Smooth Surface Curvature

Justin Solomon

6.8410: Shape Analysis
Spring 2023

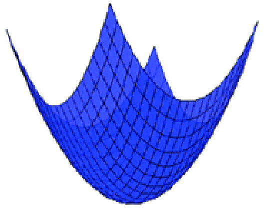
Today's Goal

Quantify how a surface

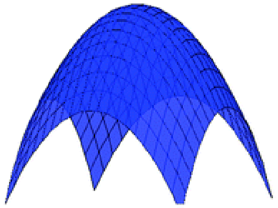
deviates from flatness.

Curvature.

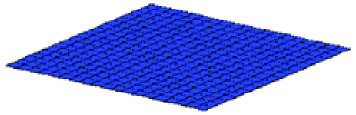
High-Level Questions



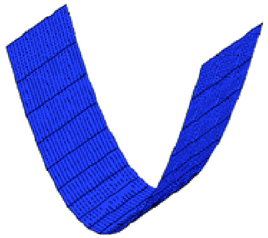
(a) $KG > 0$, $KH > 0$
elliptic concave



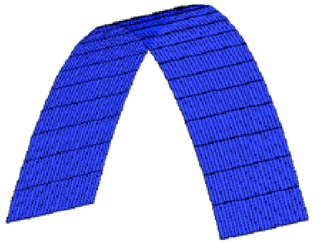
(b) $KG > 0$, $KH < 0$
elliptic convexe



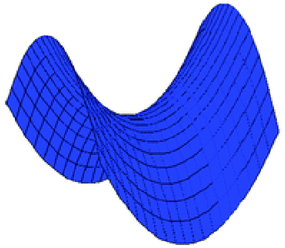
(c) $KG = 0$, $KH = 0$
plane



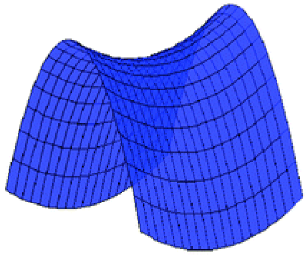
(d) $KG = 0$, $\mathrm{KH} > 0$
parabolic concave



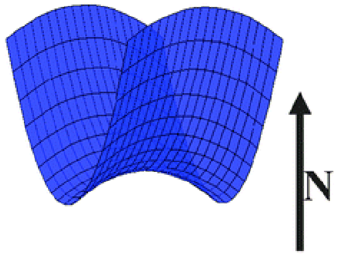
(e) $KG = 0$, $KH < 0$
parabolic convexe



(f) $KG < 0$, $KH = 0$
saddle (hyperbolic)



(g) $KG < 0$, $KH < 0$
hyperbolic-like

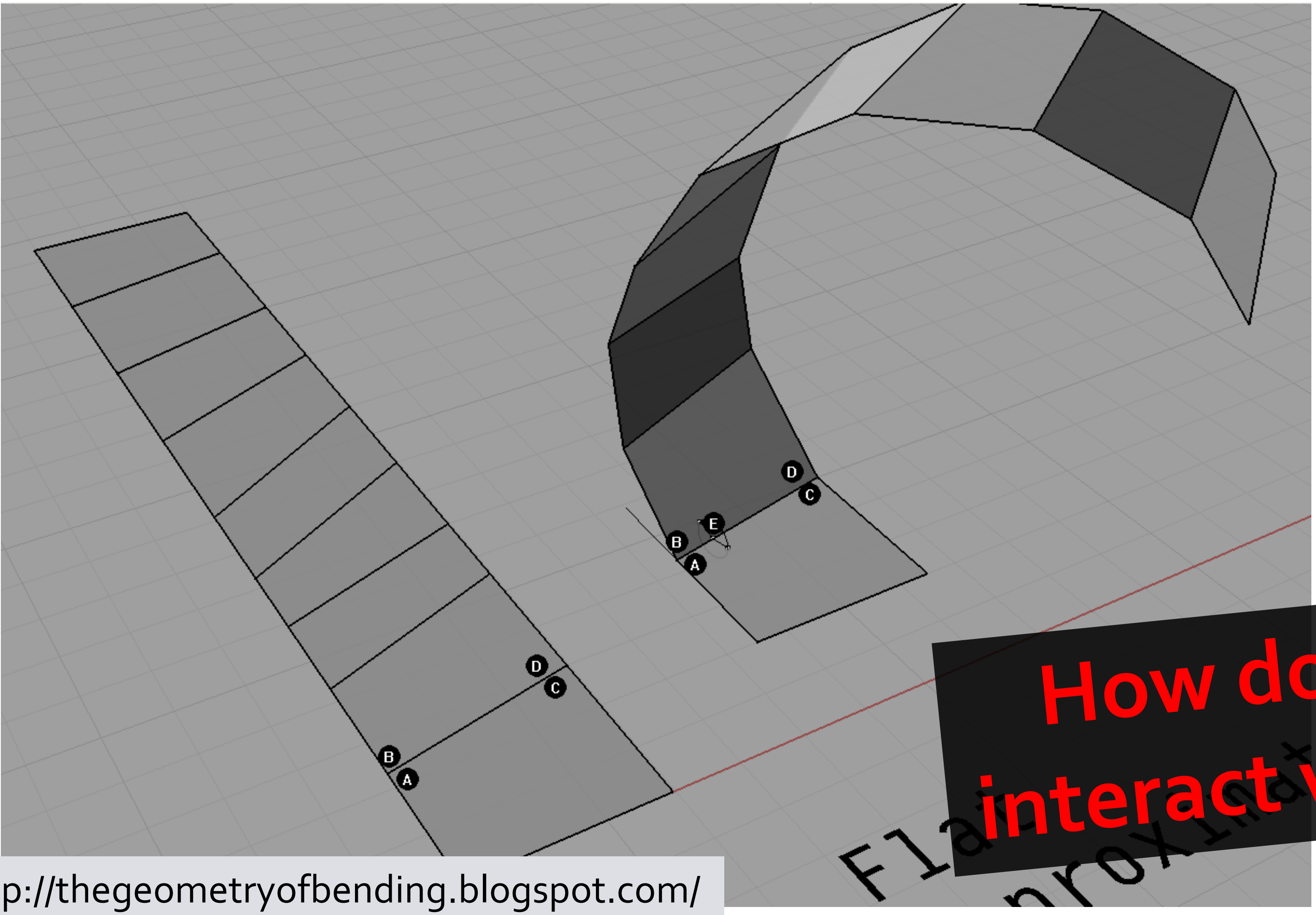


(h) $KG < 0$, $KH > 0$
hyperbolic-like

How to distinguish?

http://pubs.rsc.org/is/content/articlelanding/2013/cp/c3cp44375b

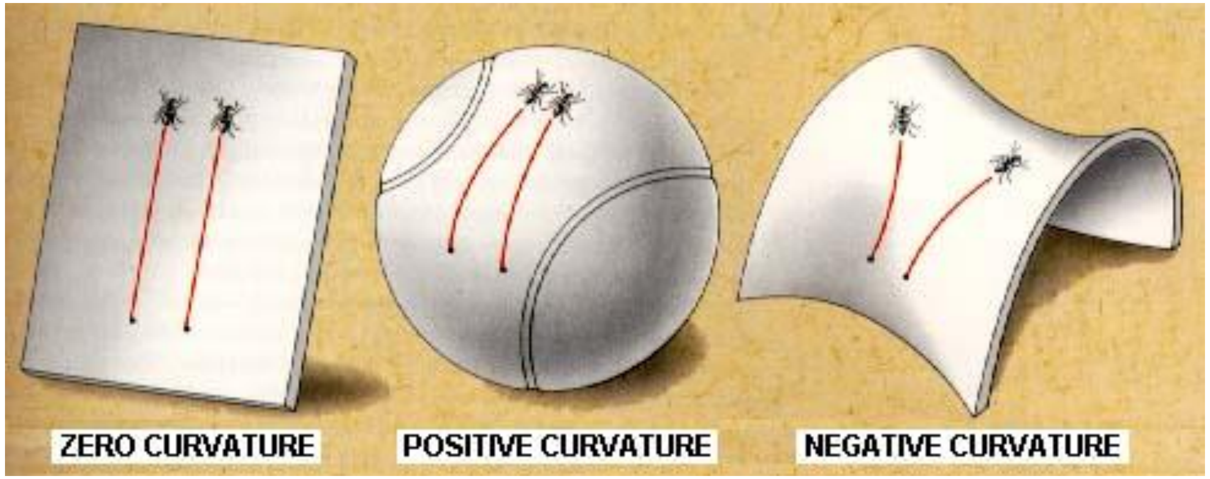
High-Level Questions



How do surfaces interact with space?

http://thegeometryofbending.blogspot.com/

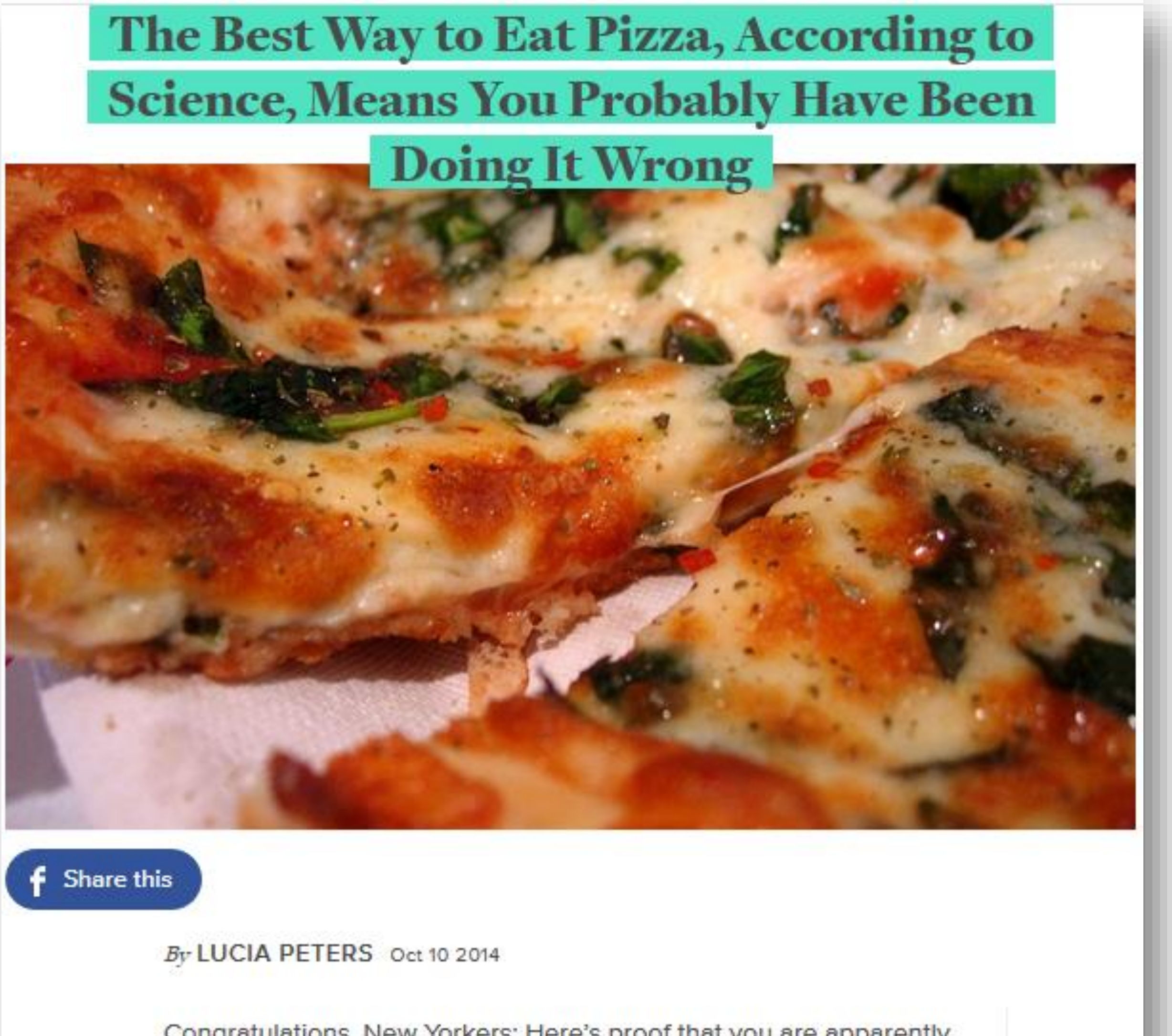
High-Level Questions



Does surrounding space matter?

http://starchild.gsfc.nasa.gov/docs/StarChild/questions/question35.html

Practical Application



The Best Way to Eat Pizza, According to Science, Means You Probably Have Been Doing It Wrong

Share this

By **LUCIA PETERS** Oct 10 2014

Congratulations, New Yorkers: Here's proof that you are apparently

Bend It Like Gauss:

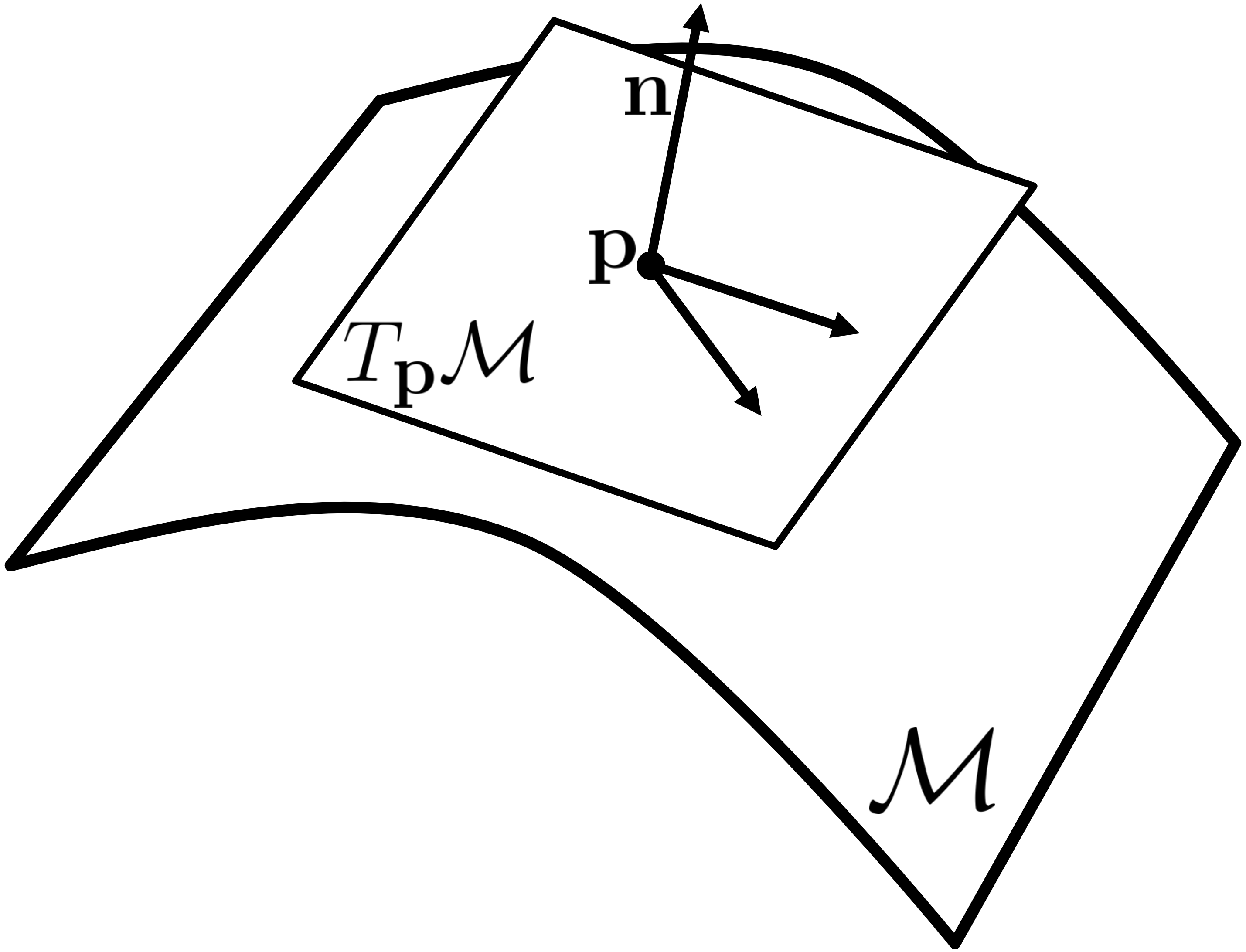
https://www.bustle.com/articles/43697-the-best-way-to-eat-pizza-according-to-science-means-you-probably-have-been-doing-it

Q:

Can curvature/torsion of a curve help us understand surfaces?

Recall:

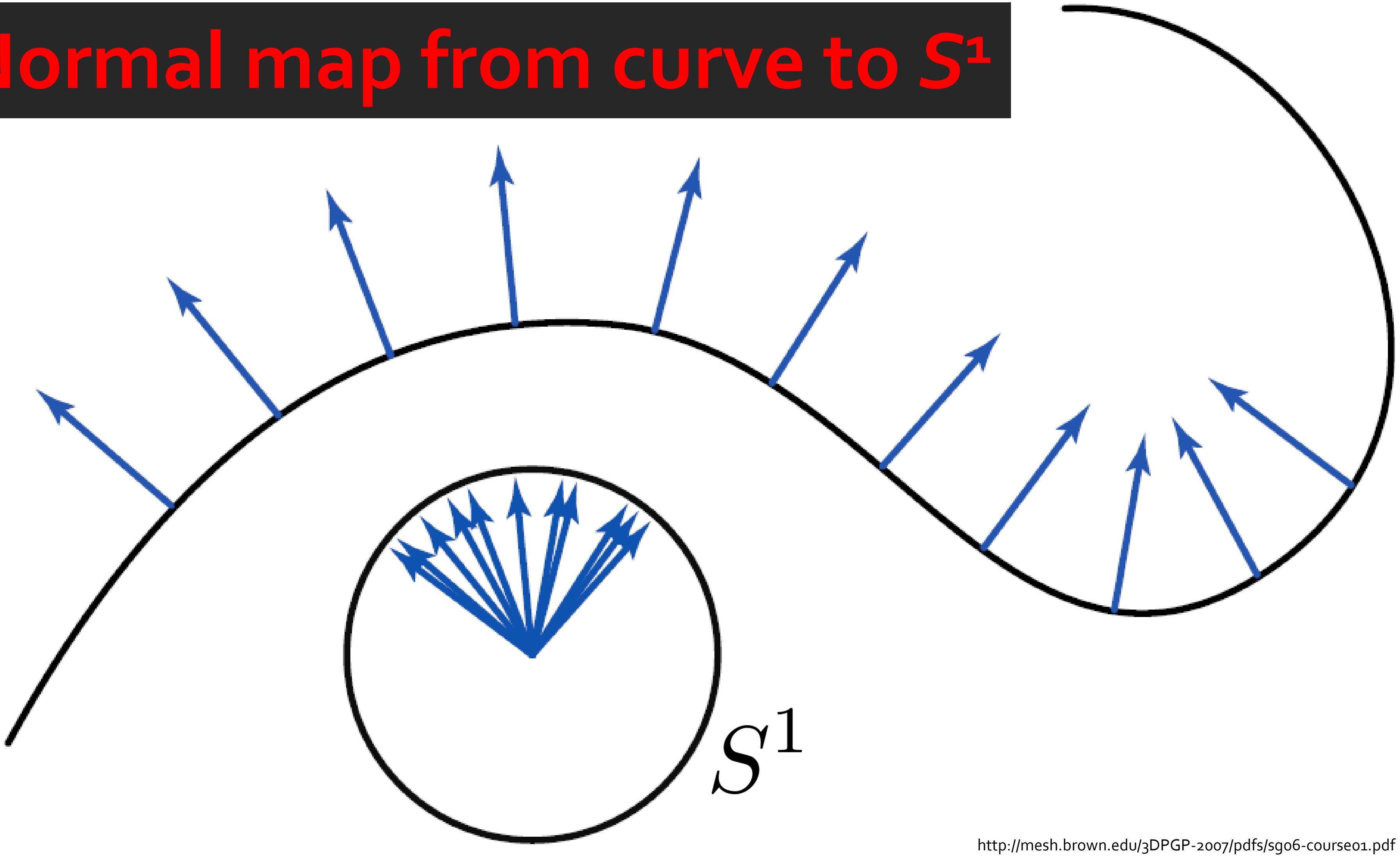
Unit Normal



Recall:

Gauss Map

Normal map from curve to S^1



http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

Recall:

