6.837 Linear Algebra Review

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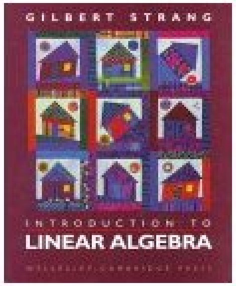
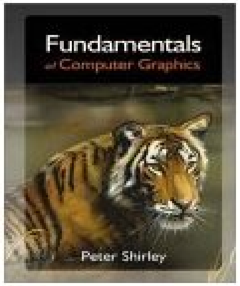
Overview

- Basic matrix operations (+, -, *)
- Cross and dot products
- Determinants and inverses
- Homogeneous coordinates
- Orthonormal basis

Additional Resources

- 18.06 Text Book
- 6.837 Text Book
- 6.837-staff@graphics.lcs.mit.edu
- Check the course website for a copy of these notes





What is a Matrix?

- A matrix is a set of elements, organized into rows and columns

rows

columns

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Basic Operations

- Addition, Subtraction, Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Just add elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Just subtract elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Multiply each row by each column

Multiplication

- Is AB = BA? Maybe, but maybe not!

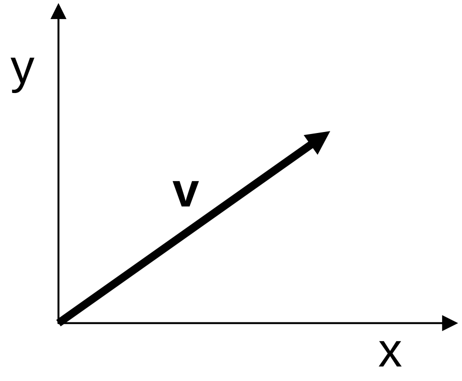
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \ldots \\ \ldots & \ldots \end{bmatrix} \qquad \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea+fc & \ldots \\ \ldots & \ldots \end{bmatrix}$$

- Heads up: multiplication is NOT commutative!

Vector Operations

- Vector: 1 x N matrix
- Interpretation: a line in N dimensional space
- Dot Product, Cross Product, and Magnitude defined on vectors only

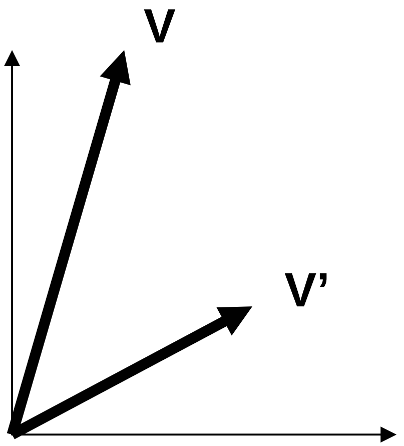
$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



Vector Interpretation

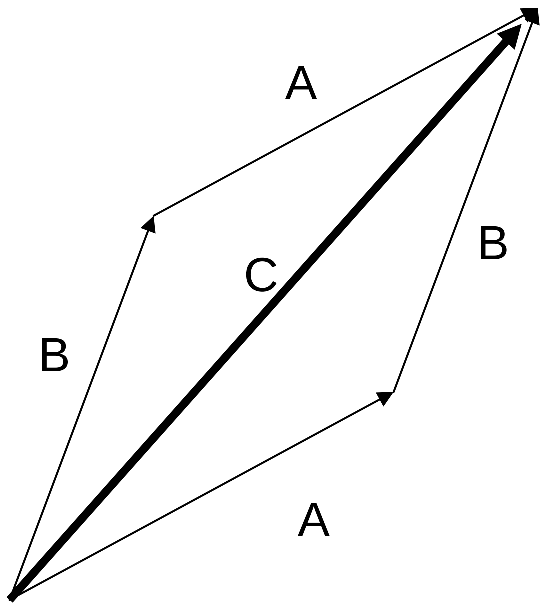
- Think of a vector as a line in 2D or 3D
- Think of a matrix as a transformation on a line or set of lines

$$\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$



Vectors: Dot Product

- Interpretation: the dot product measures to what degree two vectors are aligned



