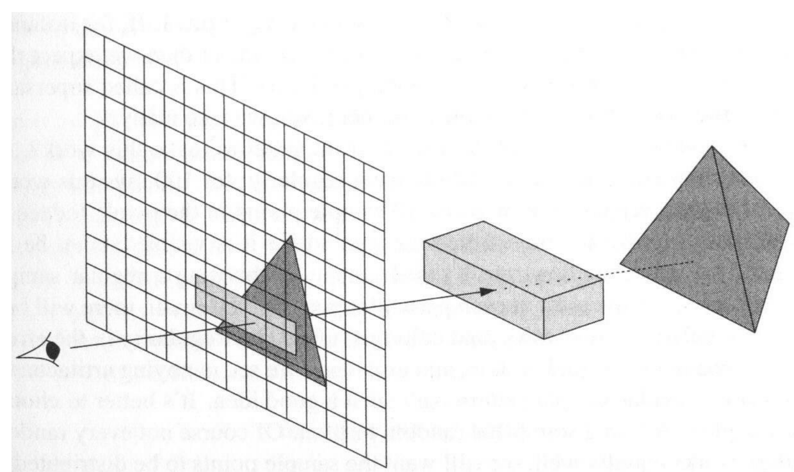
10/4/01

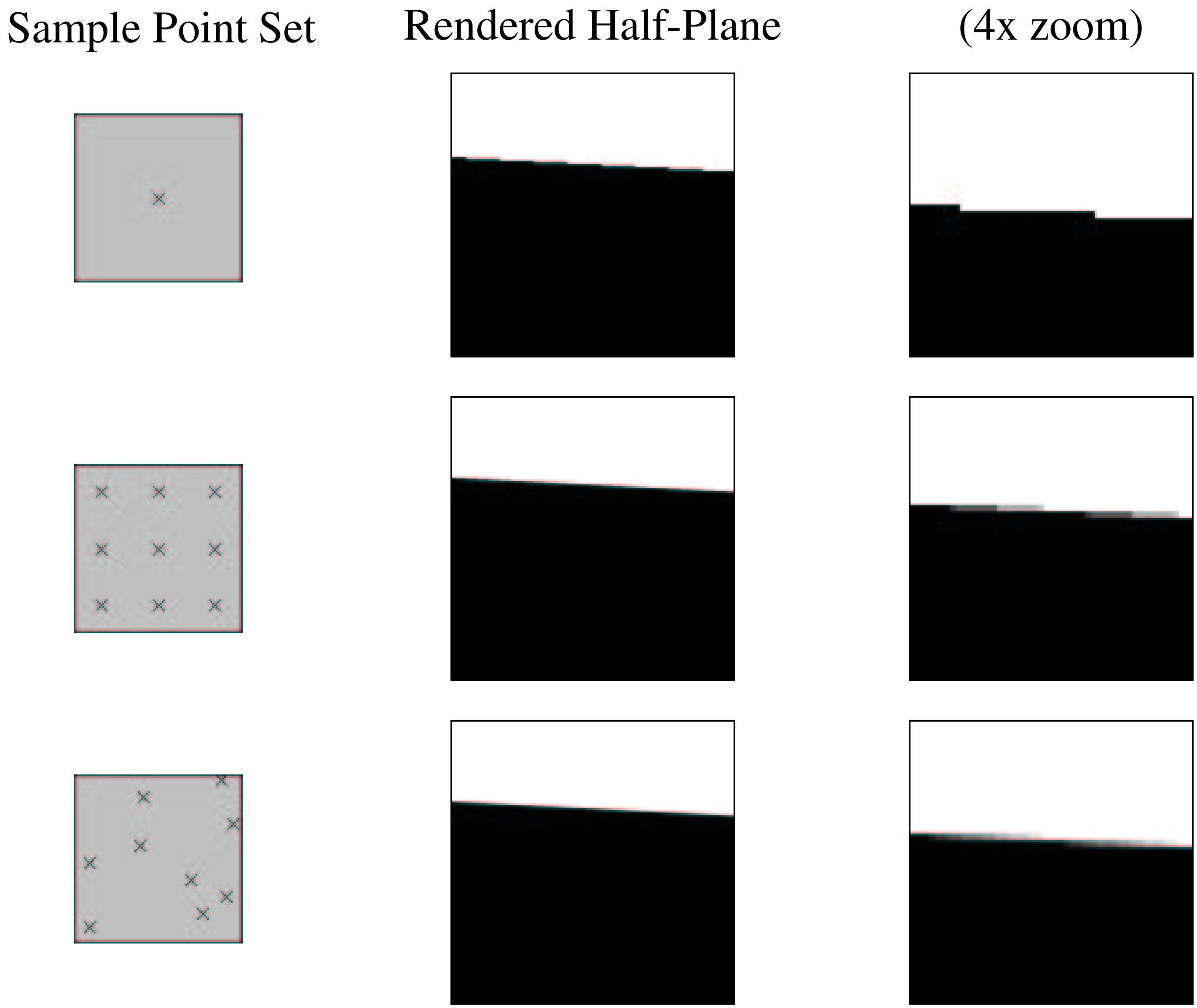
# Ray-Tracing

- Render a scene by shooting a ray from the viewer through each pixel in the scene, and determining what object it hits.
- Straight lines will have visible jaggies.
- We need to supersample



# Supersampling

- We shoot many rays through each pixel and average the results.
- How should we distribute the rays over the pixel? Regularly?
- Distributing rays regularly isn’t such a good idea. Small per-pixel error, but regularity in error across rows and columns. (Human vision is sensitive to this.)




*Wow…that really makes a difference!*

# Supersampling

- We need to choose our sample points in a somewhat random fashion.
- Finding the ideal distribution of $n$ sample points in the pixel is a very difficult mathematical problem.
- Instead we'll generate several random samplings and measure which one is best.
- How do we measure how good a distribution is?

