Discrete Laplacian Operators

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6.8410: Shape Analysis

Spring 2023

Our Focus

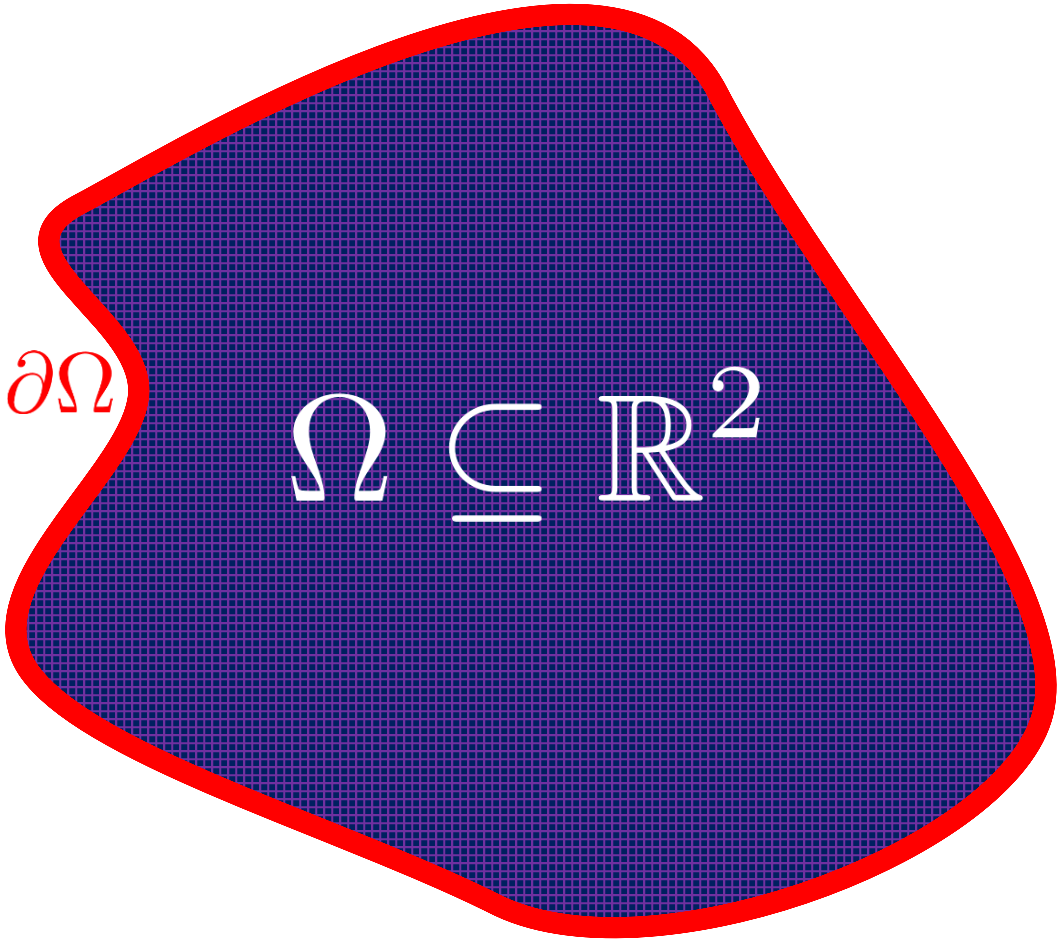
$$f \in C^\infty(\mathcal{M}) \longrightarrow \Delta \longrightarrow \Delta f \in C^\infty(\mathcal{M})$$

Computational version?

The Laplacian

Recall:

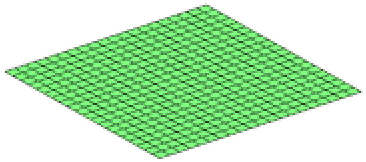
Planar Region



Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = -\Delta u$$

$$\Delta := -\sum_i \frac{\partial^2}{\partial (x^i)^2}$$



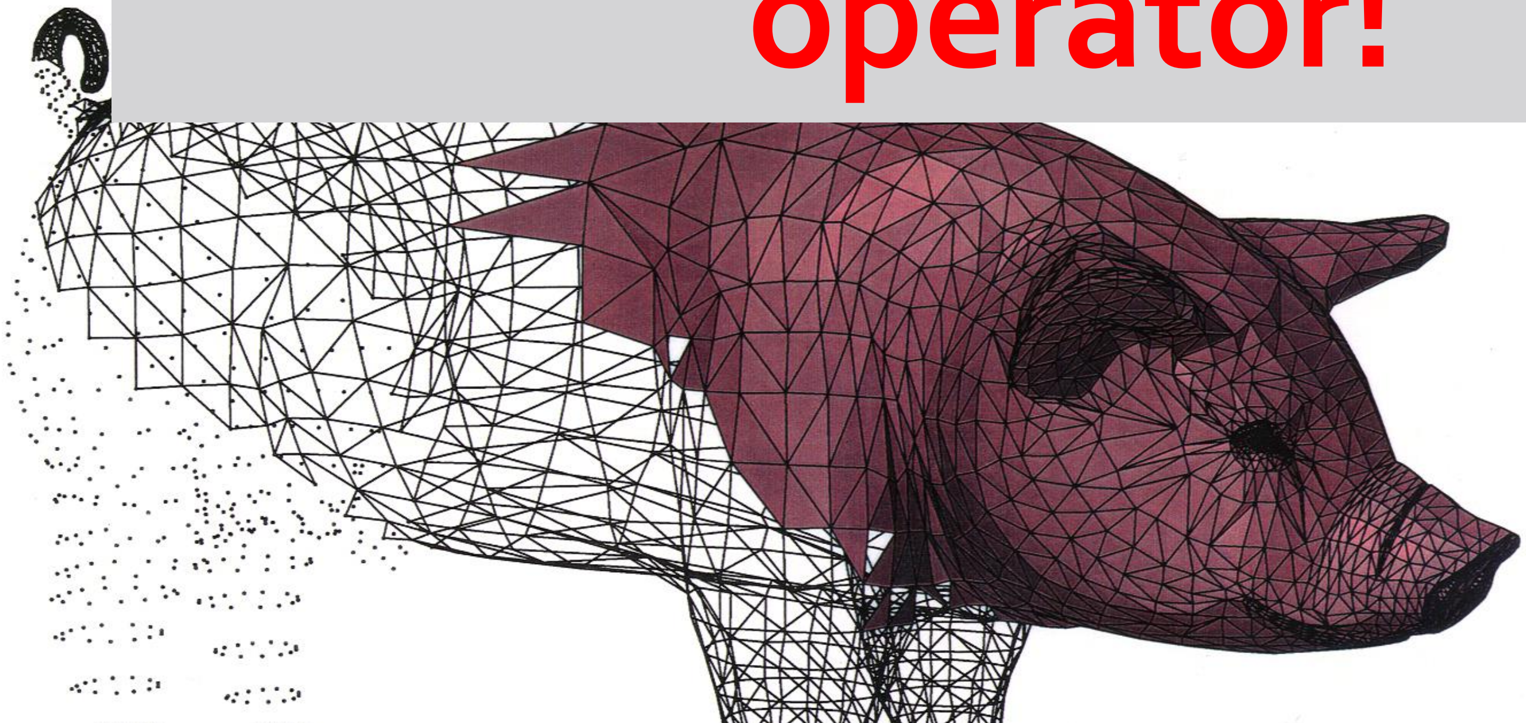
Discretizing the Laplacian

$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right)$$

?!

Problem

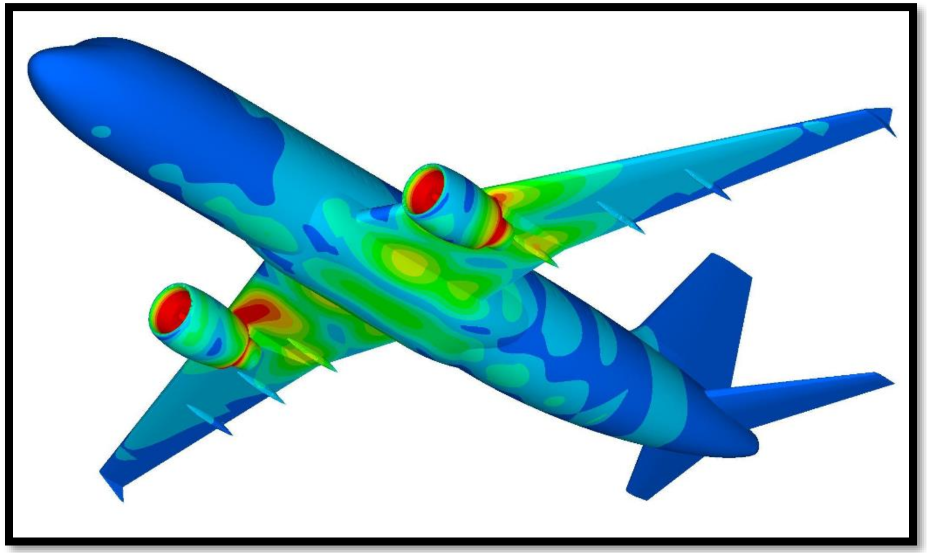
Laplacian is a *differential* operator!



Today's Approach

First-order Galerkin

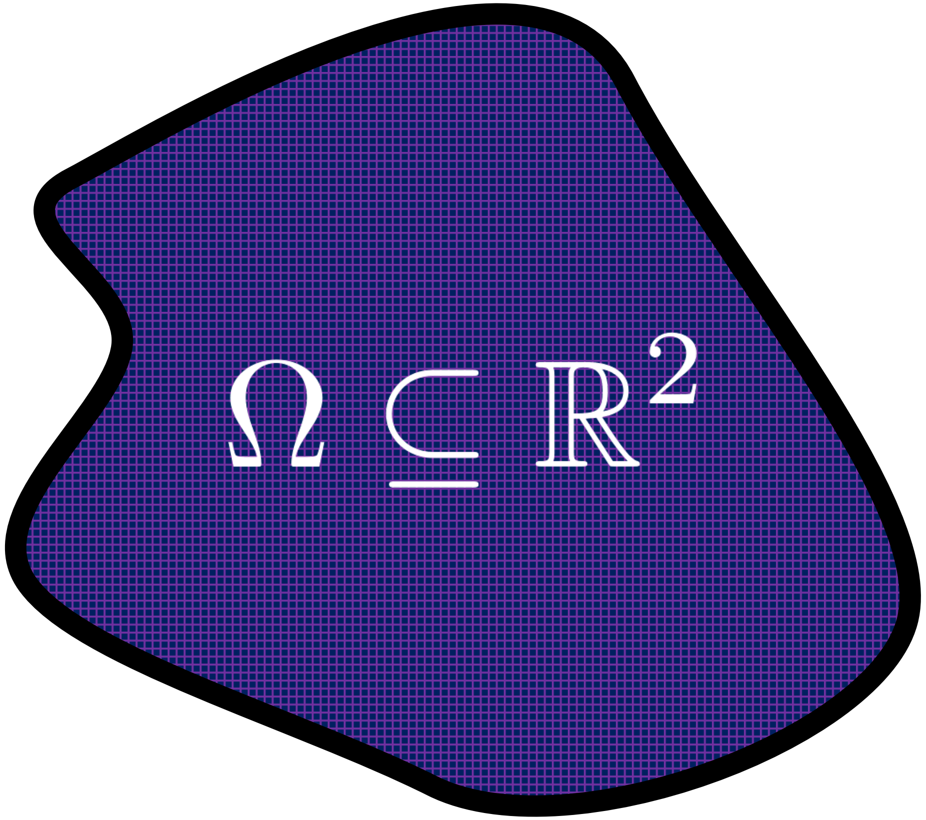
Finite element method (FEM)

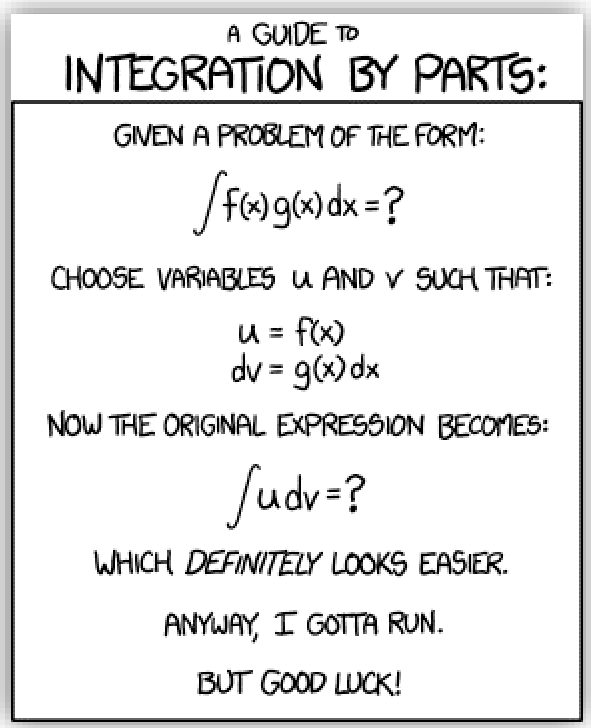


http://www.stressebook.com/wp-content/uploads/2014/08/Airbus_A320_k.jpg

Integration by Parts to the Rescue

$$\int_\Omega f \Delta g \, dA = \text{boundary terms} + \int_\Omega \nabla f \cdot \nabla g \, dA$$





https://xkcd.com/1201/

Slightly Easier?

$$\int_\Omega f \Delta g \, dA = \text{boundary terms} + \int_\Omega \nabla f \cdot \nabla g \, dA$$

Laplacian
(second derivative)

Gradient
(first derivative)

Slightly Easier?

$$\int_\Omega f \Delta g \, dA = \text{boundary terms} + \int_\Omega \nabla f \cdot \nabla g \, dA$$

One derivative,
one integral

Gradient
(first derivative)

Intuition: Cancels?

