Surfaces: Smooth and Discrete

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6.8410: Shape Analysis
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MIT EECS

What's Next?

Step up

one dimension

from curves to surfaces.

- Theoretical definition
- Discrete representations
- Higher dimensionality

Briefly Will Mention

Step up

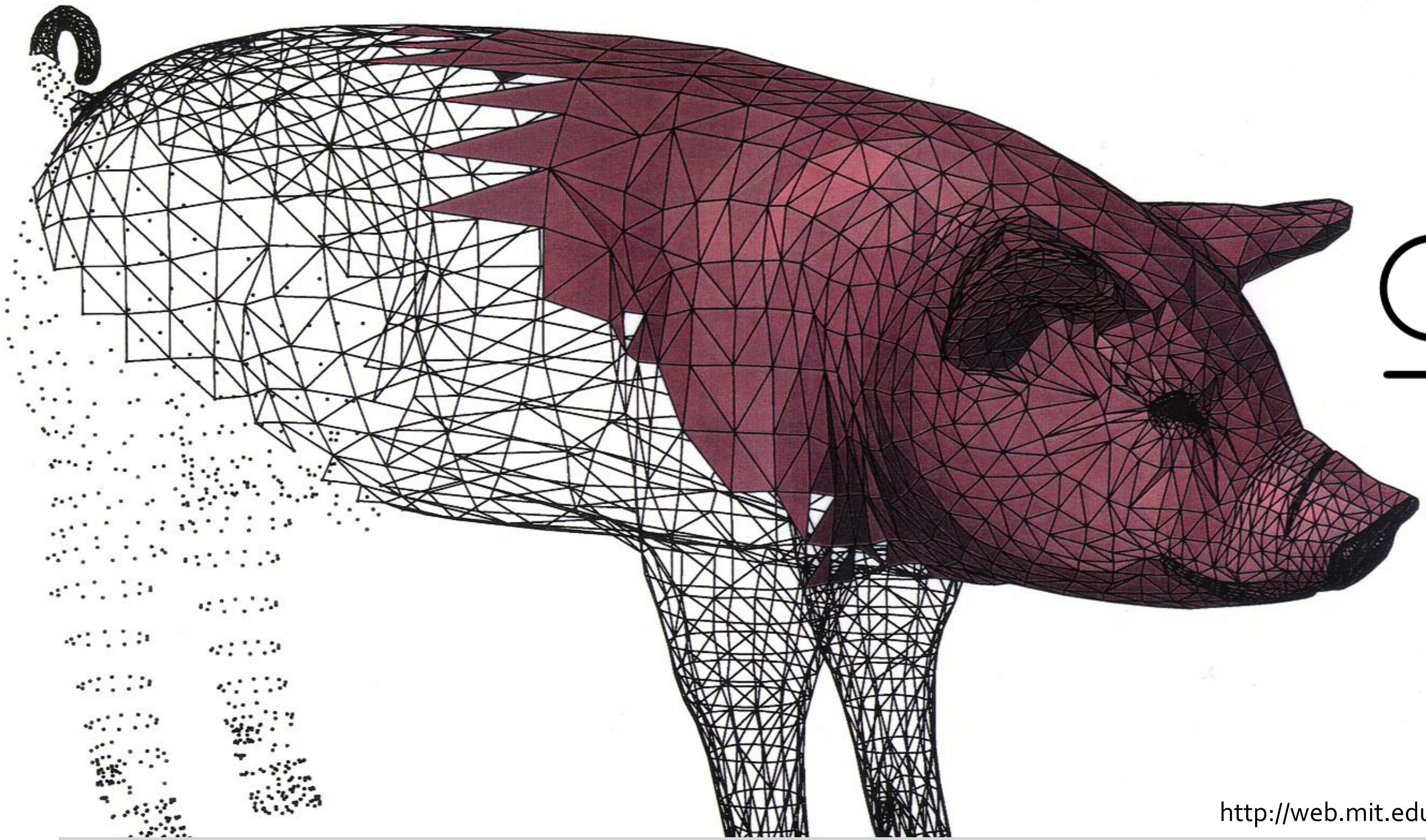
***n* dimensions**

from surfaces to (sub)manifolds.

Easier transition.

Not entirely true:
e.g. topology of 3-manifolds

Our Focus



$\subseteq \mathbb{R}^3$

http://web.mit.edu/manoli/crust/www/slides/piggy.jpg

Embedded geometry

Q:

What is an
embedded surface?