Frenet Frame: Curves in $\mathbb{R}^3$

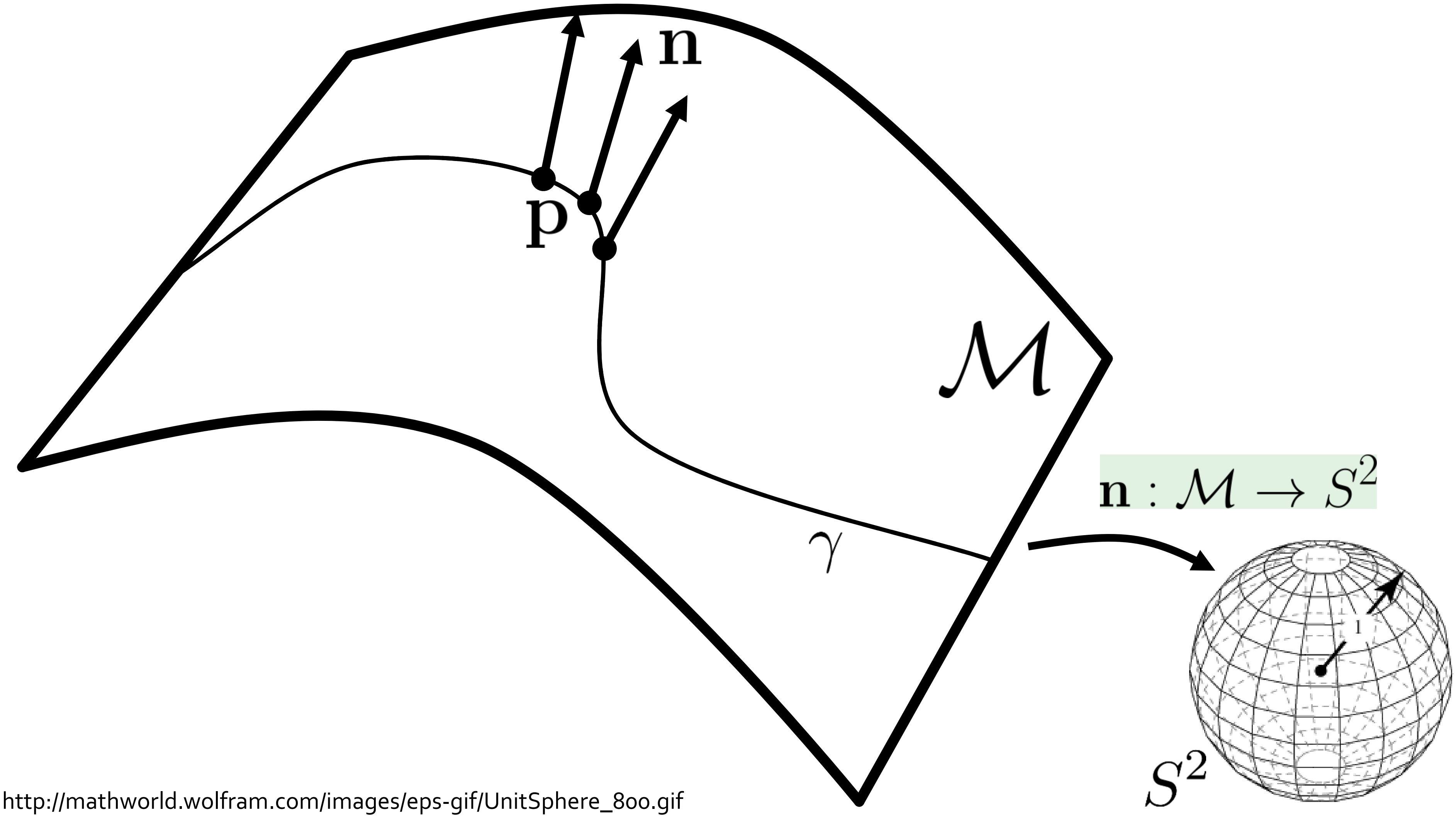
$$\frac{d}{ds}\begin{pmatrix}\mathbf{T}\\\mathbf{N}\\\mathbf{B}\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\\mathbf{N}\\\mathbf{B}\end{pmatrix}$$

- **Binormal:** $\boldsymbol{T} \times \boldsymbol{N}$
- **Curvature:** In-plane motion
- **Torsion:** Out-of-plane motion

Theorem:

Curvature and torsion determine geometry of a curve up to rigid motion.

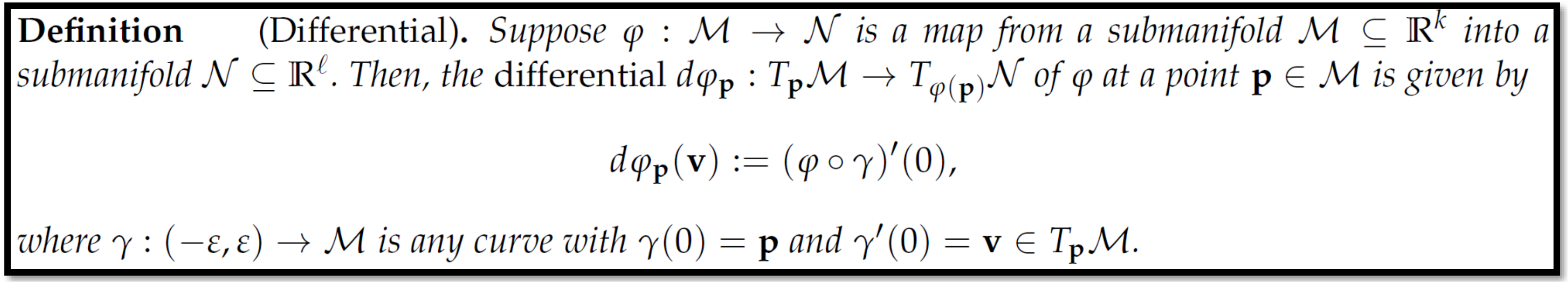
Gauss Map for an Oriented Surface



$\mathbf{n} : \mathcal{M} \to S^2$

http://mathworld.wolfram.com/images/eps-gif/UnitSphere_800.gif

Differential of a Map



Definition (Differential). *Suppose* $\varphi : \mathcal{M} \to \mathcal{N}$ *is a map from a submanifold* $\mathcal{M} \subseteq \mathbb{R}^k$ *into a submanifold* $\mathcal{N} \subseteq \mathbb{R}^\ell$*. Then, the* differential $d\varphi_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \to T_{\varphi(\mathbf{p})}\mathcal{N}$ *of* φ *at a point* $\mathbf{p} \in \mathcal{M}$ *is given by*

$$d\varphi_{\mathbf{p}}(\mathbf{v}) := (\varphi \circ \gamma)'(0),$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ *is any curve with* $\gamma(0) = \mathbf{p}$ *and* $\gamma'(0) = \mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$.

Linear map of tangent spaces

$$d\varphi_{\mathbf{p}}(\gamma'(0)) := (\varphi \circ \gamma)'(0)$$

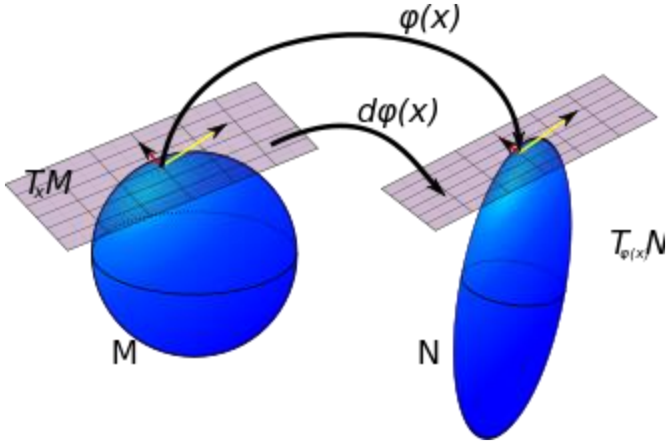


Image from Wikipedia

Calculation

Where is the

derivative of *n*?

$$d\mathbf{n}_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \to ??$$

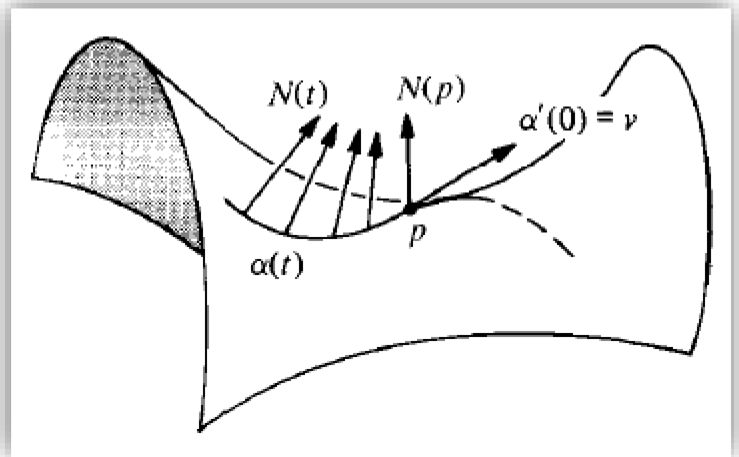
"Shape operator"

Second Fundamental Form

$$\mathbb{I}_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \times T_{\mathbf{p}}\mathcal{M} \to \mathbb{R}$$

Defined by:

$$\mathbb{I}_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) := -\mathbf{v} \cdot d\mathbf{n}_{\mathbf{p}}(\mathbf{w})$$

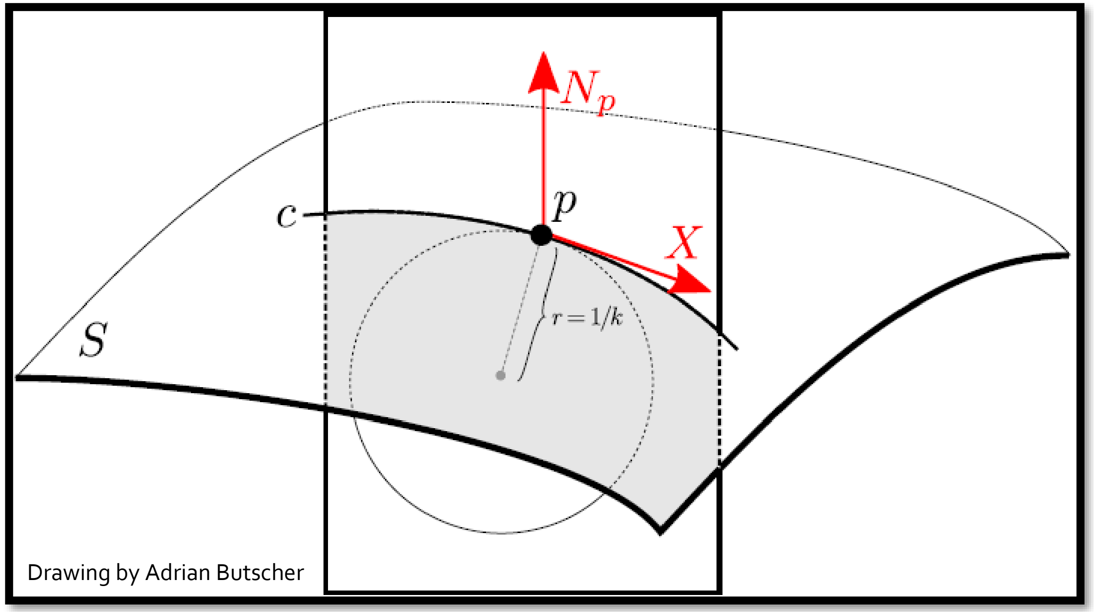


"Lower the index"

Calculation

$$\mathbb{I}(\mathbf{T}, \mathbf{T})$$

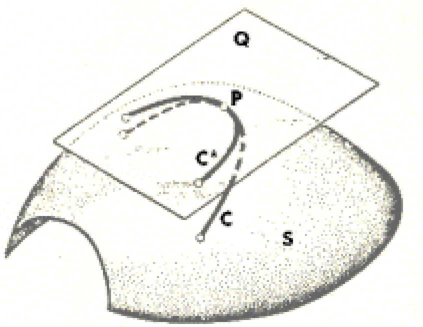
Relationship to Curvature of Curves



Drawing by Adrian Butscher

$$
\begin{aligned}
\kappa_{\mathbf{n}} &= \mathbf{K}\cdot\mathbf{n} \\
&= \mathrm{I\!I}_{\gamma(s)}(\mathbf{T},\mathbf{T})
\end{aligned}
$$

"Acceleration due to geometry"



$$\kappa_g := \mathbf{K}\cdot(\mathbf{n}\times\mathbf{T})$$

http://www.solitaryroad.com/c335.html

Geodesic curvature

Calculation

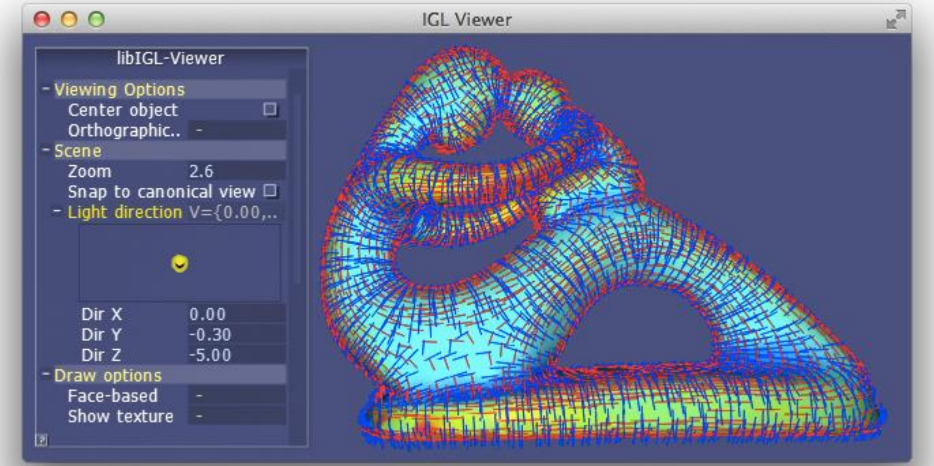
$$\mathbb{I}(\mathbf{v},\mathbf{w}) = \mathbb{I}(\mathbf{w},\mathbf{v})$$

Request for help:
How to visualize this?

Principal Curvatures/Directions

$$\kappa_{\min} := \begin{cases} \min_{\mathbf{v}\in T_{\mathbf{p}}\mathcal{M}} & \mathrm{I\!I}(\mathbf{v},\mathbf{v}) \\ \textit{subject to} & \|\mathbf{v}\|_2 = 1 \end{cases}$$

$$\kappa_{\max} := \begin{cases} \max_{\mathbf{v}\in T_{\mathbf{p}}\mathcal{M}} & \mathrm{I\!I}(\mathbf{v},\mathbf{v}) \\ \textit{subject to} & \|\mathbf{v}\|_2 = 1 \end{cases}$$



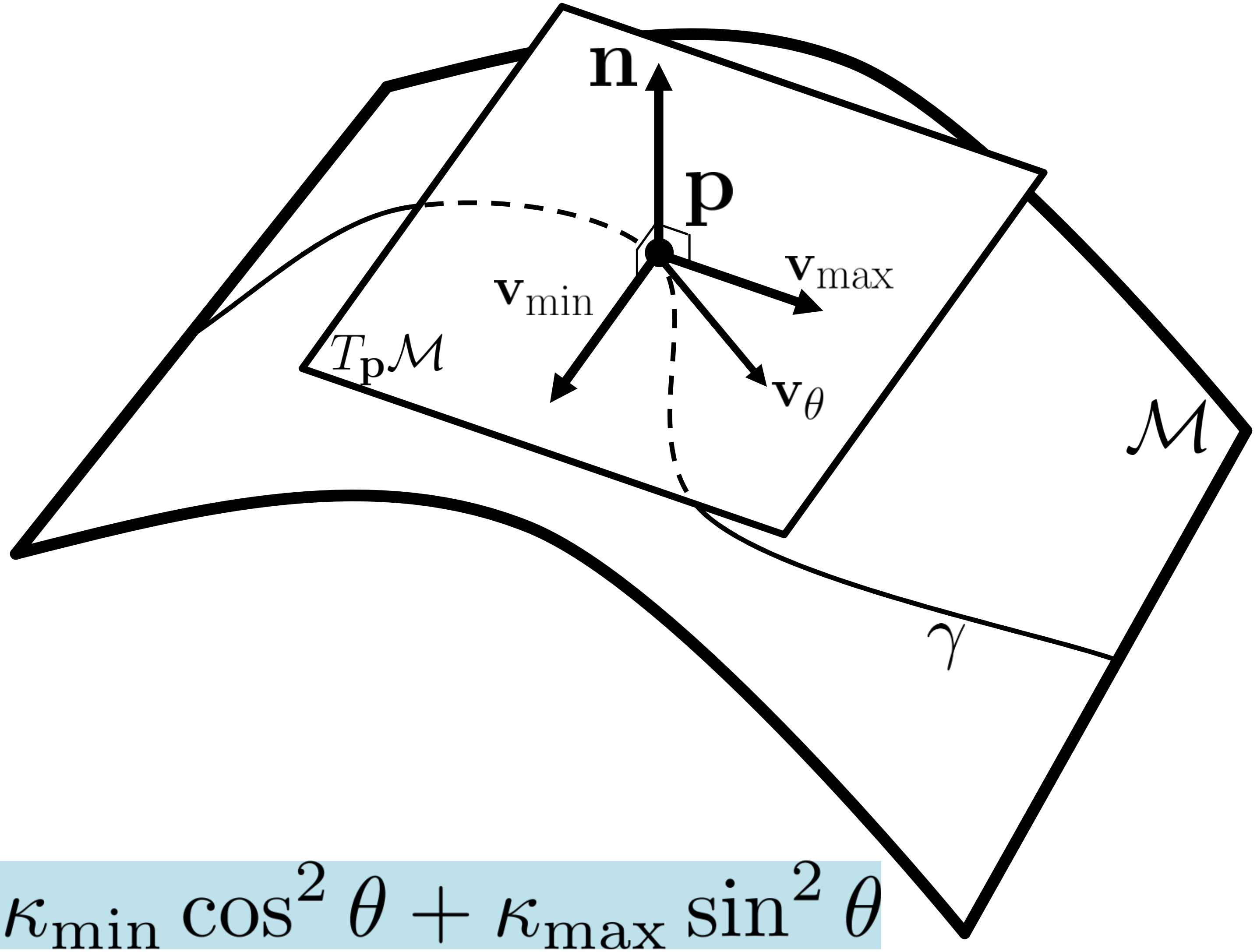
IGL Viewer

libIGL-Viewer

- Viewing Options
  - Center object ☐
  - Orthographic.. -
- Scene
  - Zoom 2.6
  - Snap to canonical view ☐
  - Light direction V={0.00,..
    - Dir X 0.00
    - Dir Y -0.30
    - Dir Z -5.00
- Draw options
  - Face-based -
  - Show texture -

https://libigl.github.io/tutorial/

Principal Directions and Curvatures



$$\kappa_\theta = \kappa_{\min} \cos^2 \theta + \kappa_{\max} \sin^2 \theta$$

Principal Curvatures

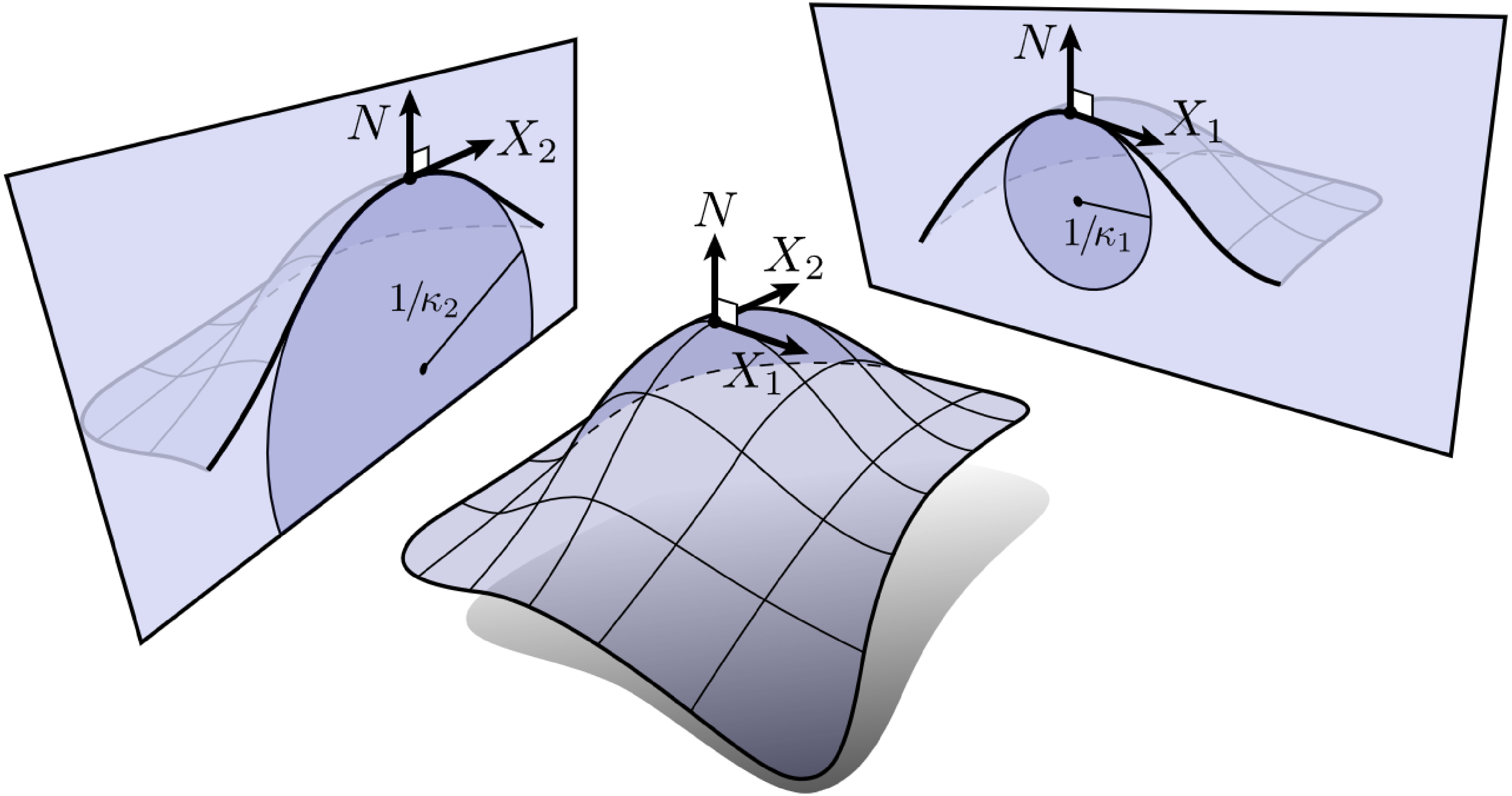
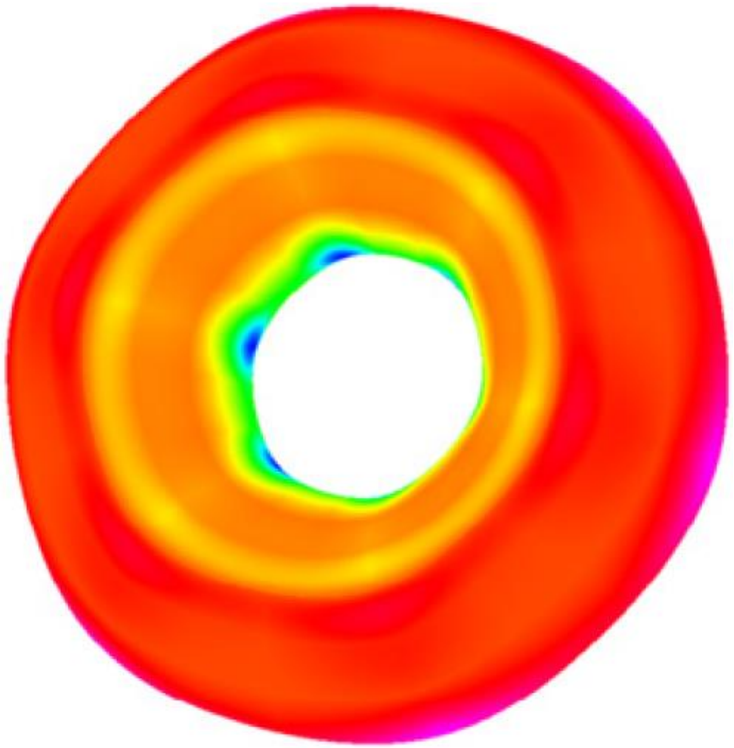