Overview: Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = [\text{boundary terms}] + \int (\nabla \psi \cdot \nabla f) \, dA$$

$$\text{Approximate } f \approx \sum_k v^k \psi_k \text{ and } g \approx \sum_k w^k \psi_k$$

$$\implies \text{Linear system } \sum_k w^k \langle \psi_i, \psi_\ell \rangle = \sum_k v^k \langle \nabla \psi_k, \nabla \psi_\ell \rangle$$

$$\text{Mass matrix: } M_{ij} := \langle \psi_i, \psi_j \rangle$$

$$\text{Stiffness matrix: } L_{ij} := \langle \nabla \psi_i, \nabla \psi_j \rangle$$

$$\implies M\mathbf{w} = L\mathbf{v}$$

Which basis?

Important to Note

Not the only way

to approximate the Laplacian operator.

- Divided differences
- Higher-order elements
- Boundary element methods
- Discrete exterior calculus
- …

But this method is worth knowing,
so we'll do it in detail!

L^2 Dual of a Function

Function $f : \mathcal{M} \to \mathbb{R}$

$$\downarrow$$

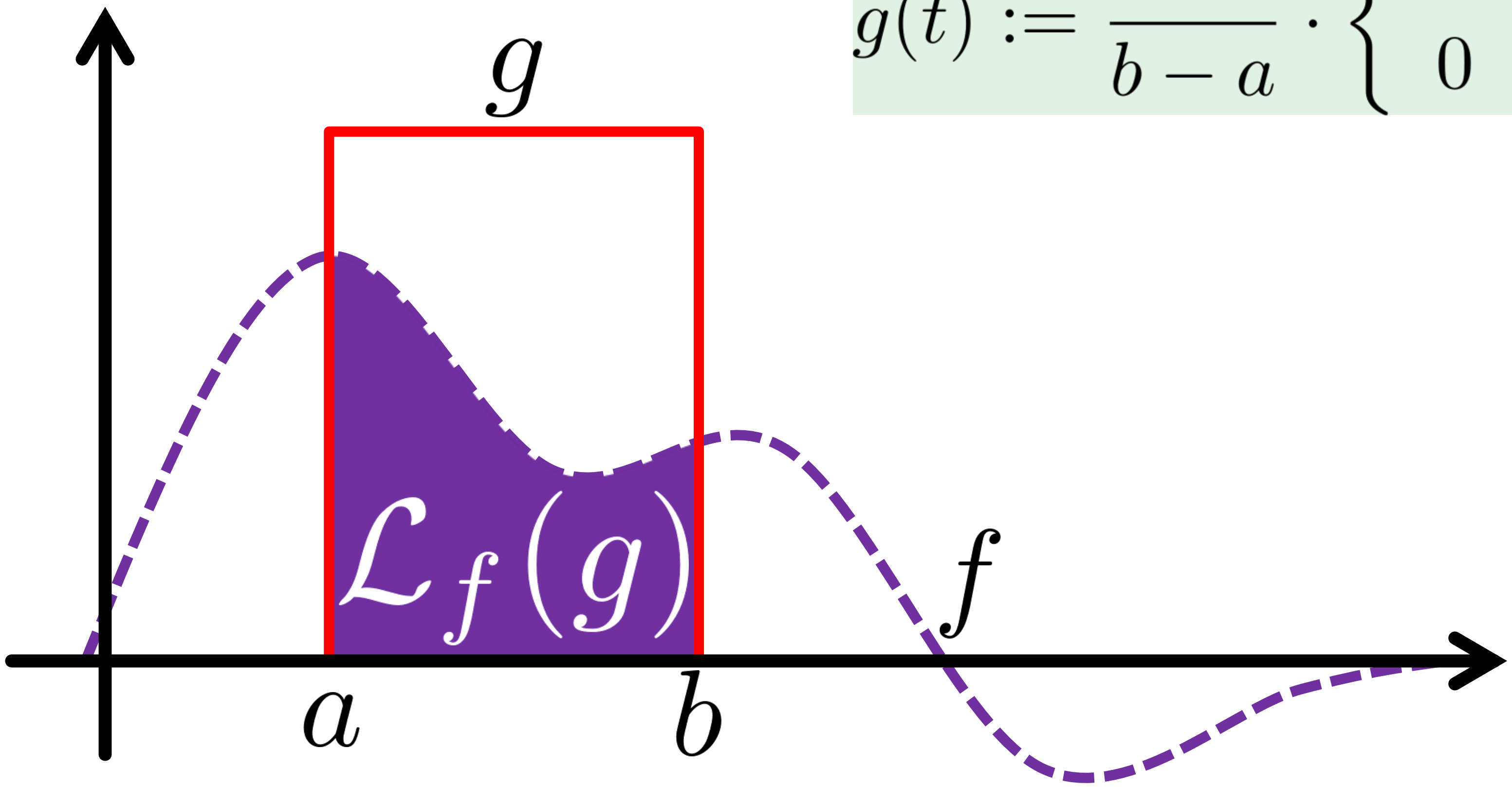
Operator $\mathcal{L}_f : L^2(\mathcal{M}) \to \mathbb{R}$

$$\mathcal{L}_f[g] := \int_{\mathcal{M}} f(\mathbf{x}) g(\mathbf{x}) \, dA(\mathbf{x})$$

"Test function"

Observation

$$g(t) := \frac{1}{b-a} \cdot \begin{cases} 1 & \text{if } t \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$



Can recover function from dual

Dual of Laplacian

Space of test functions (no boundary!):

$$\{g \in C^\infty(M) : g|_{\partial M} \equiv 0\}$$

$$\begin{aligned}\mathcal{L}_{\Delta v}[u] &= \int_{\mathcal{M}} u(\mathbf{x}) \Delta v(\mathbf{x})\, dA(\mathbf{x}) \\ &= \int_{\mathcal{M}} \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})\, dA(\mathbf{x}) \\ &\quad - \oint_{\partial \mathcal{M}} u(\mathbf{x}) \nabla v(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x})\, d\ell\end{aligned}$$

Use Laplacian without evaluating it!

Galerkin's Approach

Choose one of each:

- Function space
- Test functions

Often the same!

One Derivative is Enough

$$\mathcal{L}_{\Delta v}[u] = \int_{\mathcal{M}} \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \, dA(\mathbf{x}) - \oint_{\partial\mathcal{M}} u(\mathbf{x}) \nabla v(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) \, d\ell$$

First Order Finite Elements

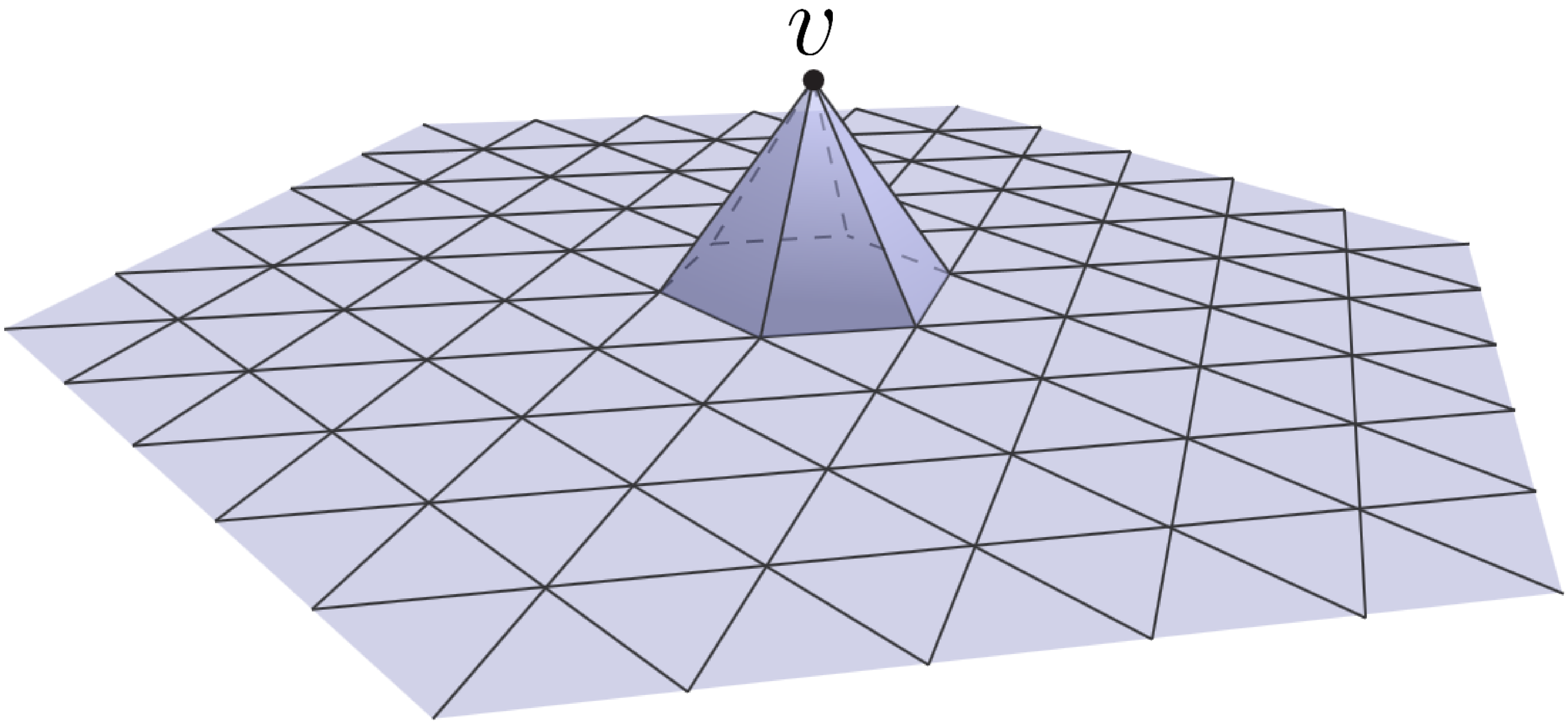
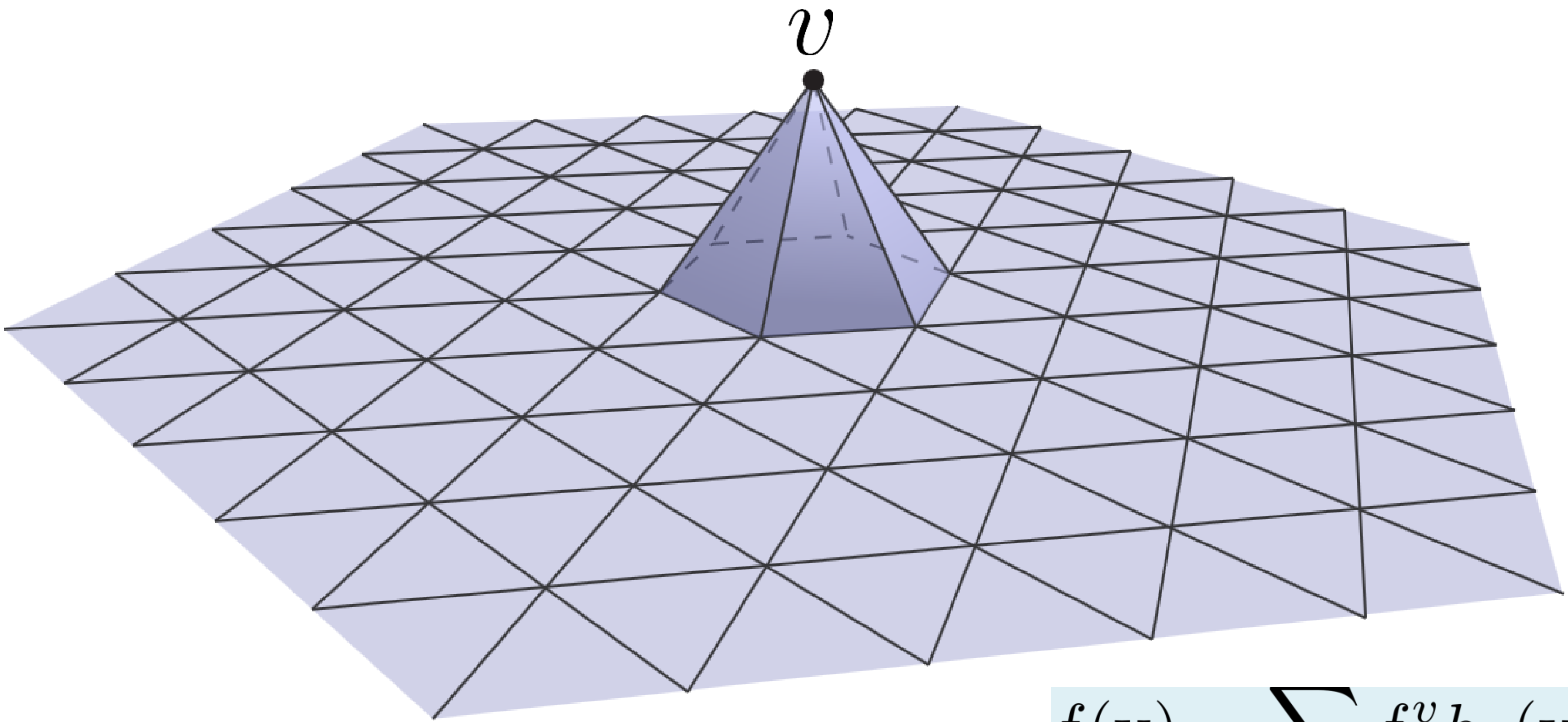