A+B = C
(use the head-to-tail method to combine vectors)

Vectors: Dot Product

$$a \cdot b = ab^T = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$$

Think of the dot product as a matrix multiplication

$$\|a\|^2 = aa^T = \sqrt{aa + bb + cc}$$

The magnitude is the dot product of a vector with itself

$$a \cdot b = \|a\|\|b\|\cos(\theta)$$

The dot product is also related to the angle between the two vectors – but it doesn't tell us the angle

Vectors: Cross Product

- The cross product of vectors A and B is a vector C which is perpendicular to A and B
- The magnitude of C is proportional to the cosine of the angle between A and B
- The direction of C follows the **right hand rule** – this why we call it a “right-handed coordinate system”

$$\left\|a \times b\right\| = \left\|a\right\|\left\|b\right\|\sin(\theta)$$

Inverse of a Matrix

- Identity matrix:
  AI = A
- Some matrices have an inverse, such that:
  $AA^{-1} = I$
- Inversion is tricky:
  $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

  Derived from non-commutativity property

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant of a Matrix

- Used for inversion
- If det(A) = 0, then A has no inverse
- Can be found using factorials, pivots, and cofactors!
- Lots of interpretations – for more info, take 18.06

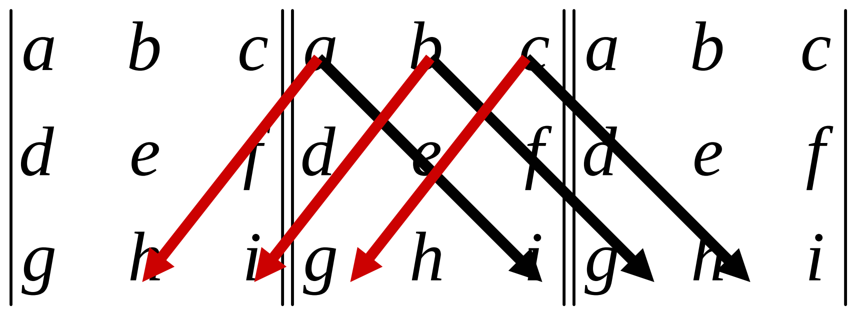
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant of a Matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$



Sum from left to right
Subtract from right to left
Note: N! terms

Inverse of a Matrix

$$\begin{bmatrix} a & b & c & 1 & 0 & 0 \\ d & e & f & +0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{bmatrix}$$

1. Append the identity matrix to A
2. Subtract multiples of the other rows from the first row to reduce the diagonal element to 1
3. Transform the identity matrix as you go
4. When the original matrix is the identity, the identity has become the inverse!

Homogeneous Matrices

- Problem: how to include translations in transformations (and do perspective transforms)
- Solution: add an extra dimension

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} x & & & \\ & y & & \\ & & z & \\ & & & 1 \end{bmatrix}$$

Orthonormal Basis

- Basis: a space is totally defined by a set of vectors – any point is a *linear combination* of the basis
- Ortho-Normal: orthogonal + normal
- Orthogonal: dot product is zero
- Normal: magnitude is one
- Example: X, Y, Z (but don't have to be!)

Orthonormal Basis

$$x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \qquad x \cdot y = 0$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \qquad x \cdot z = 0$$

$$z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \qquad y \cdot z = 0$$

X, Y, Z is an orthonormal basis. We can describe any 3D point as a linear combination of these vectors.

How do we express any point as a combination of a new basis **U, V, N**, given **X, Y, Z**?

Orthonormal Basis

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} u_1 & v_1 & n_1 \\ u_2 & v_2 & n_2 \\ u_3 & v_3 & n_3 \end{bmatrix} = \begin{bmatrix} a \cdot u + b \cdot u + c \cdot u \\ a \cdot v + b \cdot v + c \cdot v \\ a \cdot n + b \cdot n + c \cdot n \end{bmatrix}$$

(not an actual formula – just a way of thinking about it)

To change a point from one coordinate system to another, compute the dot product of each coordinate row with each of the basis vectors.

Questions?

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