Warm Up: Parametric Surface

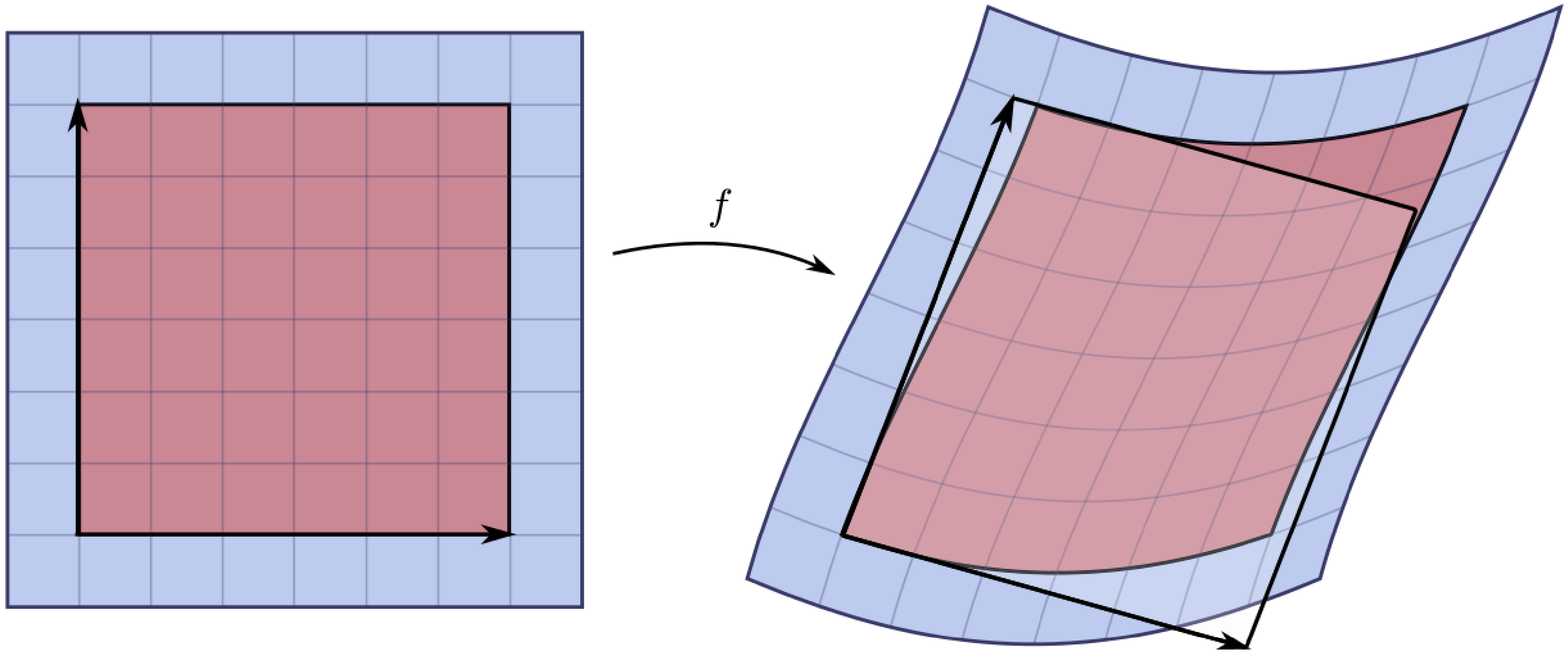


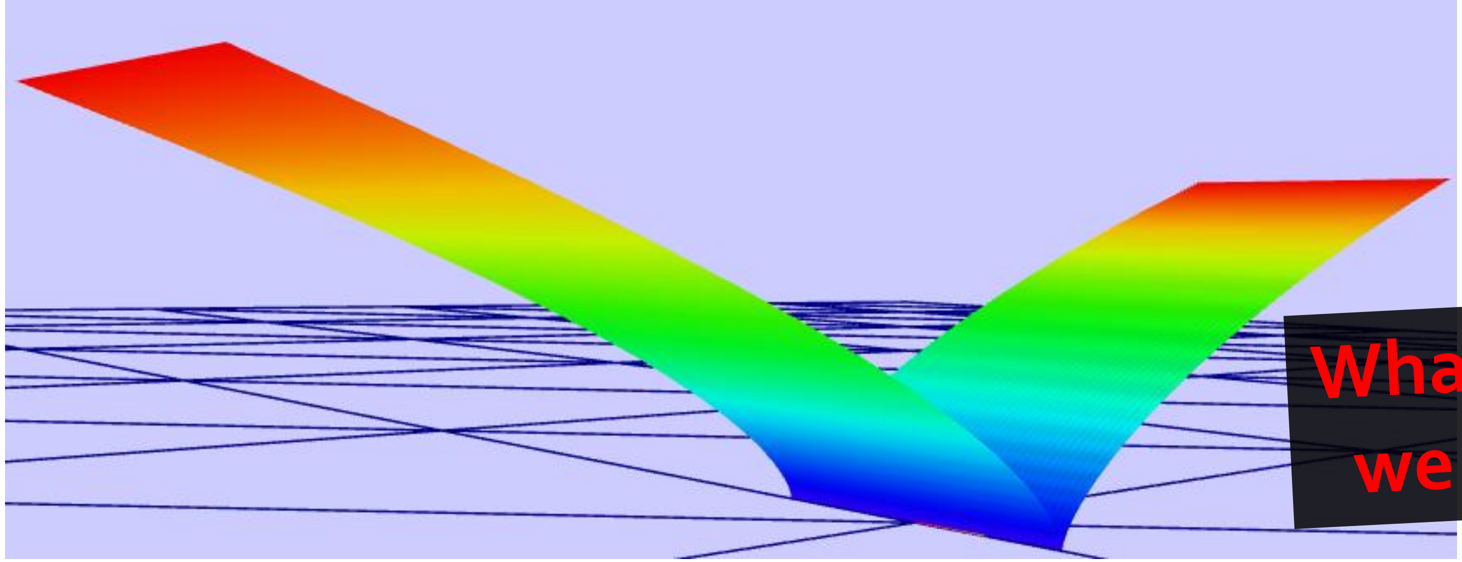
Image from Wikipedia

Pathological Cases

$$f(u,v) = (u, u^2, \cos u)$$

$$f(u,v) = (0,0,0)$$

$$f(u,v) = (u, v^3, v^2)$$



What condition do we need to add?

