The Laplacian Operator

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6.8410: Shape Analysis

Spring 2023


WARNING
SIGN
MISTAKES
LIKELY

Lots of (sloppy) math!

http://www.wgnflag.com/xcart/images/P/G-50_StopSeatBeltsStockSign.jpg

Famous Motivation

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

> "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

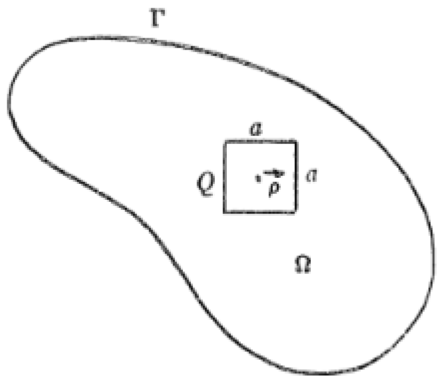
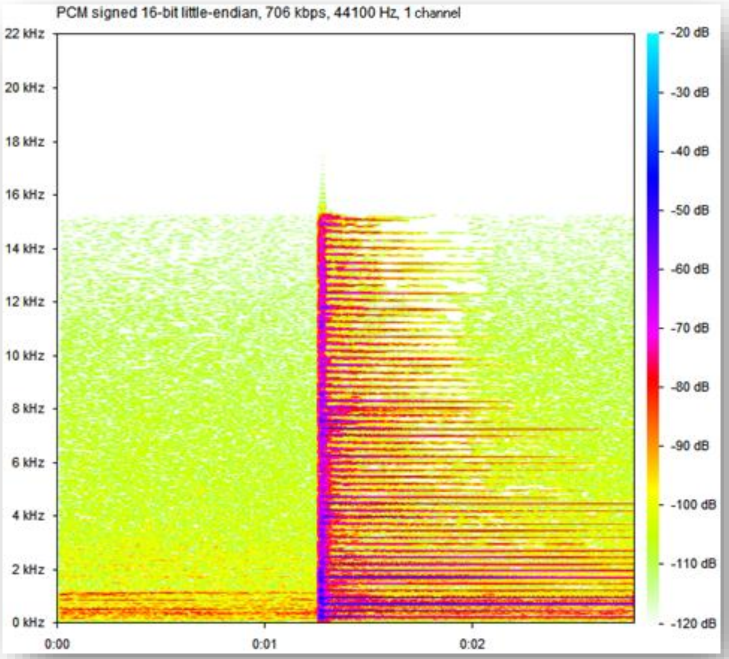


FIG. 1

An Experiment

Is this possible?

Unreasonable to Ask?

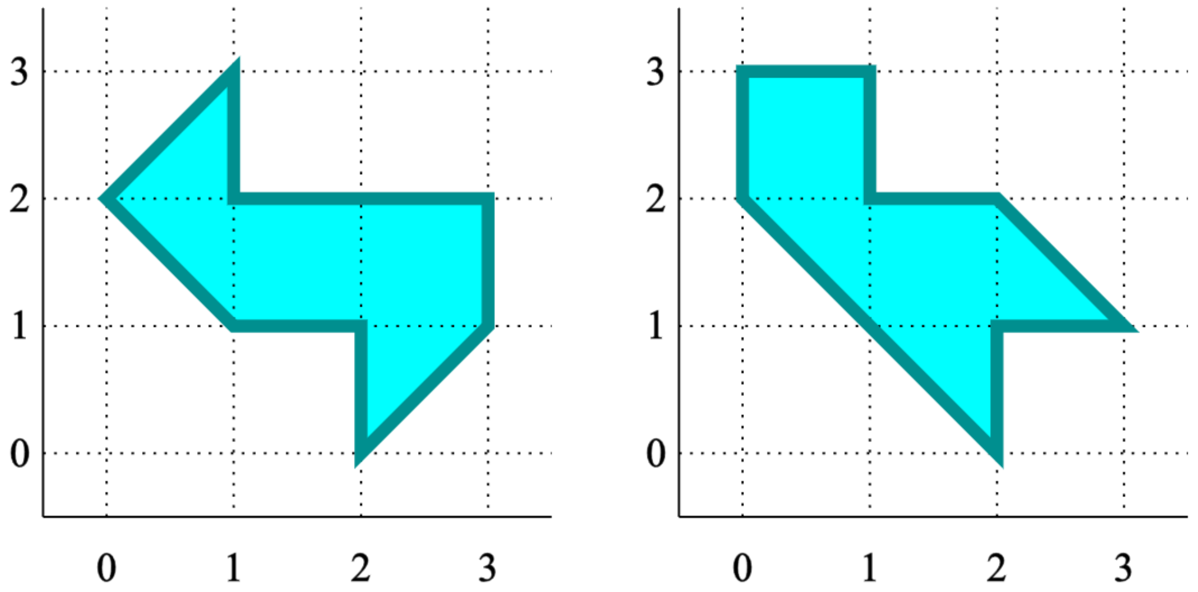


Length of string

http://www.takamine.com/templates/default/images/gclassical.png
http://firsttimeprogrammer.blogspot.com/2015/05/spectrogram-representation-of-b-note.html

Spoiler Alert

Extra credit: Make these!



"No, but…"

- Has to be a weird drum
- Spectrum tells you a lot!

https://en.wikipedia.org/wiki/Hearing_the_shape_of_a_drum

Rough Intuition

http://pngimg.com/upload/hammer_PNG3886.png

You can learn a lot about a shape by **hitting it** (lightly) **with a hammer**!

Spectral Geometry

What can you learn about its shape from **vibration frequencies and oscillation patterns?**

$$\Delta f = \lambda f$$

Objectives

- Make "vibration modes" more precise
- Progressively more complicated domains
  - Line segments
  - Regions in $\mathbb{R}^n$
  - Graphs
  - Surfaces/manifolds
- Coming up: Discretization, applications

Review:

Vector Spaces and Linear Operators

$$L[\mathbf{x}+\mathbf{y}] = L[\mathbf{x}] + L[\mathbf{y}]$$

$$L[c\mathbf{x}] = cL[\mathbf{x}]$$

$$L[\mathbf{x}] = A\mathbf{x}$$

Review:

In Finite Dimensions

$$\underbrace{A}_{\text{matrix}} \; \underbrace{\mathbf{x}}_{\text{vector}}$$

$$\underbrace{\mathbf{x} \mapsto A\mathbf{x}}_{\text{linear operator}}$$

Recall: Spectral Theorems in $\mathbb{C}^n$

Theorem. Suppose $A \in \mathbb{C}^{n \times n}$ is Hermitian. Then, A has an orthogonal basis of n eigenvectors. If A is positive definite, the corresponding eigenvalues are nonnegative.

Our Progression

- Line segments
- Regions in $\mathbb{R}^n$
- Graphs
- Surfaces/manifolds

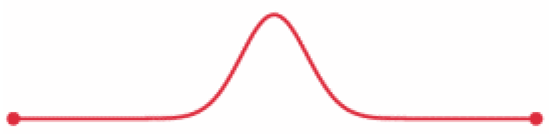
Transverse Wave: 1D Spring Network

(on the board)

Wave Equation

Minus second derivative operator!

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$



Minus Second Derivative Operator

"Dirichlet boundary conditions"

$$\{f(\cdot) \in C^\infty([a, b]) : f(0) = f(\ell) = 0\}$$

$$\mathcal{L}[\cdot] : u \mapsto -\frac{\partial^2 u}{\partial x^2}$$

Interpretation as positive (semi-)definite operator.

Eigenfunctions of Second Derivative Operator

"Dirichlet boundary conditions"

$$\{f(\cdot) \in C^\infty([a,b]) : f(0) = f(\ell) = 0\}$$

$$\mathcal{L}[\cdot] : u \mapsto -\frac{\partial^2 u}{\partial x^2}$$

Eigenfunctions:

$$\phi_k(x) = \sqrt{\frac{2}{\ell}} \sin\left(\frac{\pi k x}{\ell}\right), \quad \lambda_k = \left(\frac{\pi k}{\ell}\right)^2$$

Can you hear the length of an interval?

$$\phi_k(x) = \sqrt{\frac{2}{\ell}} \sin\left(\frac{\pi k x}{\ell}\right), \quad \lambda_k = \left(\frac{\pi k}{\ell}\right)^2$$

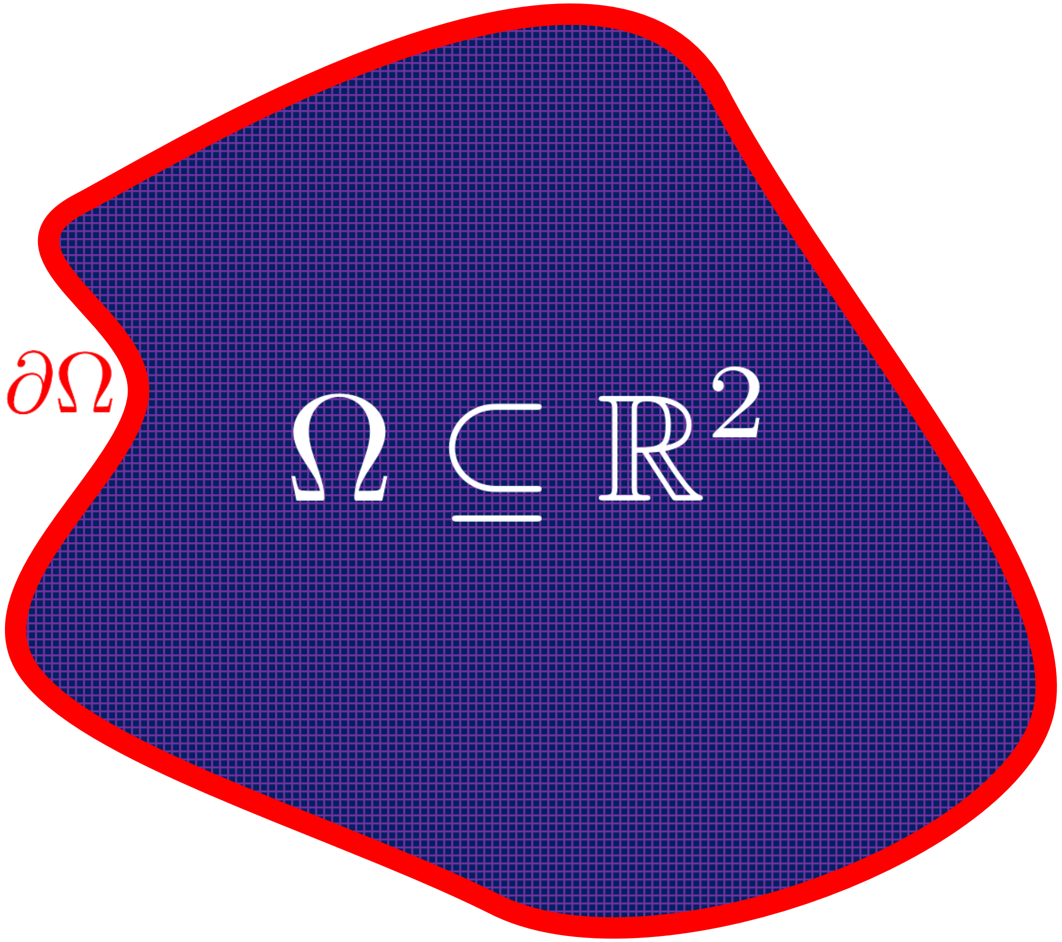
$$\lambda_k = \left(\frac{\pi k}{\ell}\right)^2$$

Yes!