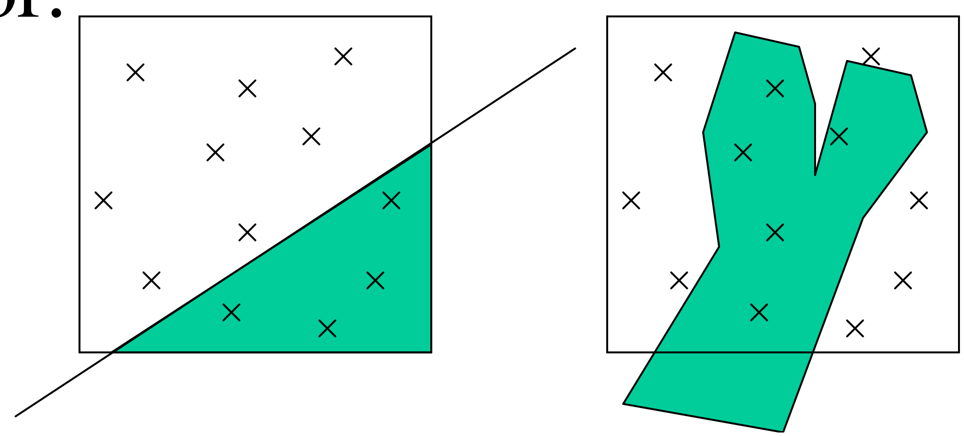
# Big Picture

- To ray-trace a pixel realistically, we need pick a good distribution of sample points in the pixel.
- We need to be able to determine how good a distribution of sample points is.

*How do we do this?*

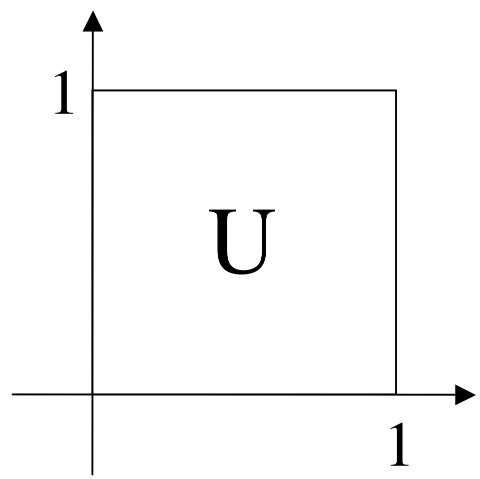
# Discrepancy

- We want to calculate the discrepancy of a distribution of sample points relative to possible scenes.
- Assume all objects project onto our screen as polygons.
- We're really only interested in the simplest case: more complex cases don't exhibit regularity of error.



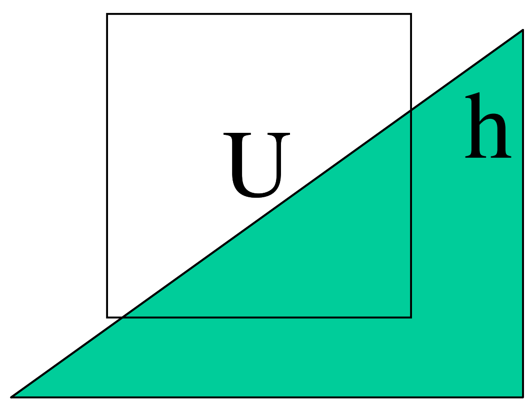
# Discrepancy

- Pixel: Unit square $U = [0:1] \times [0:1]$



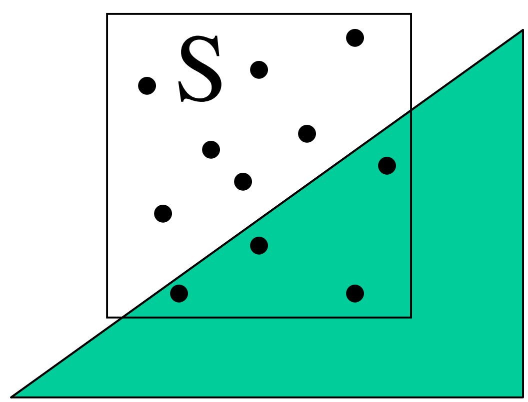
# Discrepancy

- Pixel: Unit square $U$ = [0:1] x [0:1]
- Scene: $H$ = (infinite) set of all possible half-planes $h$.



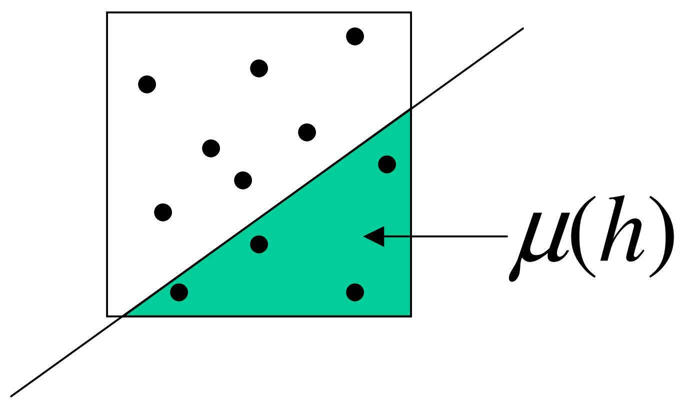
# Discrepancy

- Pixel: Unit square *U* = [0:1] x [0:1]
- Scene: *H* = set of all possible half-planes *h*.
- Distribution of sample points: set *S*



# Discrepancy

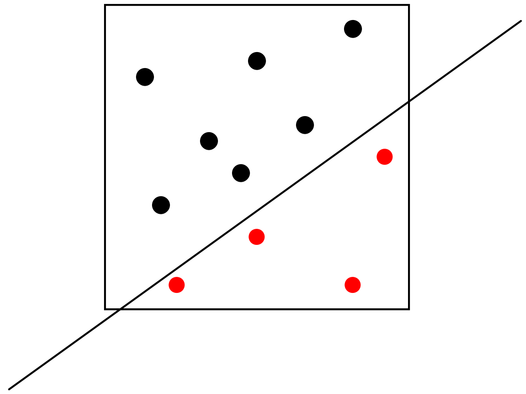
- Pixel: Unit square $U$ = [0:1] x [0:1]
- Scene: $H$ = set of all possible half-planes $h$.
- Distribution of sample points: set $S$
- Continuous Measure: $\mu(h)$ = area of $h \cap U$



# Discrepancy

- Pixel: Unit square $U$ = [0:1] x [0:1]
- Scene: $H$ = set of all possible half-planes $h$.
- Distribution of sample points: set $S$
- Continuous Measure: $\mu(h)$ = area of $h \cap U$
- Discrete Measure:

$$\mu_S(h) = \text{card}(S \cap h) \,/\, \text{card}(S)$$



# Discrepancy

- Pixel: Unit square $U$ = [0:1] x [0:1]
- Scene: $H$ = set of all possible half-planes $h$.
- Distribution of sample points: set $S$
- Continuous Measure: $\mu(h)$ = area of $h \cap U$
- Discrete Measure:

  $\mu_S(h) = \text{card}(S \cap h) / \text{card}(S)$
- Discrepancy of $h$ wrt $S$:

  $\Delta_S(h) = | \mu(h) - \mu_S(h) |$

# Discrepancy

- Pixel: Unit square $U = [0:1] \times [0:1]$
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- Continuous Measure: $\mu(h)$ = area of $h \cap U$
- Discrete Measure: $\mu_S(h) = \text{card}(S \cap h) / \text{card}(S)$
- Discrepancy of $h$ wrt $S$: $\Delta_S(h) = | \mu(h) - \mu_S(h) |$
- Half-plane discrepancy of $S$ :

$$\Delta_H(S) = \max_{\text{all } h} \Delta_S(h)$$

# Big Picture

We've defined the discrepancy of a chosen set of sample points with respect to all possible scenes as:

$$\Delta_H(S) = \max_{\text{all } h} \Delta_S(h)$$

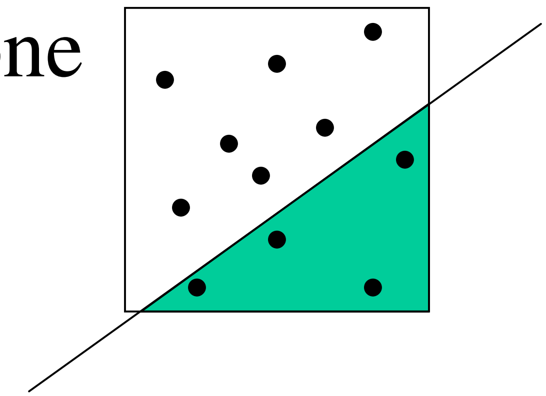
We want to pick S to minimize $\Delta_H(S)$

# Computing the Discrepancy

- $\Delta_H(S) = \max_{\text{all } h} \Delta_S(h)$
- There are an infinite number of possible half-planes…We can't just loop over all of them.

# Computing the Discrepancy

- $\Delta_H(S) = \max_{\text{all } h} \Delta_S(h)$
- There are an infinite number of possible half-planes…We can't just loop over all of them.
- But…the half-plane of maximum discrepancy must pass through one of the sample points.



# Computing the Discrepancy

- The half-plane of maximum discrepancy must pass through at least one sample point.
- It may pass through exactly one point
  - The maximum discrepancy must be at a local extremum of the continuous measure.
  - There are an infinite number of $h$ through point $p$, but only $O(1)$ of them are local extrema.
  - We can calculate the discrepancies of all $n$ points vs $O(1)$ $h$ each, in $O(n^2)$ time.

# Computing the Discrepancy

- The half-plane of maximum discrepancy must pass through at least one sample point.
- It may pass through exactly one point
- Or it may pass through two points
  - There are $O(n^2)$ possible point pairs.
  - We need some new techniques if we want to be able to compute the discrepancy in $O(n^2)$ time.

# Big Picture

We've defined the discrepancy of a chosen set of sample points with respect to all possible scenes, $\Delta_H(S)$.
We want to pick S to minimize $\Delta_H(S)$.

**We need a way to compute $O(n^2)$ discrete measures to find values of $\Delta_S(h)$.**
**We want to do this in $O(n^2)$ time.**

# New Concept: Duality

- The concept: we can map between different ways of interpreting 2D values.
- Points (x,y) can be mapped in a one-to-one manner to lines (slope,intercept) in a different space.
- There are different ways to do this, called *duality transforms*.

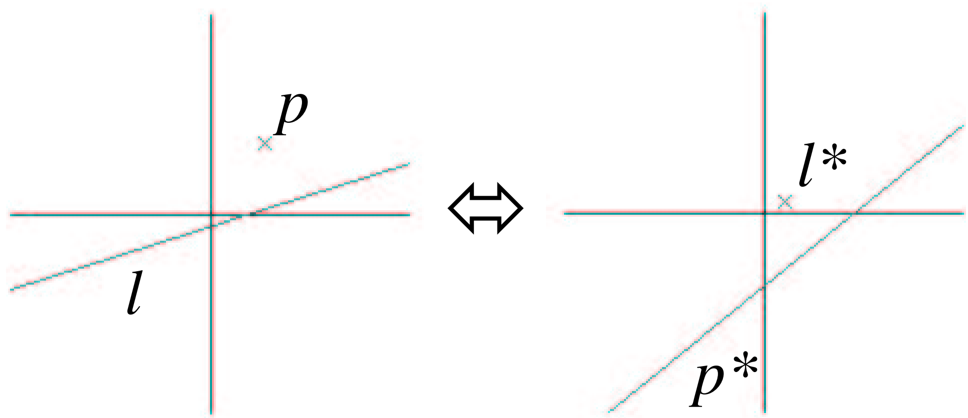
# Duality Transforms

- **A *duality transform* is a mapping which takes an element *e* in the *primal plane* to element *e** in the *dual plane*.**

- One possible duality transform:
  point $p$: $(p_x, p_y)$ ⇔ line $p^*$: $y = p_x x - p_y$
  line $l$: $y = mx + b$ ⇔ point $l^*$: $(m, -b)$

# Duality Transforms

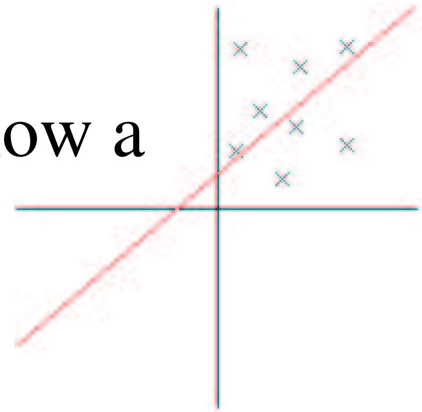
- This duality transform takes
  - points to lines, lines to points
  - line segments to double wedges
- This duality transform preserves order
  - Point $p$ lies above line $l$ ⇔ point $l^*$ lies above line $p^*$



# Back to the Discrepancy problem

To determine our discrete measure, we need to:

Determine how many sample points lie below a given line (in the primal plane).



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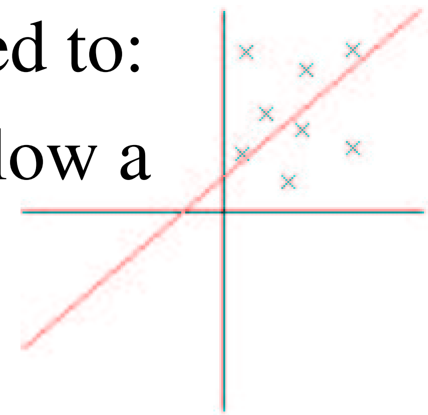
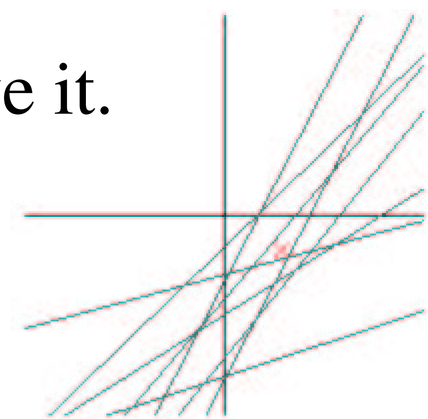


Determine how many sample points lie below a given line (in the primal plane).