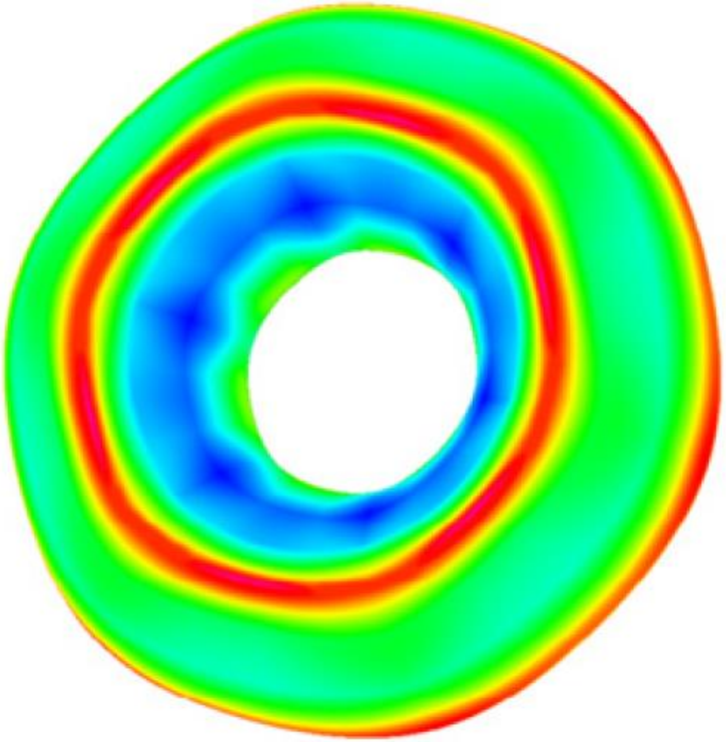
Image courtesy K. Crane, CMU

Curvature Measures



$$K := \kappa_{\min} \cdot \kappa_{\max} = \det \mathrm{I\!I}$$

Gaussian curvature

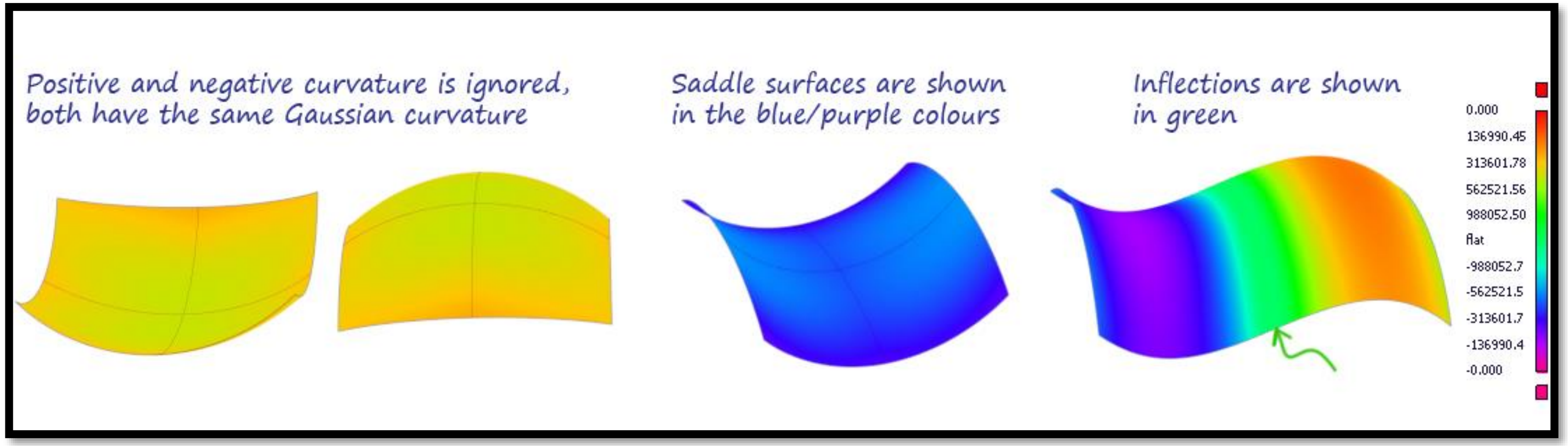


$$H := \frac{1}{2}(\kappa_{\min} + \kappa_{\max}) = \frac{1}{2}\mathrm{tr}\,\mathrm{I\!I}$$

Mean curvature

http://www.sciencedirect.com/science/article/pii/S0010448510001983

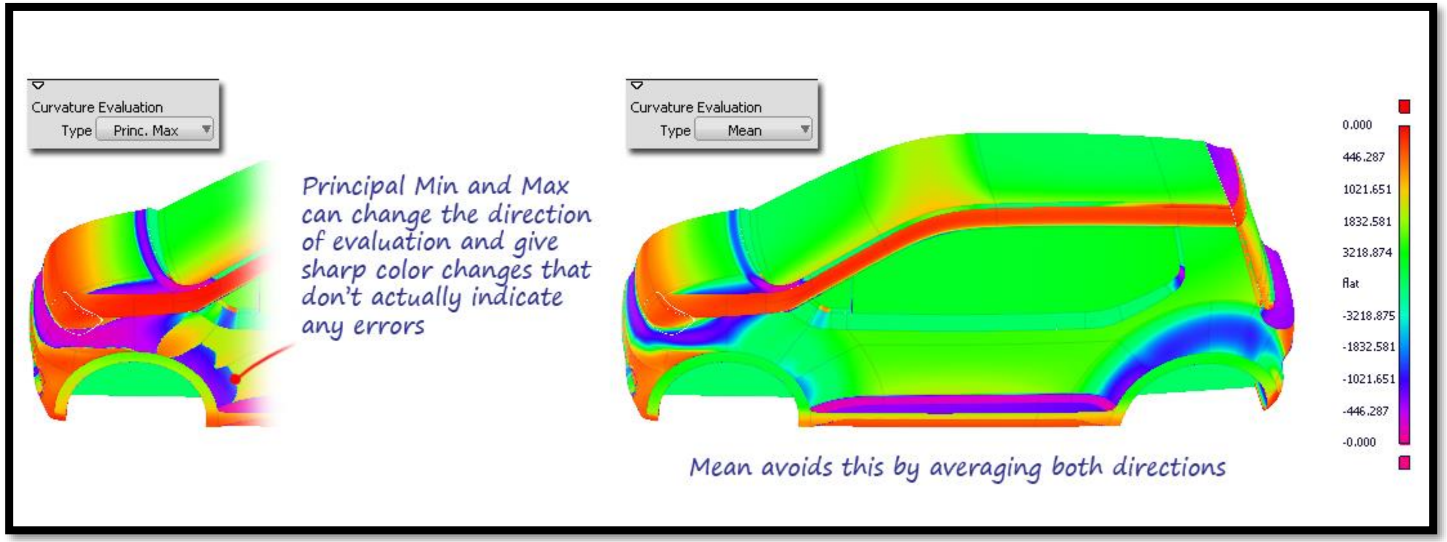
Interpretation



Positive and negative curvature is ignored, both have the same Gaussian curvature

Saddle surfaces are shown in the blue/purple colours

Inflections are shown in green



Curvature Evaluation
Type Princ. Max

Principal Min and Max can change the direction of evaluation and give sharp color changes that don't actually indicate any errors

Curvature Evaluation
Type Mean

Mean avoids this by averaging both directions

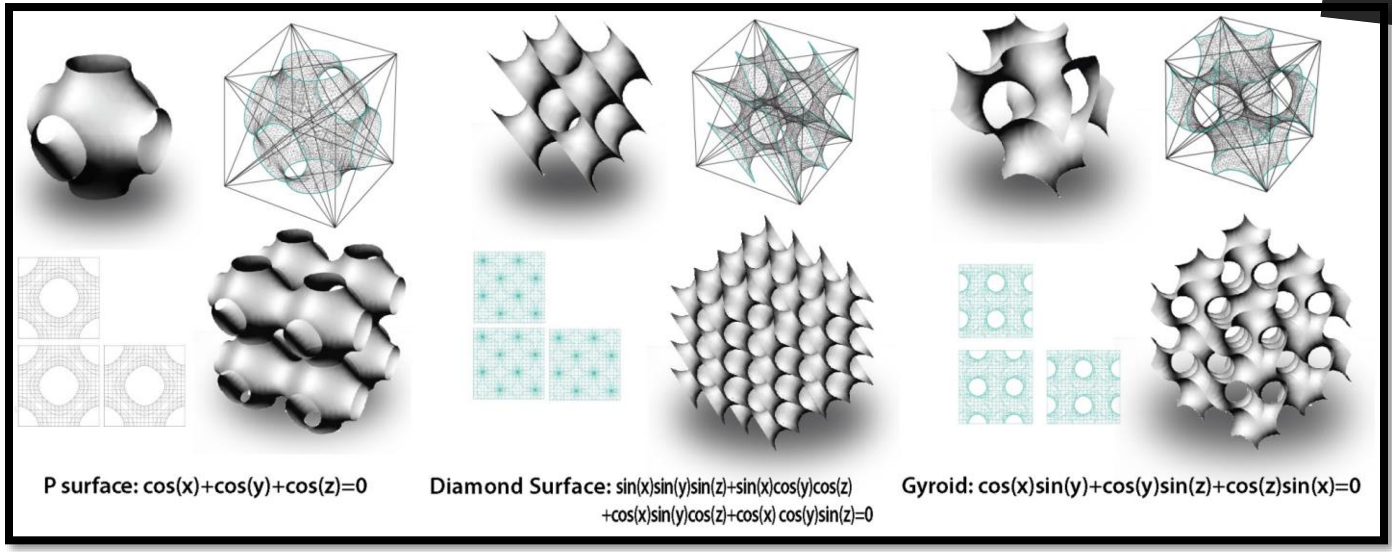
http://www.aliasworkbench.com/theoryBuilders/TB7_evaluate3.htm (Credit: Autodesk Alias Automotive)

Mean Curvature

$$H = \frac{1}{2\pi} \int_0^{2\pi} \kappa(\theta)\, d\theta$$

Byproduct of linear structure

Minimal surfaces



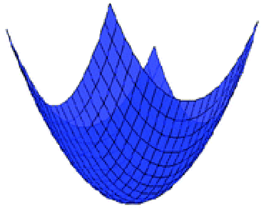
P surface: $\cos(x)+\cos(y)+\cos(z)=0$

Diamond Surface: $\sin(x)\sin(y)\sin(z)+\sin(x)\cos(y)\cos(z)$
$+\cos(x)\sin(y)\cos(z)+\cos(x)\cos(y)\sin(z)=0$

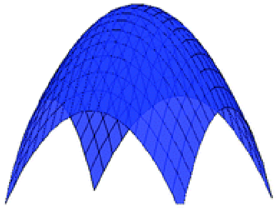
Gyroid: $\cos(x)\sin(y)+\cos(y)\sin(z)+\cos(z)\sin(x)=0$

“Form is Matter: Triply periodic minimal surfaces structures by digital design tools” (Rossi and Buratti)

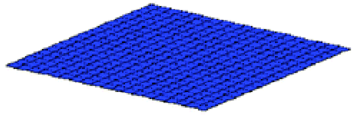
Gaussian Curvature



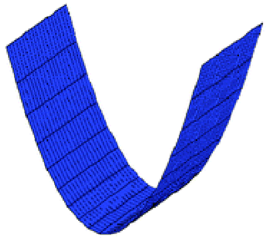
(a) $KG > 0$, $KH > 0$
elliptic concave



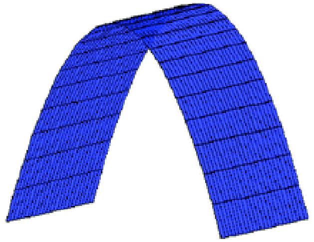
(b) $KG > 0$, $KH < 0$
elliptic convexe



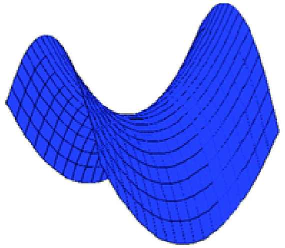
(c) $KG = 0$, $KH = 0$
plane



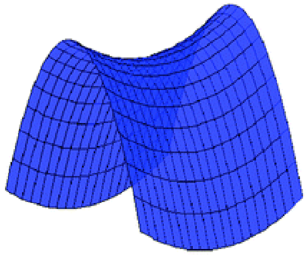
(d) $KG = 0$, $\mathrm{KH} > 0$
parabolic concave



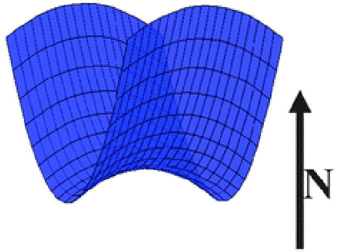
(e) $KG = 0$, $KH < 0$
parabolic convexe



(f) $KG < 0$, $KH = 0$
saddle (hyperbolic)



(g) $KG < 0$, $KH < 0$
hyperbolic-like

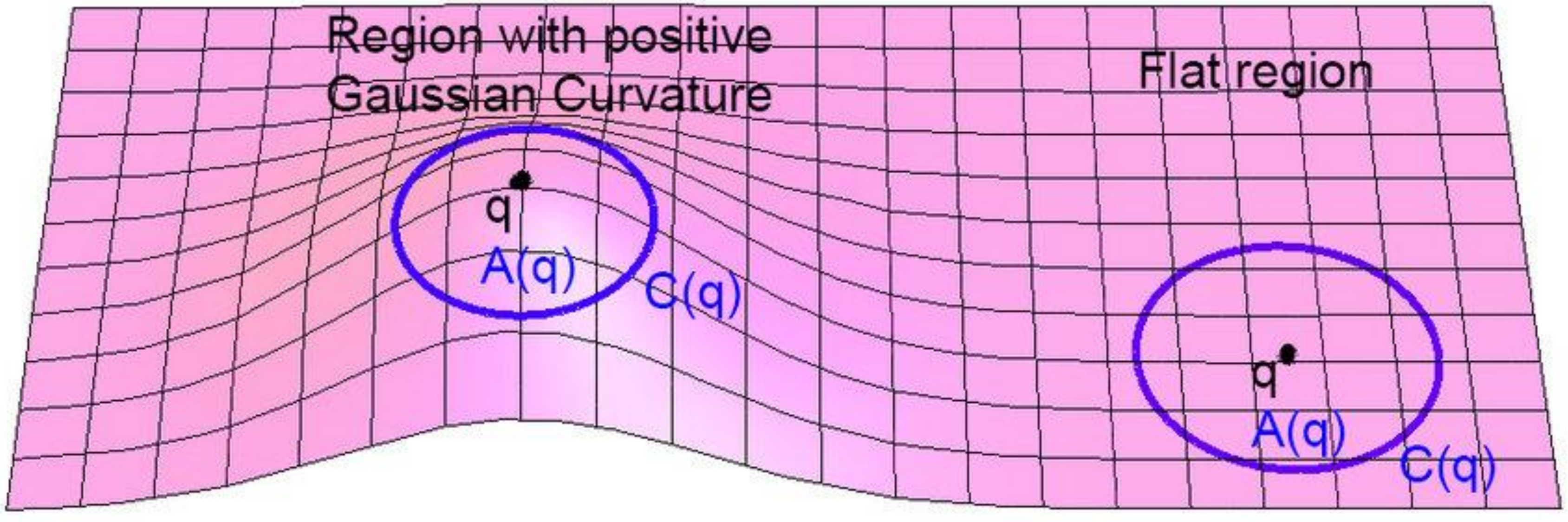


(h) $KG < 0$, $KH > 0$
hyperbolic-like

http://pubs.rsc.org/is/content/articlelanding/2013/cp/c3cp44375b

Geodesic Circle Formulae

$$K = \lim_{r \to 0^+} 3 \frac{2\pi r - C(r)}{\pi r^3} = \lim_{r \to 0^+} 12 \frac{\pi r^2 - A(r)}{\pi r^4}$$



https://www.researchgate.net/figure/The-two-blue-circles-represent-geodesic-circles-about-a-point-q-black-dot-with-both_fig8_309551474

Uniqueness Result

Theorem:

The first and second fundamental forms determine a surface up to rigid motion.

Gauss-Codazzi-Mainardi equations:
Compatibility conditions

Who Cares?

Curvature
determines
local surface geometry.