Our Progression

- Line segments
- Regions in $\mathbb{R}^n$
- Graphs
- Surfaces/manifolds

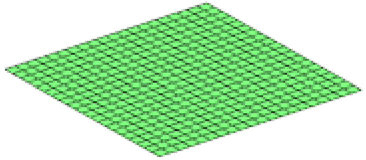
Planar Region



Wave equation:

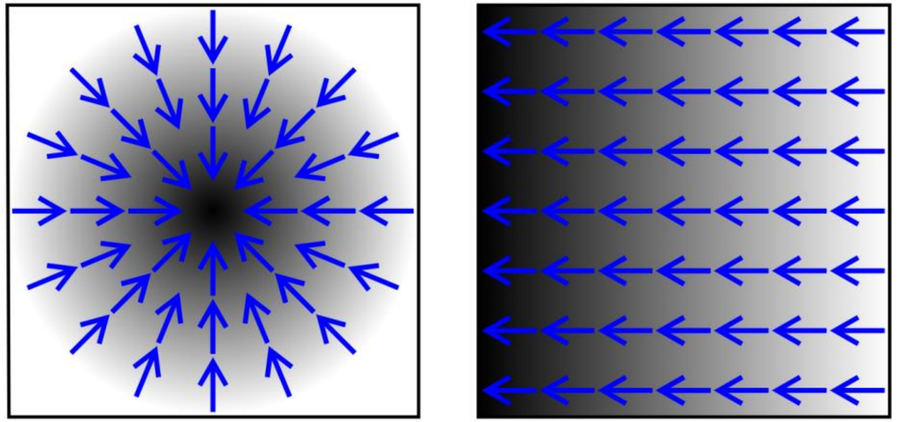
$$\frac{\partial^2 u}{\partial t^2} = -\Delta u$$

$$\Delta := -\sum_i \frac{\partial^2}{\partial (x^i)^2}$$



Typical Notation

$$\text{“}\Delta = -\underbrace{\nabla \cdot}_{\text{divergence}} \underbrace{\nabla}_{\text{gradient}}\text{”}$$

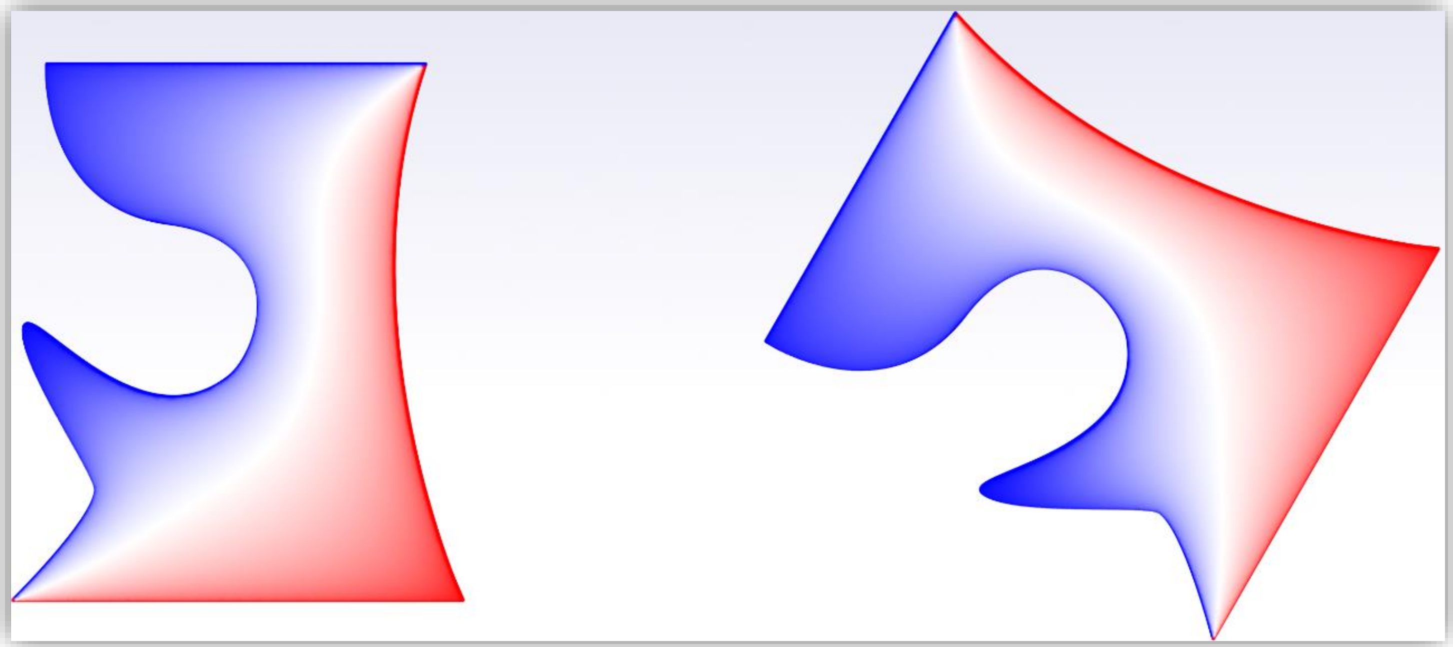


Gradient operator:

$$\nabla := \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n} \right)$$

http://www.gamasutra.com/db_area/images/feature/4164/figy.png, https://en.wikipedia.org/wiki/Gradient

Intrinsic Operator

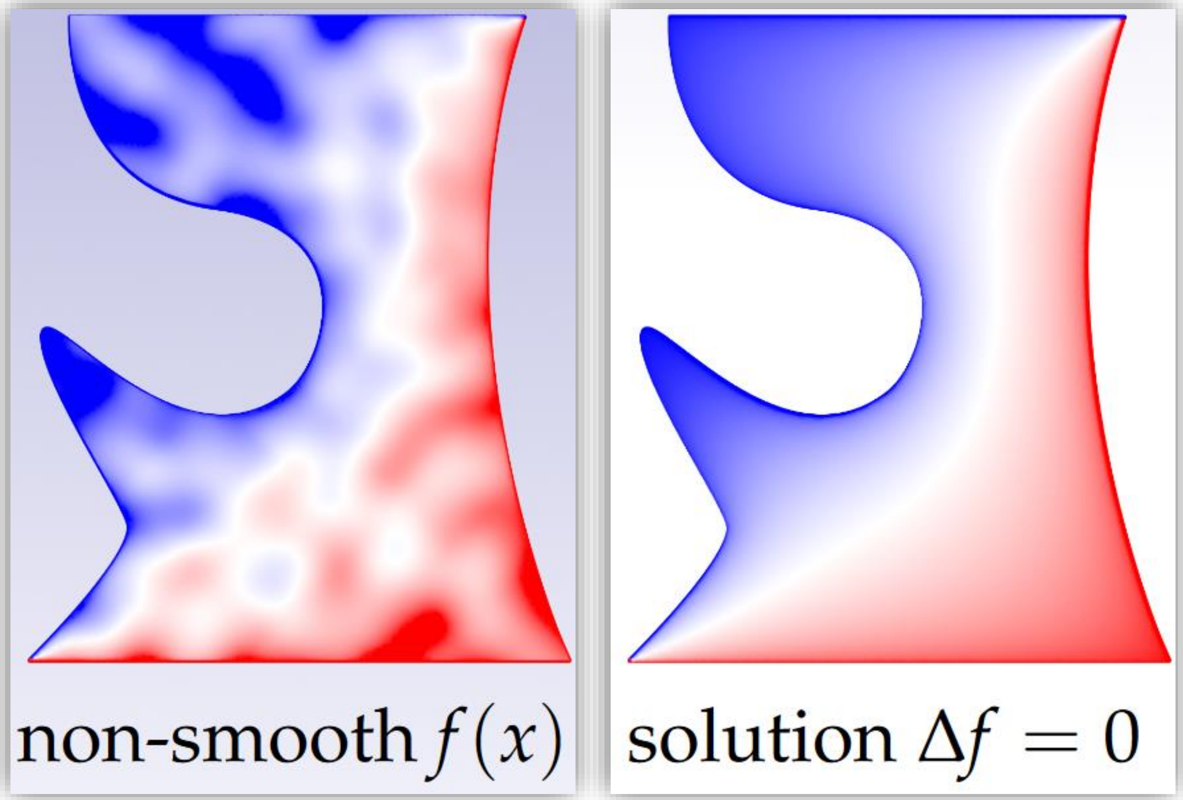


Images made by E. Vouga

Coordinate-independent

Dirichlet Energy

$$E[u] := \frac{1}{2}\int_{\Omega} \|\nabla u(\mathbf{x})\|_2^2 \, dA(\mathbf{x})$$



$$\min_{u(\mathbf{x}):\Omega\to\mathbb{R}} E[u]$$

$$\text{s.t. } u|_{\partial\Omega} \text{ prescribed}$$

$$\Downarrow$$

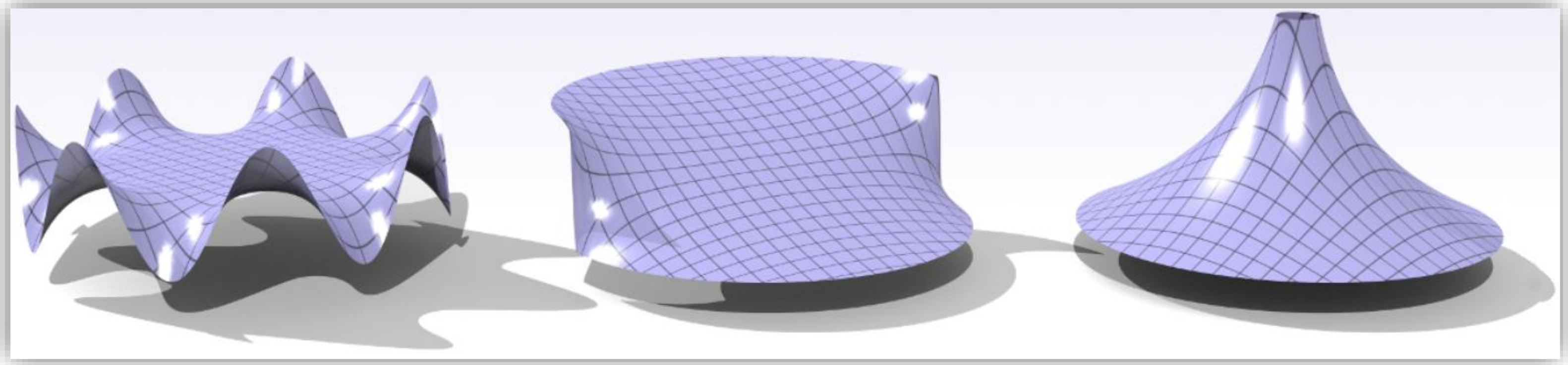
$$\Delta u \equiv 0$$

"Laplace equation"
"Harmonic function"

