Curves & Surfaces

Last Time:
- Expected value and variance
- Monte-Carlo in graphics
- Importance sampling
- Stratified sampling
- Path Tracing
- Irradiance Cache
- Photon Mapping

Questions?

Today
- Motivation
  - Limitations of Polygonal Models
  - Gouraud Shading & Phong Normal Interpolation
  - Some Modeling Tools & Definitions
- Curves
- Surfaces / Patches
- Subdivision Surfaces

Limitations of Polygonal Meshes
- Planar facets (& silhouettes)
- Fixed resolution
- Deformation is difficult
- No natural parameterization (for texture mapping)

Can We Disguise the Facets?
Gouraud Shading

- Instead of shading with the normal of the triangle, shade the vertices with the average normal and interpolate the color across each face.

Phong Normal Interpolation (Not Phong Shading)

- Interpolate the average vertex normals across the face and compute per-pixel shading

Better, but not always good enough

- Still low, fixed resolution (missing fine details)
- Still have polygonal silhouettes
- Intersection depth is planar (e.g. ray visualization)
- Collisions problems for simulation
- Solid Texturing problems
- ...

Some Non-Polygonal Modeling Tools

- Extrusion
- Surface of Revolution
- Spline Surfaces/Patches
- Quadrics and other implicit polynomials

Continuity definitions:

- C^0 continuous
  - curve/surface has no breaks/gaps/holes
- G^1 continuous
  - tangent at joint has same direction
- C^1 continuous
  - curve/surface derivative is continuous
  - tangent at joint has same direction and magnitude
- C^n continuous
  - curve/surface through n derivative is continuous
  - important for shading
Today

- Motivation
- Curves
  - What's a Spline?
  - Linear Interpolation
  - Interpolation Curves vs. Approximation Curves
  - Bézier
  - BSpine (NURBS)
- Surfaces / Patches
- Subdivision Surfaces

Questions?

Definition: What's a Spline?

- Smooth curve defined by some control points
- Moving the control points changes the curve

Linear Interpolation

- Simplest "curve" between two points

Interpolation Curves / Splines

Interpolation Curves

- Curve is constrained to pass through all control points
- Given points $P_0, P_1, ... , P_n$, find lowest degree polynomial which passes through the points

$$x(t) = a_0 t^n + a_1 t^{n-1} + ... + a_n$$
$$y(t) = b_0 t^n + b_1 t^{n-1} + ... + b_n$$

$$Q(t) = GBT(t) = Geometry \cdot Spline \ Basis \cdot Power \ Basis$$

www.abm.org
Interpolation vs. Approximation Curves

Interpolation curve must pass through control points

Approximation curve is influenced by control points

Cubic Bézier Curve

- 4 control points
- Curve passes through first & last control point
- Curve is tangent at \( P_0 \) to \((P_0 - P_1)\) and at \( P_4 \) to \((P_4 - P_3)\)

Cubic Bézier Curve

- de Casteljau's algorithm for constructing Bézier curves

Bernstein Polynomials

\[ Q(t) = (1-t)^3 P_1 + 3(1-t)^2 t P_2 + 3(1-t) t^2 P_3 + t^3 P_4 \]

\[ Q(t) = GBT(t) \]

\[ B_{\text{Bézier}} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

Connecting Cubic Bézier Curves

Asymmetric: Curve goes through some control points but misses others

- How can we guarantee \( C^0 \) continuity?
- How can we guarantee \( G^1 \) continuity?
- How can we guarantee \( C^1 \) continuity?
- Can’t guarantee higher \( C^2 \) or higher continuity
Connecting Cubic Bézier Curves

- Where is this curve
  - \( C^0 \) continuous?
  - \( G^1 \) continuous?
  - \( C^1 \) continuous?
- What’s the relationship between:
  - the \# of control points, and
  - the \# of cubic Bézier subcurves?

Higher-Order Bézier Curves

- \( > 4 \) control points
- Bernstein Polynomials as the basis functions
  \[
  B_{i,n}(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}, \quad 0 \leq i \leq n
  \]
- Every control point affects the entire curve
  - Not simply a local effect
  - More difficult to control for modeling

Cubic BSplines

- \( \geq 4 \) control points
- Locally cubic
- Curve is not constrained to pass through any control points

- Can be chained together
- Better control locally (windowing)

A BSpline curve is also bounded by the convex hull of its control points.
Connecting Cubic BSpline Curves

- What’s the relationship between
  - the # of control points, and
  - the # of cubic BSpline subcurves?