Image courtesy K. Crane, CMU

One "hat function" per vertex

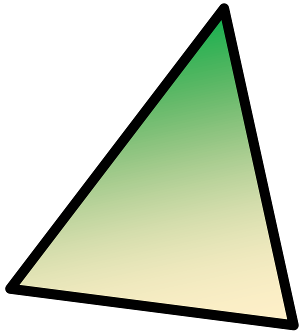
Representing Functions



$$f(\mathbf{x}) = \sum_v f^v h_v(\mathbf{x})$$

$$\mathbf{f} \in \mathbb{R}^{|V|}$$

What Do We Need

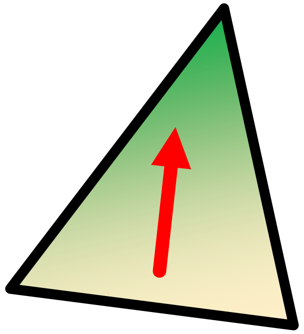


Ignoring boundary terms
(for now!)

$$\mathcal{L}_{\Delta f}[g] = \int_{\mathcal{M}} \nabla g \cdot \nabla f \, dA$$

Linear combination of hats
(piecewise linear)

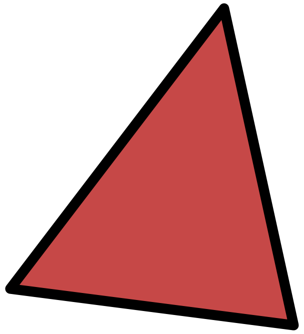
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = \int_{\mathcal{M}} \nabla g \cdot \nabla f \, dA$$

One vector per face

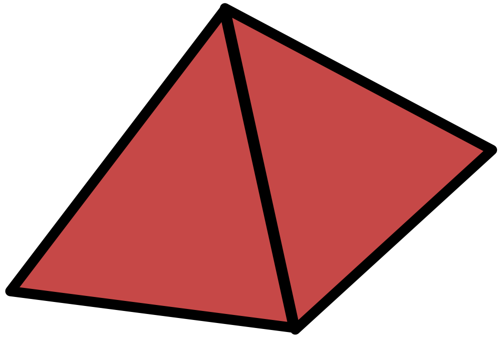
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = \int_{\mathcal{M}} \nabla g \cdot \nabla f \, dA$$