Images made by E. Vouga

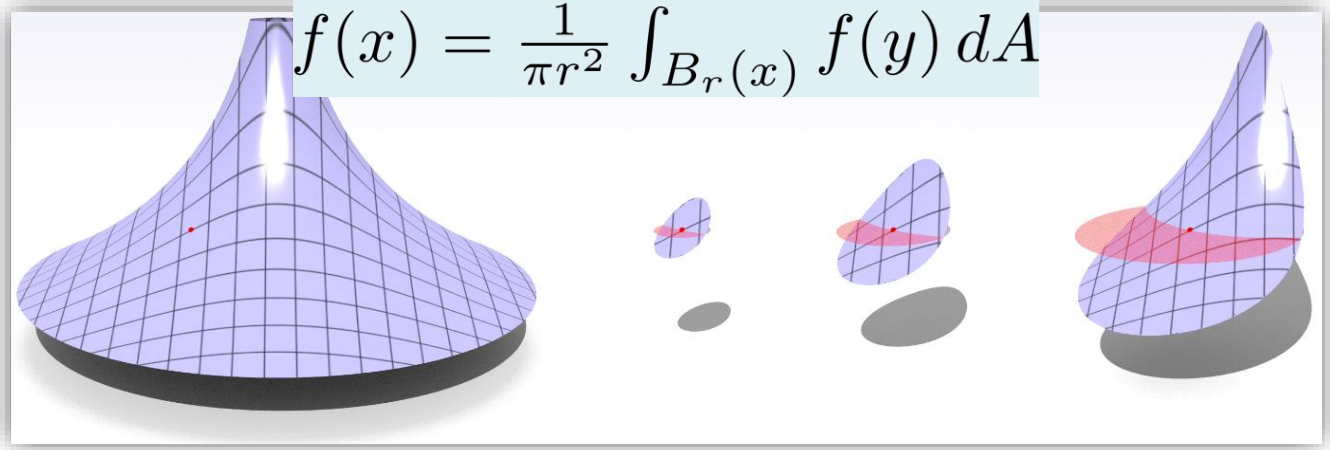
Harmonic Functions

$$\Delta f \equiv 0$$



Mean value property:

$$f(x) = \frac{1}{\pi r^2} \int_{B_r(x)} f(y)\, dA$$

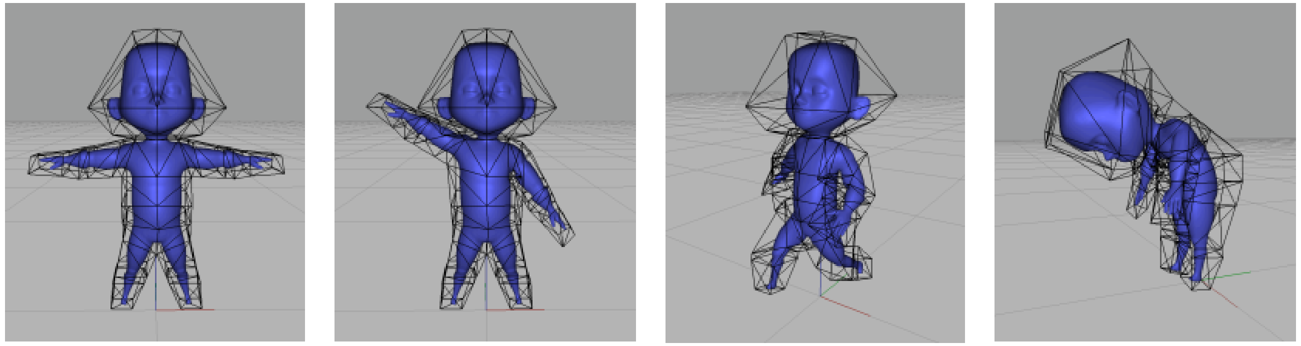


Images made by E. Vouga

Application

Harmonic Coordinates

Tony DeRose Mark Meyer
Pixar Technical Memo #06-02
Pixar Animation Studios



(a) (b) (c) (d)

Figure 1: A character (shown in blue) being deformed by a cage (shown in black) using harmonic coordinates. (a) The character and cage at bind-time; (b) - (d) the deformed character corresponding to three different poses of the cage.

Abstract

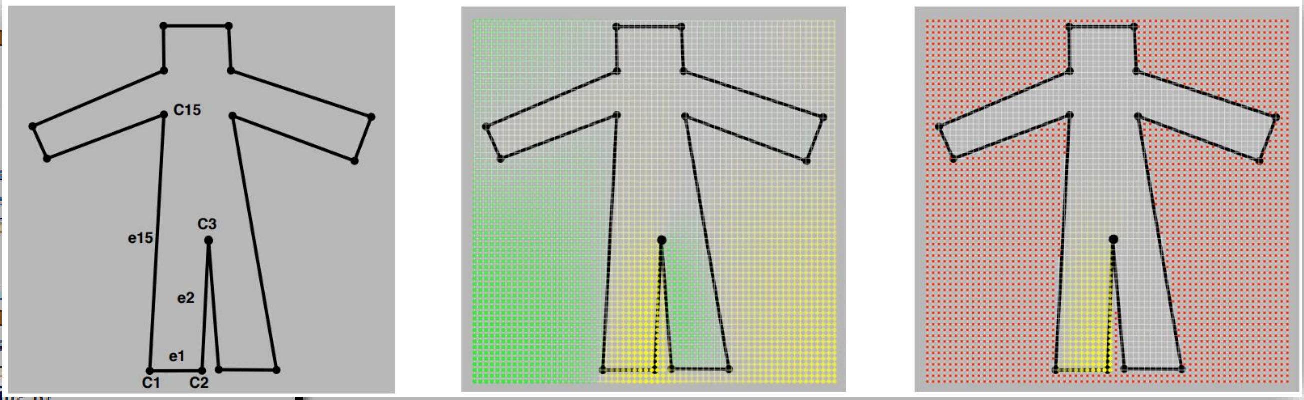
Generalizations of barycentric coordinates in two and higher dimensions have been shown to have a number of applications in recent years, including finite element analysis, the definition of S-patches (n-sided generalizations of Bézier surfaces), free-form deformations, mesh parametrization, and interpolation. In this paper we present a new form of d dimensional generalized barycentric coordinates. The new coordinates are defined as solutions to Laplace's equation subject to carefully chosen boundary conditions. Since solutions to Laplace's equation are called harmonic functions, we call the new construction harmonic coordinates. We show that harmonic coordinates possess several properties that make them more attractive than mean value coordinates when used to define two and three dimensional deformations.

ways, one of the simplest being as t
isfying the interpolation conditions:

$$\beta_i(T_j) = \delta_{i,j},$$

Similarly, barycentric coordinates
defined relative to a non-degenera
$T_1, T_2, T_3, T_4 \in \Re^3$ as the unique line
tion 2 where the indices i and j run f
3.

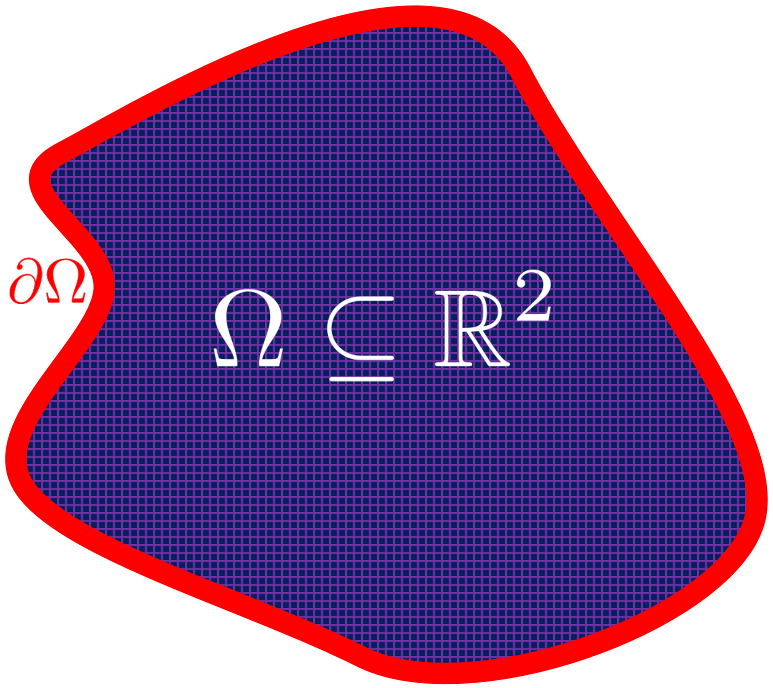
As described in Ju *et. al.* [Ju et a
barycentric coordinates stem from t
interpolating functions. Gouraud s
where colors c_1, c_2, c_3 assigned to th
terpolated across the triangle according to



Positivity, Self-Adjointness

$$\{f(\cdot) \in C^\infty(\Omega) : f|_{\partial\Omega} \equiv 0\}$$

"Dirichlet boundary conditions"