Review: Jacobian Matrix

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

Jacobian matrix:

$$(Df)^i_j = \left(\frac{\partial f^i}{\partial x^j} \right)$$

Regularity (Injectivity/One-to-One) Condition

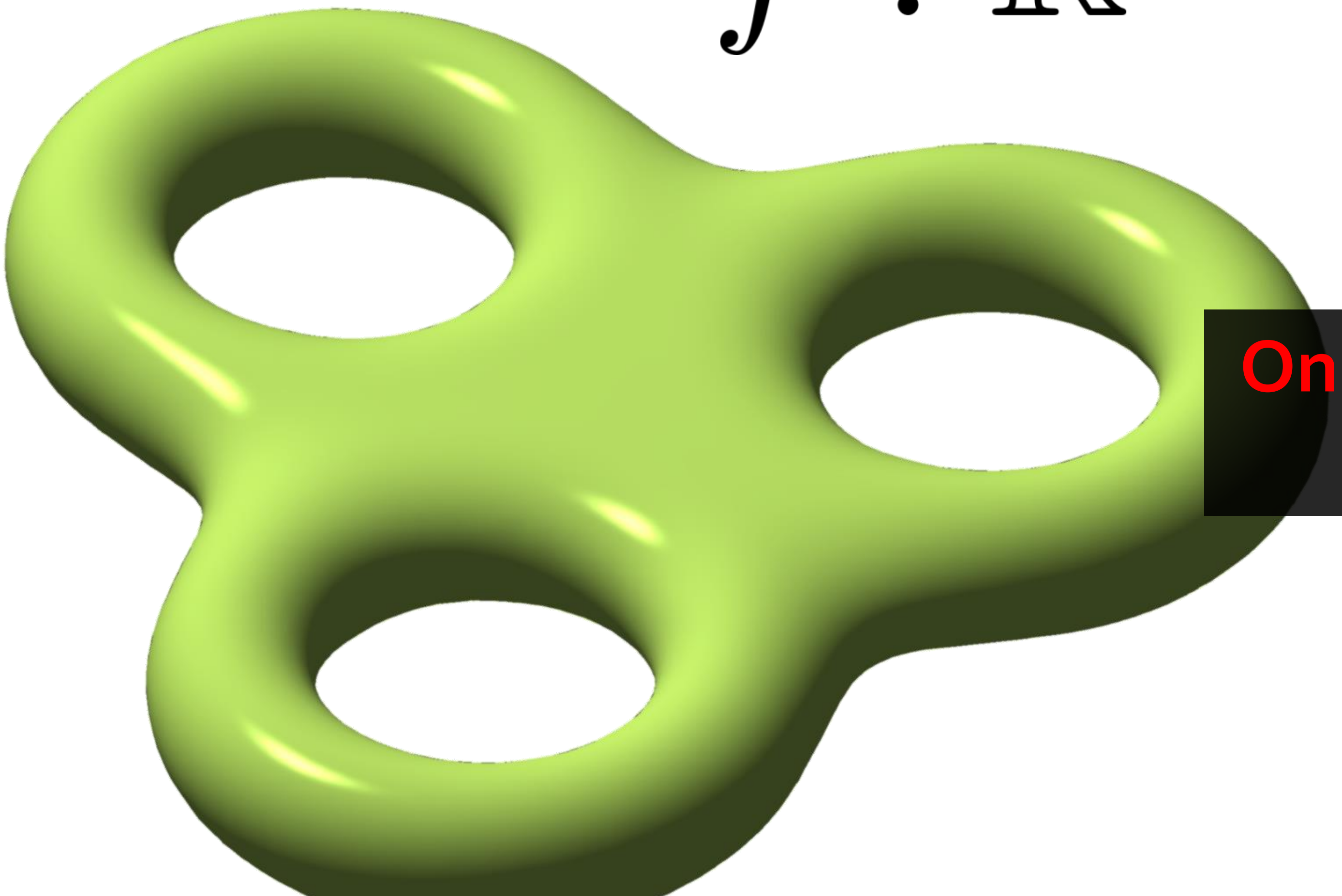
$$f : \mathbb{R}^m \to \mathbb{R}^n$$

Matrix condition:

$$Df \text{ full rank}$$

Moving Away from Parametric Surfaces

$$f : \mathbb{R}^2 \to \mathbb{R}^3 \ ?$$



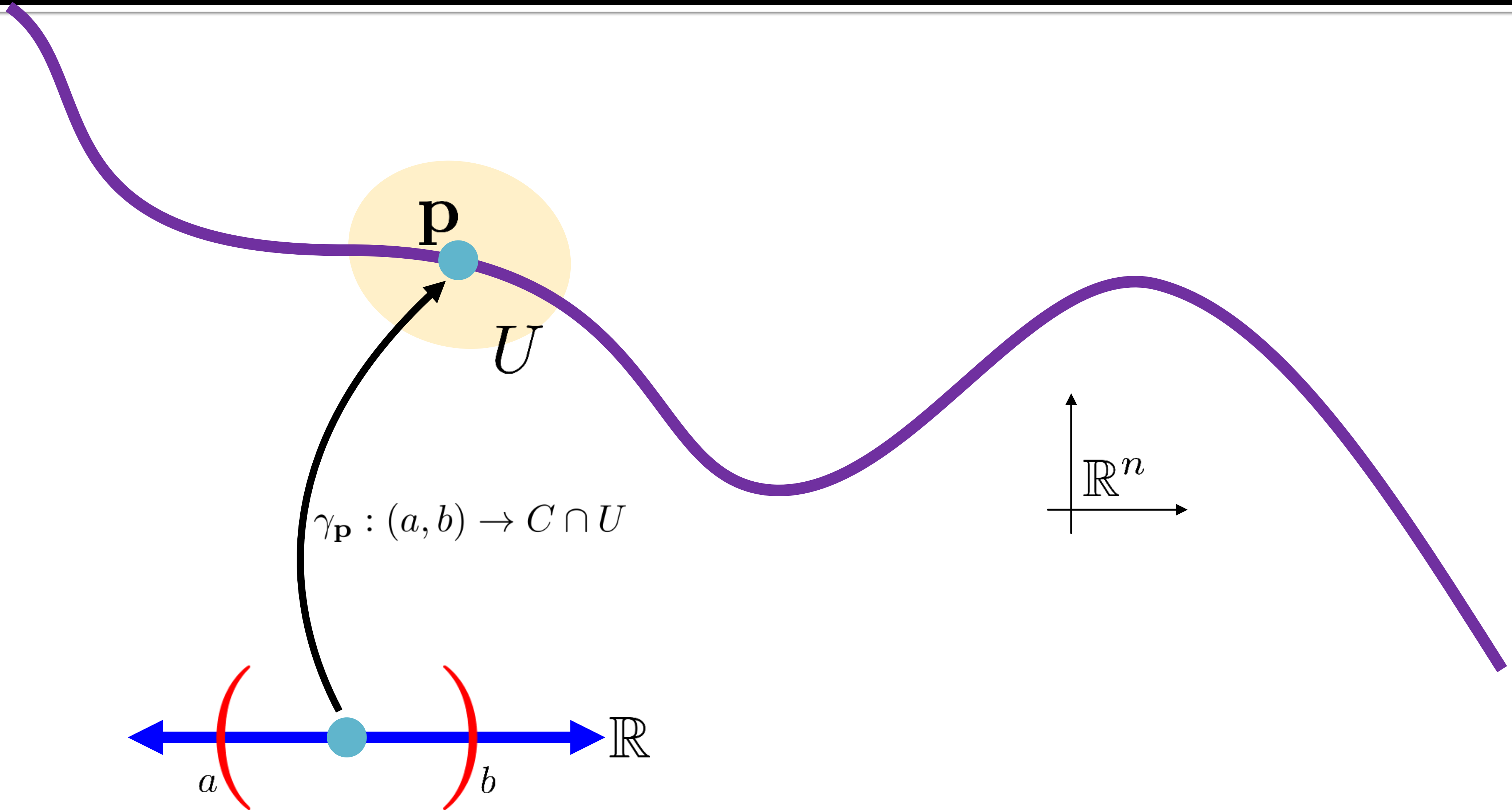
One function isn't enough!

Major difference from curves!

https://en.wikipedia.org/wiki/Triple_torus

Recall:

Differential Geometry Definition



Just Like Curves

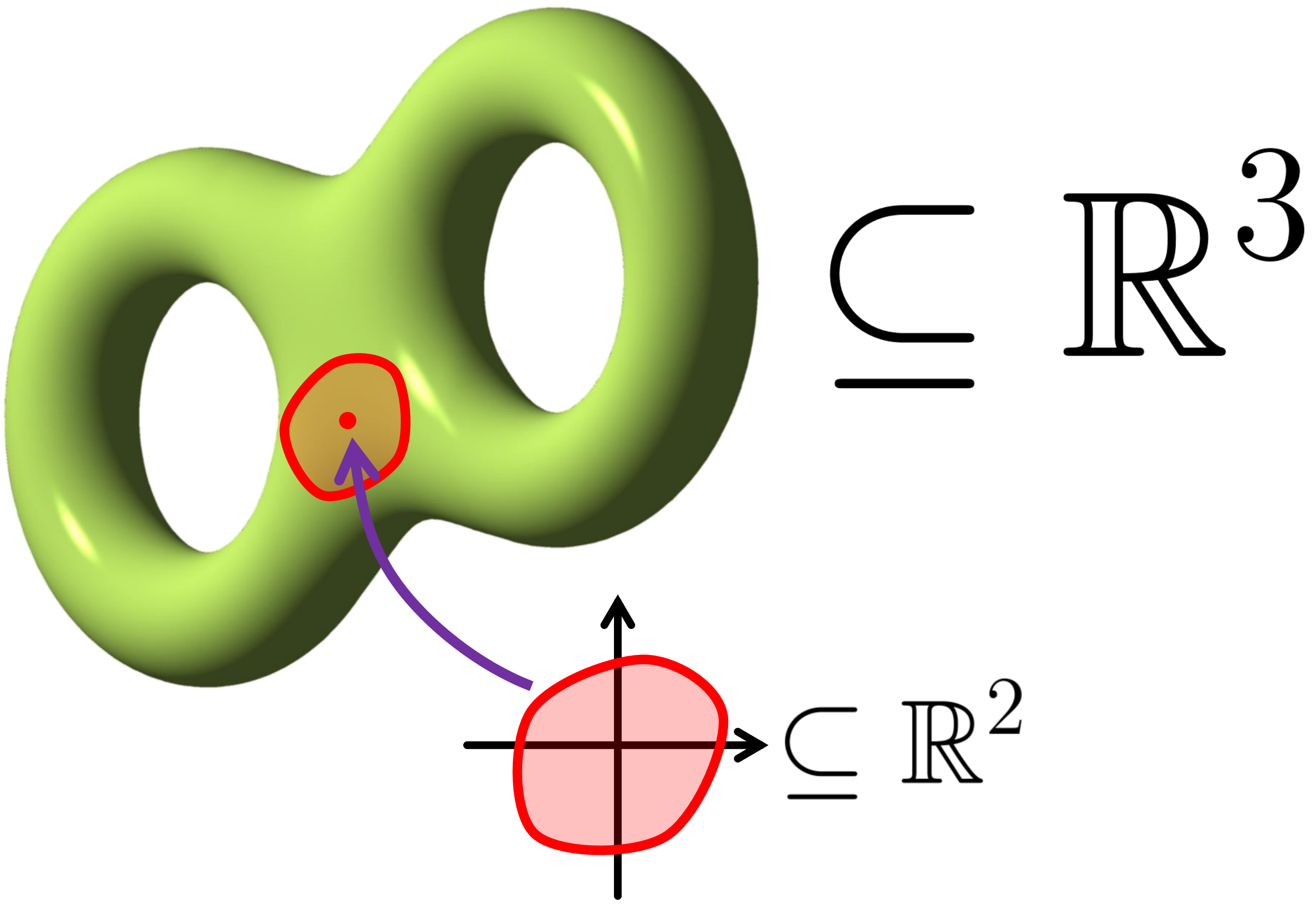
A surface is a

set of points

with certain properties.