Smooth Surface Curvature

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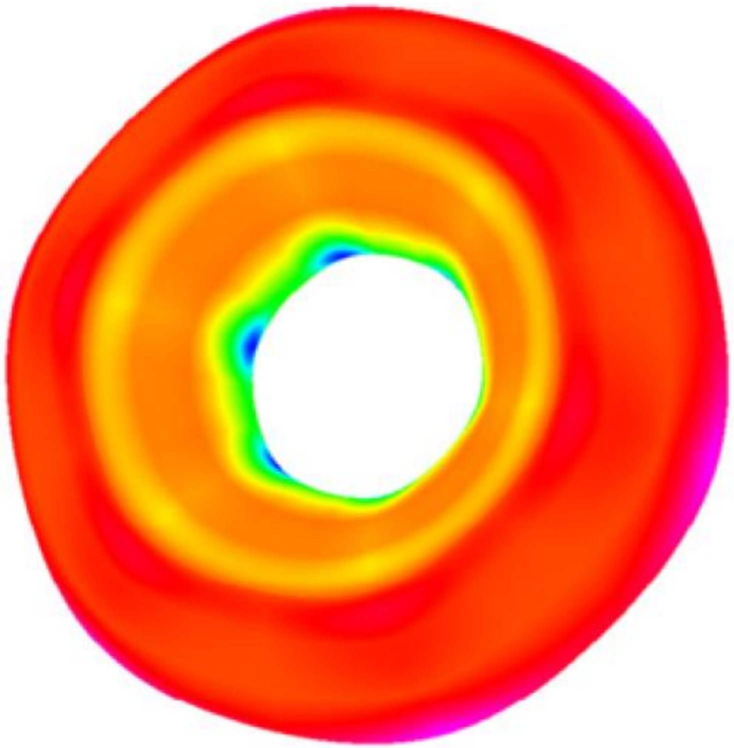
6.8410: Shape Analysis
Spring 2023

Discrete Surface Curvature

Justin Solomon

6.8410: Shape Analysis
Spring 2023

Curvature Measures



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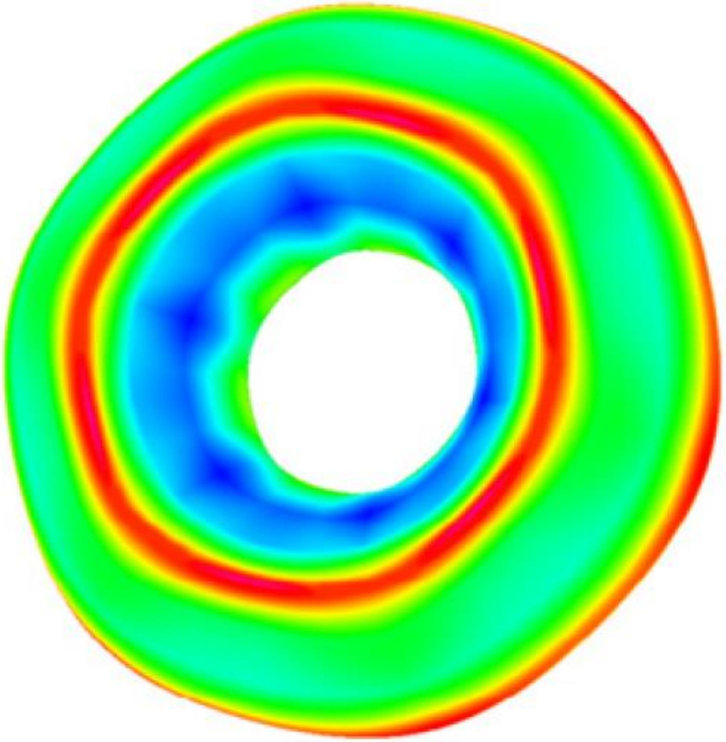
Gaussian curvature

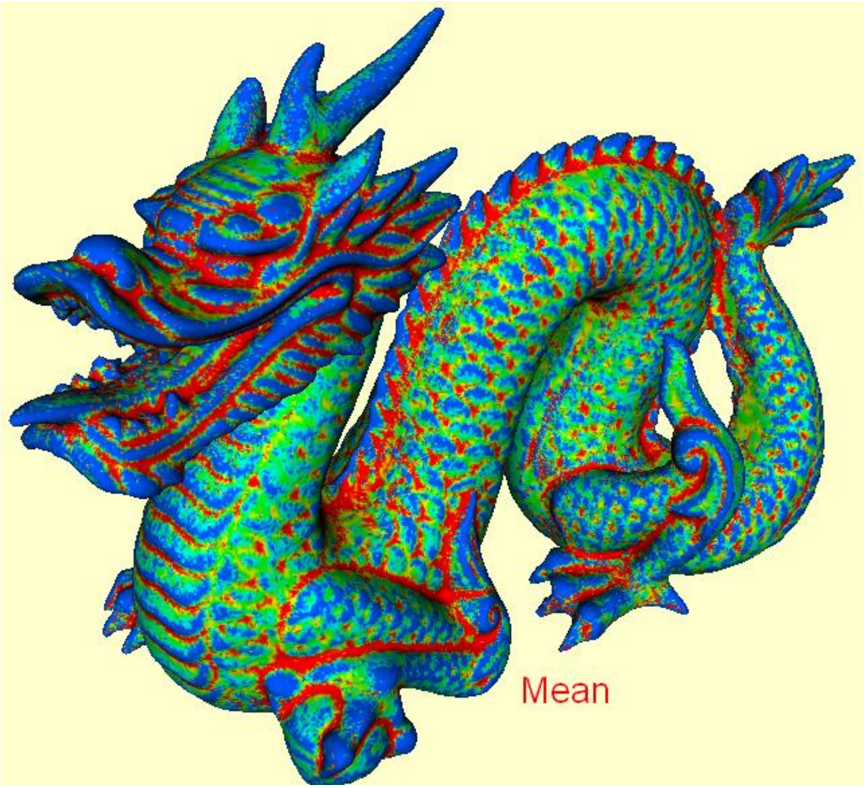


$$H := \frac{1}{2}(\kappa_{\min} + \kappa_{\max}) = \frac{1}{2}\mathrm{tr}\, \mathrm{I\!I}$$

Mean curvature

http://www.sciencedirect.com/science/article/pii/S0010448510001983

Use as a Descriptor

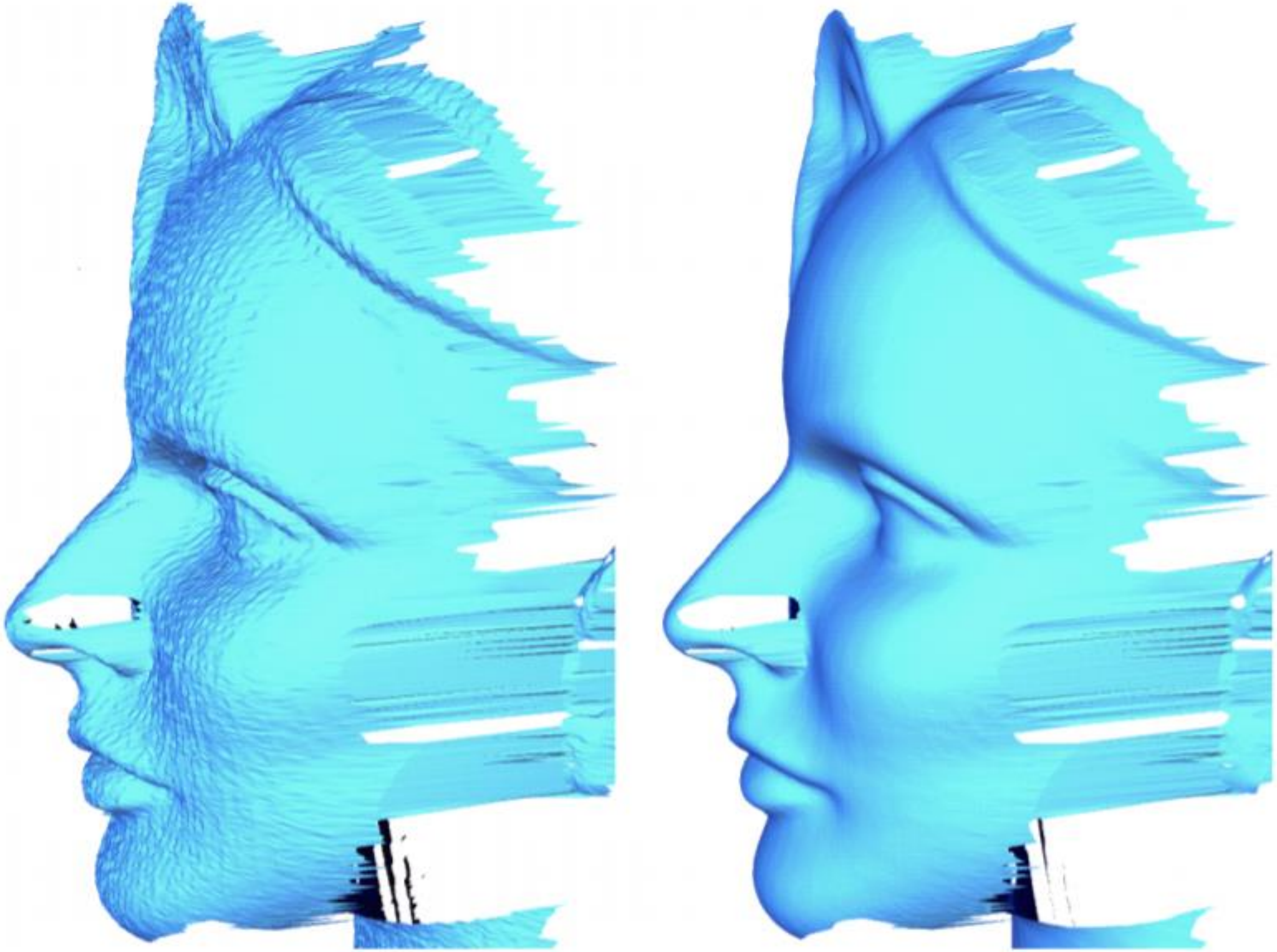


http://graphics.ucsd.edu/~iman/Curvature/

Smoothing and Reconstruction

Linear Surface Reconstruction from Discrete Fundamental Forms on Triangle Meshes
Wang, Liu, and Tong
Computer Graphics Forum 31.8 (2012)

Fairness Measure



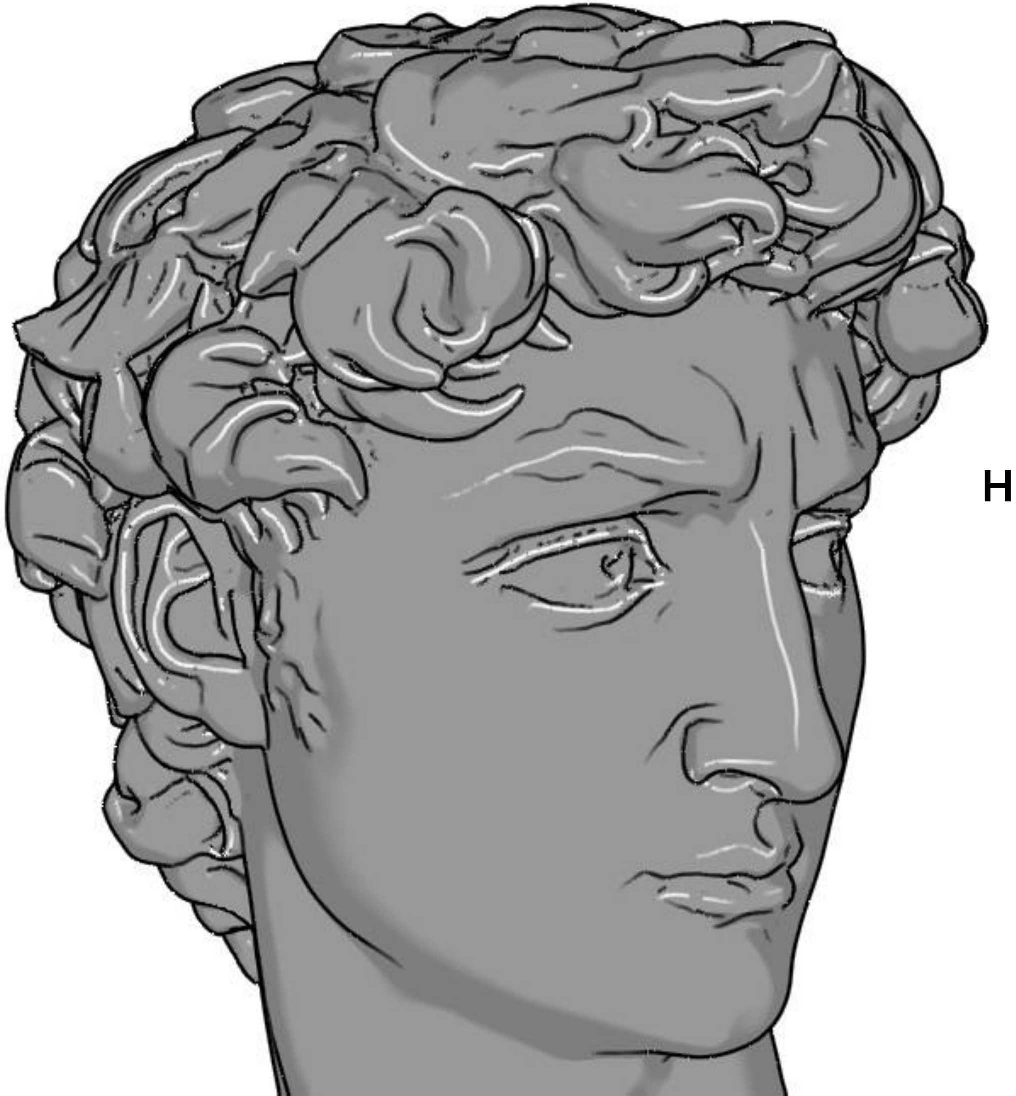
Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow

Desbrun et al.

SIGGRAPH 1999

…and many more

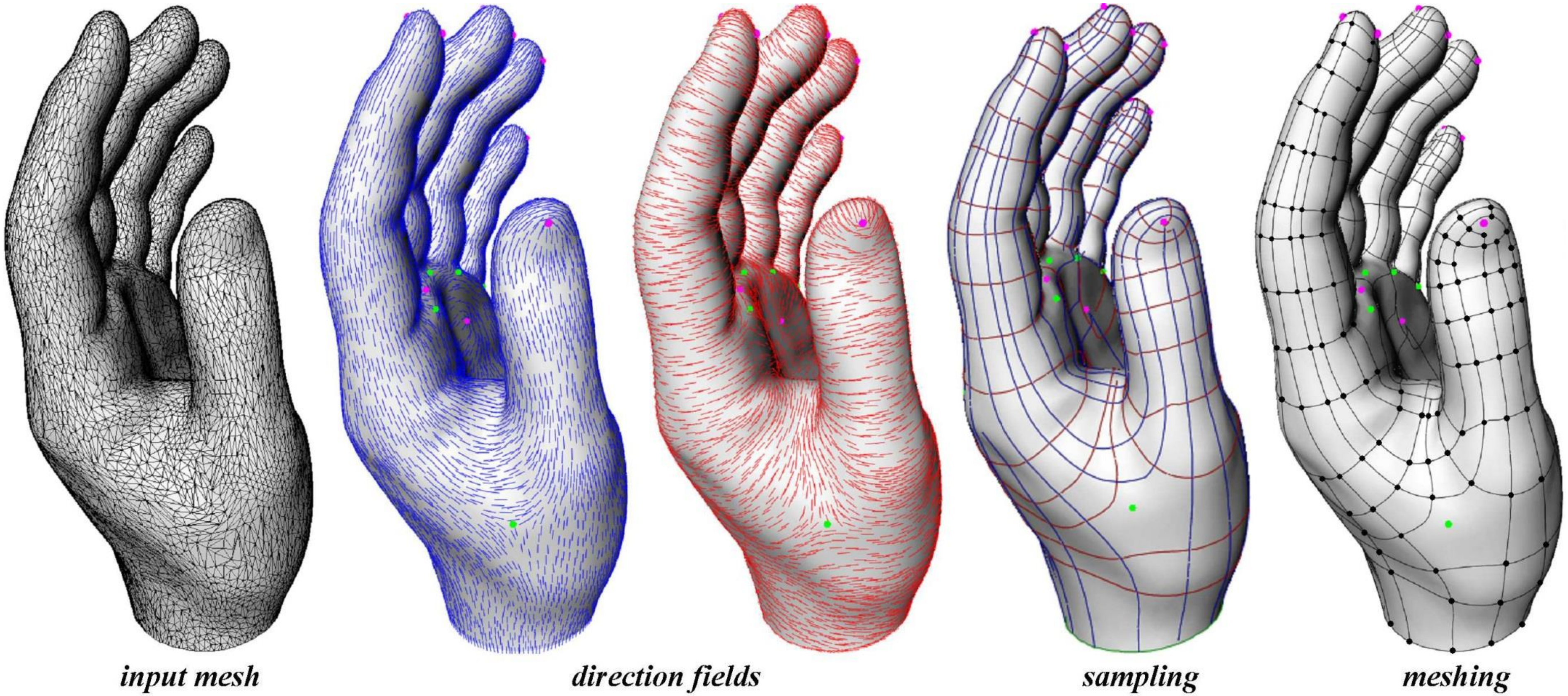
Guiding Rendering



Highlight Lines for Conveying Shape
DeCarlo, Rusinkiewicz
NPAR (2007)

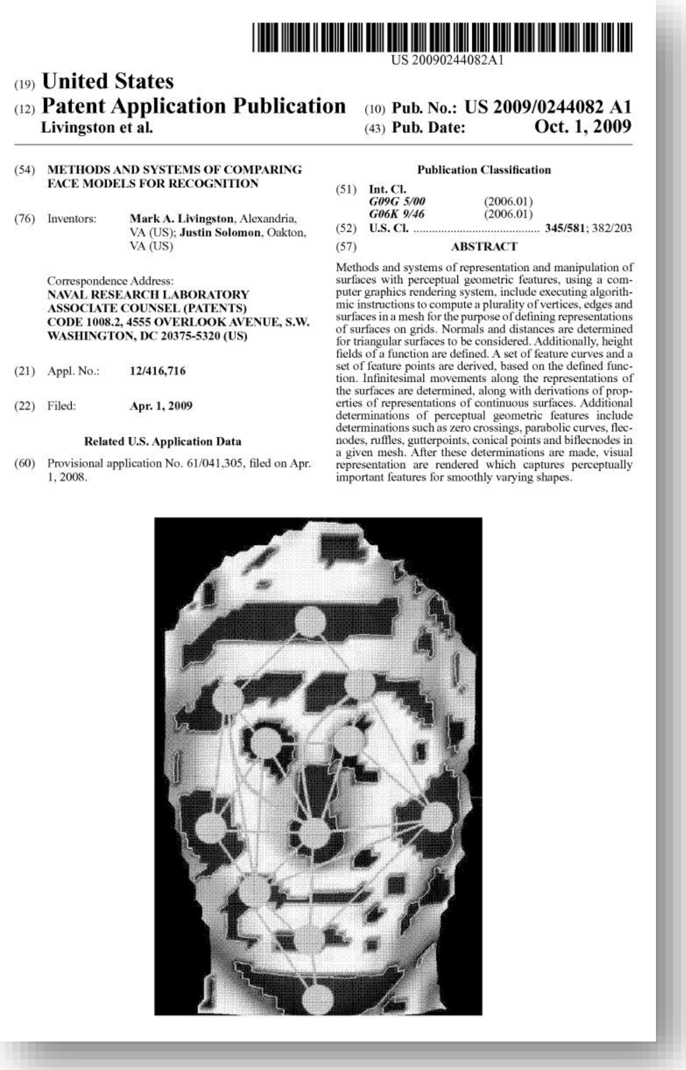
http://www.cs.rutgers.edu/~decarlo/pubs/npar07.pdf

Guiding Meshing



Anisotropic Polygonal Remeshing
Alliez et al.
SIGGRAPH (2003)

Special Topic for Me…



US 20090244082A1

(19) **United States**

(12) **Patent Application Publication**

Livingston et al.

(10) **Pub. No.: US 2009/0244082 A1**

(43) **Pub. Date: Oct. 1, 2009**

(54) **METHODS AND SYSTEMS OF COMPARING FACE MODELS FOR RECOGNITION**

(76) Inventors: **Mark A. Livingston**, Alexandria, VA (US); **Justin Solomon**, Oakton, VA (US)

Correspondence Address:
NAVAL RESEARCH LABORATORY
ASSOCIATE COUNSEL (PATENTS)
CODE 1008.2, 4555 OVERLOOK AVENUE, S.W.
WASHINGTON, DC 20375-5320 (US)

(21) Appl. No.: **12/416,716**

(22) Filed: **Apr. 1, 2009**

Related U.S. Application Data

(60) Provisional application No. 61/041,305, filed on Apr. 1, 2008.

Publication Classification

(51) **Int. Cl.**
G09G 5/00 (2006.01)
G06K 9/46 (2006.01)

(52) **U.S. Cl.** **345/581**; 382/203

(57) **ABSTRACT**

Methods and systems of representation and manipulation of surfaces with perceptual geometric features, using a computer graphics rendering system, include executing algorithmic instructions to compute a plurality of vertices, edges and surfaces in a mesh for the purpose of defining representations of surfaces on grids. Normals and distances are determined for triangular surfaces to be considered. Additionally, height fields of a function are defined. A set of feature curves and a set of feature points are derived, based on the defined function. Infinitesimal movements along the representations of the surfaces are determined, along with derivations of properties of representations of continuous surfaces. Additional determinations of perceptual geometric features include determinations such as zero crossings, parabolic curves, flecnodes, ruffles, gutterpoints, conical points and biflecnodes in a given mesh. After these determinations are made, visual representation are rendered which captures perceptually important features for smoothly varying shapes.