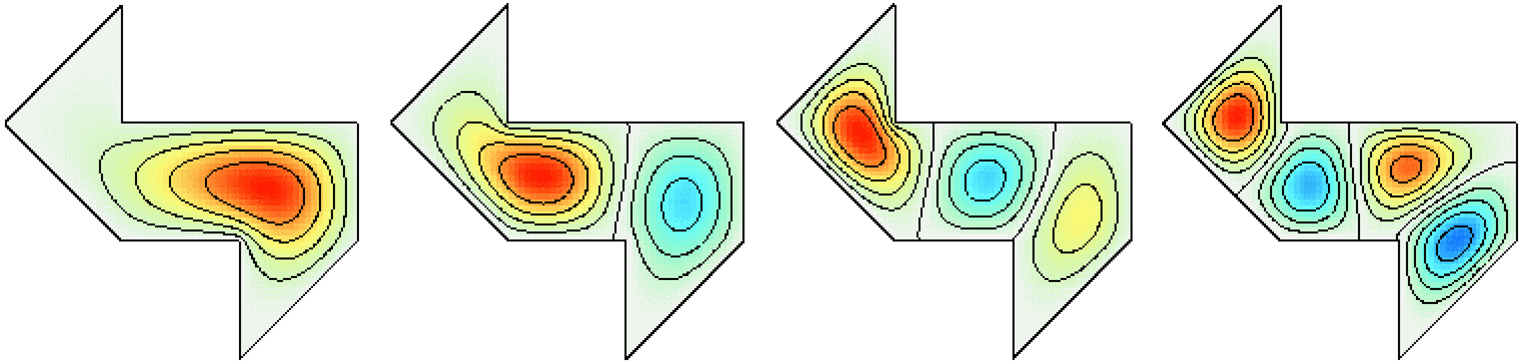


$$\mathcal{L}[f] := \Delta f$$

$$\langle f, g \rangle := \int_\Omega f(\mathbf{x}) g(\mathbf{x})\, dA(\mathbf{x})$$

1. **Positive:** $\langle f, \mathcal{L}[f] \rangle \geq 0$
2. **Self-adjoint:** $\langle f, \mathcal{L}[g] \rangle = \langle \mathcal{L}[f], g \rangle$

Laplacian Eigenfunctions



$$\min_u \frac{1}{2} \int_\Omega \|\nabla u(\mathbf{x})\|_2^2 \, d\mathbf{x}$$
$$\text{s.t.} \int_\Omega u(\mathbf{x})^2 \, d\mathbf{x} = 1$$

Critical points on the "unit sphere"

Theorem (Weyl's Law). *Let $N(\lambda)$ be the number of Dirichlet eigenvalues of the Laplacian Δ for a domain $\Omega \subseteq \mathbb{R}^d$ less than or equal to λ. Then,*

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = (2\pi)^{-d} \omega_d \text{vol}(\Omega),$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

http://www.math.udel.edu/~driscoll/research/gww1-4.gif

Small eigenvalue: Small Dirichlet Energy

Aside:

Common Misconception

$$\min_f \; E[f] \text{ s.t. } f(p) = \text{const.}$$

Point constraints are ill-advised

Possible Way Out

Fast Quasi-Harmonic Weights for Geometric Data Interpolation

YU WANG and JUSTIN SOLOMON, Massachusetts Institute of Technology, USA

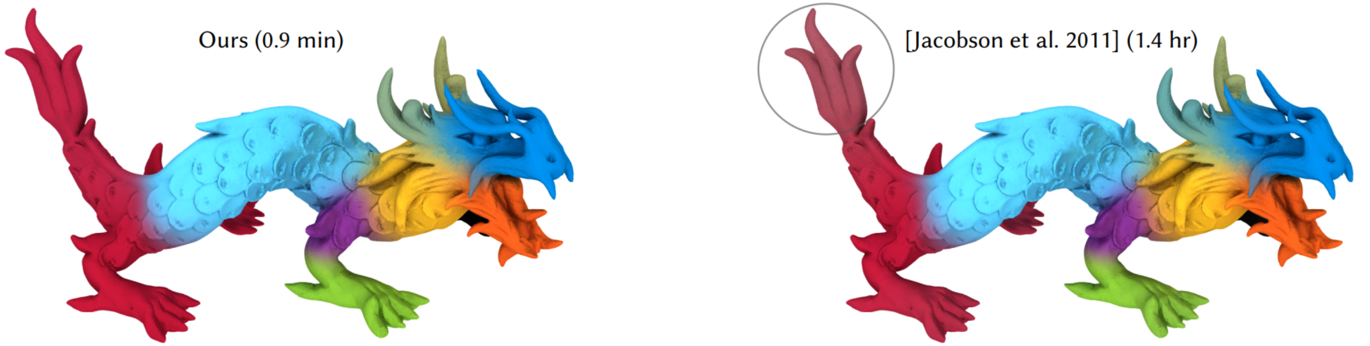


Fig. 1. Color interpolation using our skinning weights and bounded biharmonic weights [Jacobson et al. 2011]. Note the latter weights (and thus the blended color) have a local extrema at the tail of the Asian dragon. Our weights are free from local extrema and have a slightly lower smoothness energy. More importantly, our weights are many orders of magnitude faster to compute. Previous methods can take hours on this model, which has more than one million tetrahedra. Our method takes less than a minute, with potential for further speedups.

We propose *quasi-harmonic weights* for interpolating geometric data, which are orders of magnitude faster to compute than state-of-the-art. Currently, interpolation (or, skinning) weights are obtained by solving large-scale constrained optimization problems with explicit constraints to suppress oscillative patterns, yielding smooth weights only after a substantial amount of computation time. As an alternative, our weights are obtained as minima of an unconstrained problem that can be optimized quickly using straightforward numerical techniques. We consider weights that can be obtained as solutions to a parameterized family of second-order elliptic partial differential equations. By leveraging the maximum principle and careful parameterization, we pose weight computation as an inverse problem of recovering optimal anisotropic diffusivity tensors. In addition, we provide a customized ADAM solver that significantly reduces the number of gradient steps; our solver only requires inverting tens of linear systems that share the same

however, motivates automatic computation of skinning weights as a critical way to make the animation pipeline more efficient. Beyond their original application in animation, however, automatic skinning weight computation has found application in graphics pipelines beyond animation as a generic tool to interpolate geometric and physical quantities smoothly across geometric domains.

The problem of computing skinning weights amounts to designing a *partition of unity*, or set of functions that sums to 1 at every point on the domain, that satisfies a few properties. There is one skinning function per control handle of a deforming shape, which typically equals 1 at the handle and decays as we move farther away; this function indicates the influence of displacing that handle on the deformation of the rest of the shape. Other desirable properties

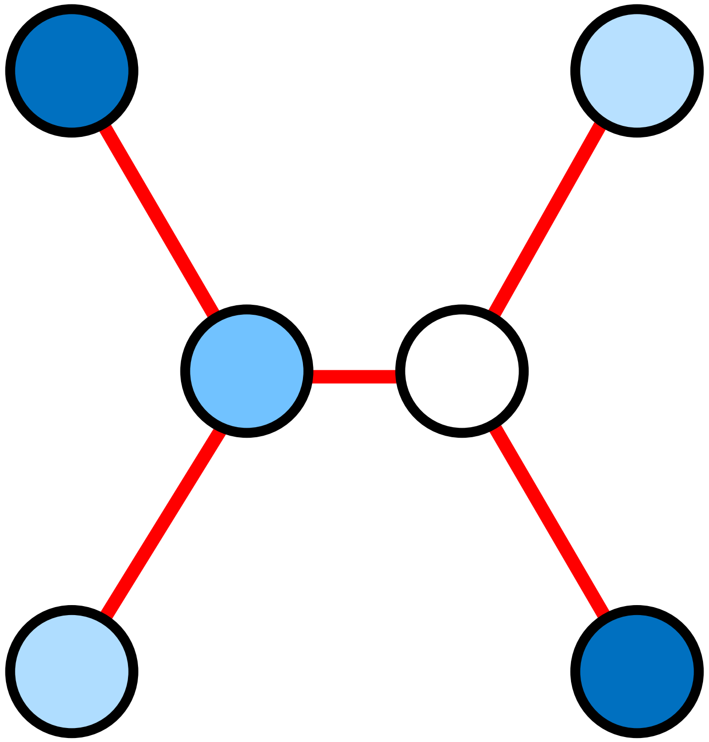
Our Progression

- Line segments
- Regions in $\mathbb{R}^n$
- Graphs
- Surfaces/manifolds

Basic Setup

- **Function:**

  One value per vertex



Q:

What is the

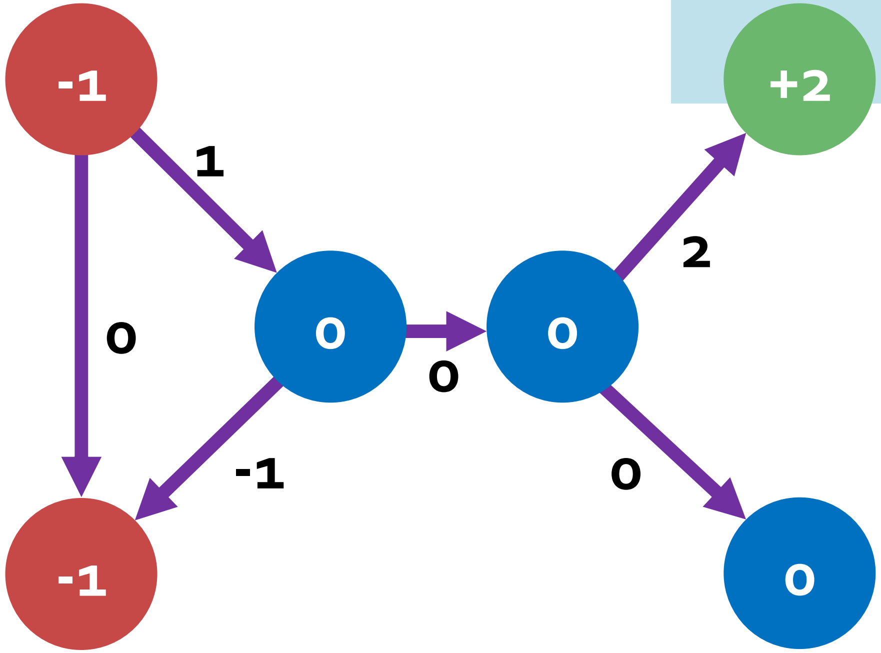
Dirichlet energy of a

function on a graph?