It is not a function.

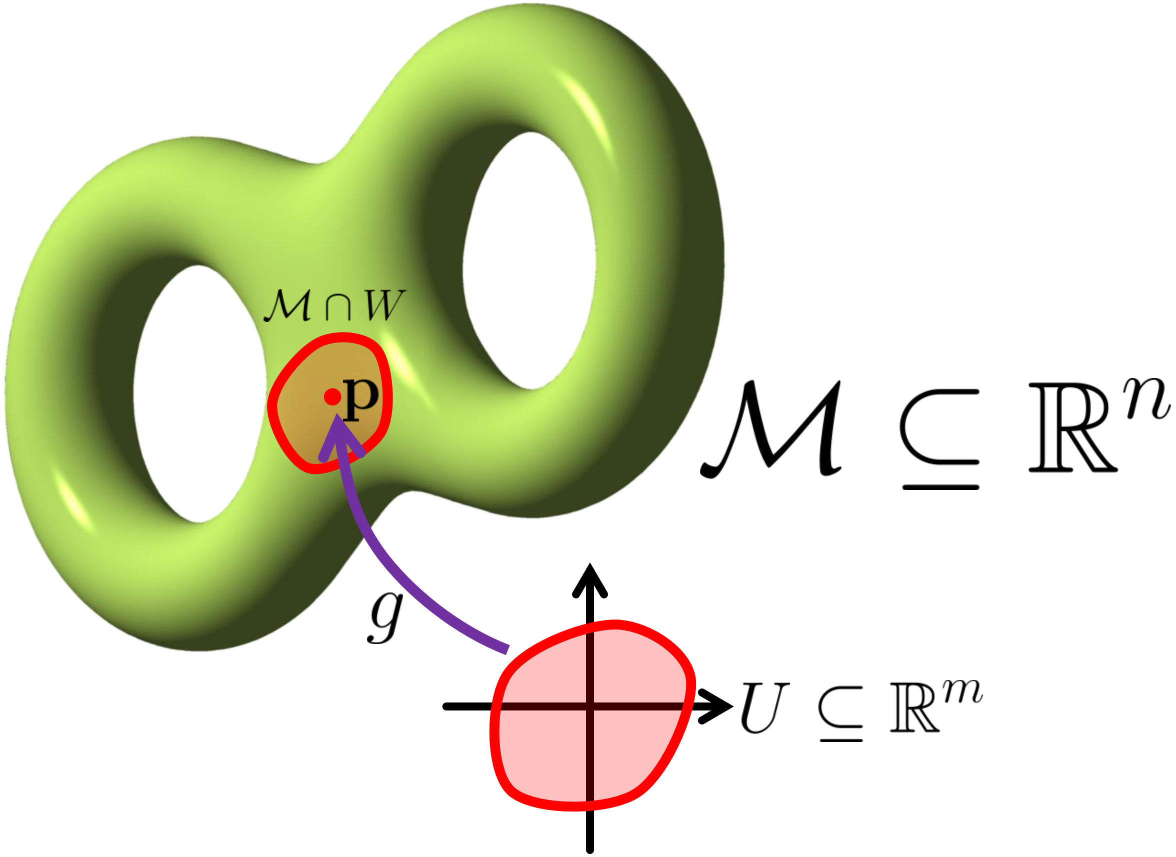
Theoretical Definition of Surface



http://upload.wikimedia.org/wikipedia/commons/b/bc/Double_torus_illustration.png

Theoretical Definition: (Sub)Manifold

Definition (Submanifold of $\mathbb{R}^n$, with and without boundary). *A set $\mathcal{M} \subseteq \mathbb{R}^n$ is an m-dimensional submanifold of $\mathbb{R}^n$ if for each $\mathbf{p} \in \mathcal{M}$ there exist open sets $U \subseteq \mathbb{R}^m$, $W \subseteq \mathbb{R}^n$ and a function $g : U \cap \mathcal{H}_m \to \mathcal{M} \cap W$ such that $\mathbf{p} \in W$ and g is a one-to-one and smooth map whose Jacobian is rank-m and admitting a continuous inverse $g^{-1} : W \cap M \to U$.*



$\mathcal{H}_m := \{\mathbf{x} \in \mathbb{R}^m : x^m \geq 0\}$

Differential Geometer's Mantra

A surface is

locally planar.