⇕ *dualizes to* ⇕

Given a point in the dual plane we want to determine how many sample lines lie above it.



*Is this easier to compute?*

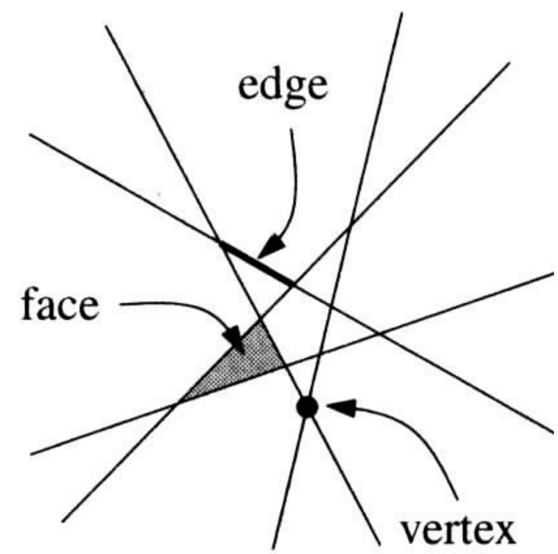
# Duality

- The dualized version of a problem is no easier or harder to compute than the original problem.

- But the dualized version may be easier to think about.

# New Concept: Arrangements of Lines

- *L* is a set of *n* lines in the plane.
- *L* induces a subdivision of the plane that consists of vertices, edges, and faces.
- **This is called the *arrangement* induced by *L*, denoted *A*(*L*)**
- The *complexity* of an arrangement is the total number of vertices, edges, and faces.

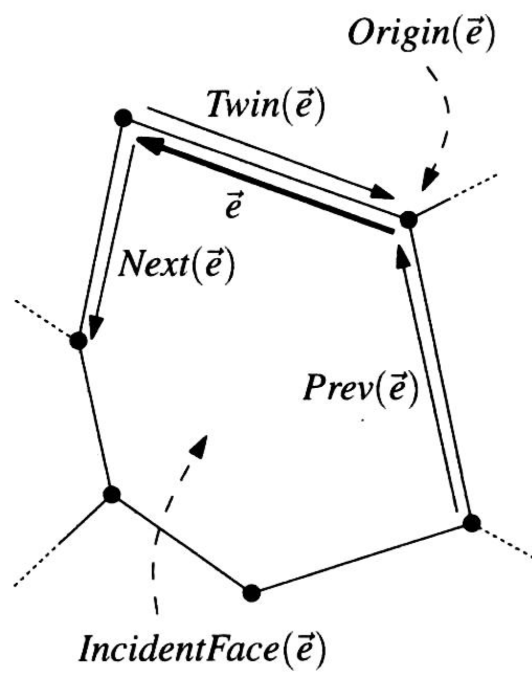


# Arrangments

- Number of vertices of $A(L) \leq \binom{n}{2}$
  - Vertices of $A(L)$ are intersections of $l_i, l_j \in L$
- Number of edges of $A(L) \leq n^2$
  - Number of edges on a single line in $A(L)$ is one more than number of vertices on that line.
- Number of faces of $A(L) \leq \frac{n^2}{2} + \frac{n}{2} + 1$
- Inductive reasoning: add lines one by one
  Each edge of new line splits a face. → $1 + \sum_{i=1}^{n} i$
- Total complexity of an arrangement is $O(n^2)$

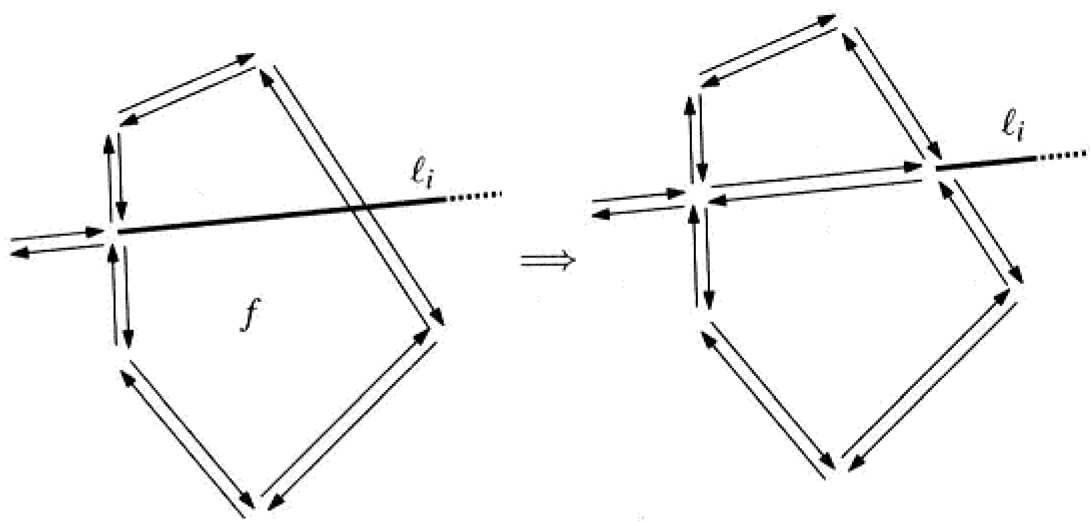
# How Do We Store an Arrangement?