One scalar per face

What Do We Need

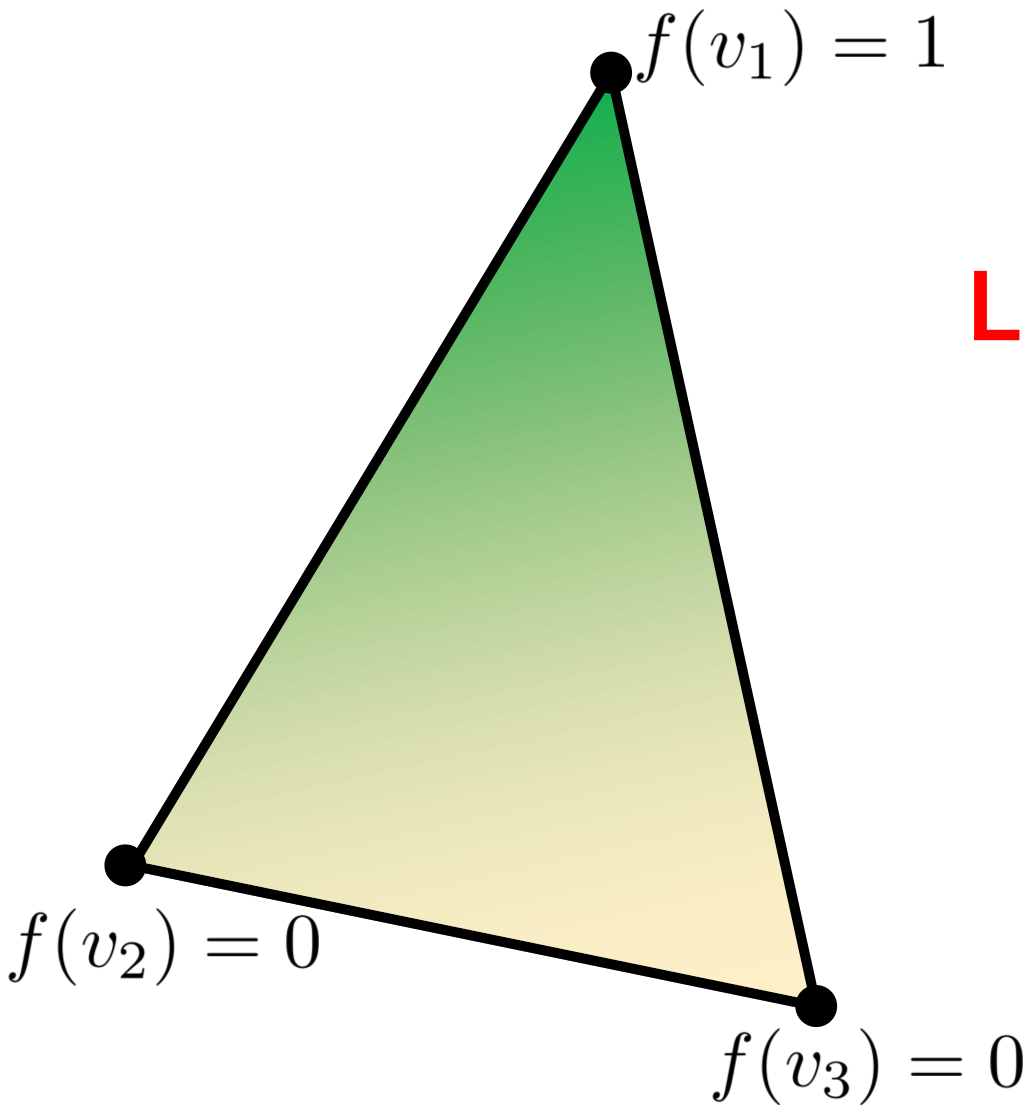


$$\mathcal{L}_{\Delta f}[g] = \int_{\mathcal{M}} \nabla g \cdot \nabla f \, dA$$

↑

Sum scalars per face
multiplied by face areas

Gradient of a Hat Function



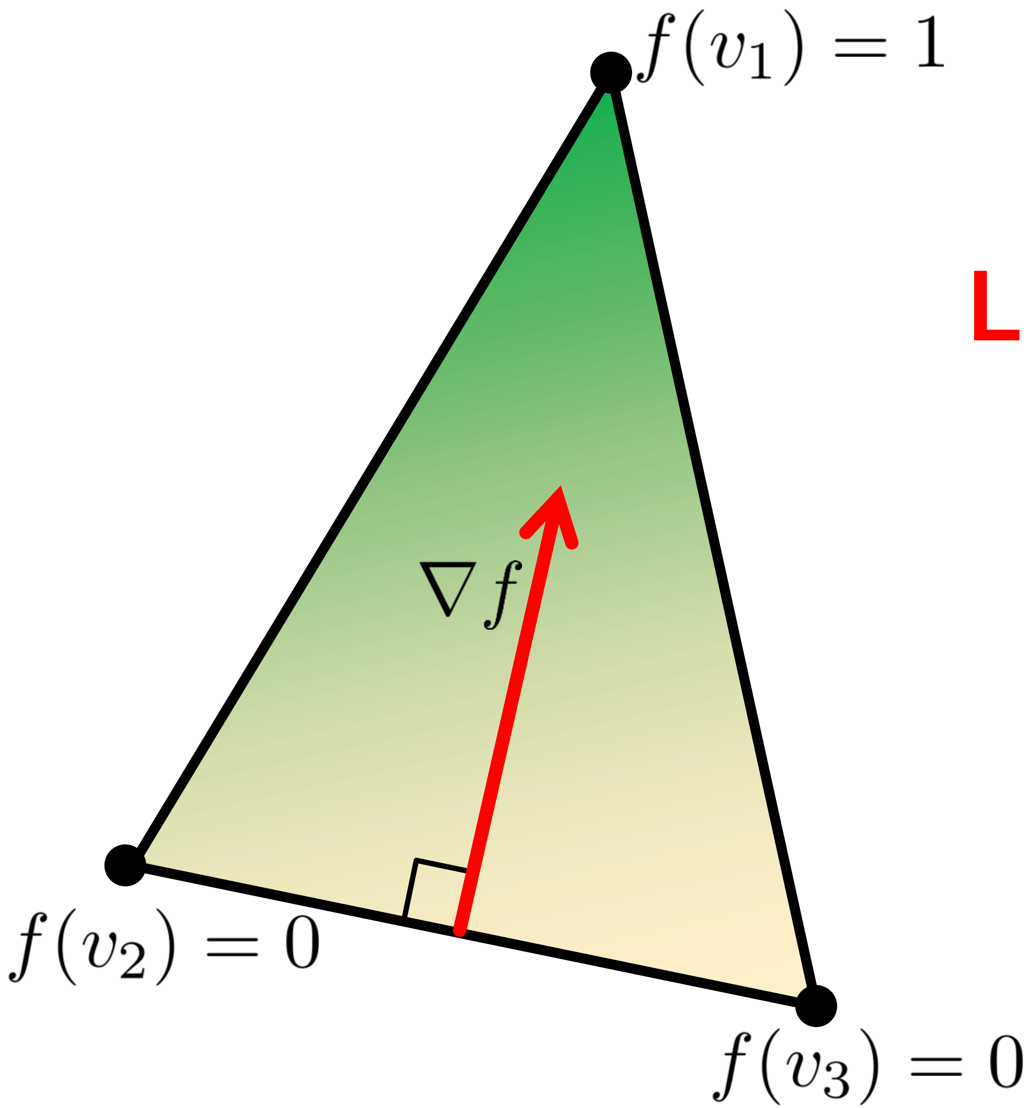
Linear along edges

$$\nabla f \cdot (v_1 - v_3) = 1$$

$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$

Gradient of a Hat Function



Linear along edges

$$\nabla f \cdot (v_1 - v_3) = 1$$

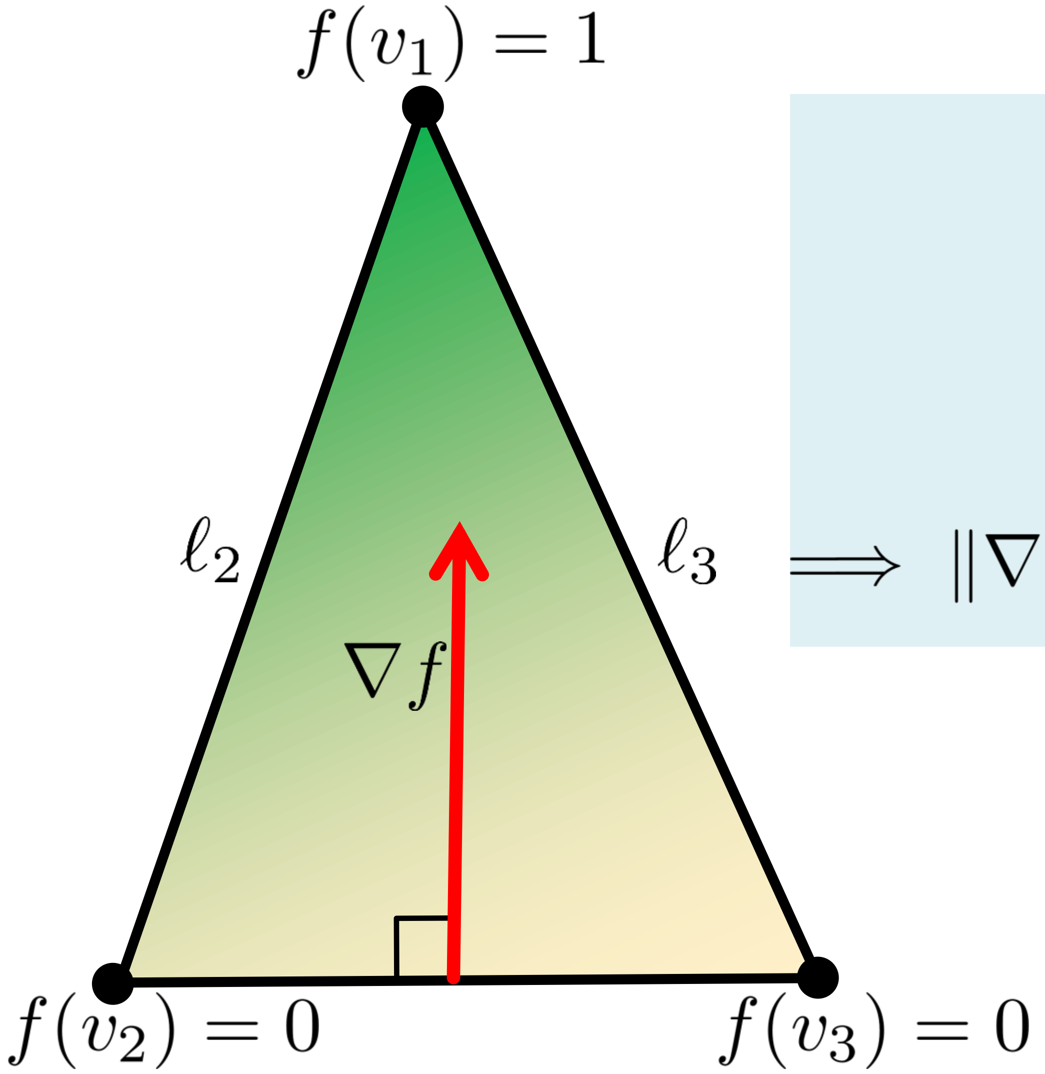
$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$

↓

$$\nabla f \cdot (v_2 - v_3) = 0$$

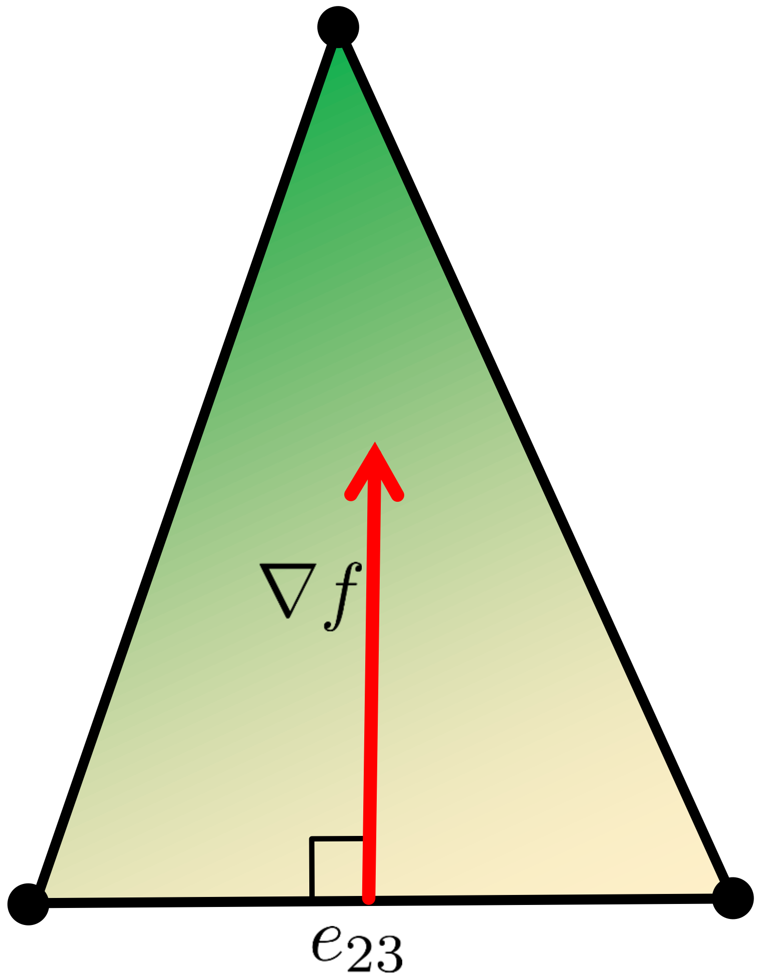
Gradient of a Hat Function



$$\begin{aligned} 1 &= \nabla f \cdot (v_1 - v_3) \\ &= \|\nabla f\| \ell_3 \cos\left(\frac{\pi}{2} - \theta_3\right) \\ &= \|\nabla f\| \ell_3 \sin\theta_3 \end{aligned}$$