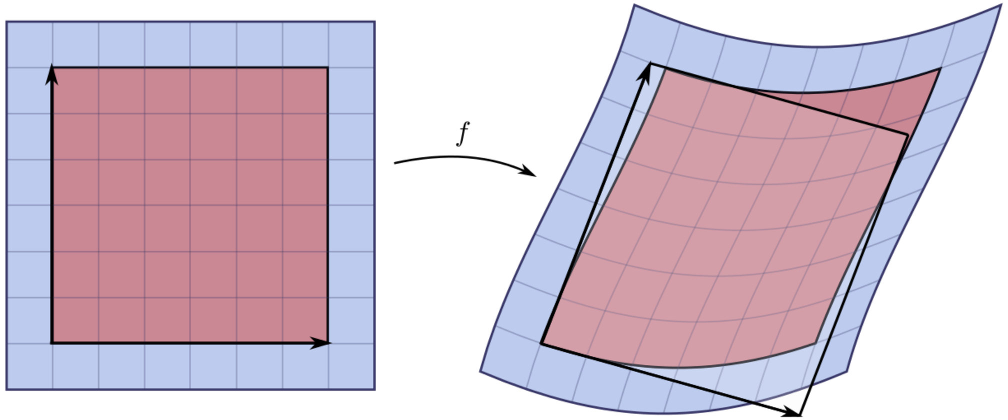
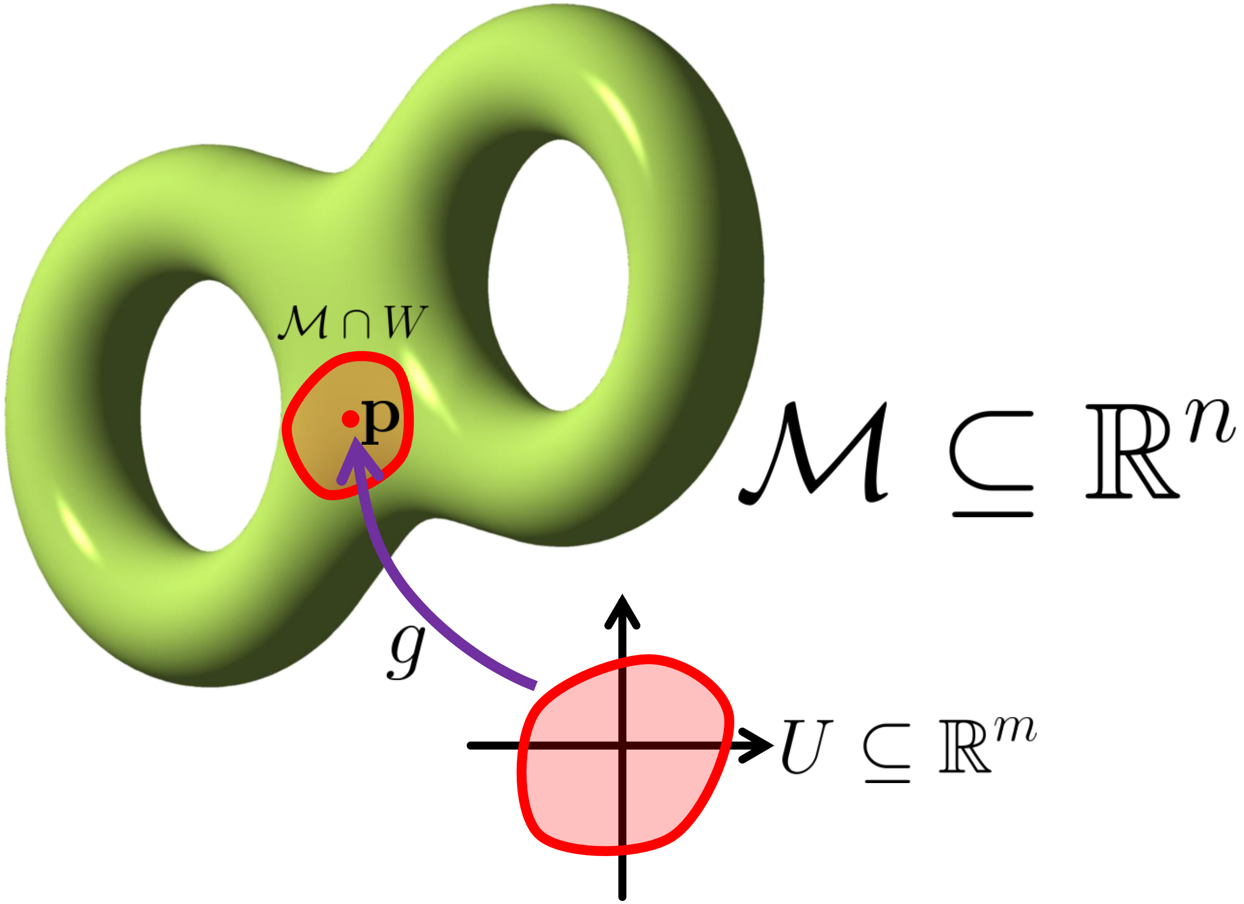


Image from Wikipedia

Tangent Space

$$T_{\mathbf{p}}\mathcal{M} = \gamma'(0), \text{ where } \gamma(0) = \mathbf{p}$$

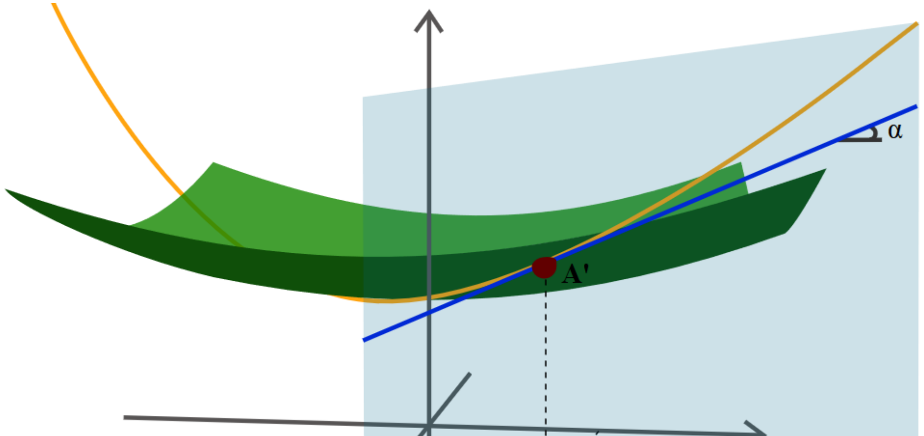


Recall: Differential

$$df_{\mathbf{x}_0}(\mathbf{v}) := \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{v}) - f(\mathbf{x}_0)}{h}$$

Proposition. df_{x_0} is a linear operator.