- Data Type: doubly-connected edge-list (DCEL)
  - Vertex:
    - Coordinates, Incident Edge
  - Face:
    - an Edge
  - Half-Edges
    - Origin Vertex
    - Twin Edge
    - Incident Face
    - Next Edge, Prev Edge



# Building the Arrangement

- Iterative algorithm: put one line in at a time.
- Start with the first edge *e* that $l_i$ intersects.
- Split that edge, and move to *Twin*(*e*)



# ConstructArrangement Algorithm

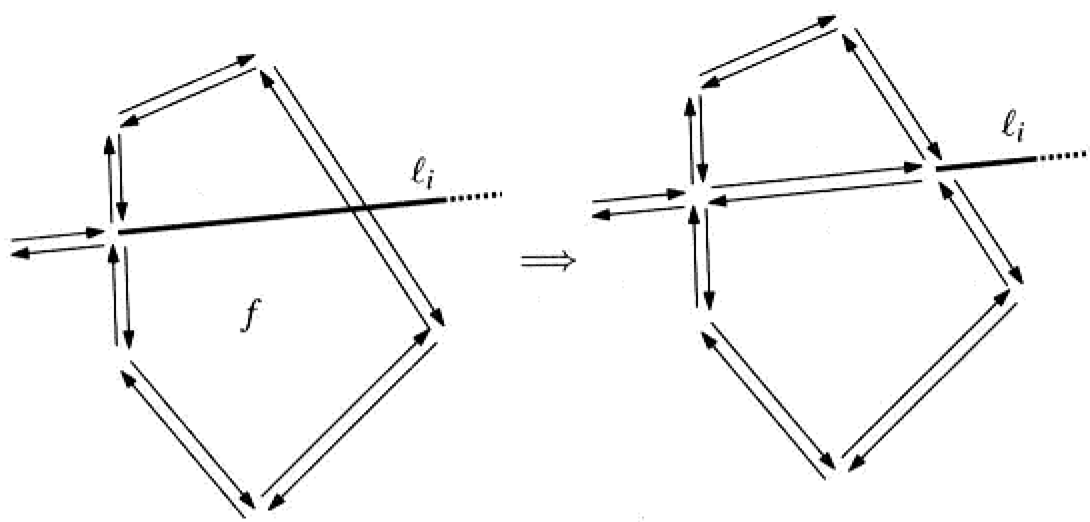
*Input*: A set $L$ of $n$ lines in the plane

*Output*: DCEL for the subdivision induced by the part of $A(L)$ inside a bounding box

1. Compute a bounding box $B(L)$ that contains all vertices of $A(L)$ in its interior
2. Construct the DCEL for the subdivision induced by $B(L)$
3. **for** $i$=1 to $n$ **do**
4. Find the edge $e$ on $B(L)$ that contains the leftmost intersection point of $l_i$ and $A_i$
5. f = the bounded face incident to $e$
6. **while** $f$ is not the face outside $B(L)$ **do**
7. Split $f$, and set $f$ to be the next intersected face

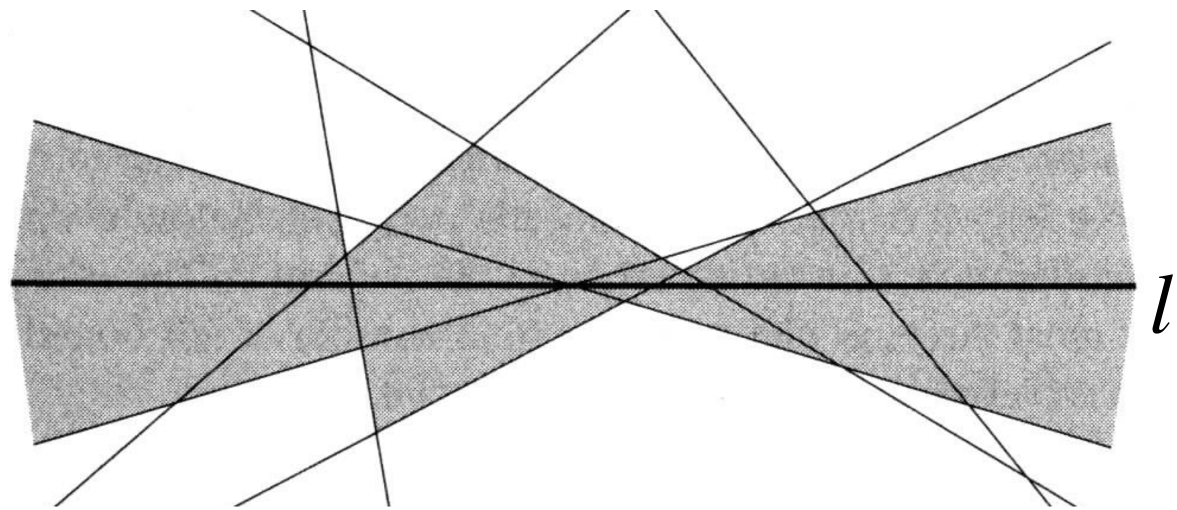
# ConstructArrangement Algorithm -Running Time-

- We need to insert *n* lines.
- Each line splits $O(n)$ edges.
- We may need to traverse $O(n)$ $Next(e)$ pointers to find the next edge to split.



# Zones

- The *zone* of a line $l$ in an arrangement $A(L)$ is the set of faces of $A(L)$ whose closure intersects $l$.




- Note how this relates to the complexity of inserting a line into a DCEL…

# Zone Complexity

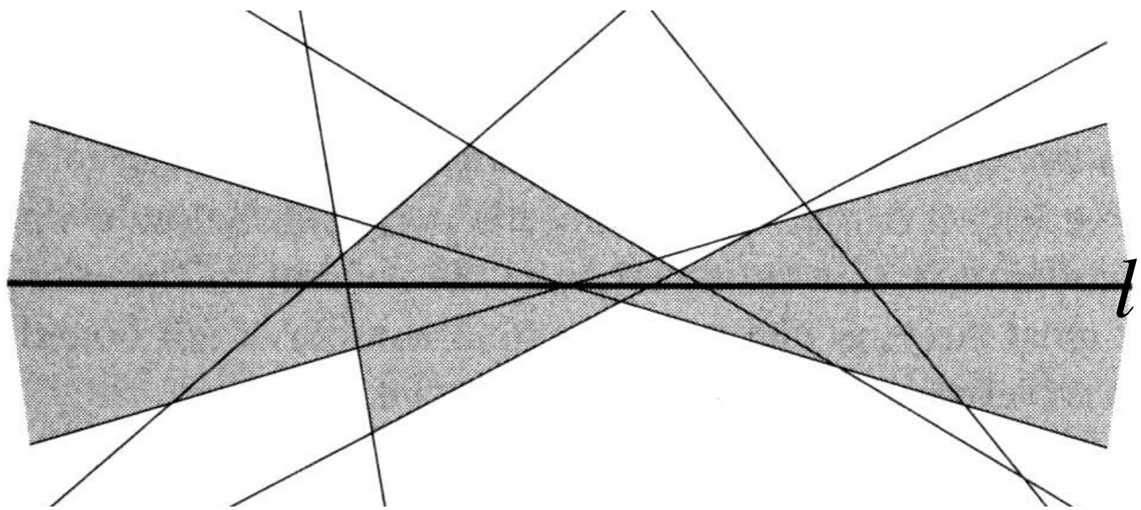
- The complexity of a zone is defined as the total complexity of all the faces it consists of, i.e. the sum of the number of edges and vertices of those faces.
- The time it takes to insert line $l_i$ into a DCEL is linear in the complexity of the zone of $l_i$ in $A(\{l_1,\ldots,l_{i-1}\})$.