Differencing Operator

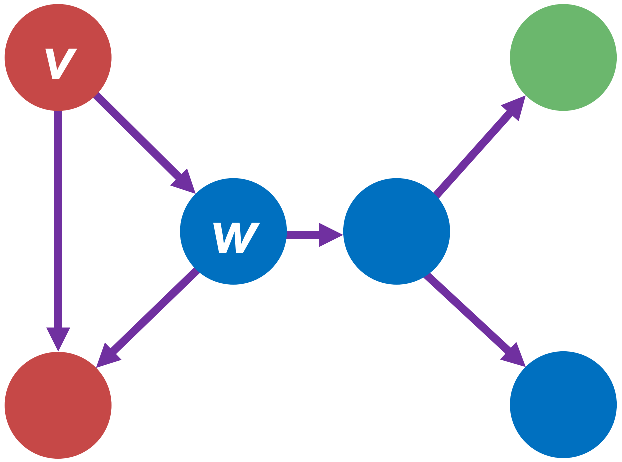
$$D_{ev} := \begin{cases} -1 & \text{if } e = (v, w) \\ 1 & \text{if } e = (w, v) \\ 0 & \text{otherwise} \end{cases}$$

$$D \in \{-1, 0, 1\}^{|E| \times |V|}$$



Orient edges arbitrarily

Dirichlet Energy on a Graph



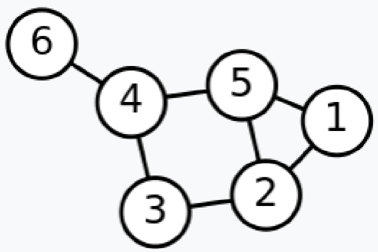
$$D_{ev} := \begin{cases} -1 & \text{if } e = (v, w) \\ 1 & \text{if } e = (w, v) \\ 0 & \text{otherwise} \end{cases}$$

$$E[\mathbf{f}] := \|D\mathbf{f}\|_2^2 = \sum_{(v,w)\in E} (f^v - f^w)^2$$

(Unweighted) Graph Laplacian

$$E[\mathbf{f}] = \|D\mathbf{f}\|_2^2 = \mathbf{f}^\top (D^\top D)\mathbf{f} := \mathbf{f}^\top L\mathbf{f}$$

$$L_{vw} = A - \overline{D} = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

Labeled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2&0&0&0&0&0 \\ 0&3&0&0&0&0 \\ 0&0&2&0&0&0 \\ 0&0&0&3&0&0 \\ 0&0&0&0&3&0 \\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&0&1&0 \\ 1&0&1&0&1&0 \\ 0&1&0&1&0&0 \\ 0&0&1&0&1&1 \\ 1&1&0&1&0&0 \\ 0&0&0&1&0&0 \end{pmatrix}$	$\begin{pmatrix} 2&-1&0&0&-1&0 \\ -1&3&-1&0&-1&0 \\ 0&-1&2&-1&0&0 \\ 0&0&-1&3&-1&-1 \\ -1&-1&0&-1&3&0 \\ 0&0&0&-1&0&1 \end{pmatrix}$

- **Symmetric**
- **Positive semidefinite**

https://en.wikipedia.org/wiki/Laplacian_matrix

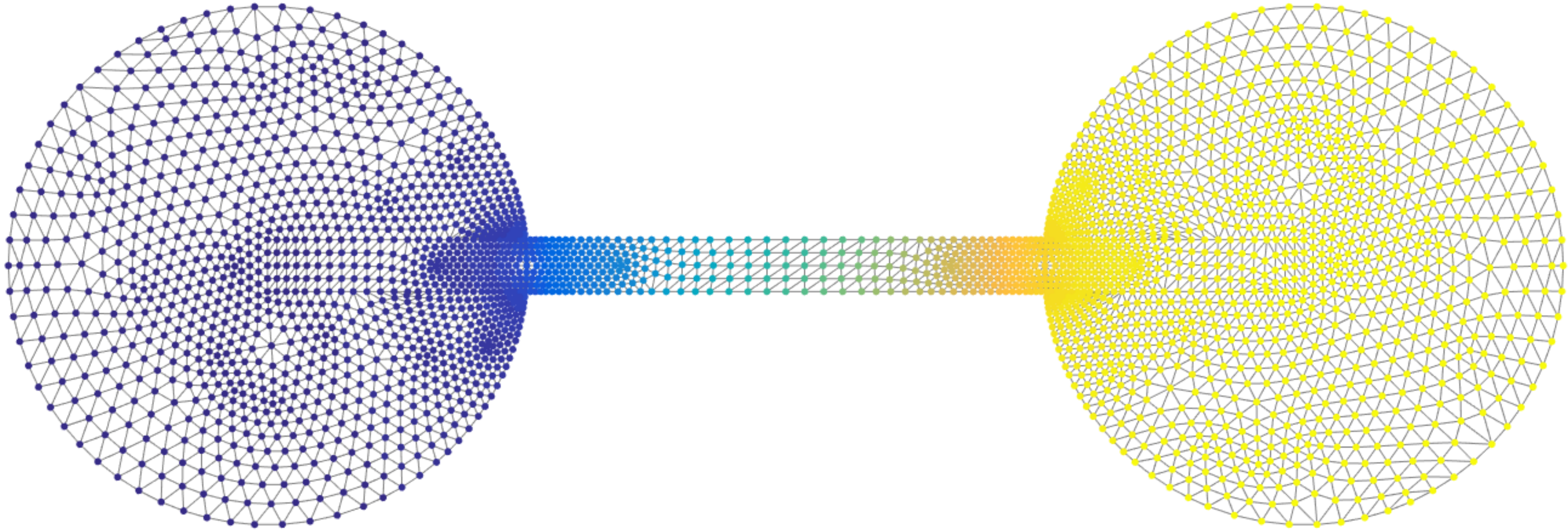
Q:

What is the

smallest eigenvalue

of the graph Laplacian?

Second-Smallest Eigenvector



$$Lx = \lambda x$$

Used for graph partitioning

Fiedler vector ("algebraic connectivity")

Mean Value Property

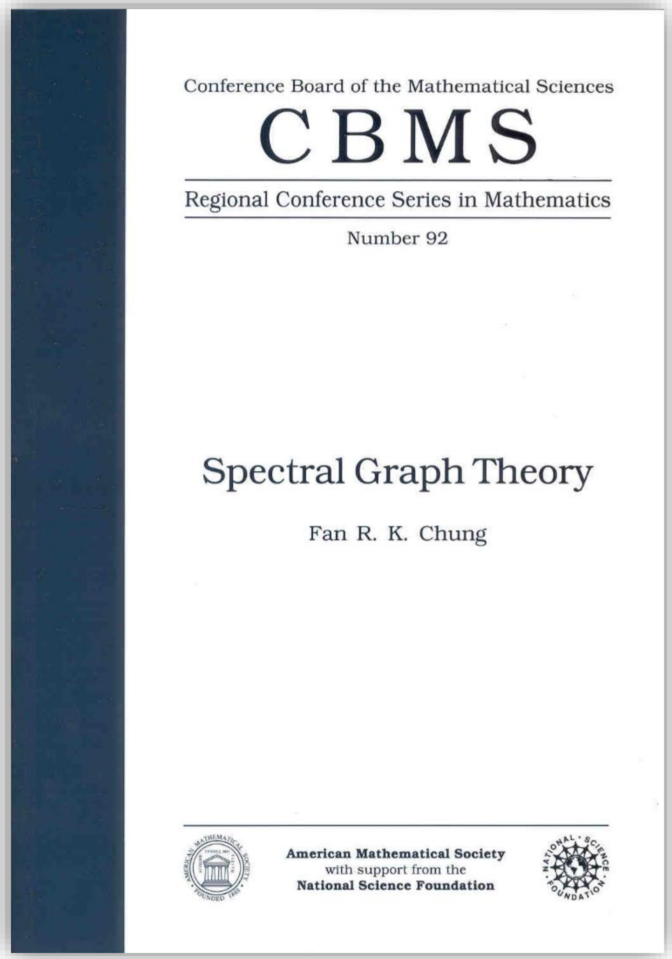
$$L_{vw} = A - D = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$$(L\mathbf{x})^v = 0$$

↓

Value at *v* is average of neighboring values

For More Information…



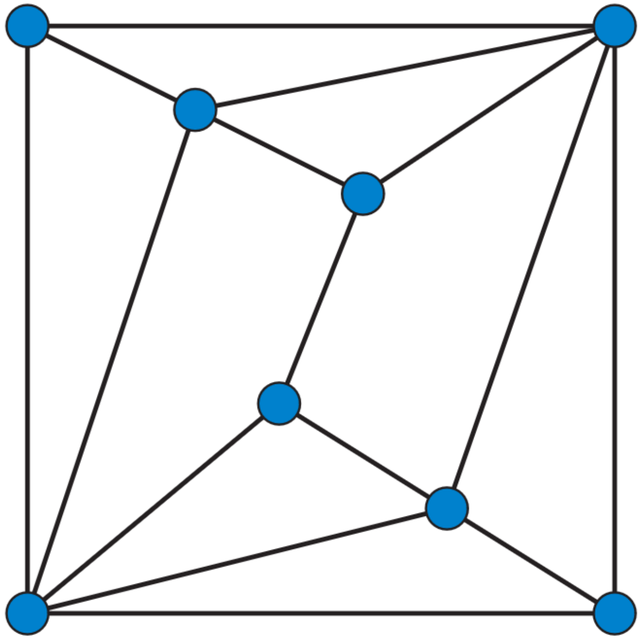
Graph Laplacian encodes lots of information!

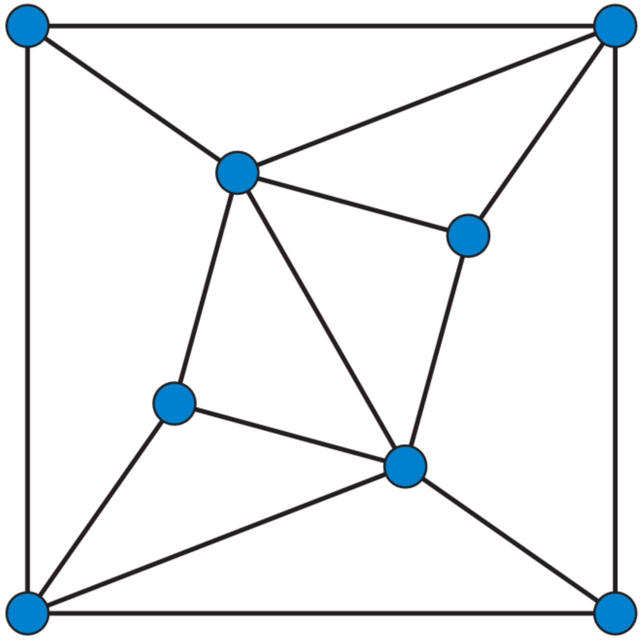
Example: Kirchoff's Theorem

Number of spanning trees equals

$$t(G) = \frac{1}{|V|} \prod_{k=2}^{|V|} \lambda_k$$

Hear the Shape of a Graph?