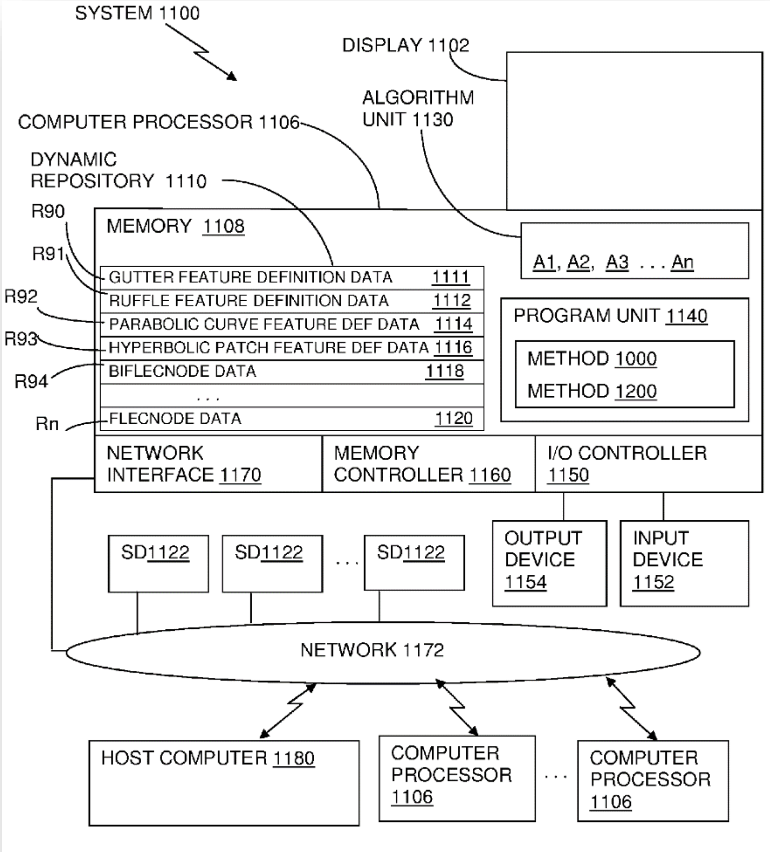
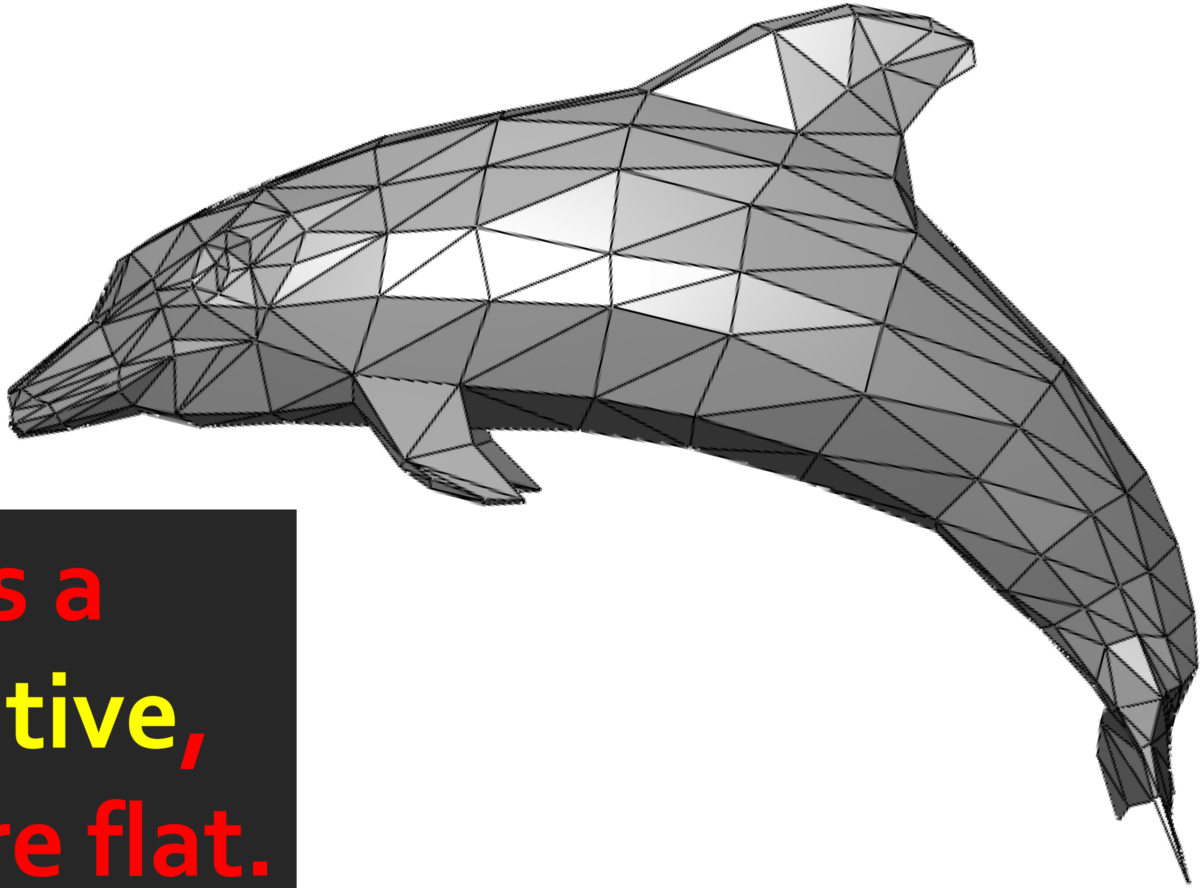


FIG. 11

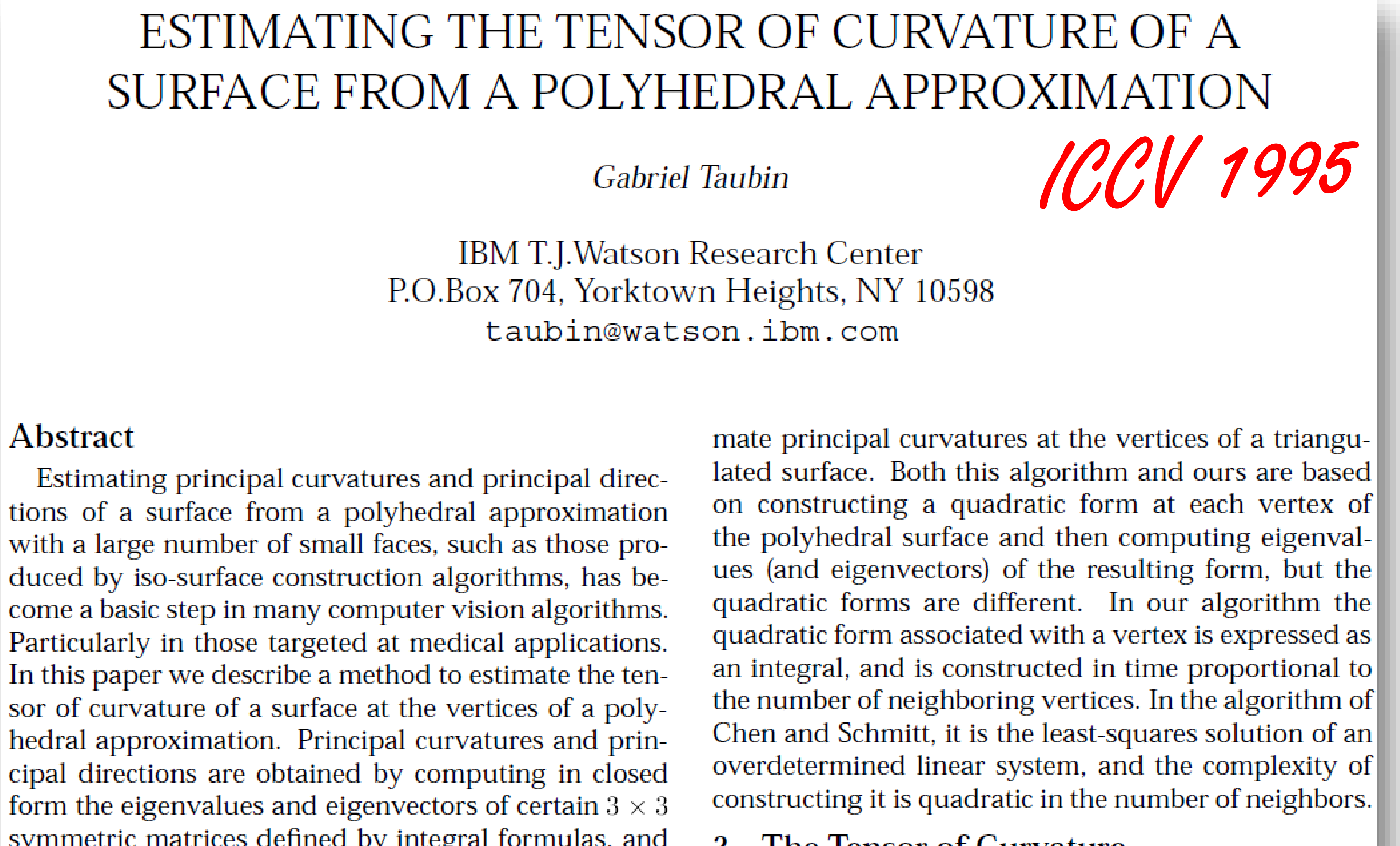
Challenge on Meshes



http://upload.wikimedia.org/wikipedia/commons/f/fb/Dolphin_triangle_mesh.png

Curvature is a second derivative, but triangles are flat.

Standard Citation



ESTIMATING THE TENSOR OF CURVATURE OF A SURFACE FROM A POLYHEDRAL APPROXIMATION

ICCV 1995

Gabriel Taubin

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`taubin@watson.ibm.com`

Abstract

Estimating principal curvatures and principal directions of a surface from a polyhedral approximation with a large number of small faces, such as those produced by iso-surface construction algorithms, has become a basic step in many computer vision algorithms. Particularly in those targeted at medical applications. In this paper we describe a method to estimate the tensor of curvature of a surface at the vertices of a polyhedral approximation. Principal curvatures and principal directions are obtained by computing in closed form the eigenvalues and eigenvectors of certain 3×3 symmetric matrices defined by integral formulas, and

mate principal curvatures at the vertices of a triangulated surface. Both this algorithm and ours are based on constructing a quadratic form at each vertex of the polyhedral surface and then computing eigenvalues (and eigenvectors) of the resulting form, but the quadratic forms are different. In our algorithm the quadratic form associated with a vertex is expressed as an integral, and is constructed in time proportional to the number of neighboring vertices. In the algorithm of Chen and Schmitt, it is the least-squares solution of an overdetermined linear system, and the complexity of constructing it is quadratic in the number of neighbors.

2 The Tensor of Curvature

Taubin Matrix

$$M_{\mathbf{p}} := \frac{1}{2\pi} \int_{-\pi}^{\pi} \kappa_\theta \mathbf{t}_\theta \mathbf{t}_\theta^\top \, d\theta$$

$$\kappa_\theta := \kappa_{\min} \cos^2\theta + \kappa_{\max} \sin^2\theta$$

$$\mathbf{t}_\theta := \mathbf{t}_{\min} \cos\theta + \mathbf{t}_{\max} \sin\theta$$

Taubin Matrix

$$M_{\mathbf{p}} := \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_\theta \mathbf{t}_\theta \mathbf{t}_\theta^\top \, d\theta$$

- Eigenvectors are $\boldsymbol{n}$, $\boldsymbol{t}_1$, and $\boldsymbol{t}_2$
- Eigenvalues are $\frac{3}{8}\kappa_{min} + \frac{1}{8}\kappa_{max}$ and $\frac{1}{8}\kappa_{min} + \frac{3}{8}\kappa_{max}$

Prove at home!

Taubin's Approximation

$$M_{\mathbf{p}} := \frac{1}{2\pi} \int_{-\pi}^{\pi} \kappa_\theta \mathbf{t}_\theta \mathbf{t}_\theta^\top \, d\theta$$

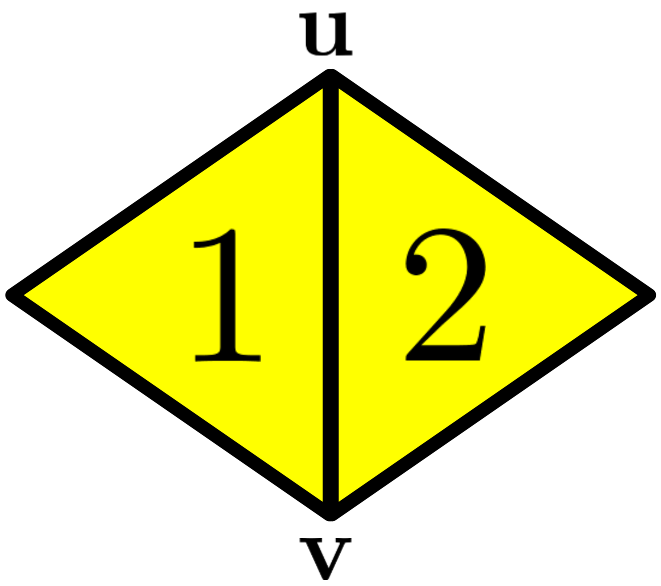
$$\Big\downarrow$$

$$M_{\mathbf{v}} \approx \sum_{\mathbf{u} \sim \mathbf{v}} w_{\mathbf{vu}} \kappa_{\mathbf{vu}} \mathbf{t}_{\mathbf{vu}} \mathbf{t}_{\mathbf{vu}}^\top$$

Taubin's Approximation

$$\mathbf{t}_{\mathbf{vu}} := \frac{(I_{3\times 3} - \mathbf{n}_{\mathbf{v}}\mathbf{n}_{\mathbf{v}}^{\top}(\mathbf{u}-\mathbf{v})}{\|(I_{3\times 3} - \mathbf{n}_{\mathbf{v}}\mathbf{n}_{\mathbf{v}}^{\top}(\mathbf{u}-\mathbf{v})\|_2}$$

$$\kappa_{\mathbf{vu}} := \frac{2\mathbf{n}_{\mathbf{v}}^{\top}(\mathbf{u}-\mathbf{v})}{\|\mathbf{u}-\mathbf{v}\|_2^2}$$

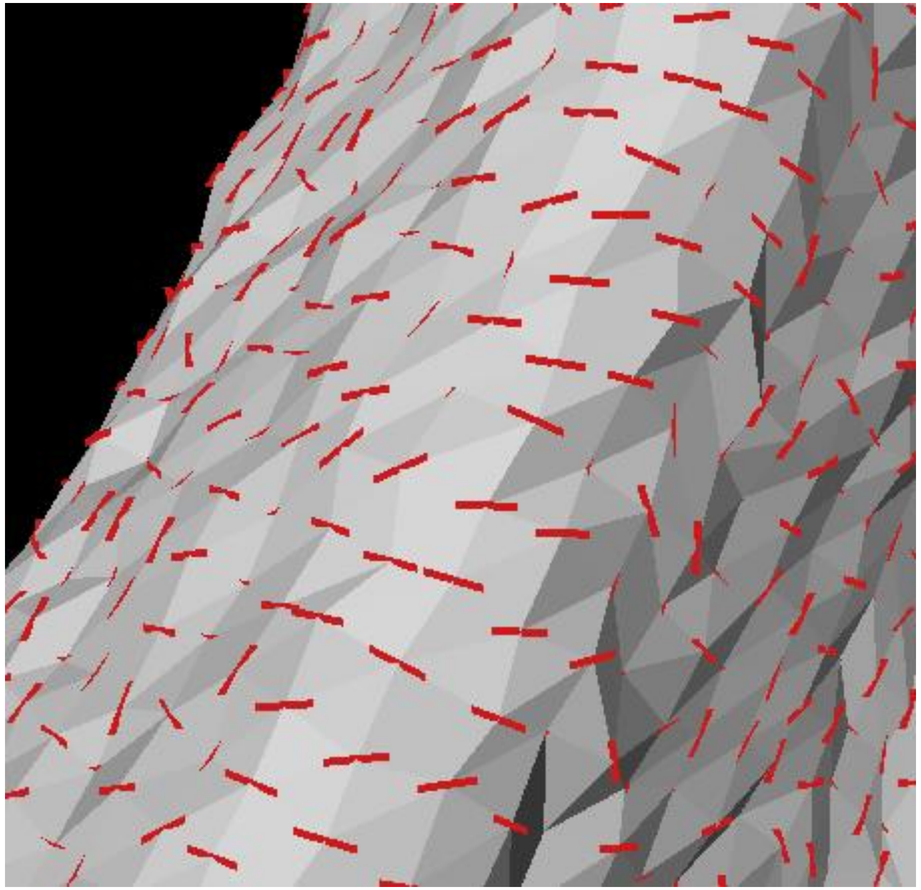


Divided difference approximation

$$M_{\mathbf{v}} \approx \sum_{\mathbf{u}\sim\mathbf{v}} w_{\mathbf{vu}}\kappa_{\mathbf{vu}}\mathbf{t}_{\mathbf{vu}}\mathbf{t}_{\mathbf{vu}}^{\top}$$

Area-based weighting

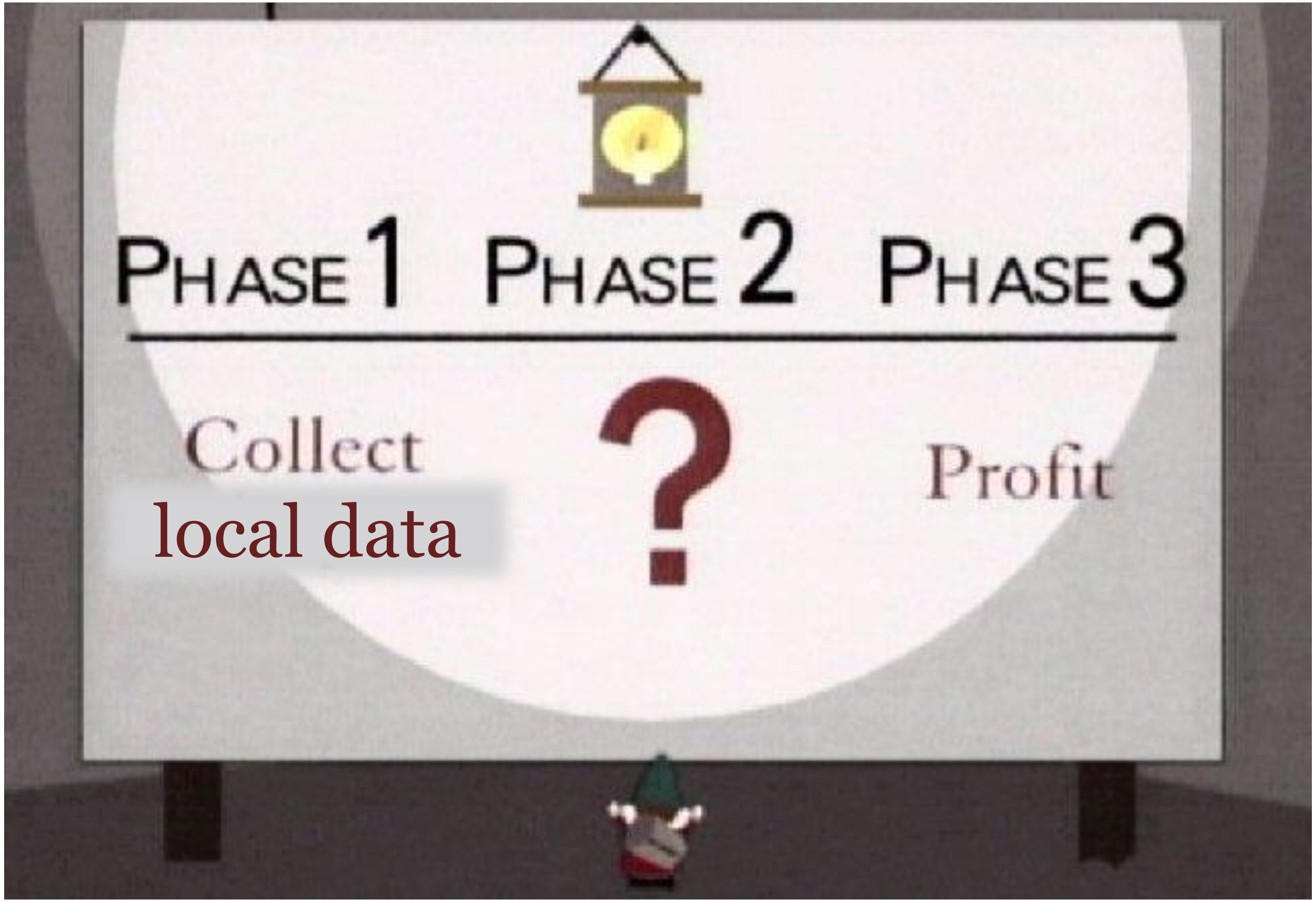
Problem

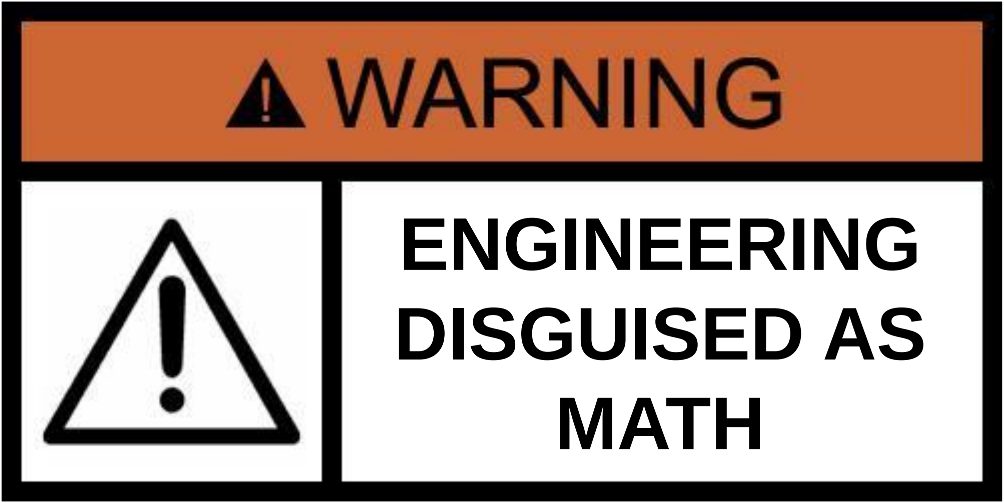


http://iristown.engr.utk.edu/~koschan/paper/CVPR01.pdf

Local estimates are noisy

General Strategy





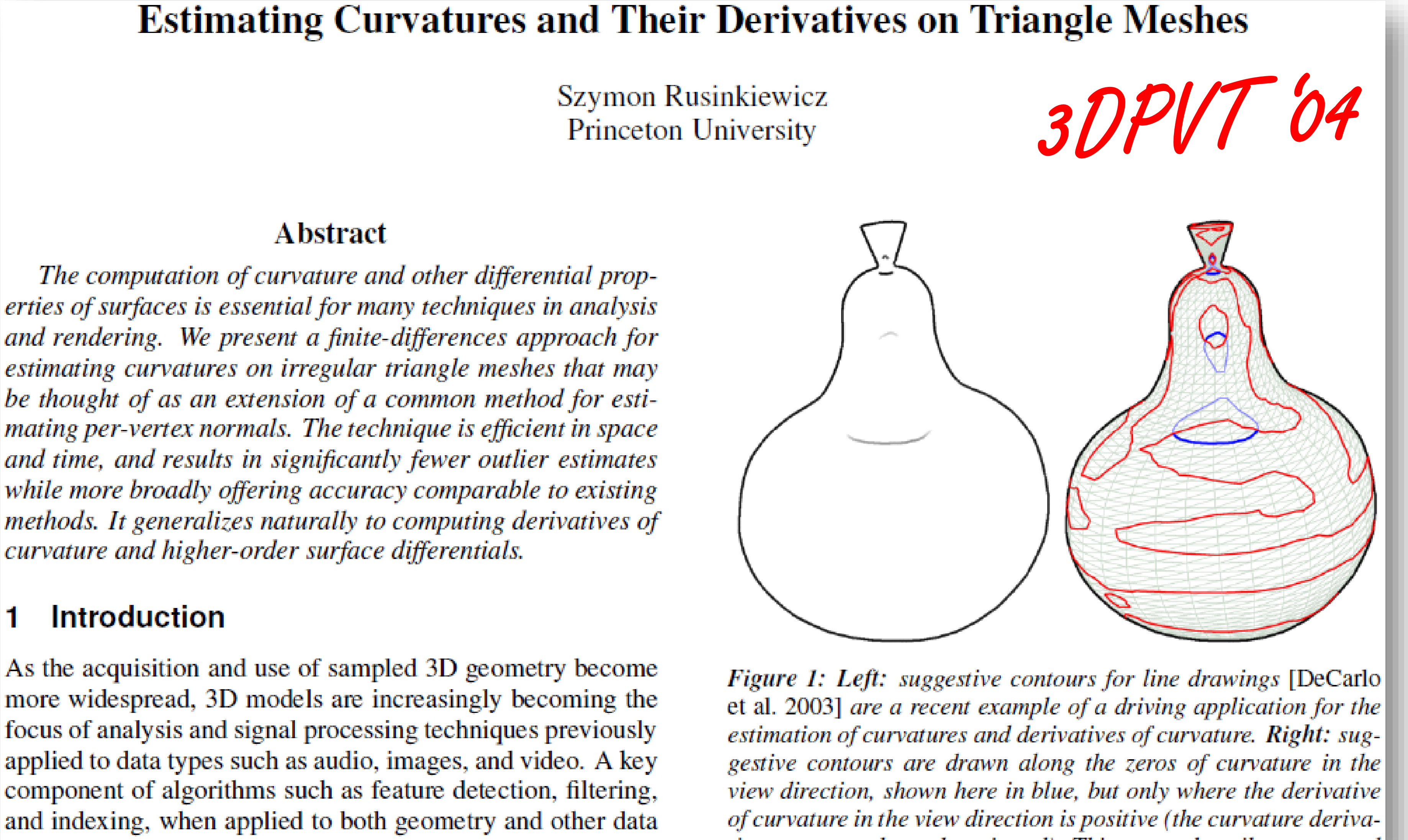
ENGINEERING DISGUISED AS MATH

Main Take-Away

Use application to motivate choice of curvature.