# Zone Theorem

- The complexity of the zone of a line in an arrangement of *m* lines on the plane is $O(m)$
- We can insert a line into an arrangement in linear time.
- **We can build an arrangement in $O(n^2)$ time.**

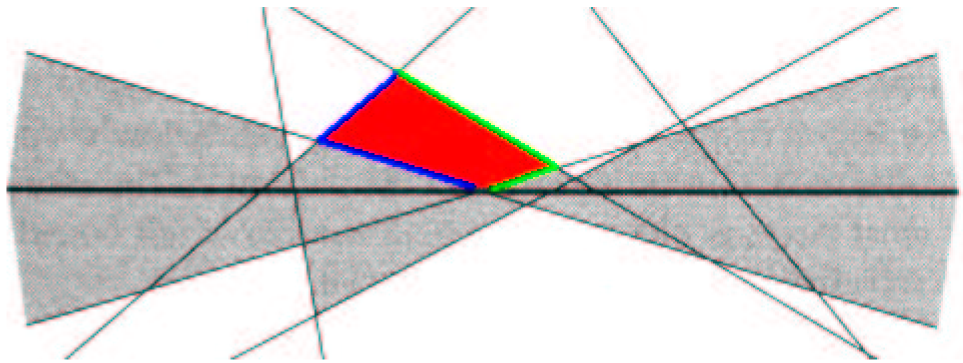
# Proof of Zone Theorem

- Given an arrangement of $m$ lines, $A(L)$, and a line $l$.
- Change coordinate system so $l$ is the x-axis.
- Assume (for now) no horizontal lines



# Proof of Zone Theorem

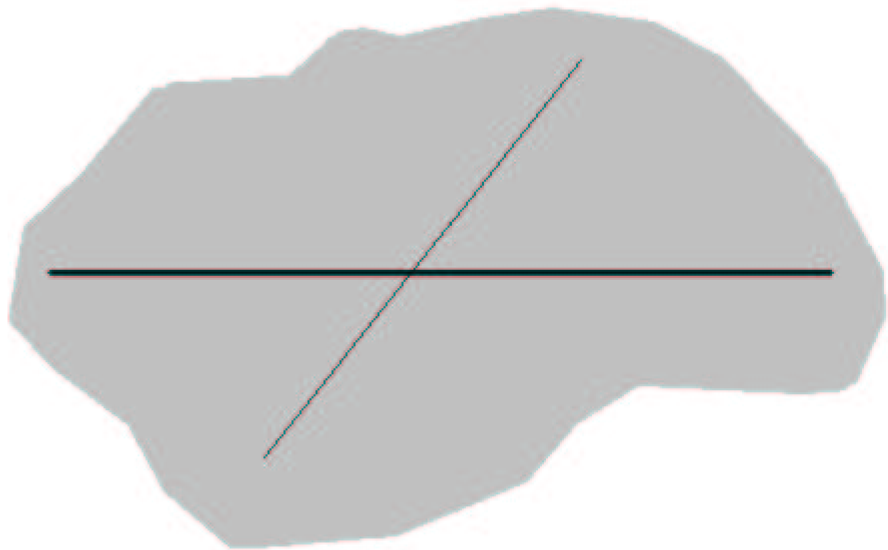
- Each edge in the zone of $l$ is a *left bounding edge* and a *right bounding edge*.



- Claim: number of left bounding edges $\leq 5m$
- Same for number of right bounding edges
  → Total complexity of *zone*($l$) is linear

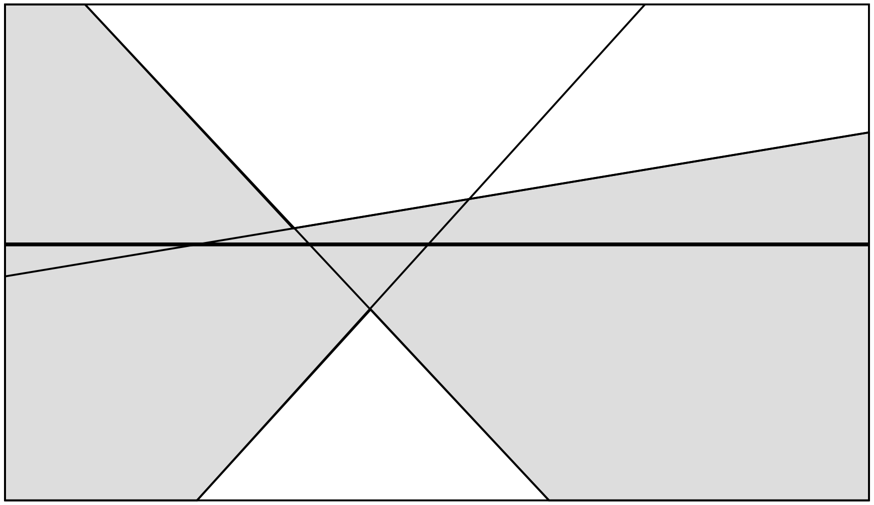
# Proof of Zone Theorem -Base Case-

- When $m$=1, this is trivially true.
  (1 left bounding edge ≤ 5)



# Proof of Zone Theorem
# -Inductive Case-

- Assume true for all but the rightmost line $l_r$: i.e. Zone of $l$ in $A(L\text{-}\{l_r\})$ has at most $5(m\text{-}1)$ left bounding edges
- Assuming no other line intersects $l$ at the same point as $l_r$ , add $l_r$