“Enneahedra”

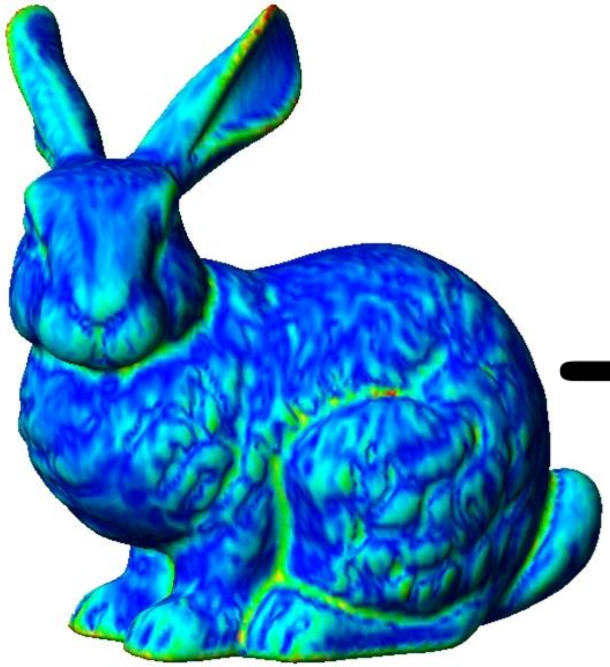
No!

Our Progression

- Line segments
- Regions in $\mathbb{R}^n$
- Graphs
- Surfaces/manifolds

Recall:

Scalar Functions

$f :$  $\longrightarrow \mathbb{R}$

http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

Map points to real numbers

Recall:

Differential of a Map

Definition (Differential). *Suppose* $\varphi : \mathcal{M} \to \mathcal{N}$ *is a map from a submanifold* $\mathcal{M} \subseteq \mathbb{R}^k$ *into a submanifold* $\mathcal{N} \subseteq \mathbb{R}^\ell$. *Then, the* differential $d\varphi_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \to T_{\varphi(\mathbf{p})}\mathcal{N}$ *of* φ *at a point* $\mathbf{p} \in \mathcal{M}$ *is given by*

$$d\varphi_{\mathbf{p}}(\mathbf{v}) := (\varphi \circ \gamma)'(0),$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ *is any curve with* $\gamma(0) = \mathbf{p}$ *and* $\gamma'(0) = \mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$.

Linear map of tangent spaces

$$d\varphi_{\mathbf{p}}(\gamma'(0)) := (\varphi \circ \gamma)'(0)$$

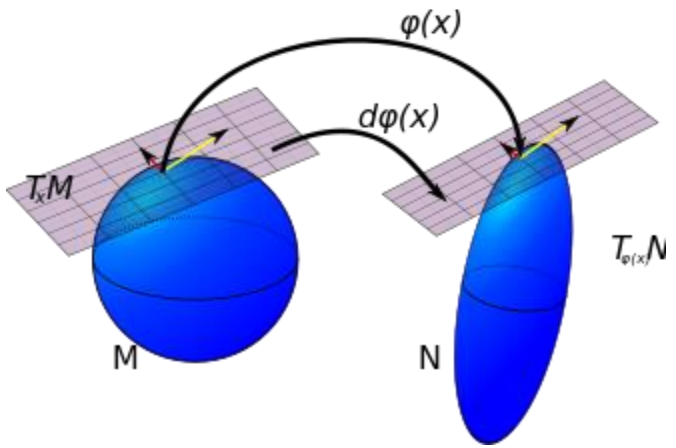
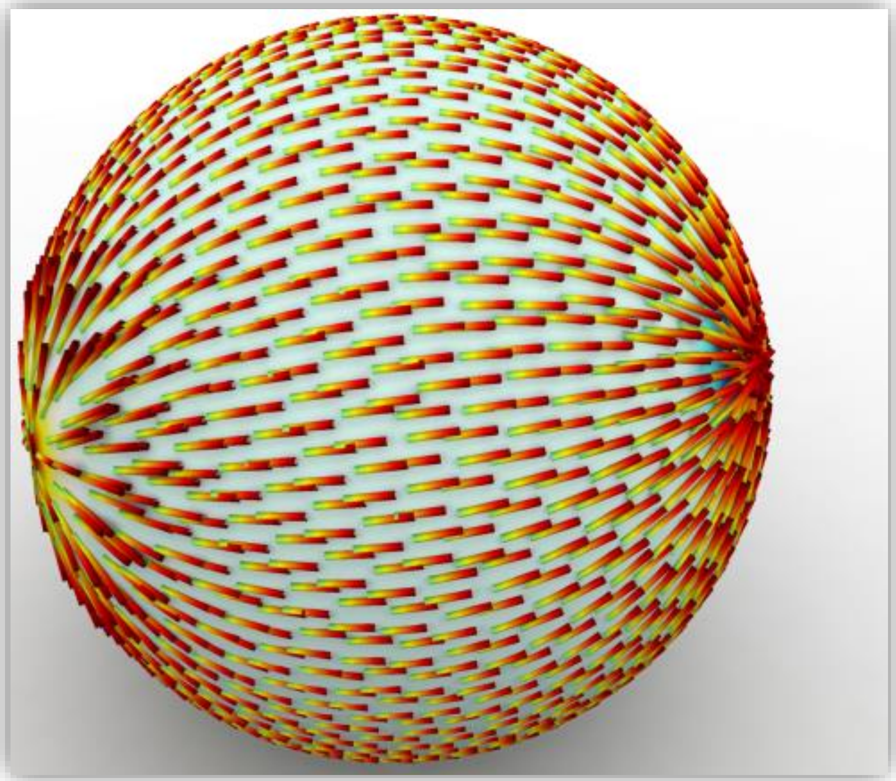


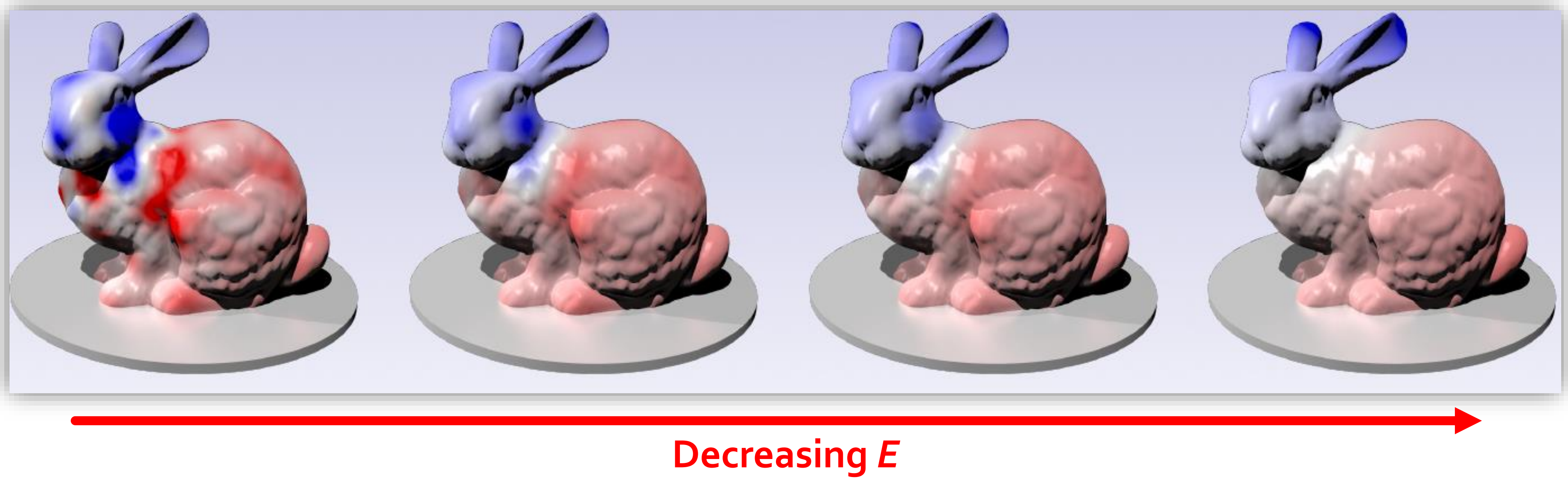
Image from Wikipedia

Gradient Vector Field

Proposition *For each* $\mathbf{p} \in \mathcal{M}$, *there exists a unique vector* $\nabla f(\mathbf{p}) \in T_{\mathbf{p}}\mathcal{M}$ *so that* $df_{\mathbf{p}}(\mathbf{v}) = \mathbf{v} \cdot \nabla f(\mathbf{p})$ *for all* $\mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$.



Dirichlet Energy



$$E[f] := \int_S \|\nabla f\|_2^2 \, dA$$

Images made by E. Vouga

From Inner Product to Operator

$$\langle f, g \rangle_\Delta := \int_S \nabla f(x) \cdot \nabla g(x)\, dA$$
$$:= \langle f, \Delta g \rangle$$

Implies
$\langle f, f \rangle \geq 0$

"Motivated" by finite-dimensional linear algebra.

Laplace-Beltrami operator

What is Divergence?

$$\mathbf{v} : \mathcal{M} \to \mathbb{R}^3 \text{ where } \mathbf{v}(\mathbf{p}) \in T_{\mathbf{p}}\mathcal{M}$$

$$d\mathbf{v}_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \to \mathbb{R}^3$$

$$\{\mathbf{e}_1, \mathbf{e}_2\} \subset T_{\mathbf{p}}\mathcal{M} \text{ orthonormal basis}$$

$$(\nabla \cdot \mathbf{v})_{\mathbf{p}} := \sum_{i=1}^{2} \langle \mathbf{e}_i, d\mathbf{v}(\mathbf{e}_i) \rangle_{\mathbf{p}}$$

Things we should check (but probably won't):

- Independent of choice of basis
- $\Delta = -\nabla \cdot \nabla$

Extra lecture segment:
Motivation for this formula

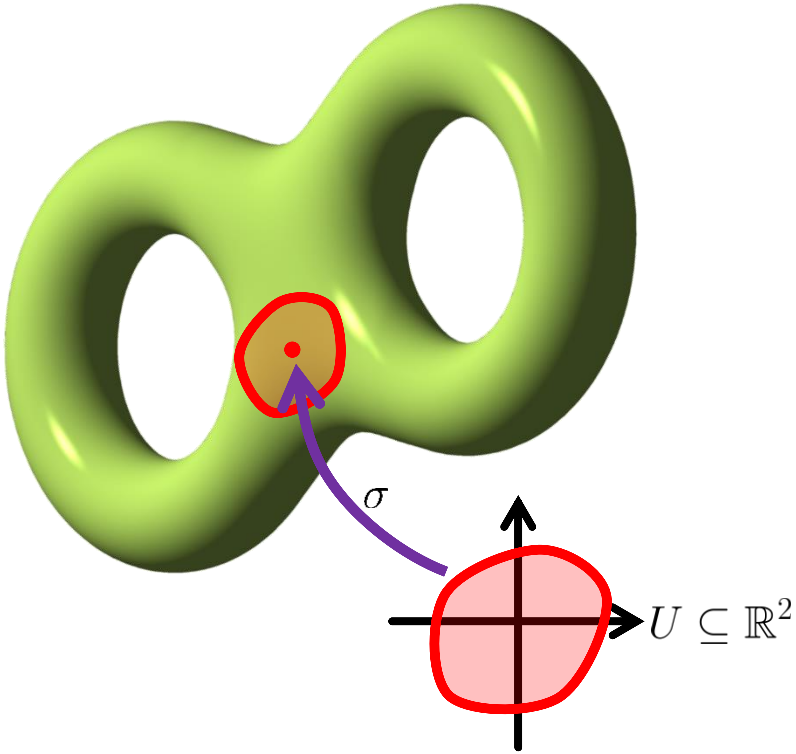
Flux Density: Backward Definition

$$\nabla \cdot \mathbf{v}(\mathbf{p}) := \lim_{r \to 0} \frac{\oint_{\partial B_r(\mathbf{p})} \mathbf{v} \cdot \mathbf{n}_{\text{tangent}} \, d\ell}{\text{vol}(B_r(\mathbf{p}))}$$