$$\implies \|\nabla f\| = \frac{1}{\ell_3 \sin\theta_3} = \frac{1}{h}$$

Gradient of a Hat Function

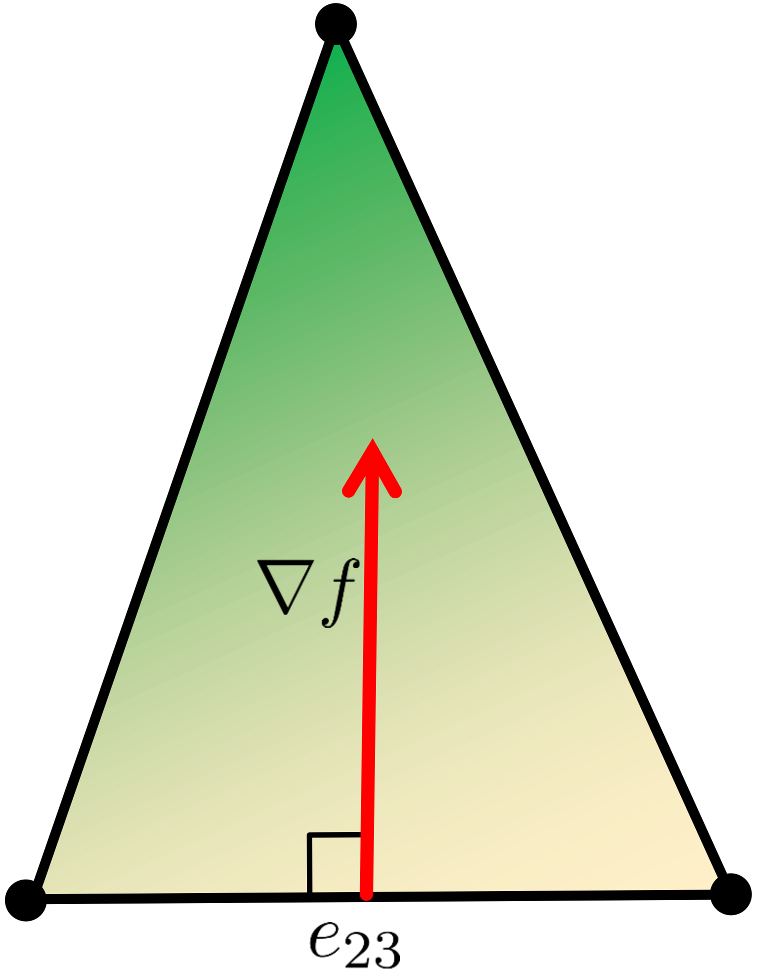


$$\|\nabla f\| = \frac{1}{\ell_3 \sin\theta_3} = \frac{1}{h}$$

$$\nabla f = \frac{e_{23}^{\perp}}{2A}$$

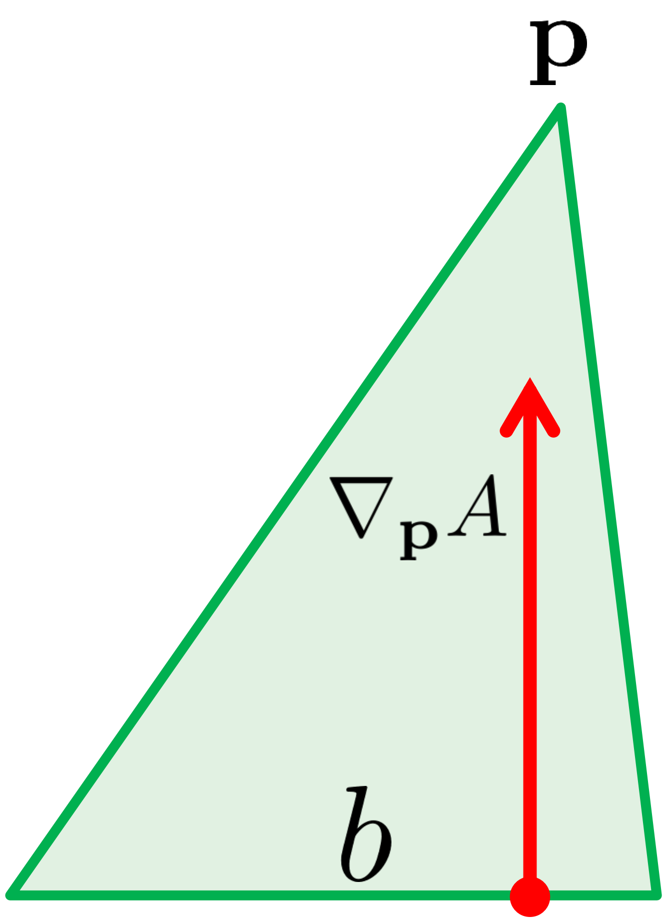
Length of e_{23} cancels "base" in A

Gradient of a Hat Function



$$\nabla f = \frac{e_{23}^{\perp}}{2A}$$

Recall: Single Triangle: Complete



$$\mathbf{p} = p_n \mathbf{n} + p_e \mathbf{e} + p_\perp \mathbf{e}_\perp$$

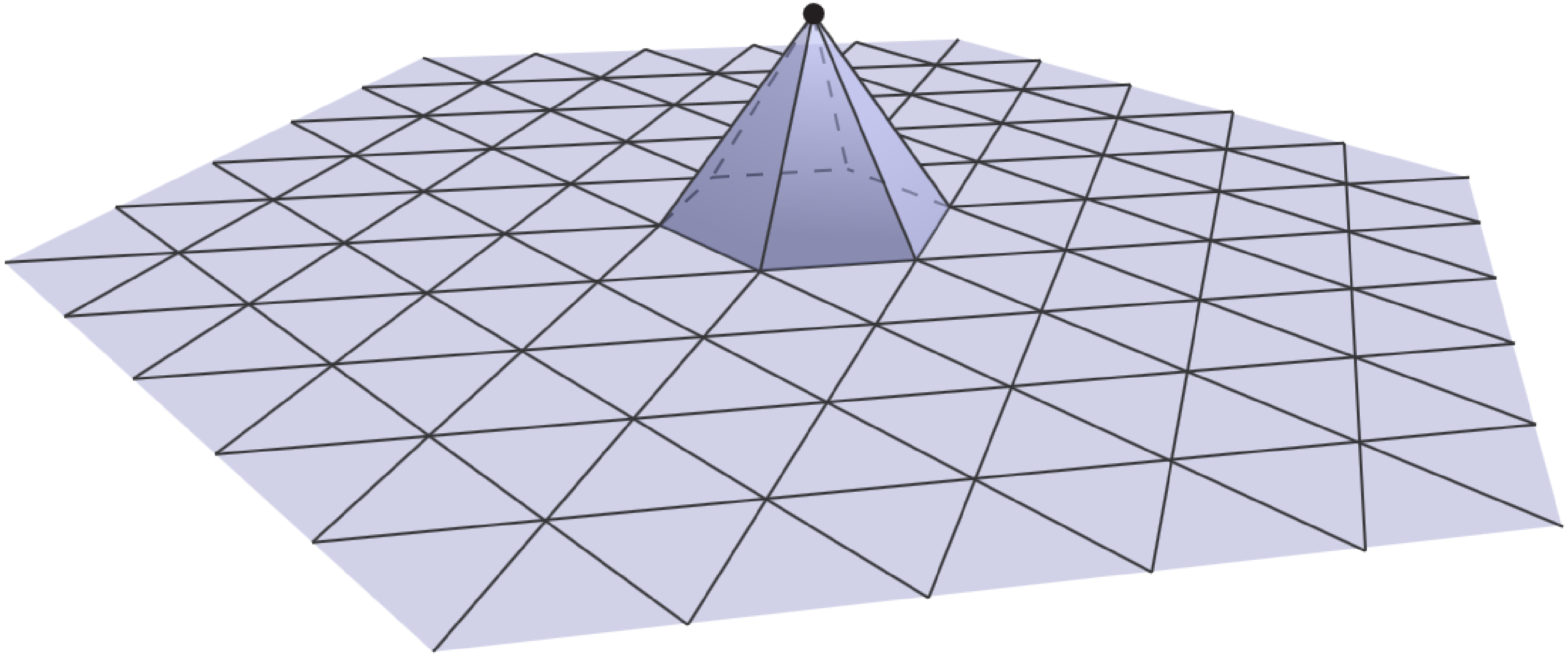
$$A = \frac{1}{2} b \sqrt{p_n^2 + p_\perp^2}$$

$$\nabla_{\mathbf{p}} A = \frac{1}{2} b \mathbf{e}_\perp$$

$$\nabla f = \frac{e_{23}^\perp}{2A} = \frac{\vec{e}_\perp}{h} = \frac{\nabla_{\vec{p}} A}{A}$$

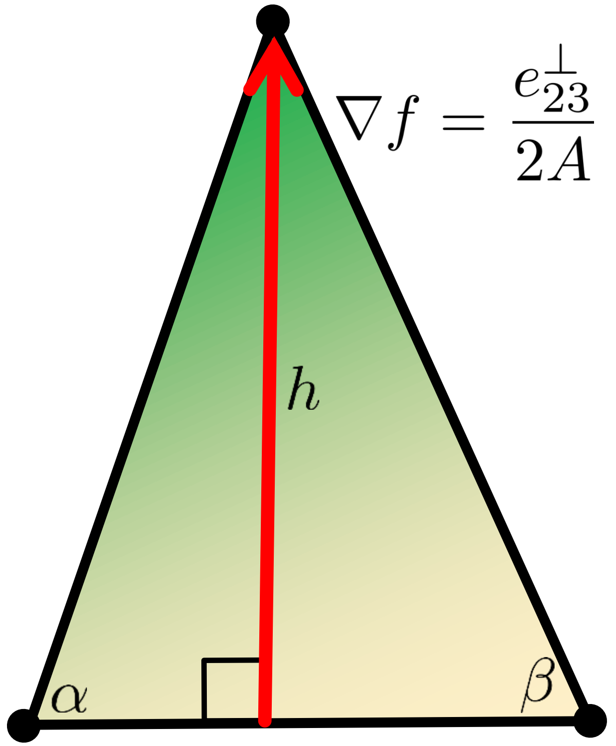
What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = \int_{\mathcal{M}} \boxed{\nabla g \cdot \nabla f} \, dA$$



What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = \int_{\mathcal{M}} \boxed{\nabla g \cdot \nabla f}\, dA$$



Case 1: Same vertex

$$\int_T \langle \nabla f, \nabla f \rangle \, dA = A \|\nabla f\|_2^2$$

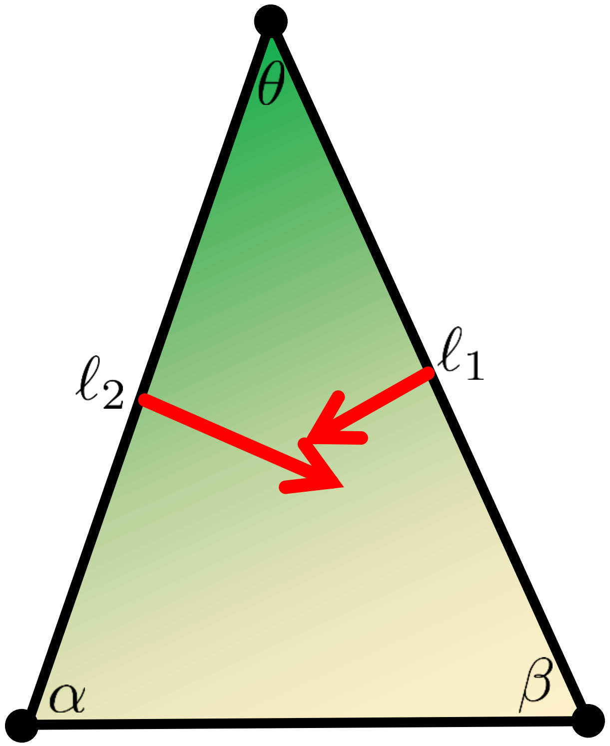
$$= \frac{A}{h^2} = \frac{b}{2h}$$

$$= \frac{1}{2}(\cot\alpha + \cot\beta)$$

What We Actually Need

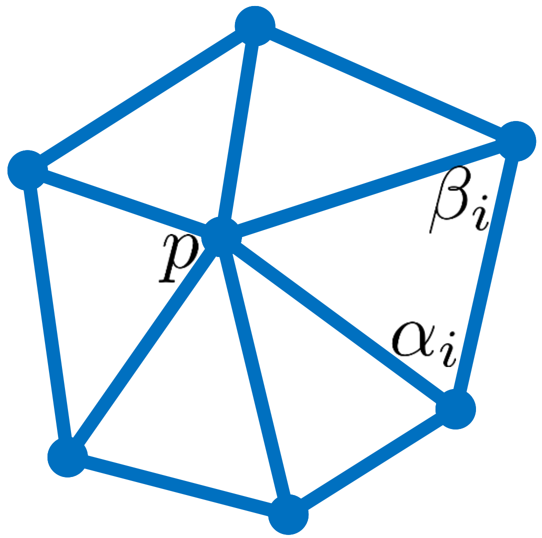
$$\mathcal{L}_{\Delta_f}[g] = \int_{\mathcal{M}} \boxed{\nabla g \cdot \nabla f}\, dA$$

Case 2: Different vertices

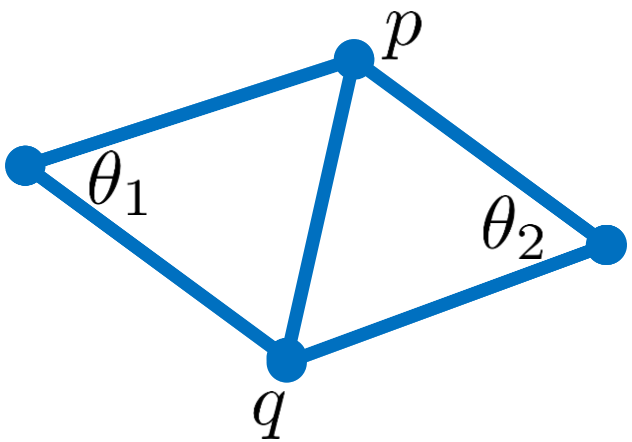


$$\begin{aligned}
\int_T \langle \nabla f_\alpha, \nabla f_\beta \rangle \, dA &= A \langle \nabla f_\alpha, \nabla f_\beta \rangle \\
&= \frac{1}{4A} \langle e_{31}^\perp, e_{32}^\perp \rangle = -\frac{\ell_1 \ell_2 \cos\theta}{4A} \\
&= -\frac{1}{2h_1} \ell_2 \cos\theta = -\frac{\cos\theta}{2\sin\theta} \\
&= -\frac{1}{2} \cot\theta
\end{aligned}$$

Summing Around a Vertex



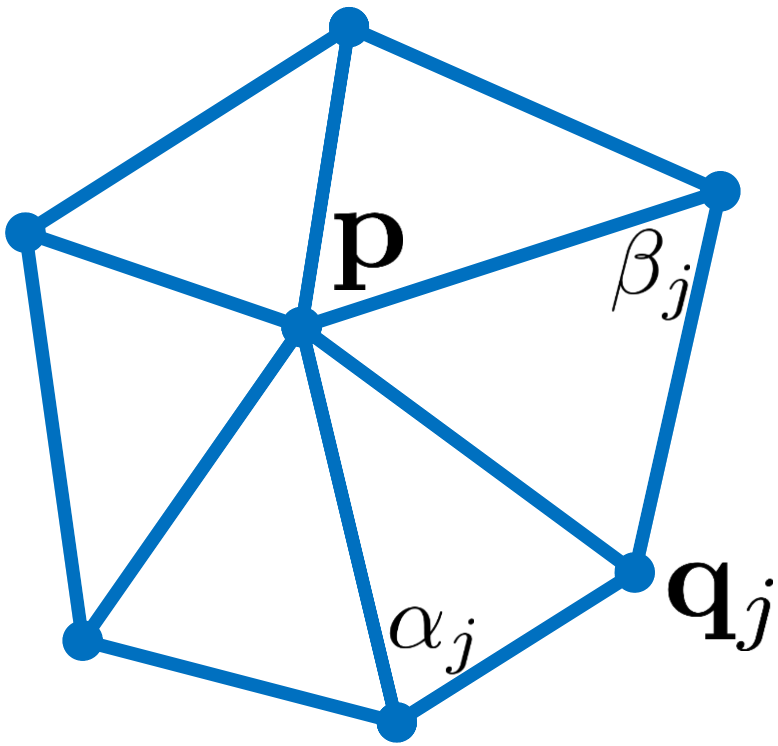
$$\langle \nabla h_p, \nabla h_p \rangle = \frac{1}{2} \sum_i (\cot \alpha_i + \cot \beta_i)$$



$$\langle \nabla h_p, \nabla h_q \rangle = -\frac{1}{2} (\cot \theta_1 + \cot \theta_2)$$