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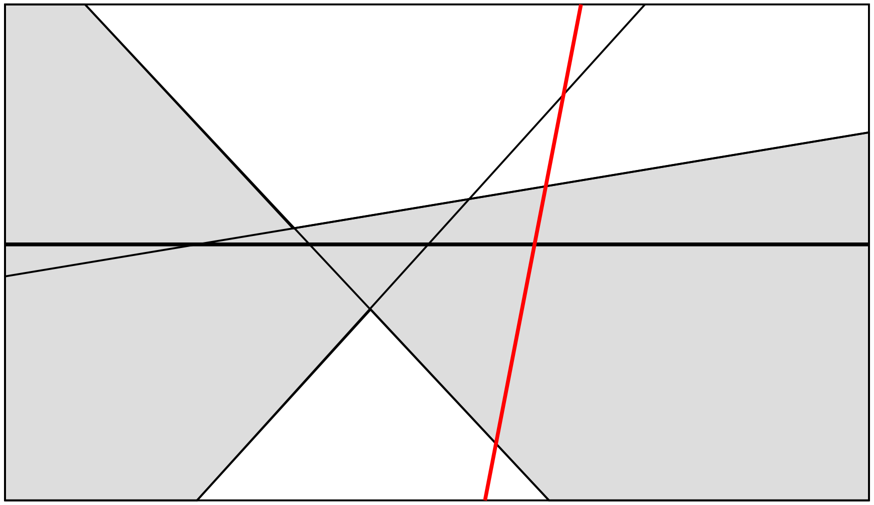
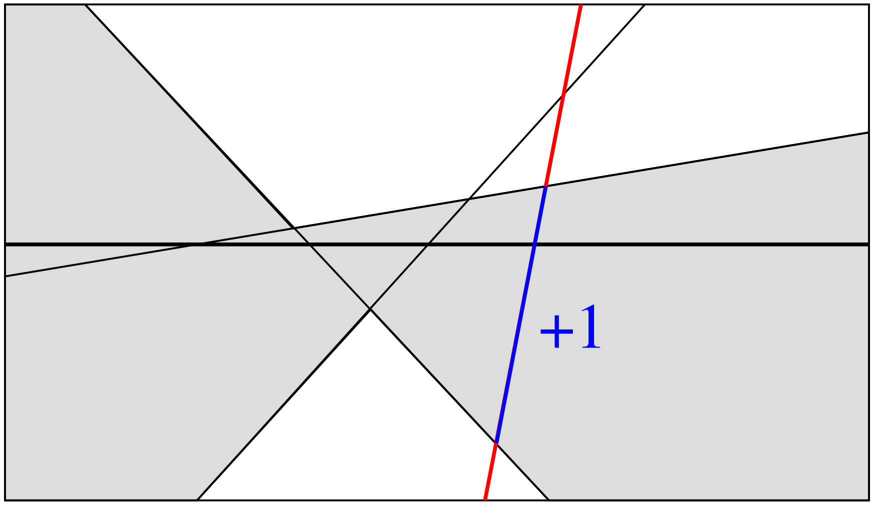


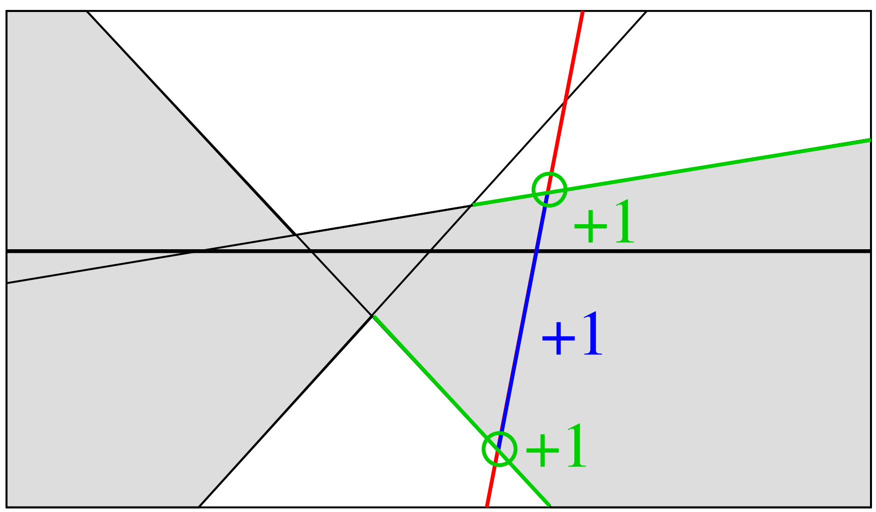
# Proof of Zone Theorem -Inductive Case-

- Assume true for all but the rightmost line $l_r$: i.e. Zone of $l$ in $A(L\text{-}\{l_r\})$ has at most $5(m\text{-}1)$ left bounding edges
- Assuming no other line intersects $l$ at the same point as $l_r$ , add $l_r$
  - $l_r$ has one left bounding edge with $l$ (+1)



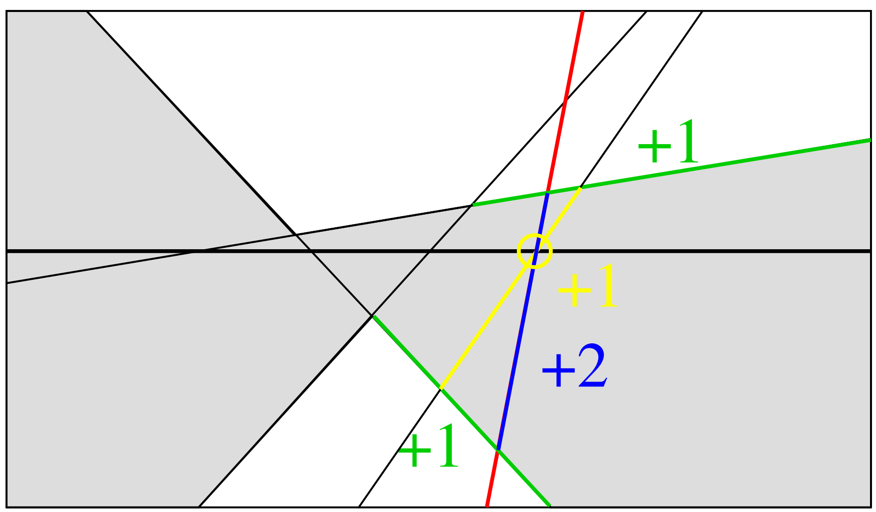
# Proof of Zone Theorem
# -Inductive Case-

- Assume true for all but the rightmost line $l_r$: i.e. Zone of $l$ in $A(L-\{l_r\})$ has at most $5(m-1)$ left bounding edges
- Assuming no other line intersects $l$ at the same point as $l_r$, add $l_r$
  - $l_r$ has one left bounding edge with $l$ (+1)
  - $l_r$ splits at most two left bounding edges (+2)



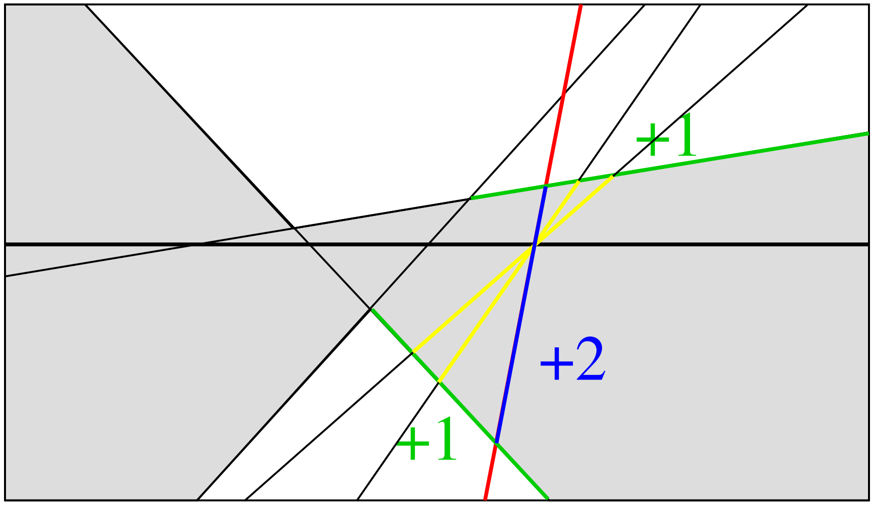
# Proof of Zone Theorem Loosening Assumptions