Sanity Check: Local Version

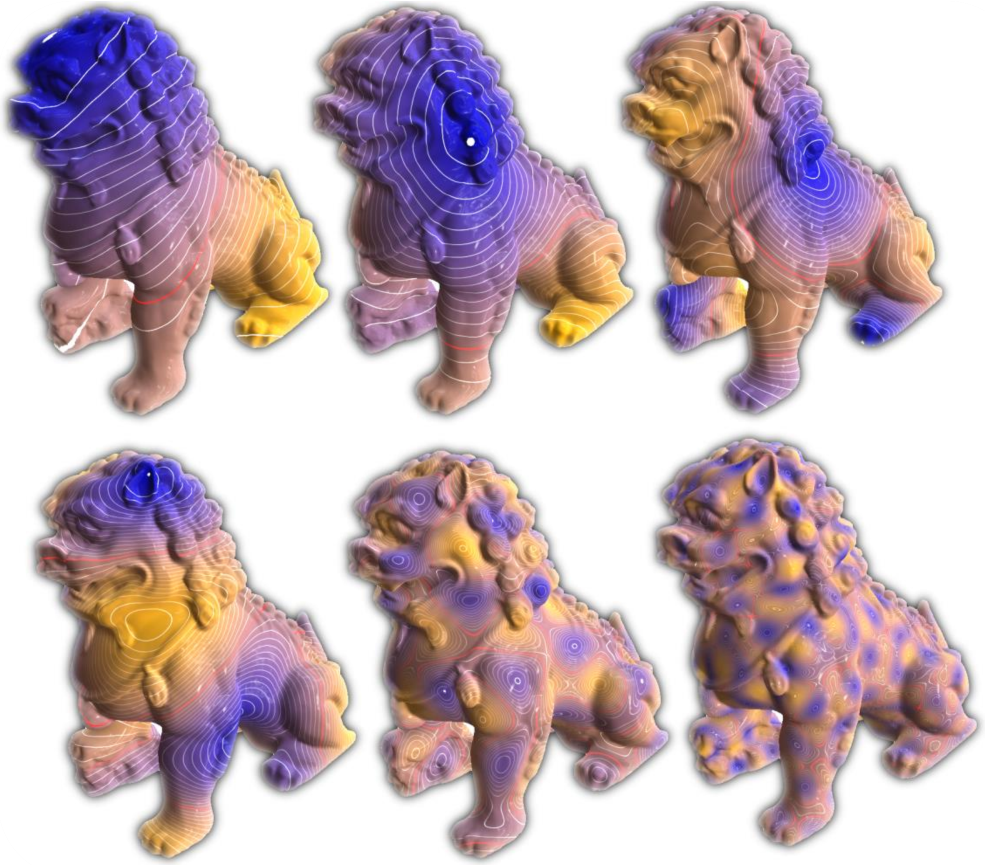
$$f : \mathcal{M} \to \mathbb{R}$$

$$\text{Pullback: } \sigma^* f := f \circ \sigma : U \to \mathbb{R}$$



Laplace-Beltrami coincides with Laplacian on $\mathbb{R}^2$ when σ takes x, y axes to orthonormal vectors.

Eigenfunctions



$$\Delta\psi_i = \lambda_i\psi_i$$

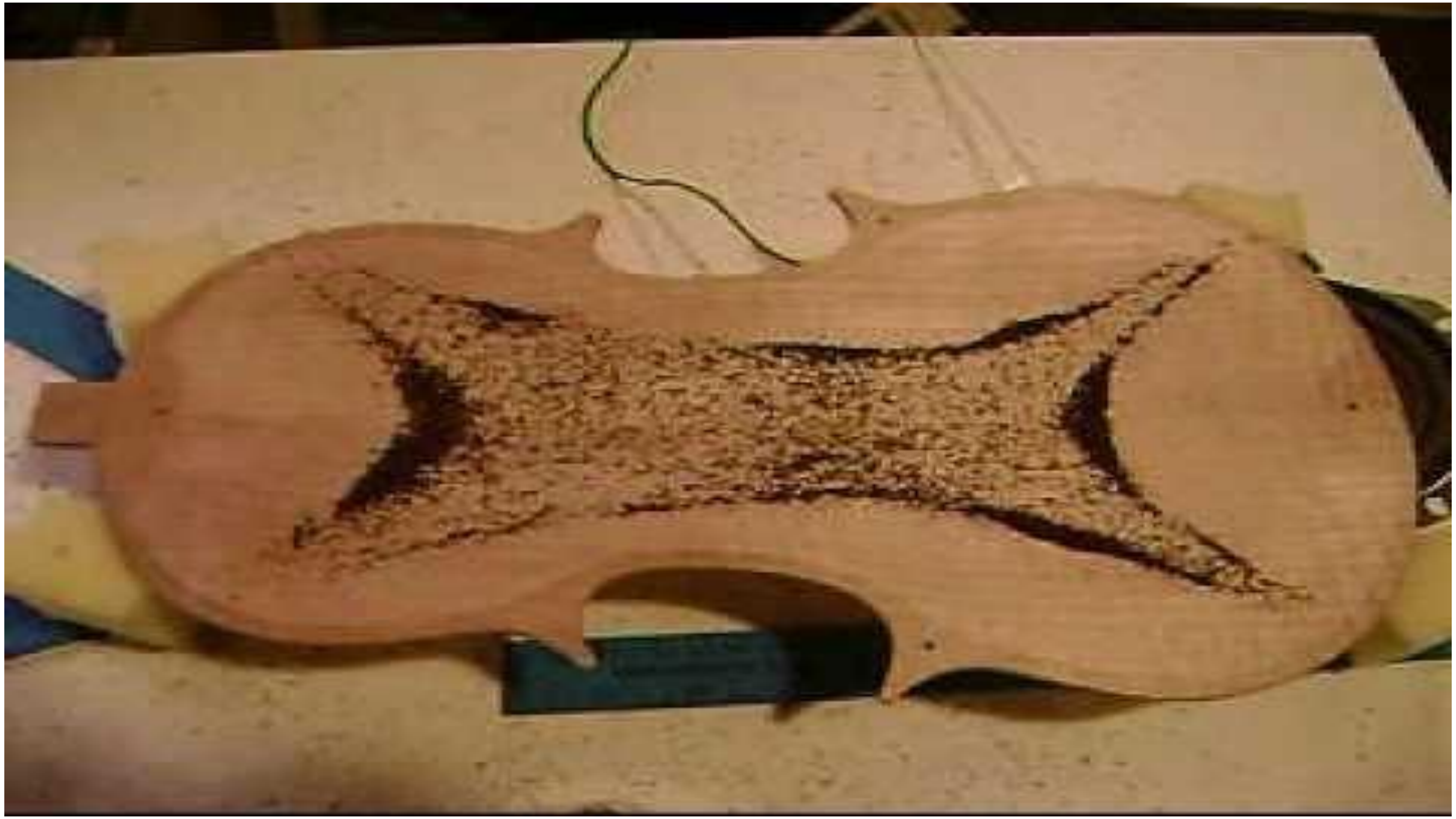
Vibration modes of surface (not volume!)

http://alice.loria.fr/publications/papers/2008/ManifoldHarmonics//photo/dragon_mhb.png