Simulation, smoothing, analysis, meshing, nonphotorealistic rendering, …

Another Example



Estimating Curvatures and Their Derivatives on Triangle Meshes

Szymon Rusinkiewicz
Princeton University

3DPVT '04

Abstract

The computation of curvature and other differential properties of surfaces is essential for many techniques in analysis and rendering. We present a finite-differences approach for estimating curvatures on irregular triangle meshes that may be thought of as an extension of a common method for estimating per-vertex normals. The technique is efficient in space and time, and results in significantly fewer outlier estimates while more broadly offering accuracy comparable to existing methods. It generalizes naturally to computing derivatives of curvature and higher-order surface differentials.

1 Introduction

As the acquisition and use of sampled 3D geometry become more widespread, 3D models are increasingly becoming the focus of analysis and signal processing techniques previously applied to data types such as audio, images, and video. A key component of algorithms such as feature detection, filtering, and indexing, when applied to both geometry and other data

Figure 1: Left: *suggestive contours for line drawings* [DeCarlo et al. 2003] *are a recent example of a driving application for the estimation of curvatures and derivatives of curvature.* ***Right:*** *suggestive contours are drawn along the zeros of curvature in the view direction, shown here in blue, but only where the derivative of curvature in the view direction is positive (the curvature deriva-*

Second Fundamental Form Matrix

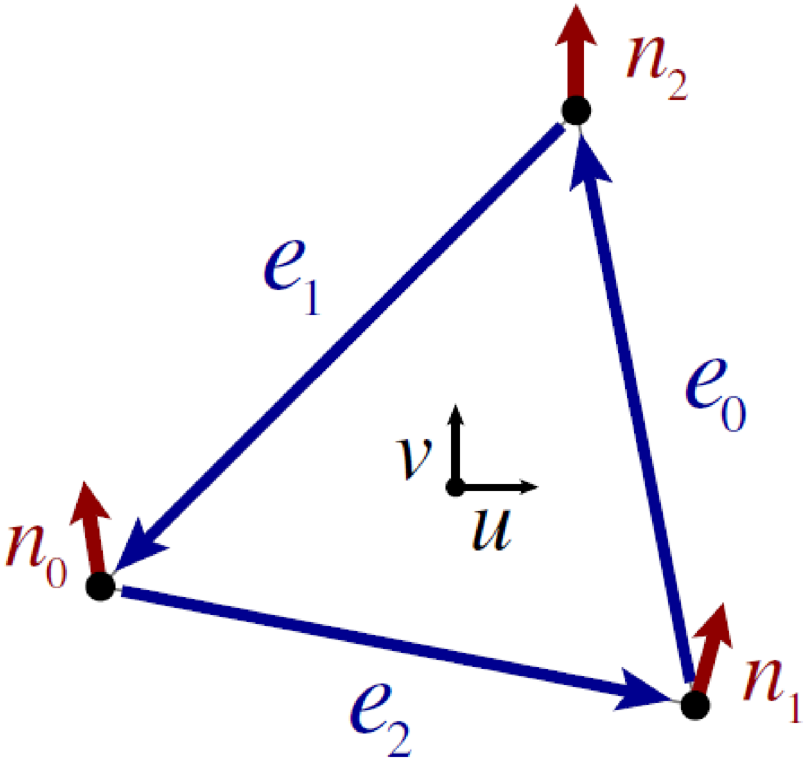
$$\mathbb{I}_{\mathbf{p}} = \begin{pmatrix} d\mathbf{n}_{\mathbf{p}}(\mathbf{u}) \cdot \mathbf{u} & d\mathbf{n}_{\mathbf{p}}(\mathbf{v}) \cdot \mathbf{u} \\ d\mathbf{n}_{\mathbf{p}}(\mathbf{u}) \cdot \mathbf{v} & d\mathbf{n}_{\mathbf{p}}(\mathbf{v}) \cdot \mathbf{v} \end{pmatrix}$$

$$\mathbf{w} = c^1\mathbf{u} + c^2\mathbf{v}$$

$$\Longrightarrow \mathbb{I}_{\mathbf{p}} \cdot \begin{pmatrix} c^1 \\ c^2 \end{pmatrix} = d\mathbf{n}_{\mathbf{p}}(\mathbf{w})$$

Assume *u*, *v* are orthogonal

Finite Difference Per-Face



$$\mathrm{II}\begin{pmatrix} e_0 \cdot u \\ e_0 \cdot v \end{pmatrix} = \begin{pmatrix} (n_2 - n_1) \cdot u \\ (n_2 - n_1) \cdot v \end{pmatrix}$$

$$\mathrm{II}\begin{pmatrix} e_1 \cdot u \\ e_1 \cdot v \end{pmatrix} = \begin{pmatrix} (n_0 - n_2) \cdot u \\ (n_0 - n_2) \cdot v \end{pmatrix}$$

$$\mathrm{II}\begin{pmatrix} e_2 \cdot u \\ e_2 \cdot v \end{pmatrix} = \begin{pmatrix} (n_1 - n_0) \cdot u \\ (n_1 - n_0) \cdot v \end{pmatrix}$$

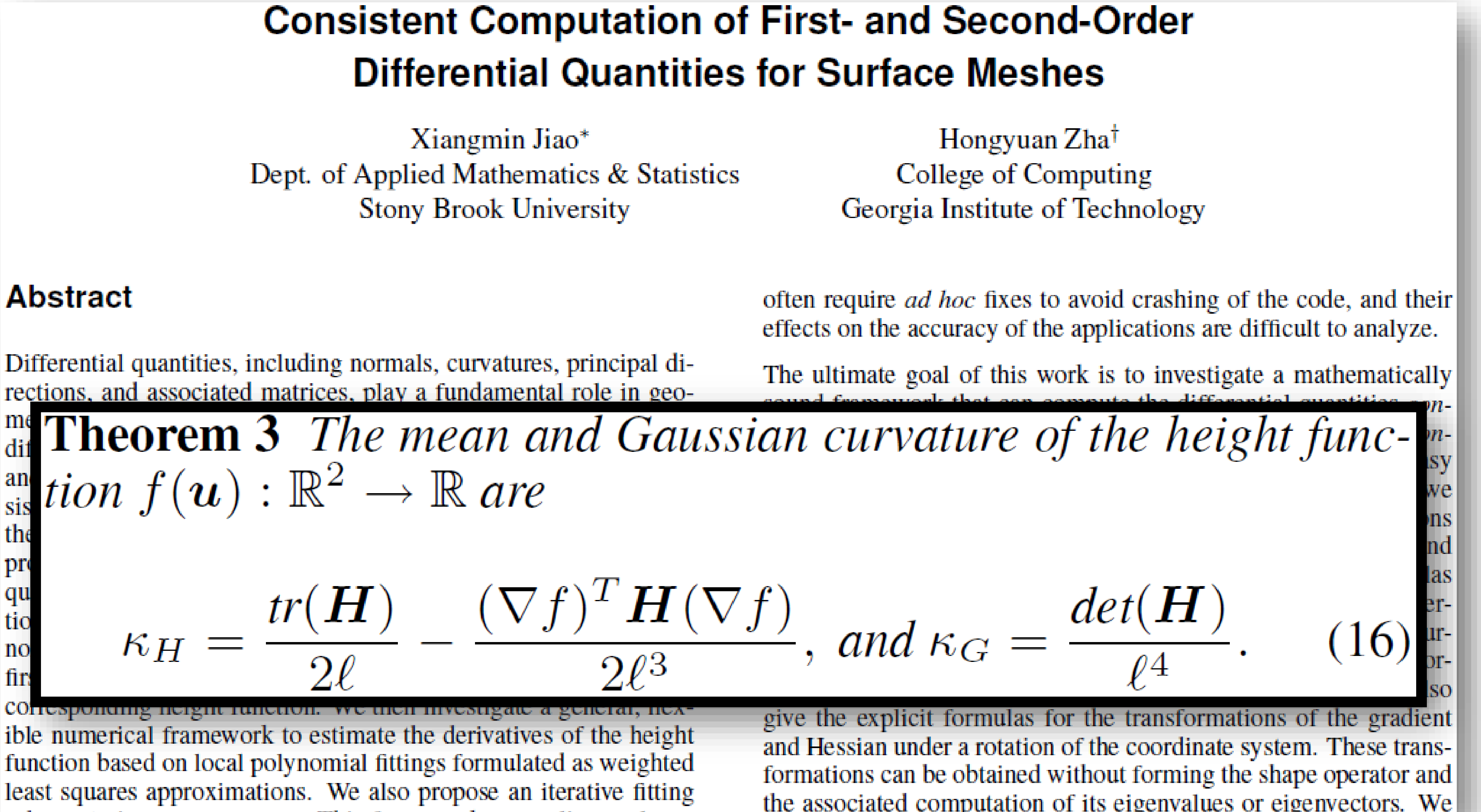
Figure from the paper

Per-triangle II

Average for Per-Vertex

- **Rotate** tangent plane about cross product of normals
- **Average** using Voronoi weights

Completely Different Formula



Consistent Computation of First- and Second-Order Differential Quantities for Surface Meshes

Xiangmin Jiao*
Dept. of Applied Mathematics & Statistics
Stony Brook University

Hongyuan Zha†
College of Computing
Georgia Institute of Technology

Abstract

Differential quantities, including normals, curvatures, principal directions, and associated matrices, play a fundamental role in geo-

corresponding height function. We then investigate a general, flexible numerical framework to estimate the derivatives of the height function based on local polynomial fittings formulated as weighted least squares approximations. We also propose an iterative fitting

often require *ad hoc* fixes to avoid crashing of the code, and their effects on the accuracy of the applications are difficult to analyze.

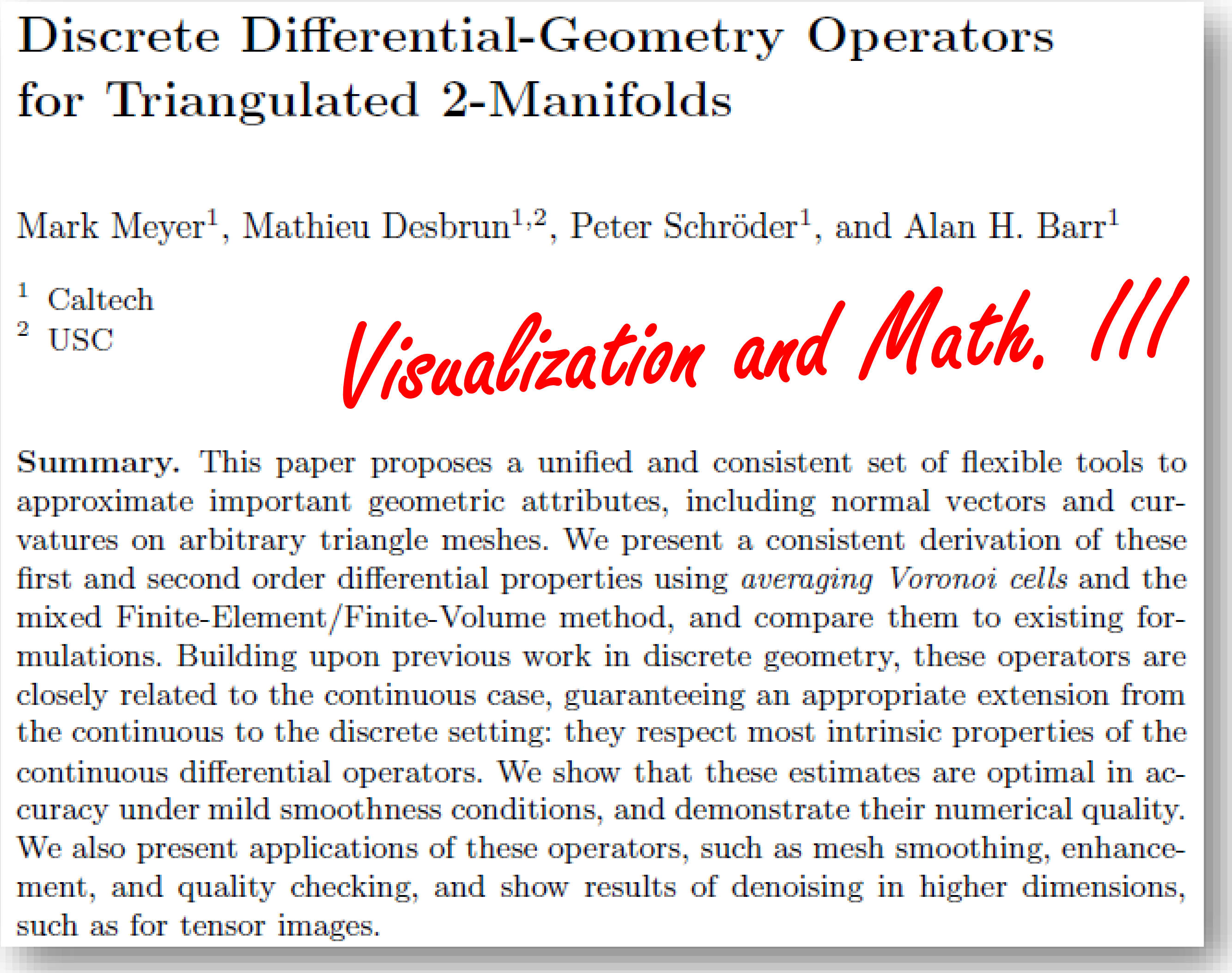
The ultimate goal of this work is to investigate a mathematically sound framework that can compute the differential quantities con-

give the explicit formulas for the transformations of the gradient and Hessian under a rotation of the coordinate system. These transformations can be obtained without forming the shape operator and the associated computation of its eigenvalues or eigenvectors. We

Theorem 3 *The mean and Gaussian curvature of the height function* $f(\boldsymbol{u}) : \mathbb{R}^2 \rightarrow \mathbb{R}$ *are*

$$\kappa_H = \frac{tr(\boldsymbol{H})}{2\ell} - \frac{(\nabla f)^T \boldsymbol{H} (\nabla f)}{2\ell^3}, \text{ and } \kappa_G = \frac{det(\boldsymbol{H})}{\ell^4}. \qquad (16)$$

Conserved Quantity Approach



Discrete Differential-Geometry Operators for Triangulated 2-Manifolds

Mark Meyer[1], Mathieu Desbrun[1,2], Peter Schröder[1], and Alan H. Barr[1]

[1] Caltech
[2] USC

Visualization and Math. ///

Summary. This paper proposes a unified and consistent set of flexible tools to approximate important geometric attributes, including normal vectors and curvatures on arbitrary triangle meshes. We present a consistent derivation of these first and second order differential properties using *averaging Voronoi cells* and the mixed Finite-Element/Finite-Volume method, and compare them to existing formulations. Building upon previous work in discrete geometry, these operators are closely related to the continuous case, guaranteeing an appropriate extension from the continuous to the discrete setting: they respect most intrinsic properties of the continuous differential operators. We show that these estimates are optimal in accuracy under mild smoothness conditions, and demonstrate their numerical quality. We also present applications of these operators, such as mesh smoothing, enhancement, and quality checking, and show results of denoising in higher dimensions, such as for tensor images.

Recall:

Structure preservation

[**struhk**-cher pre-zur-**vey**-sh*uh*n]**:**

Keeping properties from the continuous abstraction exactly true in a discretization.

Gauss-Bonnet Theorem

$$\int_M K\,dA + \int_{\partial M} k_g\,ds = 2\pi\chi(\mathcal{M})$$

Gaussian curvature

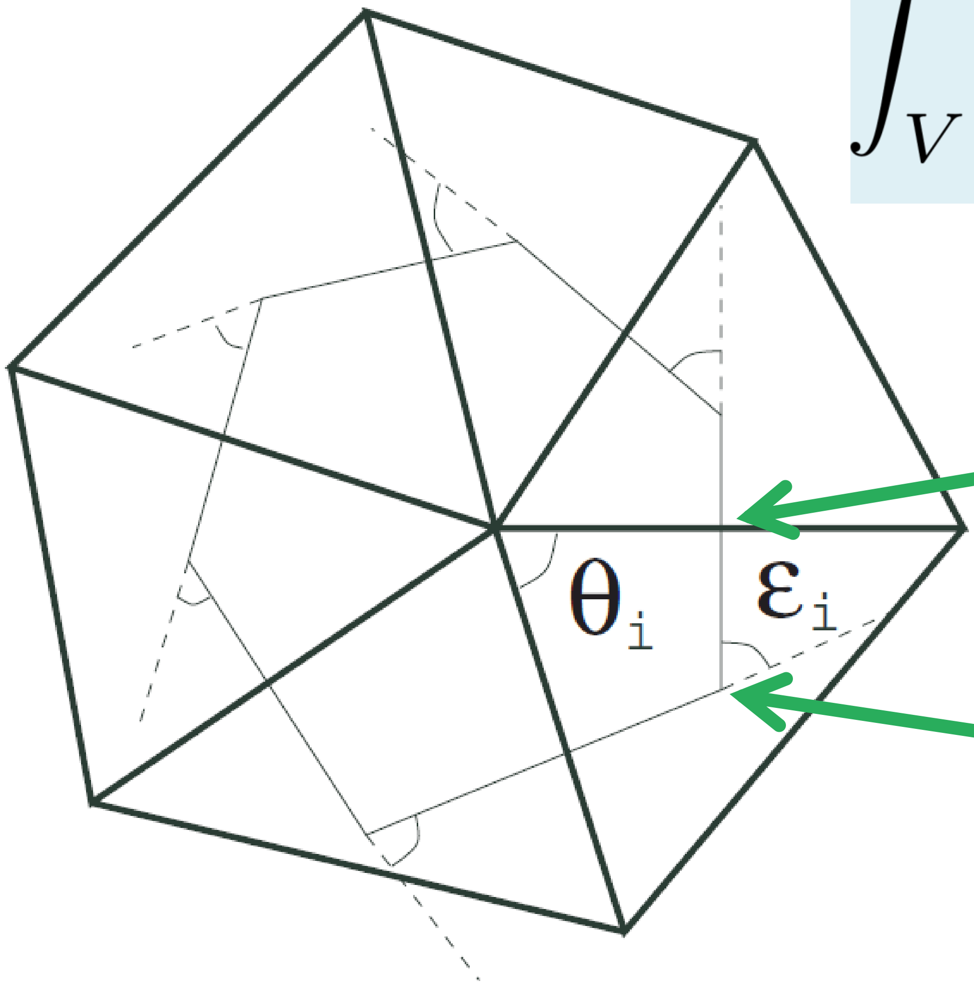
Geodesic curvature (curvature projected on tangent plane)