BSpline Curve Control Points

- Default BSpline
- BSpline with Discontinuity
- BSpline which passes through end points
  - Repeat interior control point
  - Repeat end points

Bézier is not the same as BSpline

- Relationship to the control points is different

Bézier

- BSpline

Converting between Bézier & BSpline

- Using the basis functions:

\[ B_{\text{Bézier}} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]

\[ B_{\text{BSpline}} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ 6 & -9 & 6 & 0 \\ -1 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]

\[ Q(t) = G(t) \cdot B_{\text{Bézier}} \cdot \text{Geometry} + B_{\text{BSpline}} \cdot \text{BSpline Basis} + T(t) \cdot \text{Power Basis} \]
NURBS (generalized BSplines)

- BSpline: uniform cubic BSpline
- NURBS: Non-Uniform Rational BSpline
  - non-uniform = different spacing between the blending functions, a.k.a. knots
  - rational = ratio of polynomials (instead of cubic)

Questions?

Today

- Motivation
- Spline Curves
- Spline Surfaces / Patches
  - Tensor Product
  - Bilinear Patches
  - Bezier Patches
- Subdivision Surfaces

Tensor Product

- Of two vectors:
  \[ [a_1, a_2, a_3] \otimes [b_1, b_2, b_3] = [a_1b_1, a_2b_1, a_3b_1; a_1b_2, a_2b_2, a_3b_2; a_1b_3, a_2b_3, a_3b_3] \]
- Similarly, we can define a surface as the tensor product of two curves....

Farin, Curves and Surfaces for Computer Aided Geometric Design

Bilinear Patch

- Smooth version of quadrilateral with non-planar vertices...
  - But will this help us model smooth surfaces?
  - Do we have control of the derivative at the edges?
Bicubic Bezier Patch

Notation: \[ \mathbf{C}(P_1, P_2, P_3, P_4, \alpha) \] is Bezier curve with control points \( P_i \) evaluated at \( \alpha \)

Define: “Tensor-product” Bezier surface

\[ Q(u, v) = \mathbf{C}(\mathbf{B}(P_{11}, P_{12}, P_{13}, P_{14}), \mathbf{B}(P_{21}, P_{22}, P_{23}, P_{24}), \mathbf{B}(P_{31}, P_{32}, P_{33}, P_{34})), \mathbf{B}(P_{41}, P_{42}, P_{43}, P_{44})), \alpha), \]

Bicubic Bezier Patch Tessellation

- Assignment 8: Given 16 control points and a tessellation resolution, create a triangle mesh

<table>
<thead>
<tr>
<th>Resolution</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>5x5 vertices</td>
<td>resolution:</td>
</tr>
<tr>
<td>11x11 vertices</td>
<td></td>
</tr>
<tr>
<td>41x41 vertices</td>
<td></td>
</tr>
</tbody>
</table>

Modeling Headaches

- Original Teapot model is not "watertight": intersecting surfaces at spout & handle, no bottom, a hole at the spout tip, a gap between lid & base

Modeling with Bicubic Bezier Patches

- Original Teapot specified with Bezier Patches

Trimming Curves for Patches

Shirley, Fundamentals of Computer Graphics
Questions?

- Bezier Patches?
  - or
- Triangle Mesh?

Henrik Wann Jensen

Today

- Review
- Motivation
- Spline Curves
- Spline Surfaces / Patches
  - Subdivision Surfaces

Chaikin's Algorithm

Möbius transform for each face
At the midpoint of old vertex, face centroid

Doo-Sabin Subdivision

Möbius transform for each face
At the midpoint of old vertex, face centroid

Doo-Sabin Subdivision

http://www.ke.ics.saitama-u.ac.jp/xuz/pic/doo-sabin.gif

Loop Subdivision

Shirley, Fundamentals of Computer Graphics
Loop Subdivision

- Some edges can be specified as crease edges

http://grail.cs.washington.edu/projects/subdivision/

Questions?

Justin Legakis

Neat Bezier Spline Trick

- A Bezier curve with 4 control points:
  \[- P_0 \quad P_1 \quad P_2 \quad P_3 \]
- Can be split into 2 new Bezier curves:
  \[- P_0 \quad P'_1 \quad P'_2 \quad P'_3 \]
  \[- P'_3 \quad P'_4 \quad P'_5 \quad P_3 \]

A Bezier curve is bounded by the convex hull of its control points.

Next Tuesday: (no class Thursday!)

Animation I:
Particle Systems