Recall: Summing Around a Vertex

$$\nabla_{\mathbf{p}} A = \frac{1}{2} \sum_j (\cot \alpha_j + \cot \beta_j)(\mathbf{p} - \mathbf{q}_j)$$

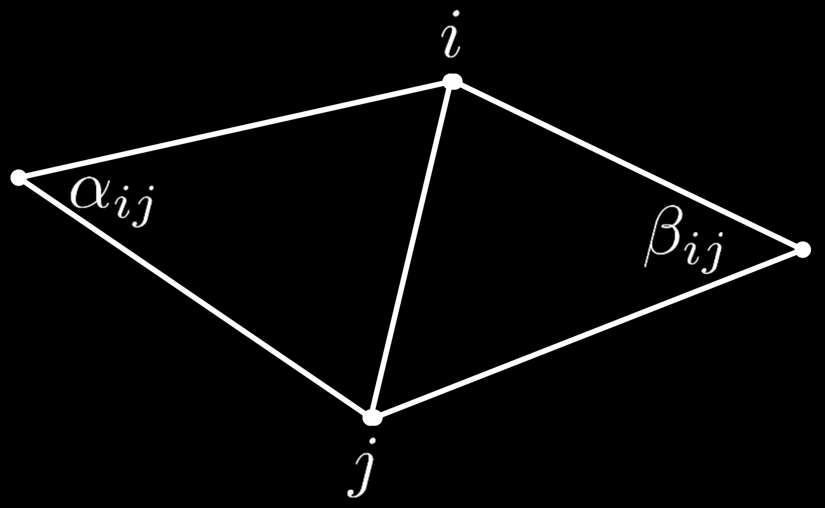


$$\nabla_{\mathbf{p}} A = \frac{1}{2}((\mathbf{p} - \mathbf{r}) \cot \alpha + (\mathbf{p} - \mathbf{q}) \cot \beta)$$

Same operator!

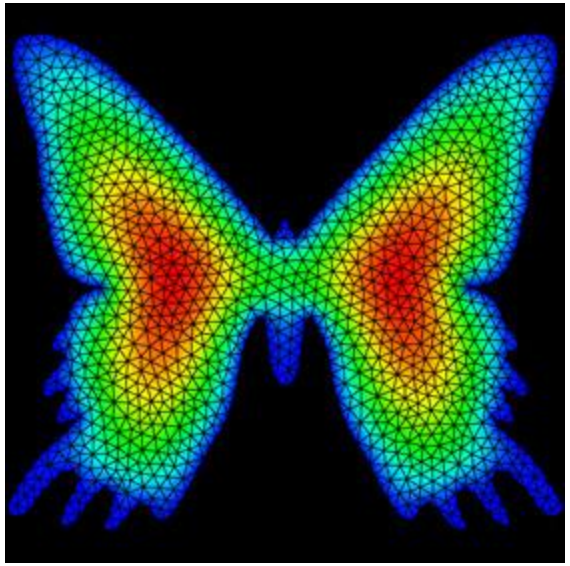
The Cotangent Laplacian

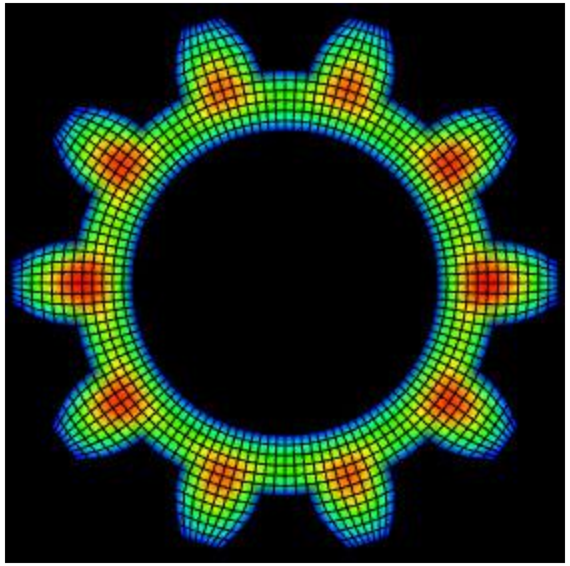
$$L_{ij} = \begin{cases} \frac{1}{2}\sum_{i\sim k}(\cot\alpha_{ik} + \cot\beta_{ik}) & \text{if } i = j \\ -\frac{1}{2}(\cot\alpha_{ij} + \cot\beta_{ij}) & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$



Poisson Equation

$$\Delta f = g$$

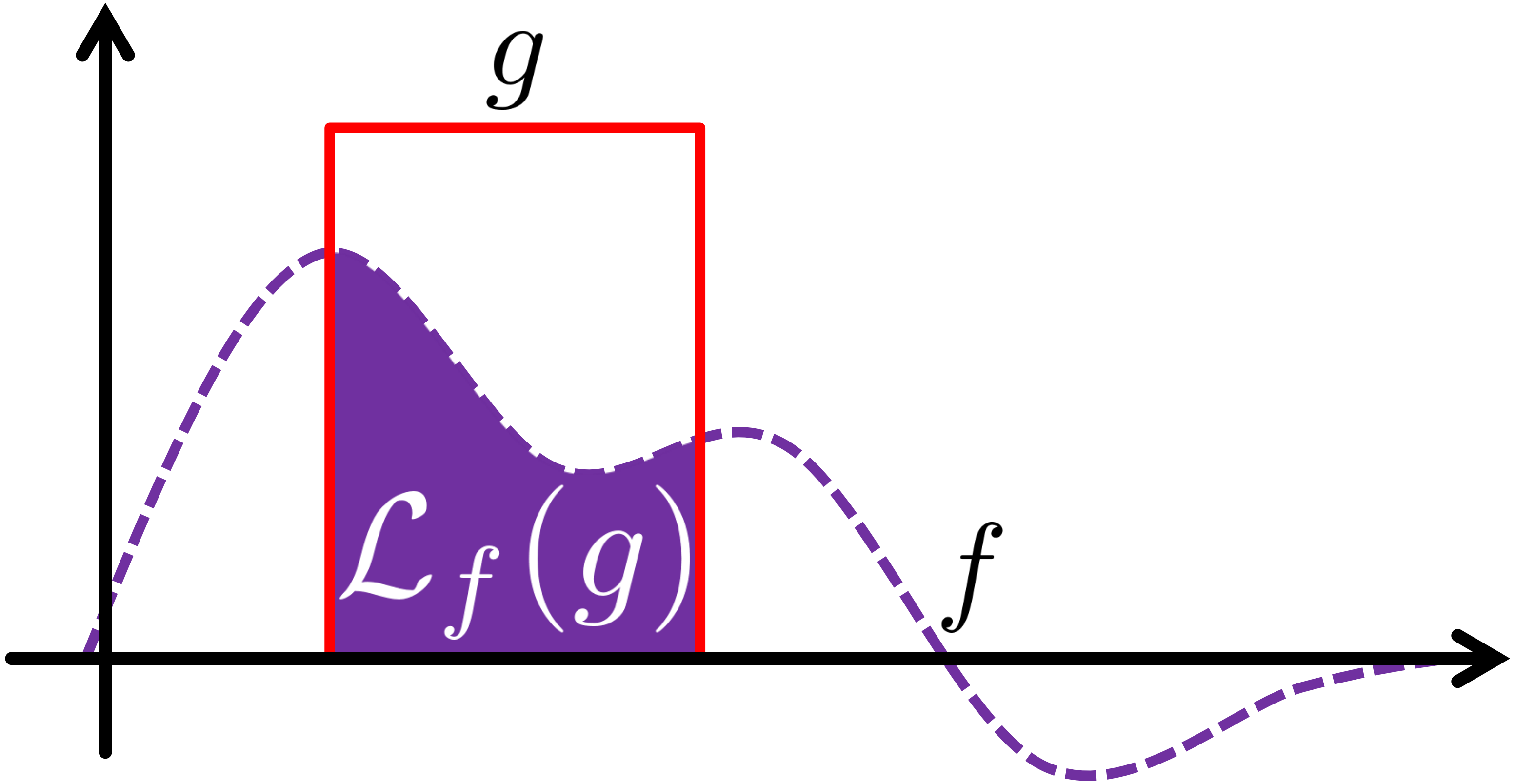




http://nylander.wordpress.com/2006/05/24/finite-element-method-fem-solution-to-poisson%E2%80%99s-equation-on-triangular-mesh/

Weak Solutions

$$\int_M \phi \Delta f \, dA = \int_M \phi g \, dA \;\forall \text{ test functions } \phi$$



FEM Hat Weak Solutions

$$\int_{\mathcal{M}} h_i \Delta f \, dA = \int_{\mathcal{M}} h_i g \, dA \; \forall \text{ hat functions } h_i$$

$$\begin{aligned}\int_{\mathcal{M}} h_\ell \Delta f \, dA &= \int_{\mathcal{M}} \nabla h_\ell \cdot \nabla f \, dA \\ &= \int_{\mathcal{M}} \nabla h_\ell \cdot \nabla \sum_k v^k h_k \, dA \\ &= \sum_k v^k \int_{\mathcal{M}} \nabla h_\ell \cdot \nabla h_k \, dA \\ &= \sum_k L_{\ell k} v^k\end{aligned}$$

Approximate $f \approx \sum_k v^k \psi_k$ and $g \approx \sum_k w^k \psi_k$

$\Longrightarrow$ Linear system $\sum_k w^k \langle \psi_i, \psi_\ell \rangle = \sum_k v^k \langle \nabla \psi_k, \nabla \psi_\ell \rangle$

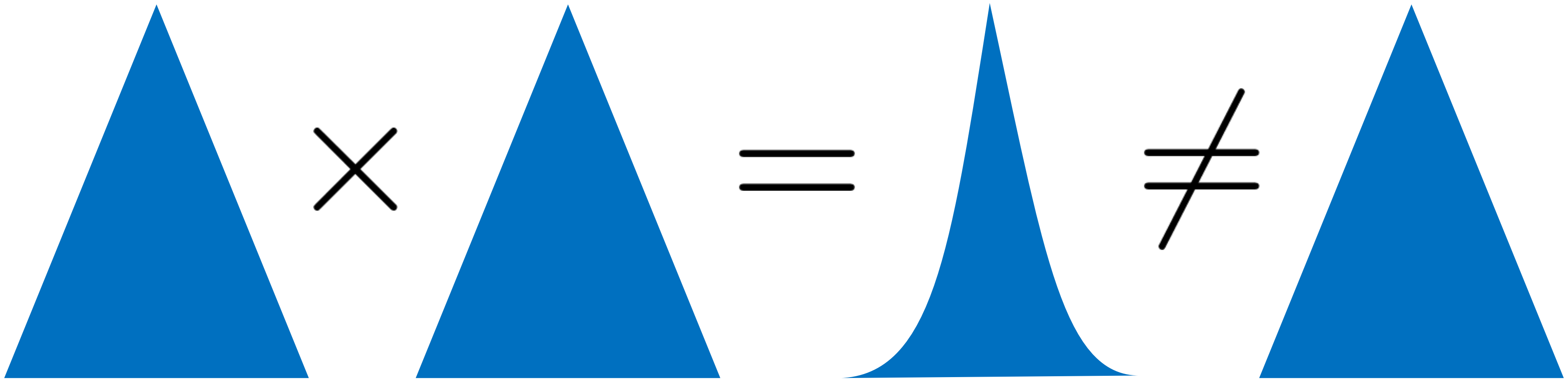
Stacking Integrated Products

$$\begin{pmatrix} \int_{\mathcal{M}} h_1 \Delta f \, dA \\ \int_{\mathcal{M}} h_2 \Delta f \, dA \\ \vdots \\ \int_{\mathcal{M}} h_{|V|} \Delta f \, dA \end{pmatrix} = \begin{pmatrix} \sum_k L_{1k} v^k \\ \sum_k L_{2k} v^k \\ \vdots \\ \sum_k L_{|V|k} v^k \end{pmatrix} = L\mathbf{v}$$

Multiply by Laplacian matrix!