2-2g-n

`\omit{proof}`

For Polygonal Voronoi Cells

$$\int_V K\, dA = 2\pi - \sum_j \varepsilon_j$$

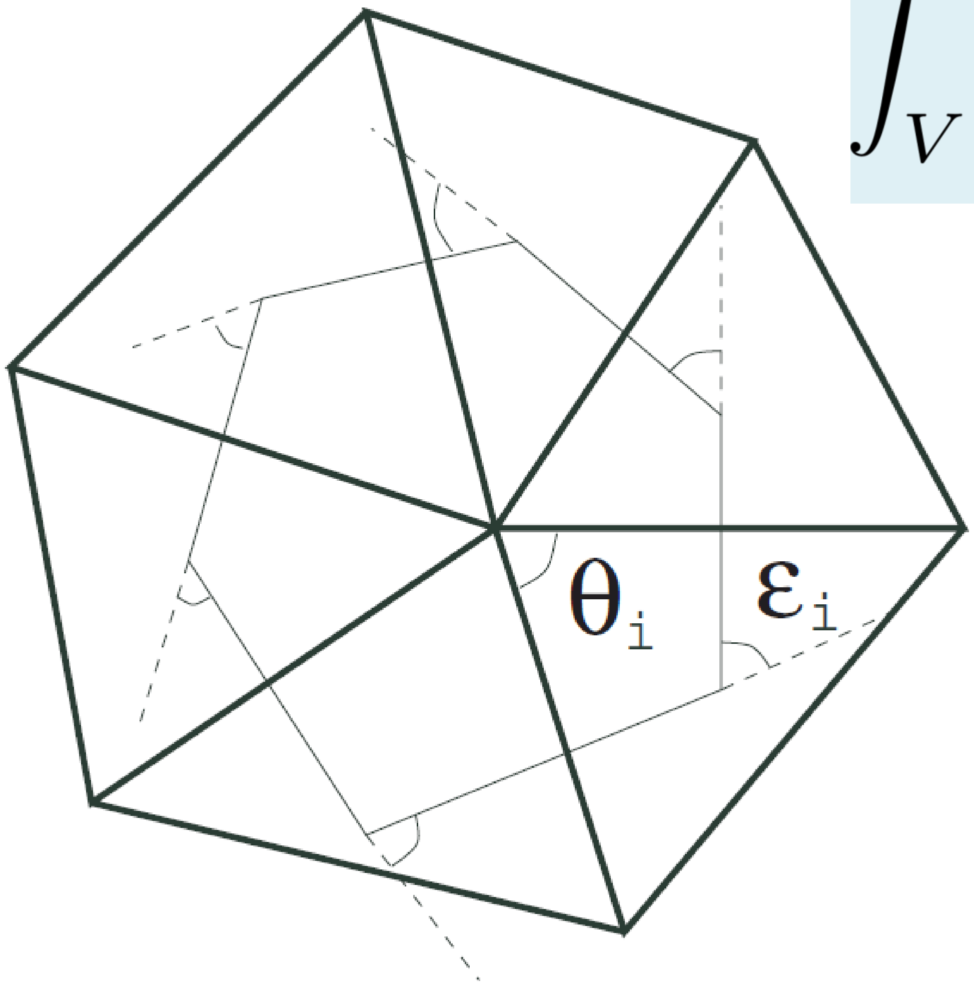


Change is in normal direction

Turning angle integrated curvature

Figure from the paper

Simplification



$$\int_V K\,dA = 2\pi - \sum_j \theta_j$$

Figure from the paper

Flip Things Backward

DEFINITION:

Gaussian curvature integrated over Voronoi region *V* is given by

$$\int_V K\, dA = 2\pi - \sum_j \theta_j$$

Divide by area for curvature estimate

Recall:

Euler Characteristic

$$V - E + F := \chi$$

$$\chi = 2 - 2g$$


$g = 0$


$g = 1$

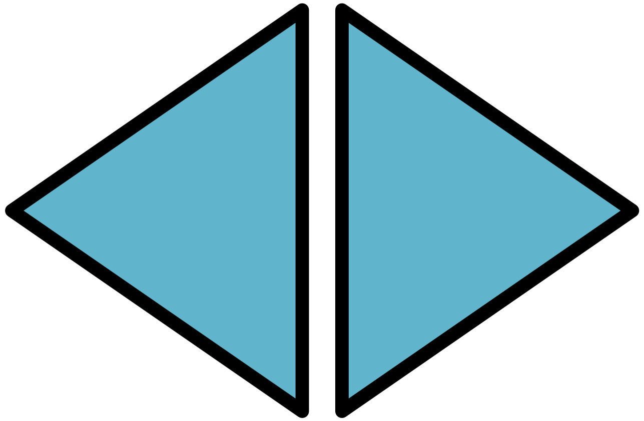

$g = 2$

Recall:

Consequences for Triangle Meshes

$$V - E + F := \chi$$

"Each edge is adjacent to two faces. Each face has three edges."



$$2E = 3F$$

Closed mesh: Easy estimates!

Discrete Gauss-Bonnet

$$\int_M K\,dA = \sum_i \int_{V_i} K\,dA$$

Partition the surface

Discrete Gauss-Bonnet

$$\int_M K\,dA = \sum_i \int_{V_i} K\,dA$$

$$= \sum_i \left(2\pi - \sum_j \theta_{ij} \right)$$

Apply our definition

Discrete Gauss-Bonnet

$$\begin{aligned}\int_M K\,dA &= \sum_i \int_{V_i} K\,dA \\ &= \sum_i \left(2\pi - \sum_j \theta_{ij} \right) \\ &= 2\pi V - \sum_{ij} \theta_{ij}\end{aligned}$$

Pull out constants

Discrete Gauss-Bonnet

$$\begin{aligned}\int_M K\,dA &= \sum_i \int_{V_i} K\,dA \\ &= \sum_i \left(2\pi - \sum_j \theta_{ij}\right) \\ &= 2\pi V - \sum_{ij} \theta_{ij} \\ &= 2\pi V - \pi F\end{aligned}$$

Consider sum over triangles

Discrete Gauss-Bonnet

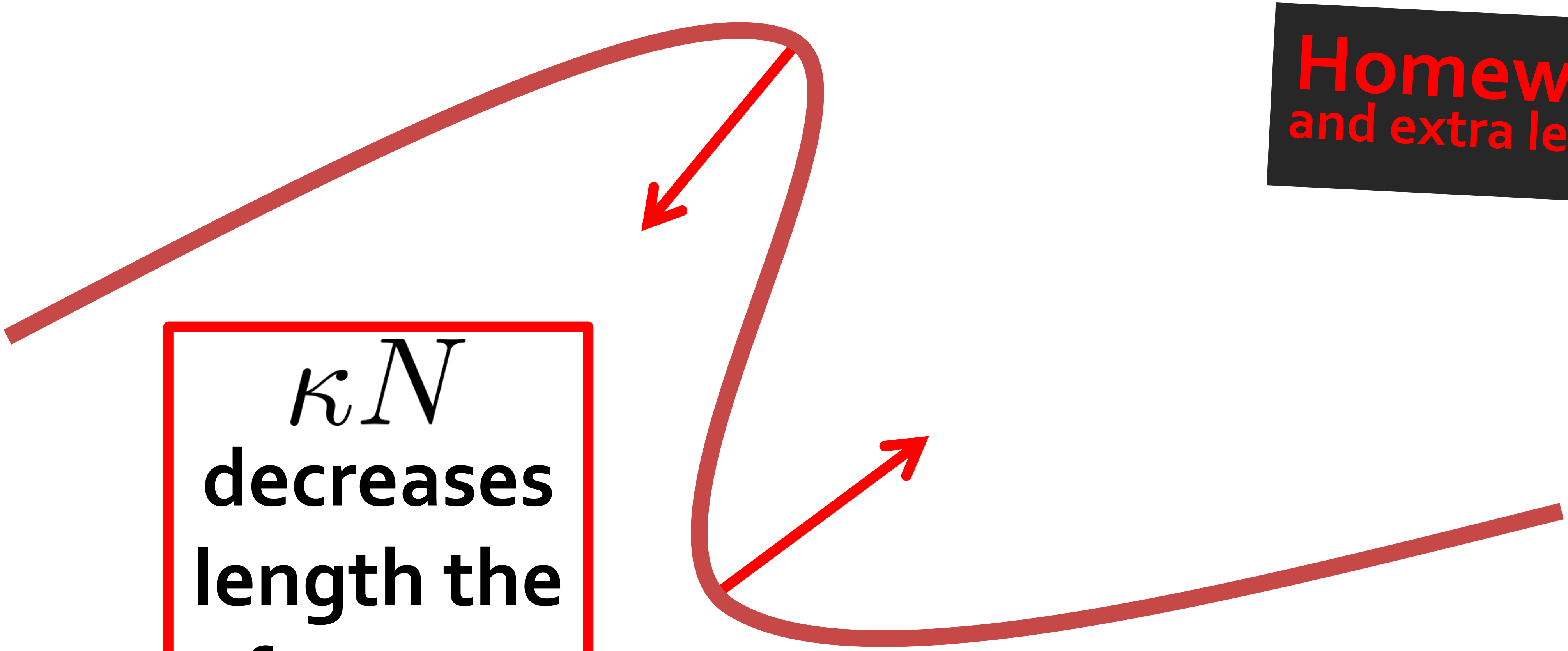
$$\begin{aligned}\int_M K\,dA &= \sum_i \int_{V_i} K\,dA \\ &= \sum_i \left(2\pi - \sum_j \theta_{ij}\right) \\ &= 2\pi V - \sum_{ij} \theta_{ij} \\ &= 2\pi V - \pi F \\ &= \pi(2V - F) \\ &= 2\pi\chi\end{aligned}$$

By definition

<qed/>

Recall: Alternative Definition

Homework
and extra lecture



κN

decreases length the fastest.

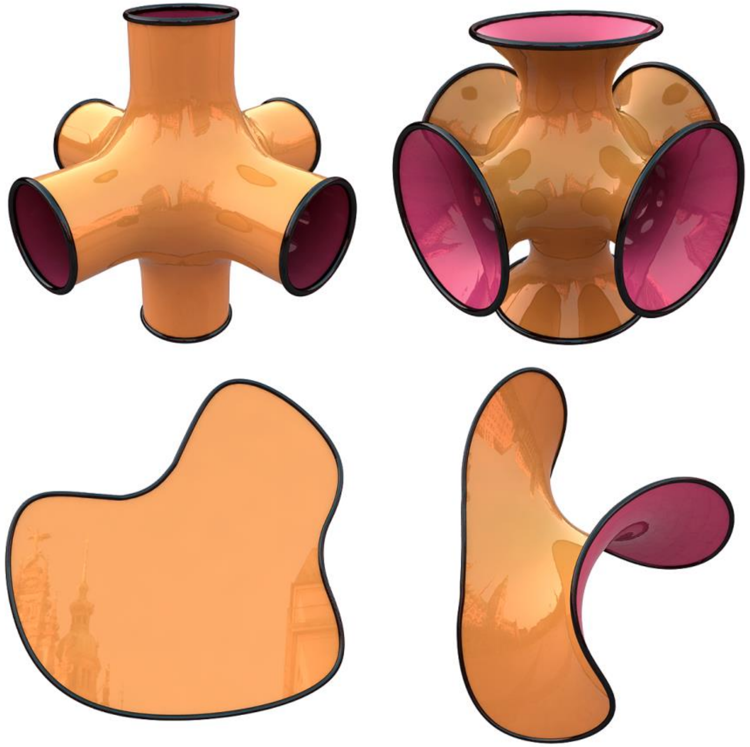
Mean Curvature Normal

Derived in extra lecture video.

$$E(\mathcal{M}) = \text{Area}(\mathcal{M})$$

$$\text{``}\nabla E(\mathbf{p})\text{''} = H\mathbf{n}$$

“Variational derivative”

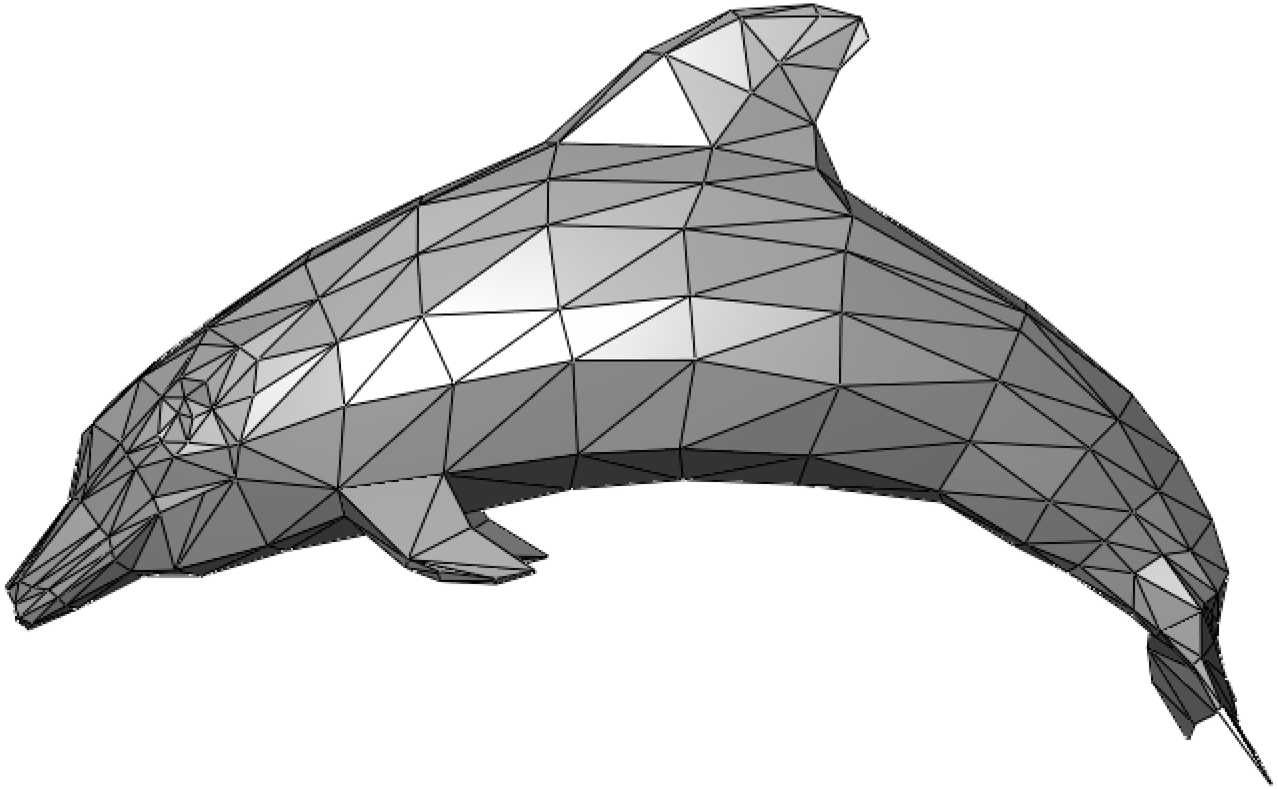


$$\nabla E(\mathbf{p}) \equiv \mathbf{0} \;\forall p \in \text{int}\,\mathcal{M}$$

Minimal surfaces

Image courtesy K. Crane

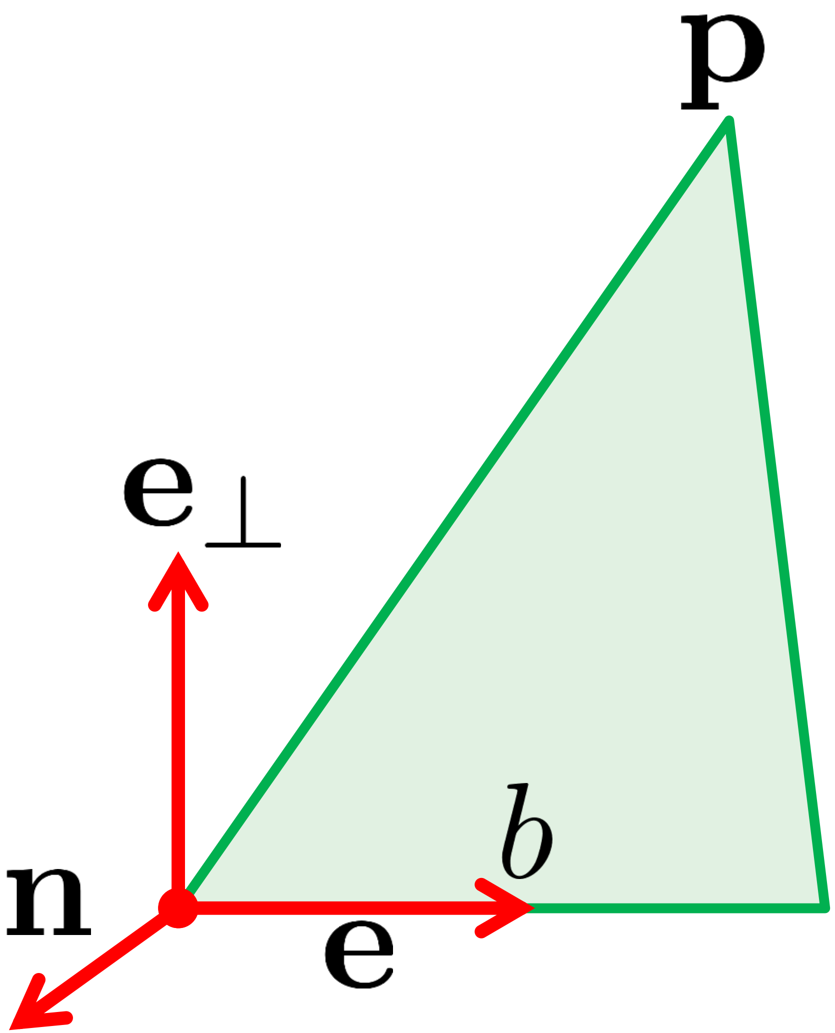
Area Functional for Meshes



$$\text{Area} : \mathbb{R}^{3V} \to \mathbb{R}$$

http://upload.wikimedia.org/wikipedia/commons/f/fb/Dolphin_triangle_mesh.png

Single Triangle



$$\mathbf{p} = p_n\mathbf{n} + p_e\mathbf{e} + p_\perp\mathbf{e}_\perp$$

$$A(\mathbf{p}) = \frac{1}{2}b\sqrt{p_n^2 + p_\perp^2}$$

As a function of *p*

Single Triangle: Derivatives

$$\mathbf{p} = p_n \mathbf{n} + p_e \mathbf{e} + p_\perp \mathbf{e}_\perp$$

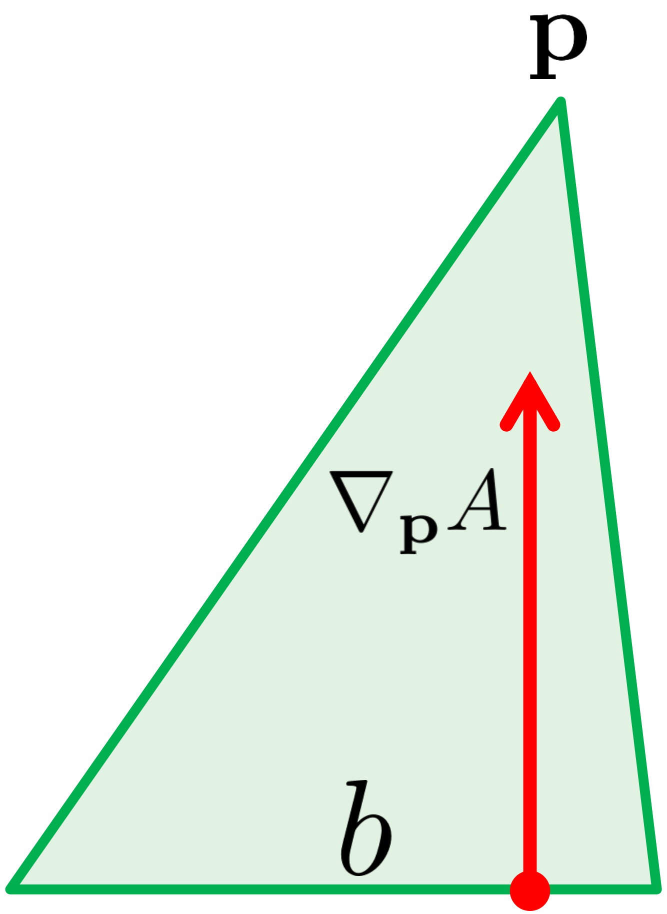
$$A = \frac{1}{2} b \sqrt{p_n^2 + p_\perp^2}$$

$$\frac{\partial A}{\partial p_e} = 0$$

$$\frac{\partial A}{\partial p_n} = \frac{b \cancelto{0}{p_n}}{2\sqrt{p_n^2 + p_\perp^2}} = 0 \implies \nabla_{\mathbf{p}} A = \frac{1}{2} b \mathbf{e}_\perp$$

$$\frac{\partial A}{\partial p_\perp} = \frac{b p_\perp}{2\sqrt{p_n^2 + p_\perp^2}}$$