- What if $l_r$ intersects $l$ at the same point as another line, $l_i$ does?
  - $l_r$ has two left bounding edges (+2)
  - $l_i$ is split into two left bounding edges (+1)
  - As in simpler case, $l_r$ splits two other left bounding edges (+2)



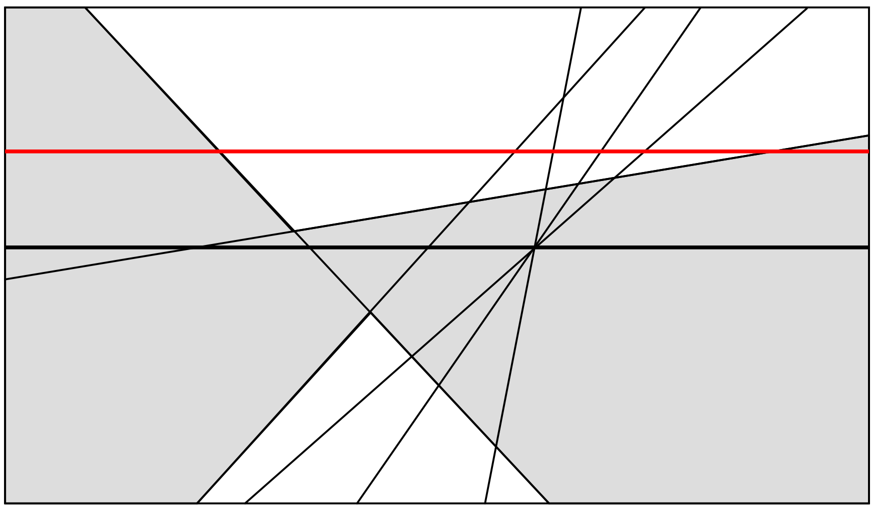
# Proof of Zone Theorem
# Loosening Assumptions

- What if $l_r$ intersects $l$ at the same point as another line, $l_i$ does? (+5)
- What if >2 lines ($l_i$, $l_j$, …) intersect $l$ at the same point?
  - Like above, but $l_i$, $l_j$, … are already split in two (+4)



# Proof of Zone Theorem
# -Loosening Assumptions-

- What if there are horizontal lines in *L*?
- A horizontal line introduces *less* complexity into A(L) than a non-horizontal line.



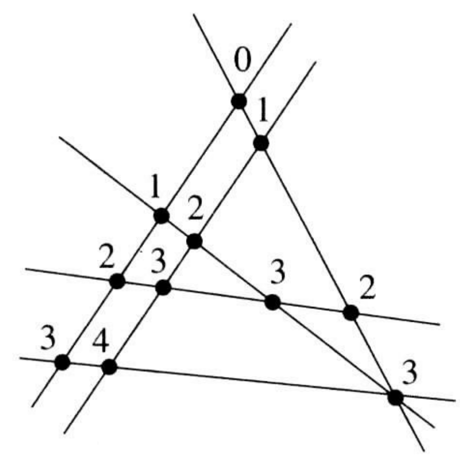
Done proving the Zone Theorem

# Back to Discrepancy (Again)

- For every line between two sample points, we want to determine how many sample points lie below that line.

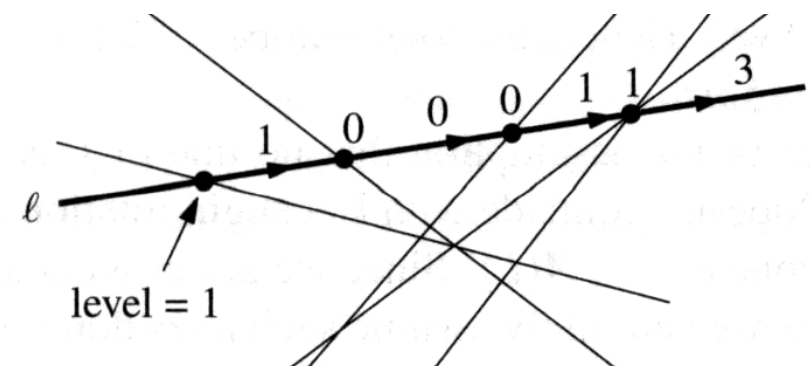
  -or-

- For every vertex in the dual plane, we want to determine how many sample lines lie above it.
- We build the arrangement $A(S^*)$ and use that to determine, for each vertex, how many lines lie above it.
  Call this the *level* of a vertex.



# Levels and Discrepancy

- For each line *l* in *S**
  - Compute the level of the leftmost vertex. $O(n)$
    - Check, for all other lines $l_i$, whether $l_i$ is above that vertex
  - Walk along *l* from left to right to visit the other vertices on *l*, using the DCEL.
    - Walk along *l,* maintaining the level as we go (by inspecting the edges incident to each vertex we encounter).
  - $O(n)$ per line



# What did we just do?

- Given the level of a vertex in the (dualized) arrangement, we can compute the discrete measure of $S$ wrt the $h$ that vertex corresponds to in $O(1)$ time.
- **We can compute all the interesting discrete measures in $O(n^2)$ time.**
- Thus we can compute all $\Delta_S(h)$, and hence $\Delta_H(S)$, in $O(n^2)$ time.

# Summary

- Problem regarding points $S$ in ray-tracing
- Dualize to a problem of lines $L$.
- Compute arrangement of lines $A(L)$.
- Compute level of each vertex in $A(L)$.
- Use this to compute discrete measures in primal space.
- We can determine how good a distribution of sample points is in $O(n^2)$ time.

# Further

- Zone Theorem has an analog in higher dimensions
  - Zone of a hyperplane in an arrangement of n hyperplanes in d-dimensional space has complexity $O(n^{d\text{-}1})$
- There are other point-line dualities
  - $$p_i = \vec{\mathbf{f}} + \left( \frac{\hat{\mathbf{h}}_{\mathbf{i}}}{d_i} \right)$$