Problematic Right Hand Side

$$\int_{\mathcal{M}} h_\ell \Delta f \, dA = \int_{\mathcal{M}} h_\ell g \, dA \ \forall \text{ hat functions } h_\ell$$



Product of hats is quadratic

Some Ways Out

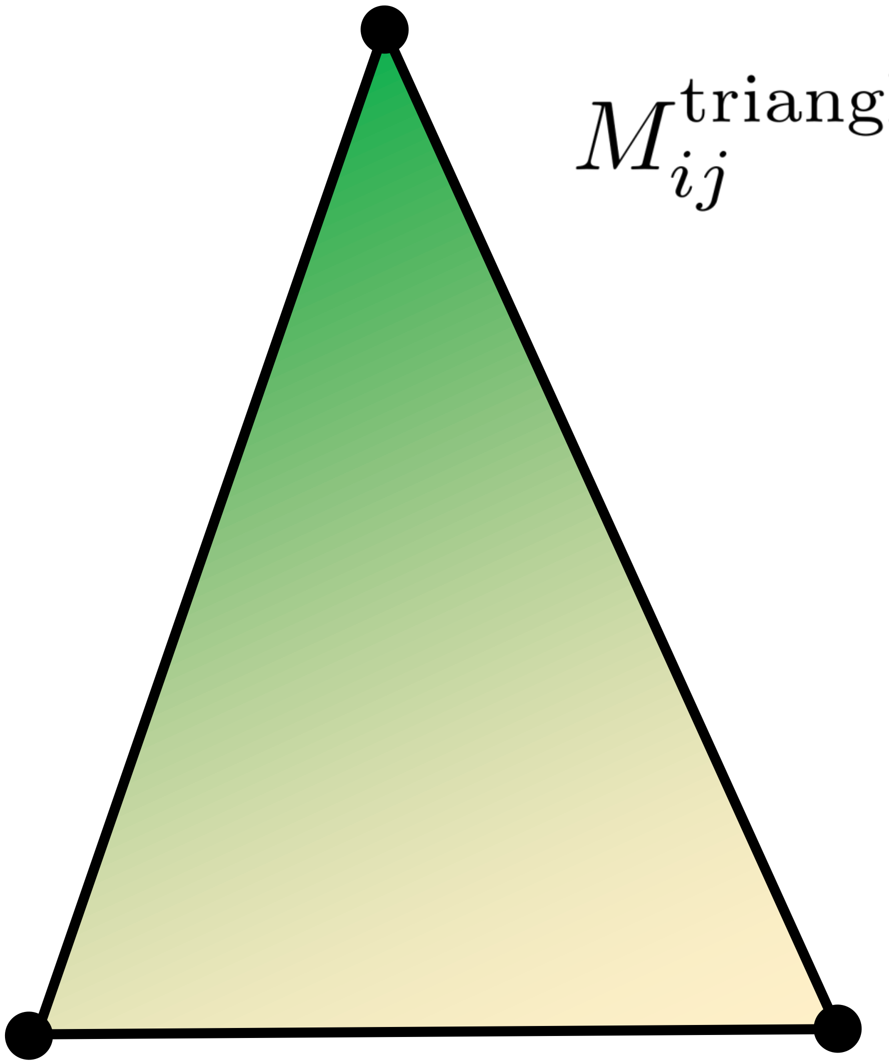
- Just do the integral
  - “Consistent” approach
- Approximate some more

The Mass Matrix

$$M_{ij} := \int_{\mathcal{M}} h_i h_j \, dA$$

- **Diagonal** elements:
  Norm of h_i
- **Off-diagonal** elements:
  Overlap between h_i and h_j

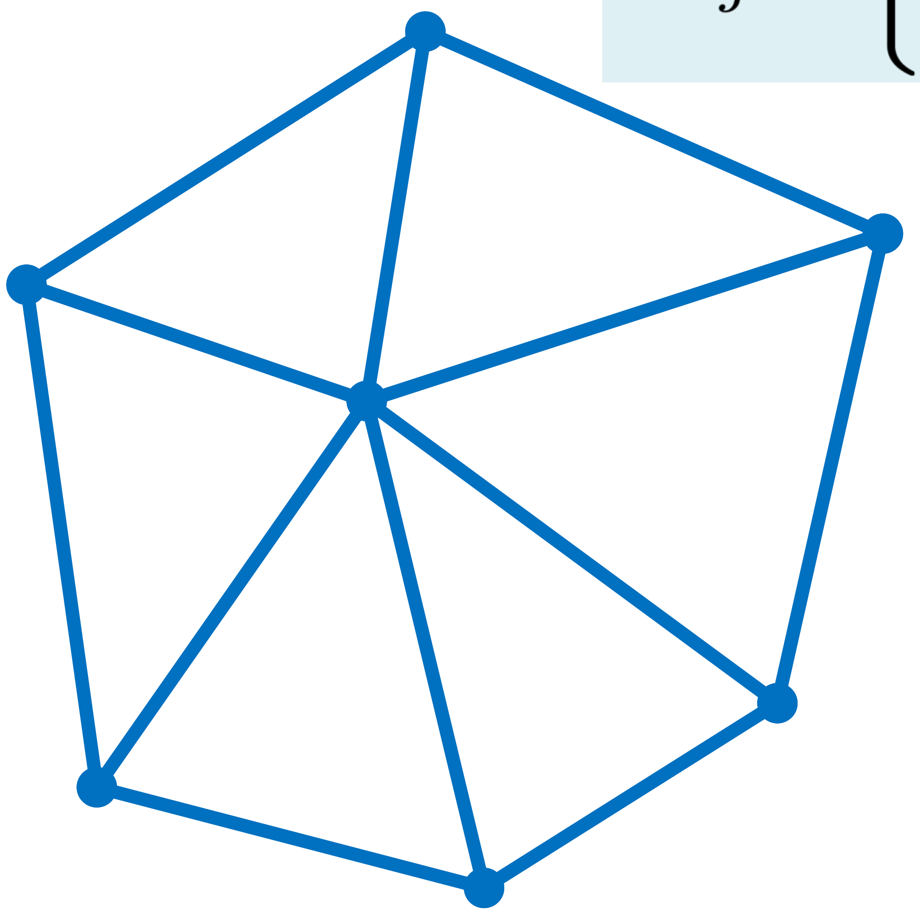
Consistent Mass Matrix



$$M_{ij}^{\text{triangle}} = \begin{cases} \frac{\text{area}}{6} & \text{if } i = j \\ \frac{\text{area}}{12} & \text{if } i \neq j \end{cases}$$

Non-Diagonal Mass Matrix

$$M_{ij} = \begin{cases} \frac{\text{one-ring area}}{6} & \text{if } i = j \\ \frac{\text{adjacent area}}{12} & \text{if } i \neq j \end{cases}$$



Properties of Mass Matrix

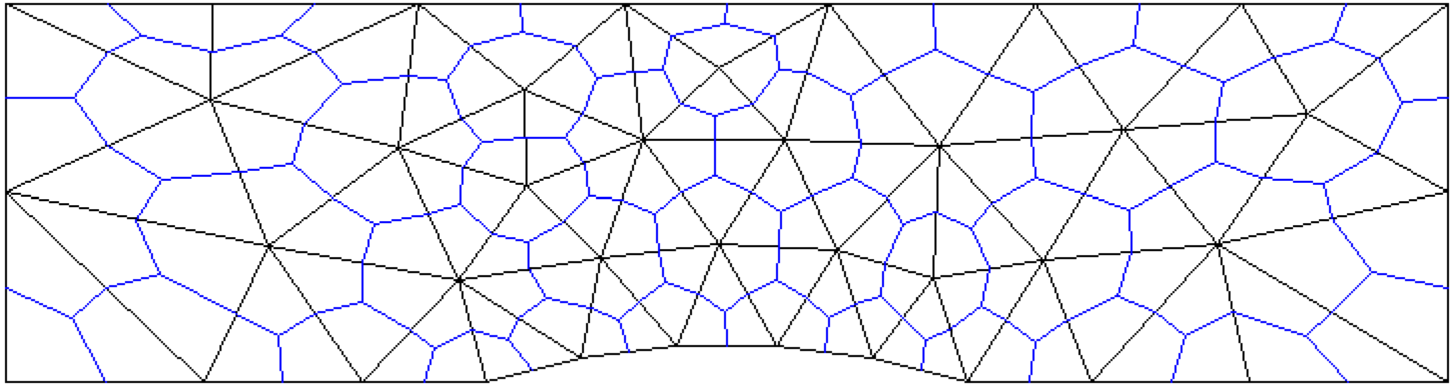
- Rows sum to one ring area / 3
- Involves only vertex and its neighbors
- Partitions surface area

Issue: Not diagonal!

Use for Integration

$$\begin{aligned}
\int_{\mathcal{M}} f\, dA &= \int_{\mathcal{M}} \left[\sum_k v^k h_k(\mathbf{x}) \cdot 1 \right] dA(\mathbf{x}) \\
&= \int_{\mathcal{M}} \left[\sum_k v^k h_k(\mathbf{x}) \cdot \sum_i h_i(\mathbf{x}) \right] dA(\mathbf{x}) \\
&= \sum_{ki} M_{ki} v^k \\
&= \mathbf{1}^\top M \mathbf{v}
\end{aligned}$$

Lumped Mass Matrix



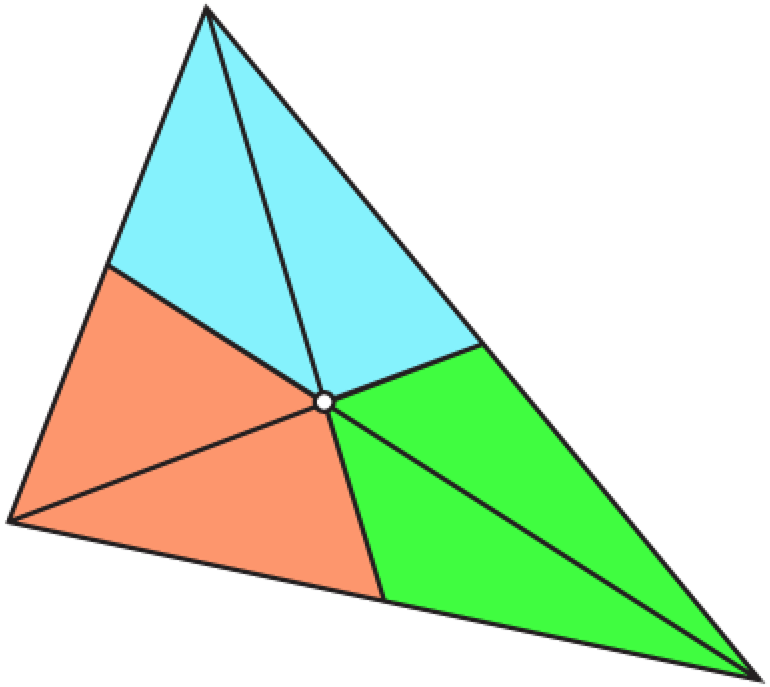
$$\tilde{a}_{ii} := \text{Area}(\text{cell } i)$$

