

# Useful Formulas for 6.8410

## Basic Geometry and Trigonometry

$$\begin{aligned}
 A &= \frac{1}{2}bh \\
 \tan \theta &= \frac{\sin \theta}{\cos \theta} \\
 \cot \theta &= (\tan \theta)^{-1} \\
 \cos^2 \theta + \sin^2 \theta &= 1 \\
 \sin(-\theta) &= -\sin \theta \\
 \cos(-\theta) &= \cos \theta \\
 \sin(\theta \pm \frac{\pi}{2}) &= \pm \cos \theta \\
 \cos(\theta \pm \frac{\pi}{2}) &= \mp \sin \theta \\
 \sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\
 \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \\
 e^{i\theta} &= \cos \theta + i \sin \theta \\
 \frac{d}{dt} \sin t &= \cos t \\
 \frac{d}{dt} \cos t &= -\sin t
 \end{aligned}$$

## Linear Algebra

$$\begin{aligned}
 (AB)^{-1} &= B^{-1}A^{-1} \\
 (AB)^T &= B^T A^T \\
 \text{tr}(A) &= \sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i \\
 \text{tr}(A) &= \text{tr}(A^T) \\
 \text{tr}(AB) &= \text{tr}(BA) \\
 \langle A, B \rangle &= \sum_{ij} a_{ij} b_{ij} = \text{tr}(A^T B) \\
 \|A\|_{\text{Fro}} &= \sqrt{\langle A, A \rangle} \\
 v \cdot w &= v^T w = \text{tr}(v^T w) = \text{tr}(wv^T) \\
 \|v\|_2^2 &= v \cdot v = v^T v \\
 \|v\|_p &= (\sum_i |v_i|^p)^{1/p} \\
 \det(A) &= \prod_{i=1}^n \lambda_i \\
 \det(A^{-1}) &= 1/\det(A)
 \end{aligned}$$

## Differential Vector Calculus

See [this Wikipedia page](#) for many vector calculus identities.

$$\begin{aligned}
 df_x(v) &= \lim_{h \rightarrow 0} \frac{f(x+hv) - f(x)}{h} = \nabla f \cdot v \\
 \nabla f &= \left( \frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n} \right) \\
 \text{div } F &= \nabla \cdot F = \sum_i \frac{\partial F^i}{\partial x^i} \\
 \text{curl } F &= \nabla \times F = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (F^x, F^y, F^z) \text{ for } F: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \\
 \Delta f &= -\nabla^2 f = -\nabla \cdot \nabla f = -\sum_{i=1}^n \frac{\partial^2 f}{\partial (x^i)^2} \\
 &\text{(in 6.8410 we use a positive semidefinite Laplacian)}
 \end{aligned}$$

## Matrix Calculus

Check out [matrixcalculus.org](http://matrixcalculus.org) for a handy matrix derivative calculation tool. The [Matrix Cookbook](#) also contains a comprehensive list of identities.

$$\begin{aligned}
 \frac{dY^{-1}}{dt} &= -Y^{-1} \frac{dY}{dt} Y^{-1} \\
 \nabla_x (x^T b) &= b \\
 \nabla_X (a^T X b) &= ab^T \\
 \nabla_x (x^T A x + b^T x) &= (A + A^T)x + b \\
 \nabla_X \text{tr}(X) &= I \\
 \nabla_X \text{tr}(XB) &= B^T \\
 \nabla_X \text{tr}(X^T BXC) &= BXC + B^T X C^T \\
 \nabla_X \det(X) &= \det(X) \cdot X^{-T} \\
 e^A &= \sum_n \frac{1}{n!} A^n \\
 e^{ABA^{-1}} &= A e^B A^{-1} \\
 A e^A &= e^A A \\
 e^A e^B &= e^{A+B+1/2[A,B]+\dots} \\
 \frac{d}{dt} e^A(t) &= A'(t) e^A(t)
 \end{aligned}$$

## Derivatives and Integrals, Integration by Parts, Stokes, etc.

$$\begin{aligned}
 \frac{d}{dt} \left( \int_{a(t)}^{b(t)} f(x, t) dx \right) &= f(b(t), t) \frac{db(t)}{dt} - f(a(t), t) \frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x, t) dx \\
 \frac{d}{dt} \int_{D(t)} F(x, t) dV &= \int_{D(t)} \frac{\partial F}{\partial t}(x, t) dV + \oint F(x, t) v_b \cdot \hat{n} dA \\
 \int_a^b u(x) v'(x) dx &= [u(x)v(x)]_a^b - \int_a^b u'(x)v(x) dx \\
 \int_{\Omega} u \nabla \cdot V dA &= \oint_{\partial \Omega} u V \cdot \hat{n} d\ell - \int_{\Omega} \nabla u \cdot V dA \\
 \int_{\Omega} (\psi \nabla \cdot \Gamma + \Gamma \cdot \nabla \psi) dA &= \oint_{\partial \Omega} \psi (\Gamma \cdot \hat{n}) d\ell
 \end{aligned}$$

$$\begin{aligned}
 \int_{\Omega} (\psi \nabla \cdot (\varepsilon \nabla \phi) - \phi \nabla \cdot (\varepsilon \nabla \psi)) dV &= \oint_{\partial \Omega} \varepsilon (\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}) dA \\
 \int_{\Omega} [G \cdot (\nabla \times F) - F \cdot (\nabla \times G)] dV &= \oint_{\partial \Omega} (F \times G) \cdot \hat{n} dA \\
 \int_{\Omega} G \cdot \nabla f dV &= \oint_{\partial \Omega} (fG) \cdot \hat{n} dA - \int_{\Omega} f(\nabla \cdot G) dV \\
 \oint_{\partial \Omega} F \cdot \hat{n} dA &= \int_{\Omega} \nabla \cdot F dV \\
 \int_{\Omega} [F \cdot \nabla g + g(\nabla \cdot F)] dV &= \oint_{\partial \Omega} gF \cdot \hat{n} dA
 \end{aligned}$$