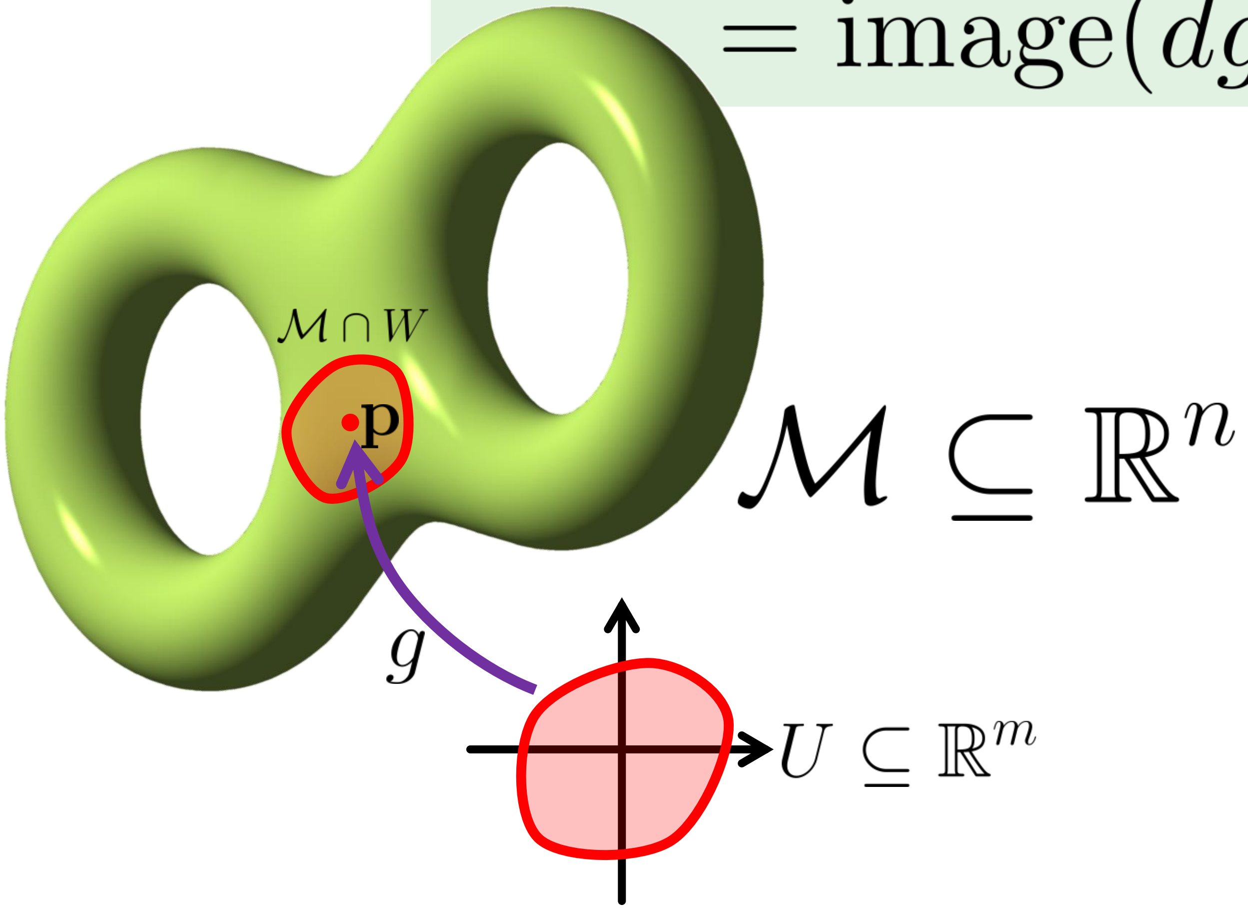
$$df_{\mathbf{x}_0}(\mathbf{v}) = Df(\mathbf{x}_0) \cdot \mathbf{v}$$



Note: Technically we derived the 1D version. Nothing changes!