Chladni Plates

https://www.youtube.com/watch?v=CGiiSlMFFlI

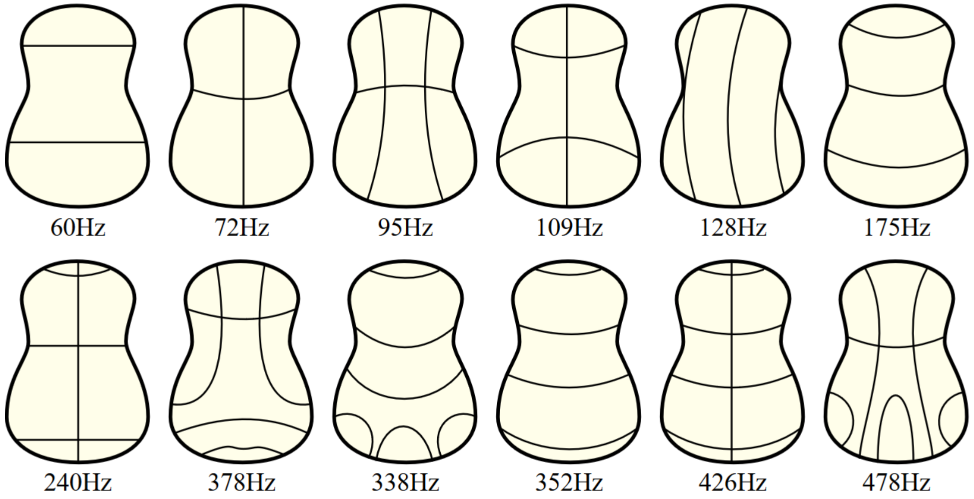
Practical Application



https://www.youtube.com/watch?v=3uMZzVvnSiU

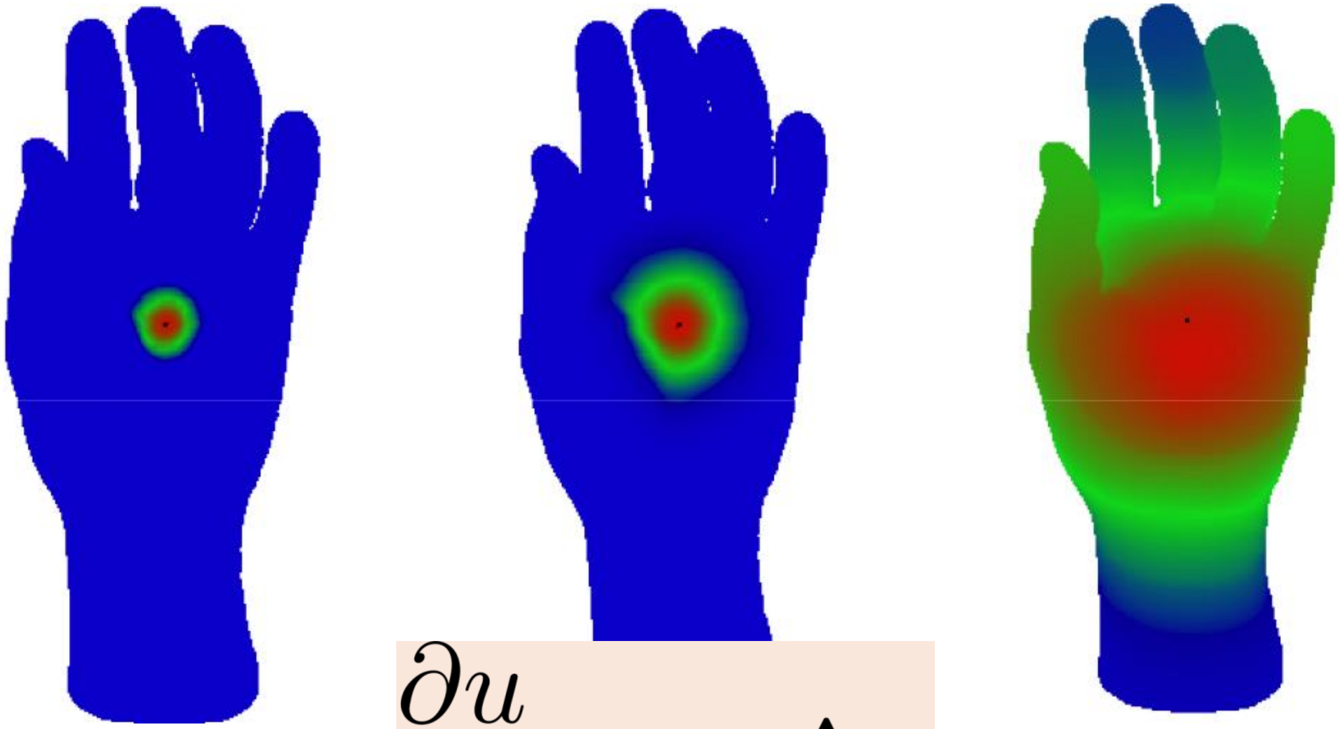
Nodal Domains

Theorem (Courant). The *n*-th eigenfunction of the Dirichlet boundary value problem has at most *n* nodal domains.



https://en.wikipedia.org/wiki/Ernst_Chladni

Additional Connection to Physics

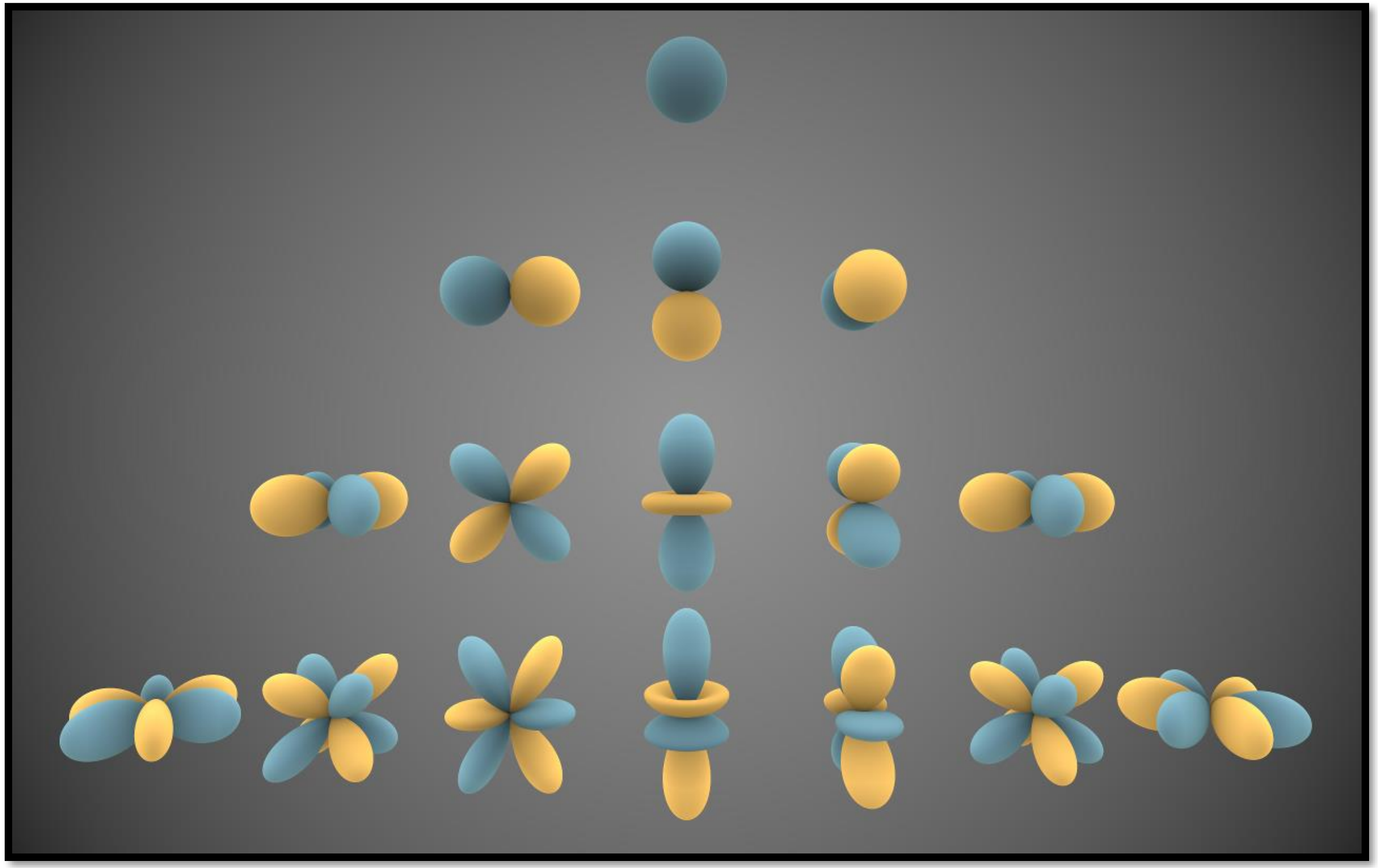


$$\frac{\partial u}{\partial t} = -\Delta u$$

http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

Spherical Harmonics



https://en.wikipedia.org/wiki/Spherical_harmonics

Weyl's Law

$$N(\lambda) := \#\ \text{eigenfunctions} \leq \lambda$$

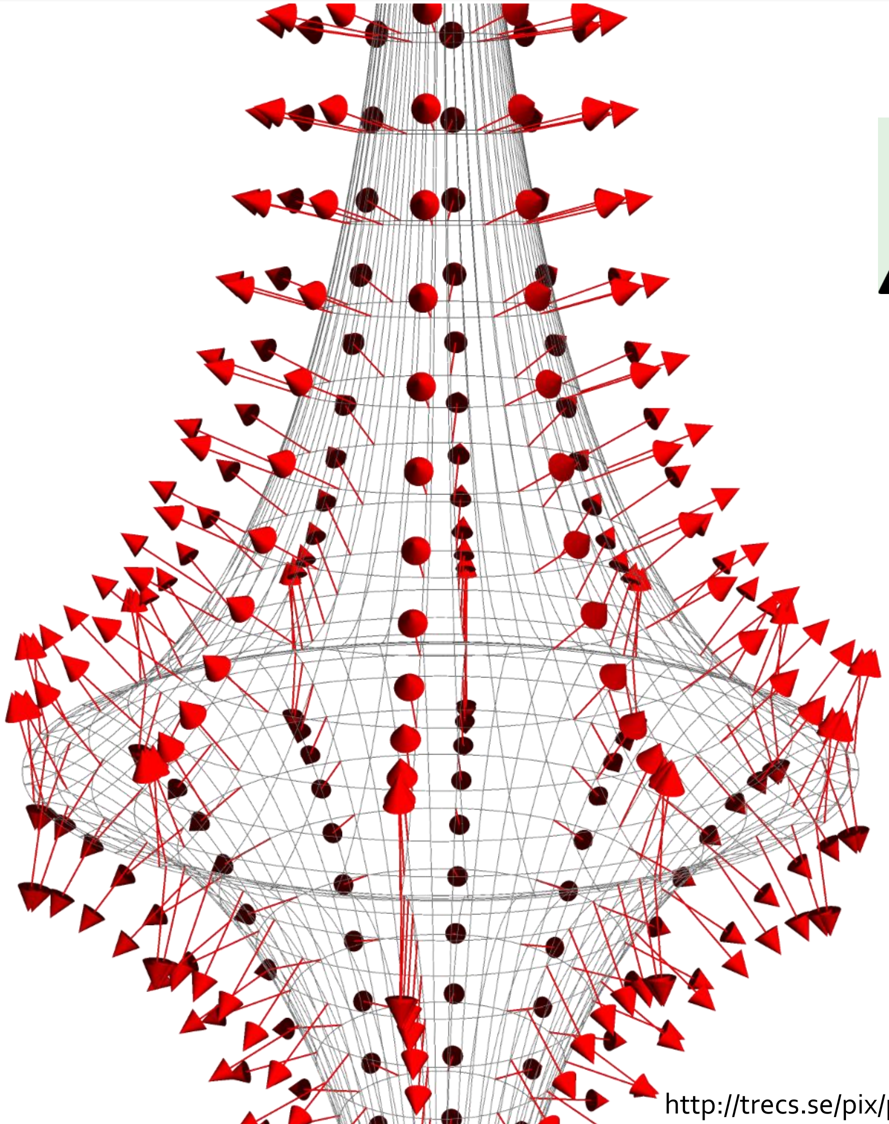
$$\omega_d := \text{volume of unit ball in } \mathbb{R}^d$$

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = (2\pi)^{-d} \omega_d \text{vol}(\Omega)$$

Corollary: $\text{vol}(\Omega) = (2\pi)^d \lim_{R \to \infty} \frac{N(R)}{R^{d/2}}$

$$\text{For surfaces: } \lambda_n \sim \frac{4\pi}{\text{vol}(\Omega)} n$$

Laplacian of *xyz* function



$$\Delta \mathbf{x} = H\mathbf{n}$$

Intuition:
Laplacian measures difference with neighbors.

http://trecs.se/pix/pseudosphereNormalVectorField4096.png

The Laplacian Operator

Justin Solomon

6.8410: Shape Analysis
Spring 2023