Single Triangle: Complete

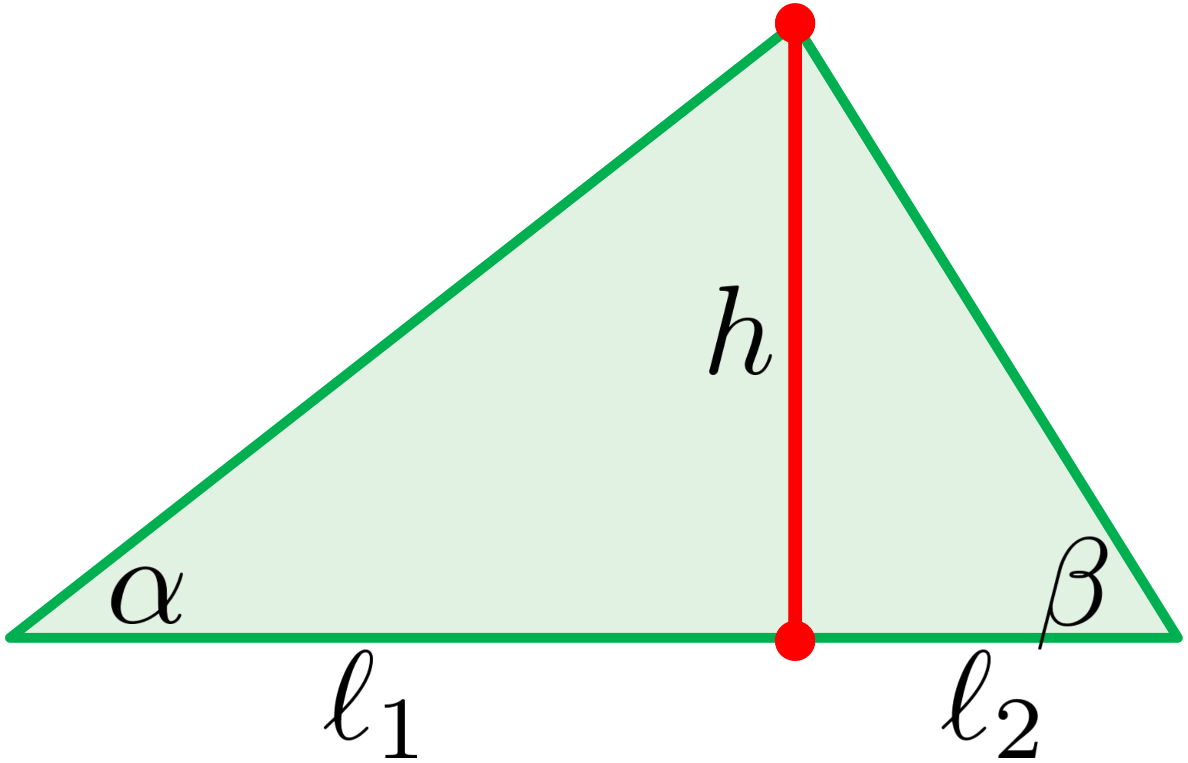


$$\mathbf{p} = p_n \mathbf{n} + p_e \mathbf{e} + p_\perp \mathbf{e}_\perp$$

$$A(\mathbf{p}) = \frac{1}{2} b \sqrt{p_n^2 + p_\perp^2}$$

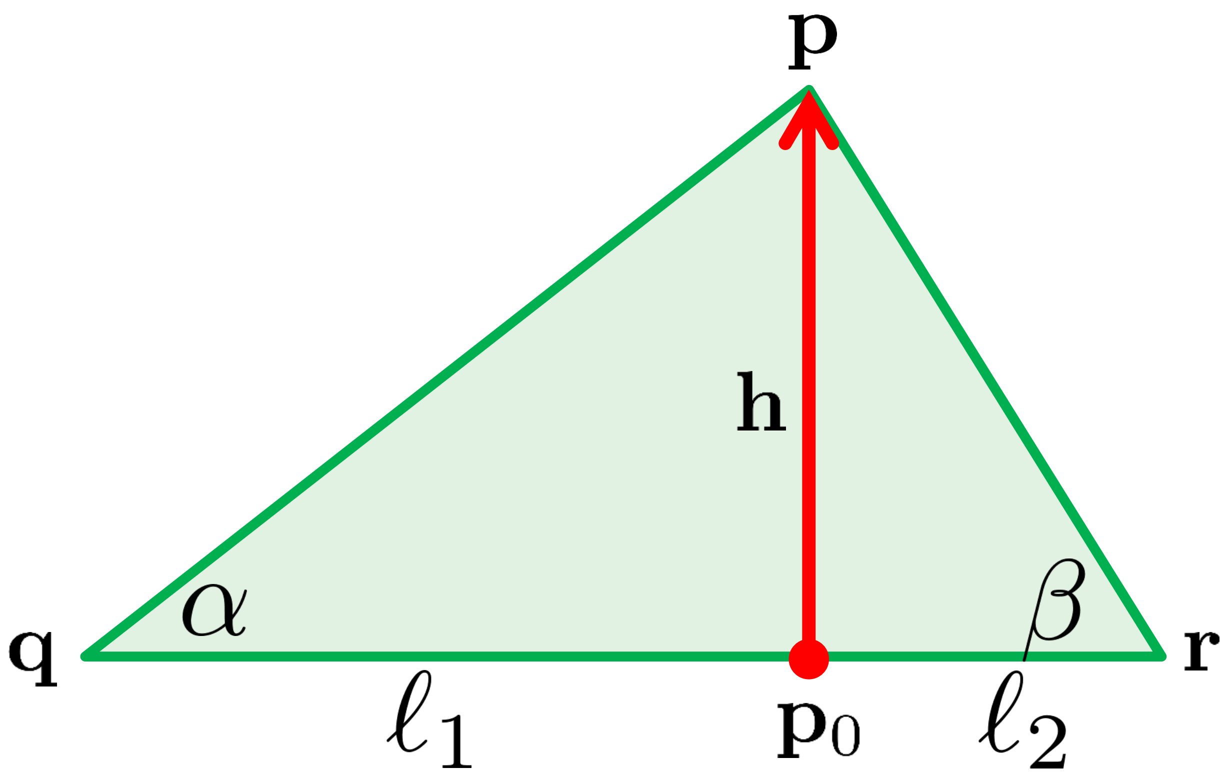
$$\nabla_{\mathbf{p}} A = \frac{1}{2} b \mathbf{e}_\perp$$

Ratio of Base to Height



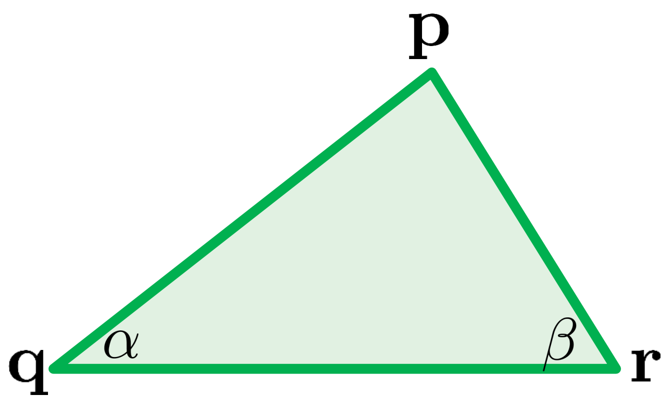
$$\frac{b}{h} = \frac{\ell_1 + \ell_2}{h} = \frac{\ell_1}{h} + \frac{\ell_2}{h} = \cot\alpha + \cot\beta$$

Height Vector



$$\mathbf{h} = \mathbf{p} - \mathbf{p}_0 = \mathbf{p} - \frac{\mathbf{r}\cot\alpha + \mathbf{q}\cot\beta}{\cot\alpha + \cot\beta}$$

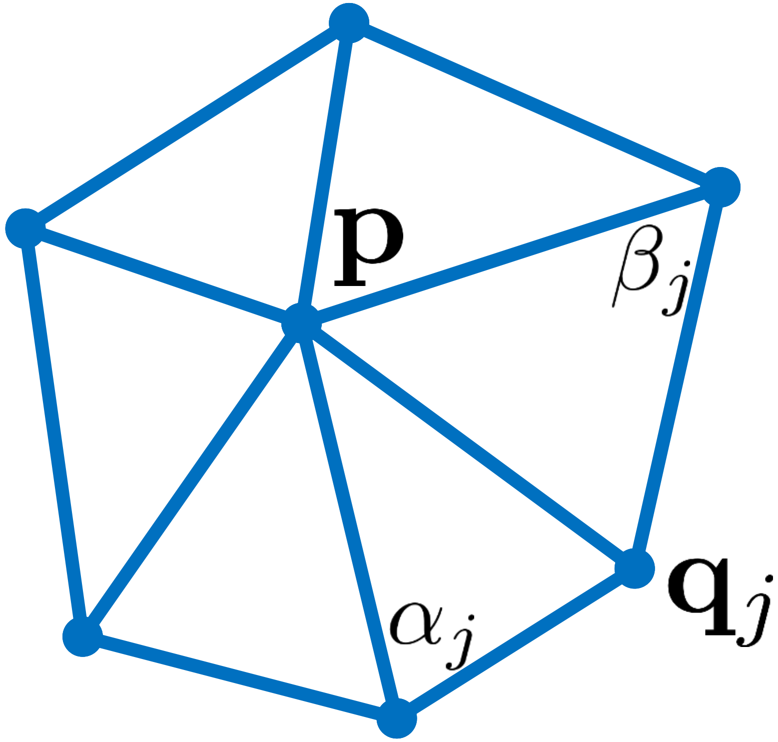
Alternative Gradient Formula



$$
\begin{aligned}
\nabla_{\mathbf{p}} A &= \frac{1}{2} b \mathbf{e}_{\perp} \\
&= \frac{1}{2} \frac{b}{\|\mathbf{h}\|_2} \mathbf{h} \\
&= \frac{1}{2} (\cot \alpha + \cot \beta) \left[\mathbf{p} - \frac{\mathbf{r} \cot \alpha + \mathbf{q} \cot \beta}{\cot \alpha + \cot \beta} \right] \\
&= \frac{1}{2} ((\mathbf{p} - \mathbf{r}) \cot \alpha + (\mathbf{p} - \mathbf{q}) \cot \beta)
\end{aligned}
$$

Summing Around a Vertex

$$\nabla_{\mathbf{p}} A = \frac{1}{2} \sum_j (\cot \alpha_j + \cot \beta_j)(\mathbf{p} - \mathbf{q}_j)$$



$$\nabla_{\mathbf{p}} A = \frac{1}{2}((\mathbf{p} - \mathbf{r}) \cot \alpha + (\mathbf{p} - \mathbf{q}) \cot \beta)$$

Vanishes as you refine the mesh

Integrated Mean Curvature Normal

DEFINITION:

The discrete mean curvature normal integrated over region *V* is given by

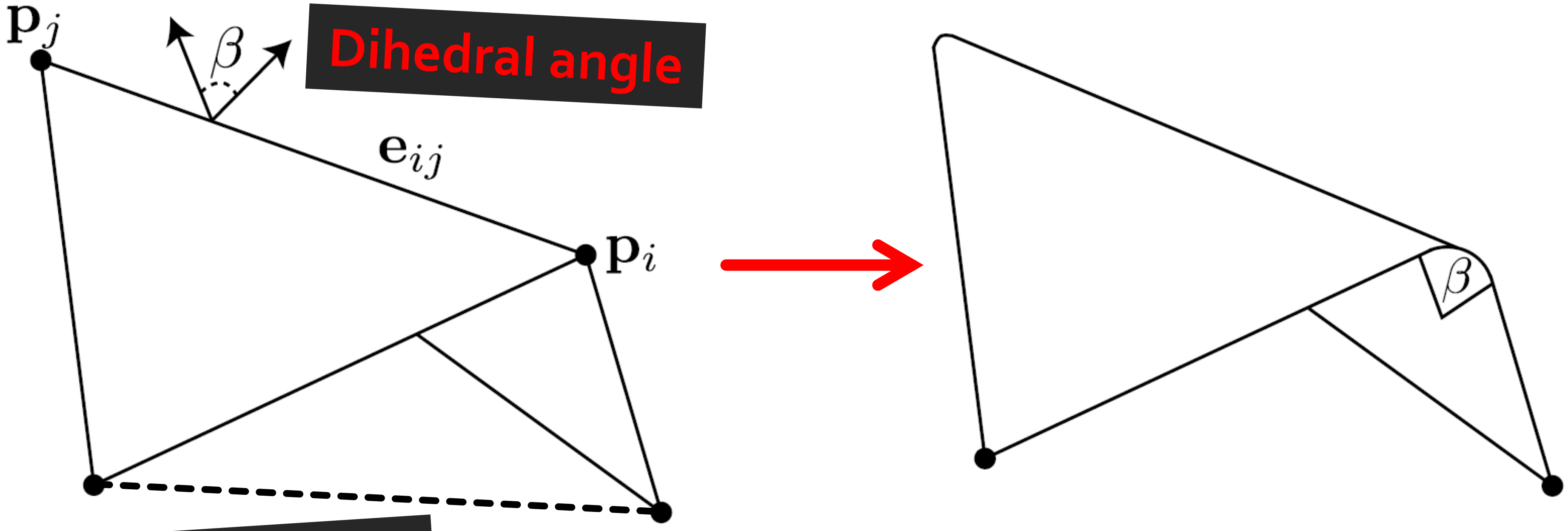
$$\nabla_{\mathbf{p}} A = \frac{1}{2} \sum_j (\cot \alpha_j + \cot \beta_j)(\mathbf{p} - \mathbf{q}_j)$$

Divide by area for curvature estimate

Pipeline

- Compute integrated *H*, *K*
- Divide by area of cell for estimated value

Another Mean Curvature



Dihedral angle

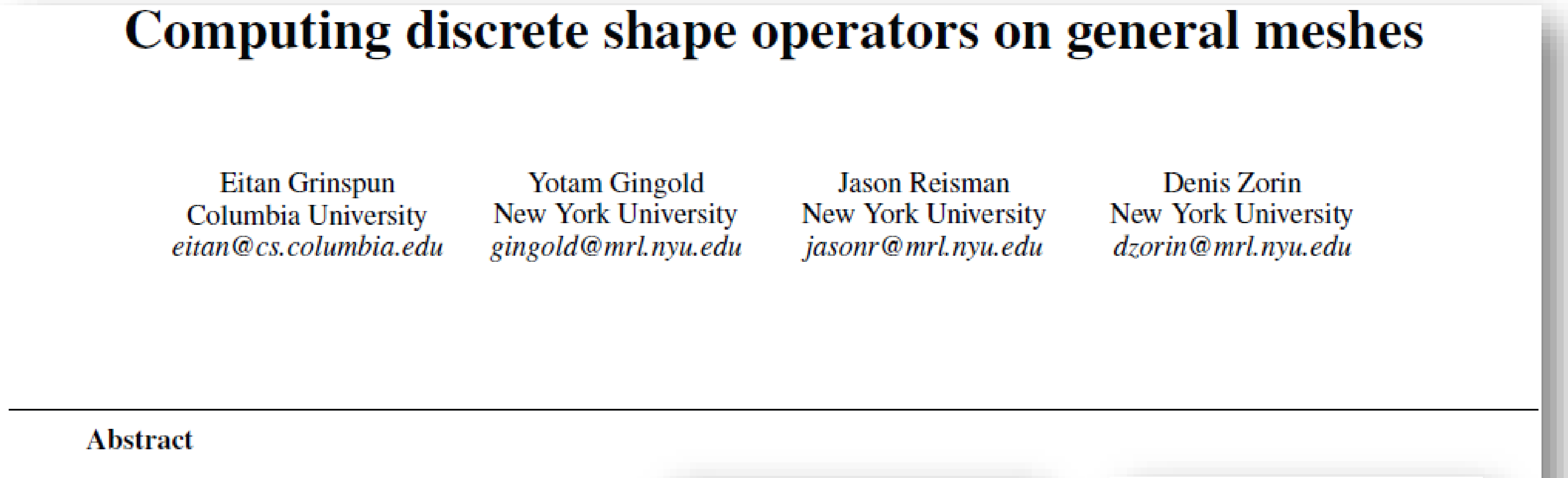
Same vertices, lower curvature

$$\int_B H = \frac{1}{2}\beta\|\mathbf{e}\|$$

J.A. Bærentzen et al., *Guide to Computational Geometry Processing* (2012)

Used for triangulation applications

Tuned for Variational Applications



Computing discrete shape operators on general meshes

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Abstract

Discrete curvature and shape operators, which c
are essential in a variety of applications: simulat
geometric data processing. In many of these appl
approaches for formulating curvature operators j
expensive methods used in engineering applicati
computer graphics.

We propose a simple and efficient formulation for
degrees of freedom associated with normals. On
curvature operators commonly used in graphics;
and produces consistent results for different types

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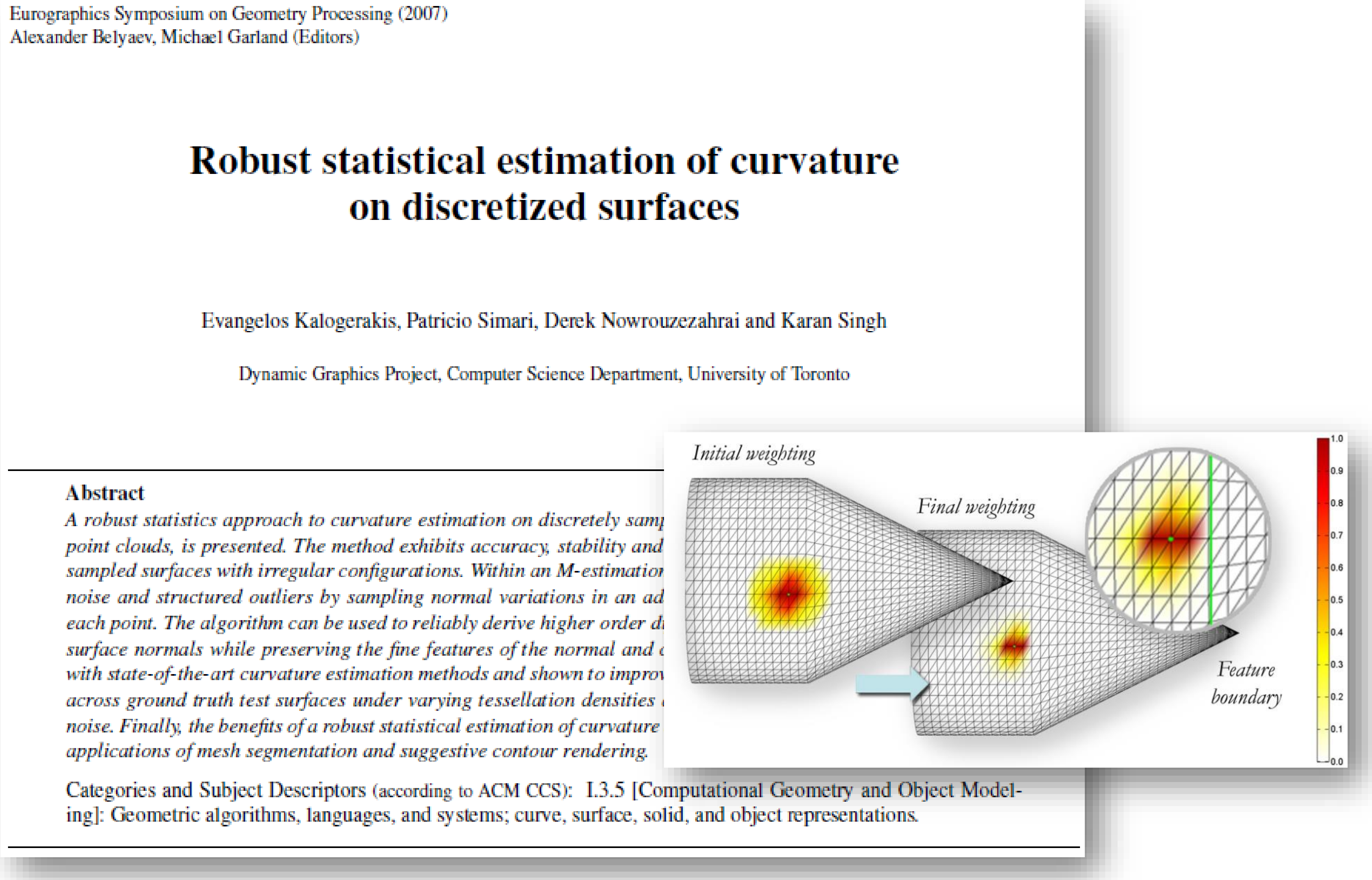
Tuned for Robustness



Eurographics Symposium on Geometry Processing (2007)
Alexander Belyaev, Michael Garland (Editors)

Robust statistical estimation of curvature on discretized surfaces

Evangelos Kalogerakis, Patricio Simari, Derek Nowrouzezahrai and Karan Singh

Dynamic Graphics Project, Computer Science Department, University of Toronto

Abstract

A robust statistics approach to curvature estimation on discretely samp
point clouds, is presented. The method exhibits accuracy, stability and
sampled surfaces with irregular configurations. Within an M-estimation
noise and structured outliers by sampling normal variations in an ad
each point. The algorithm can be used to reliably derive higher order d
surface normals while preserving the fine features of the normal and c
with state-of-the-art curvature estimation methods and shown to impro
across ground truth test surfaces under varying tessellation densities a
noise. Finally, the benefits of a robust statistical estimation of curvature
applications of mesh segmentation and suggestive contour rendering.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computational Geometry and Object Modeling]: Geometric algorithms, languages, and systems; curve, surface, solid, and object representations.

Alternative Strategies

- **Locally fit** a smooth surface

  What type of surface? How to fit?

- Different **formula**

  Function of curvature? Where on mesh?
  Convergence of approximation?

- **Learn** curvature computation

  Tune for application? Training data?

Practical Advice

Try as many as you can.

Most are easy to implement!

Discrete Surface Curvature

Justin Solomon

6.8410: Shape Analysis
Spring 2023