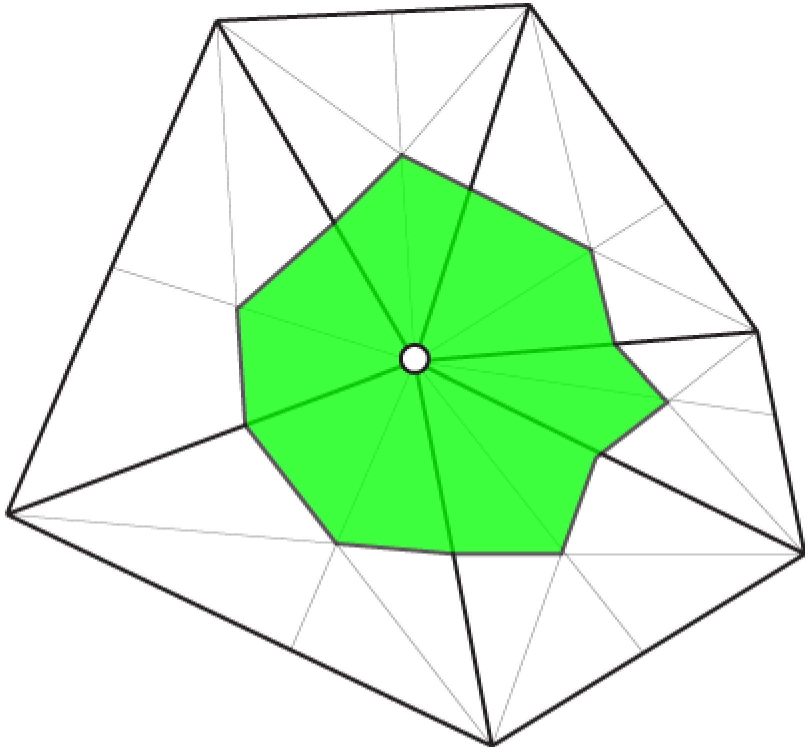
Won't make big difference for smooth functions

http://users.led-inc.eu/~phk/mesh-dualmesh.html

Approximate with diagonal matrix

Simplest: Barycentric Lumped Mass





http://www.alecjacobson.com/weblog/?p=1146

Area/3 to each vertex

Ingredients

- **Cotangent Laplacian *L***

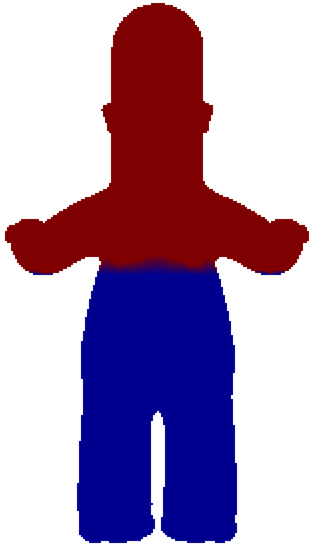
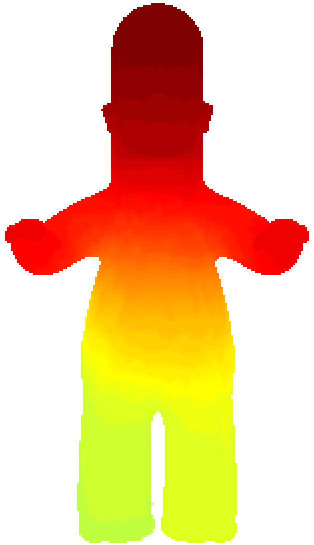
  Per-vertex function to integral of its Laplacian against each hat

- **Mass matrix *M***

  Integrals of pairwise products of hats
  (or approximation thereof)

Solving the Poisson Equation

$$\Delta f = g \longrightarrow M\mathbf{w} = L\mathbf{v}$$

g   f

Must integrate to zero

Determined up to constant

Important Detail: Boundary Conditions

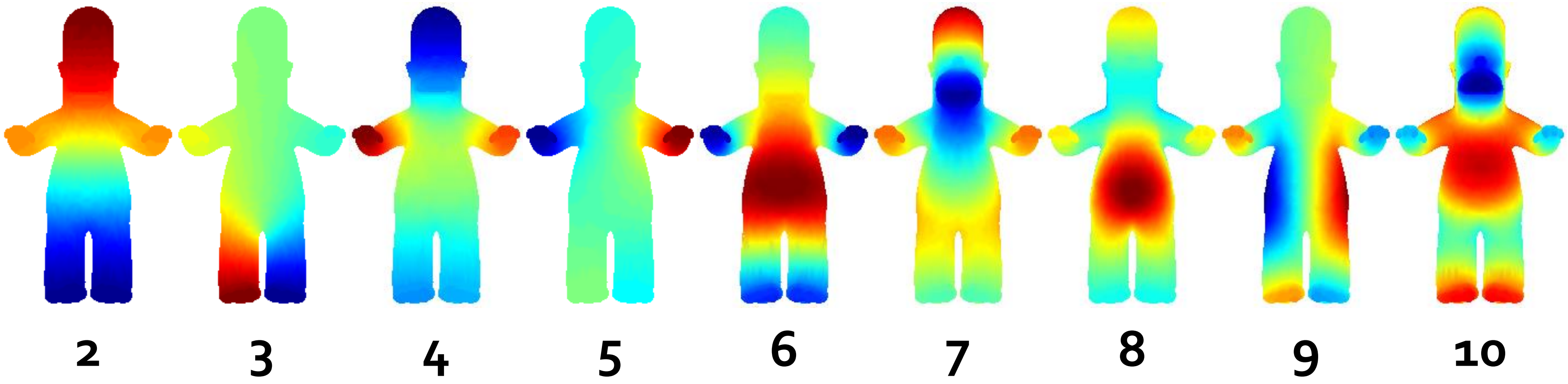
$$\Delta f(x) = g(x) \ \forall x \in \Omega$$
$$f(x) = u(x) \ \forall x \in \Gamma_D$$
$$\nabla f \cdot n = v(x) \ \forall x \in \Gamma_N$$

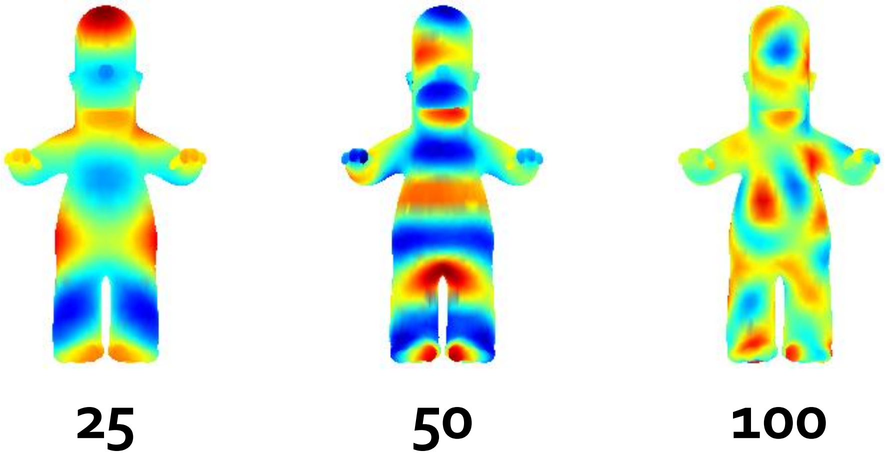
Strong form

$$\int_\Omega \nabla f \cdot \nabla \phi = \int_{\Gamma_N} v(x)\phi(x)\, d\Gamma - \int_\Omega f(x)\phi(x)\, d\Omega$$
$$f(x) = u(x) \ \forall x \in \Gamma_D$$

Weak form

Eigenhomers

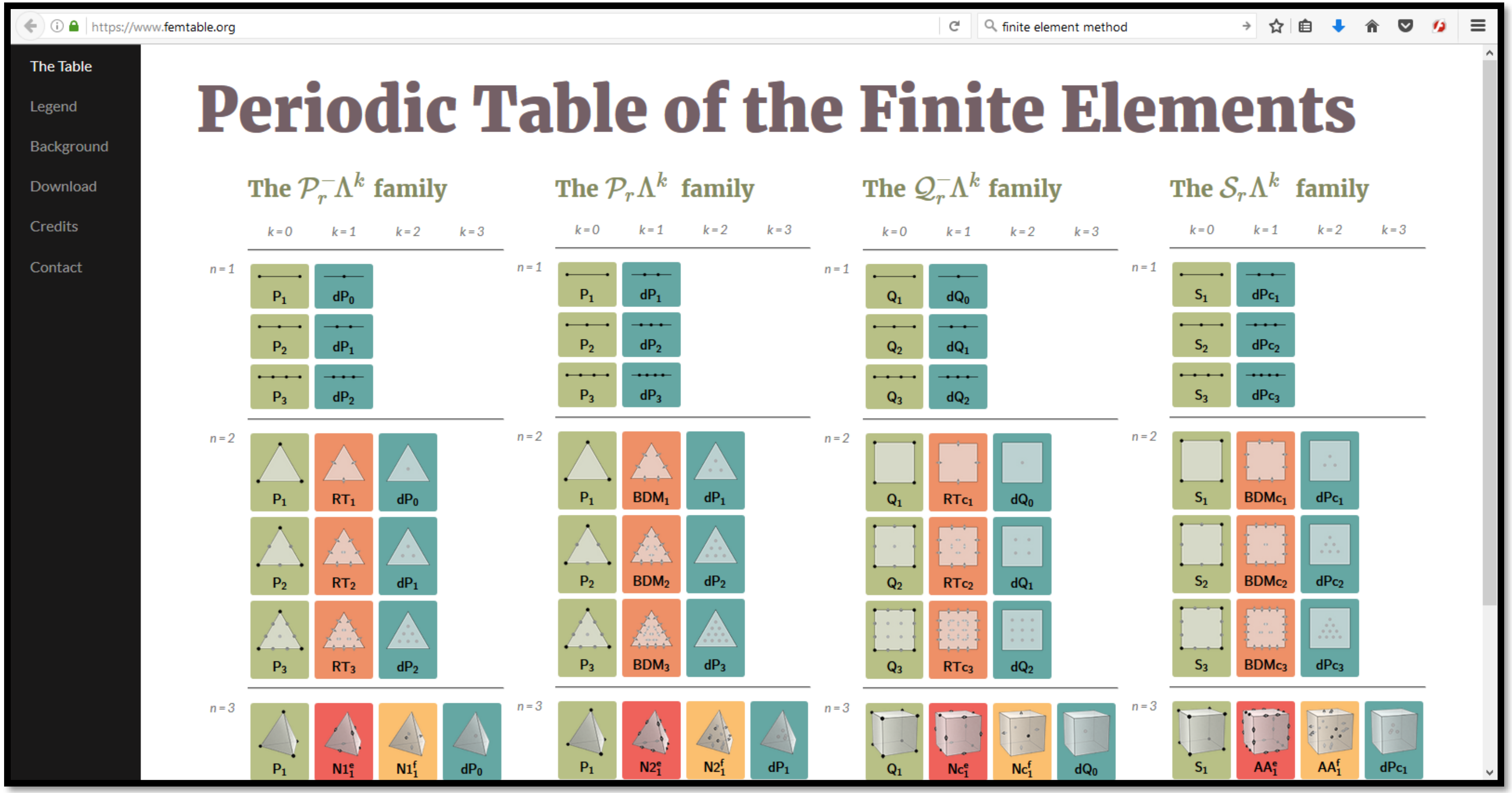




What is smallest eigenvalue?

FEM approach?

Higher-Order Elements



https://www.femtable.org/

Point Cloud Laplace: Easiest Option

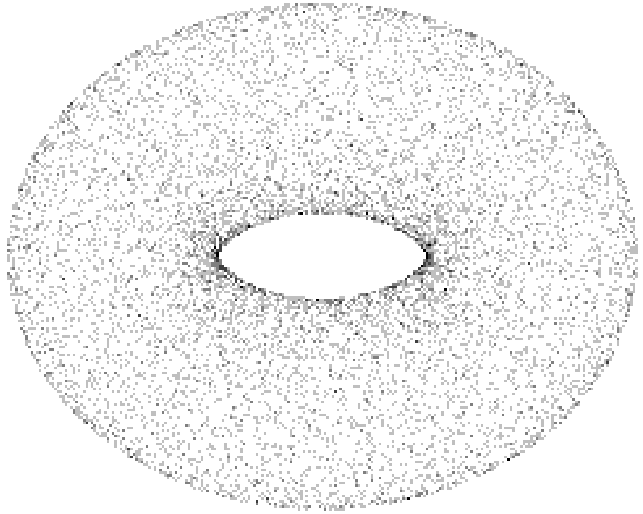
$$W_{ij} = \exp\left(-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{4t}\right)$$

Tricky parameter to choose

$$D_{ii} = \sum_j W_{ji}$$

$$L = D - W$$

$$Lf = \lambda D f$$



Interesting recent alternative for surfaces:
"A Laplacian for Nonmanifold Triangle Meshes"
Sharp & Crane, SGP 2020

Extra: Motivation

"Laplacian Eigenmaps for Dimensionality Reduction and Data Representation"
Belkin & Niyogi 2003

Discrete Laplacian Operators

Justin Solomon

6.8410: Shape Analysis

Spring 2023

Extra: Point Cloud Laplacian

Justin Solomon

6.8410: Shape Analysis

Spring 2023