Tangent Space

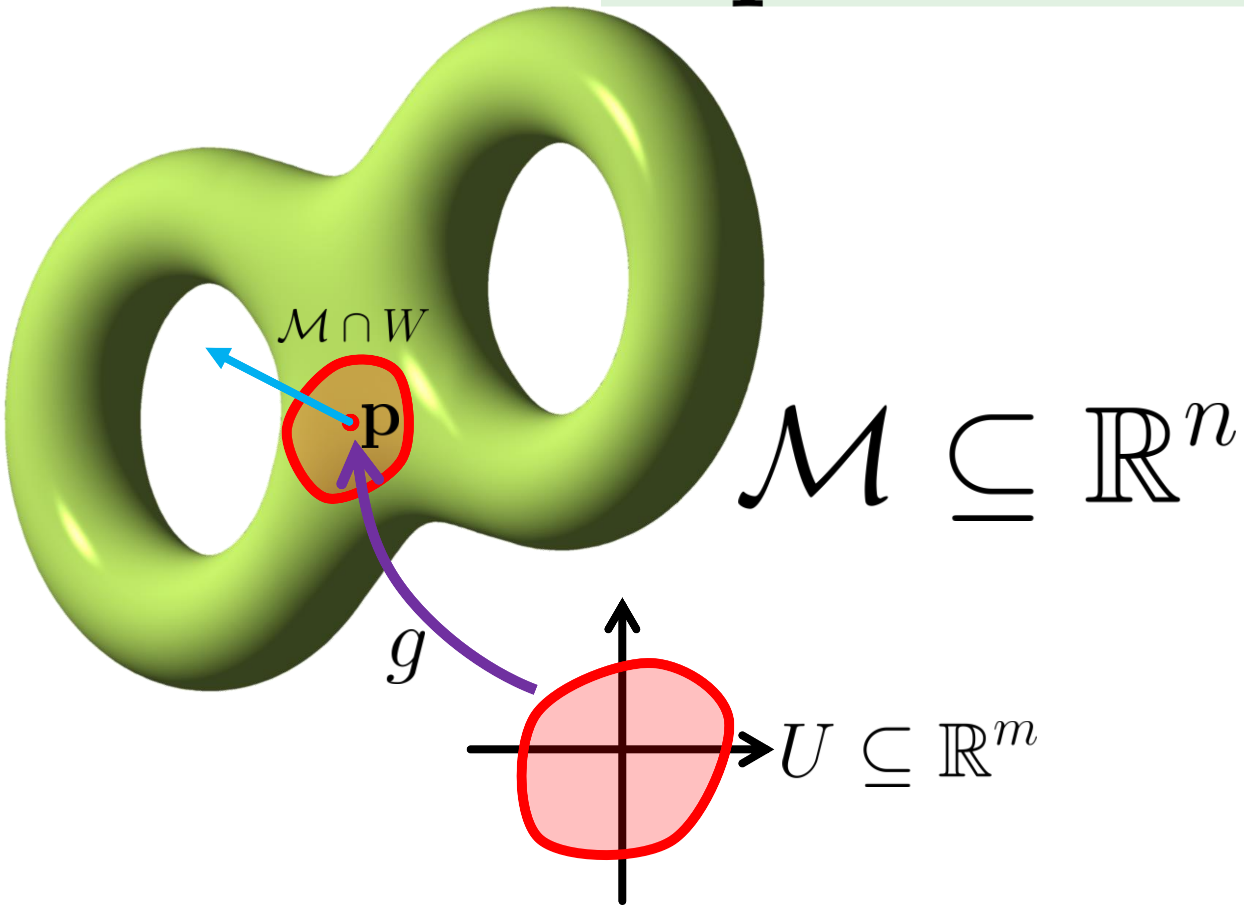
$$T_{\mathbf{p}}\mathcal{M} = \gamma'(0), \text{ where } \gamma(0) = \mathbf{p}$$
$$= \text{image}(dg_{g^{-1}(\mathbf{p})})$$



Skipping:
Independence of choice of g.

Normal Space

$$N_{\mathbf{p}}\mathcal{M} := (T_{\mathbf{p}}\mathcal{M})^{\perp}$$



Orientable Submanifold

Admits a continuous map

$$\mathbf{n}(\mathbf{p}) : \mathcal{M} \backslash \partial \mathcal{M} \rightarrow \mathcal{S}^{n-1}$$

with

$$\mathbf{n}(\mathbf{p}) \in N_{\mathbf{p}}\mathcal{M}$$

Toroidal Closures of Scherk Towers

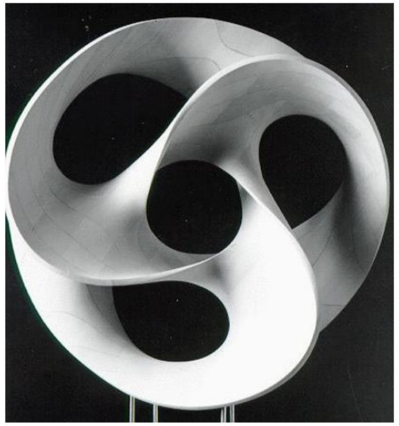
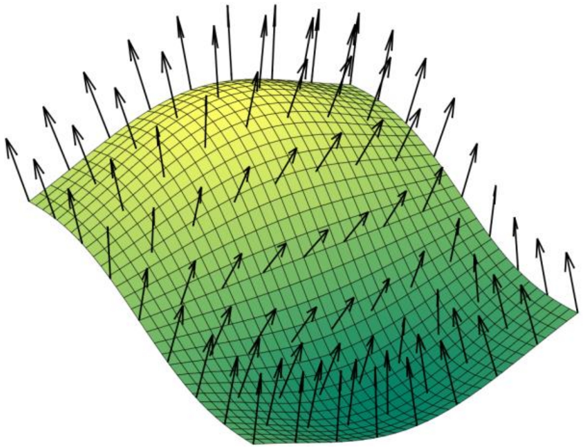
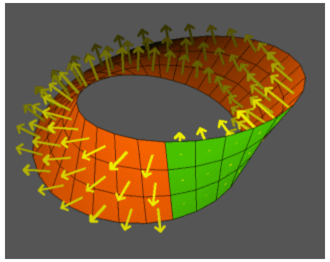


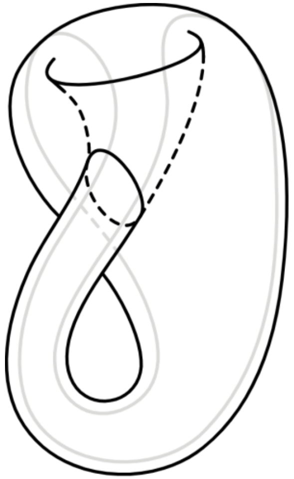
Figure 3. Non-orientable surface with second order saddles.
(all photos, Phillip Geller)

Brent Collins (Gower, Missouri) is a sculptor who has presented his work at AM93, AM95, and AM97. His recent collaboration with Carlo Séquin has resulted in new forms that represent the leading edge of sculpture influenced by mathematics.



Orientable

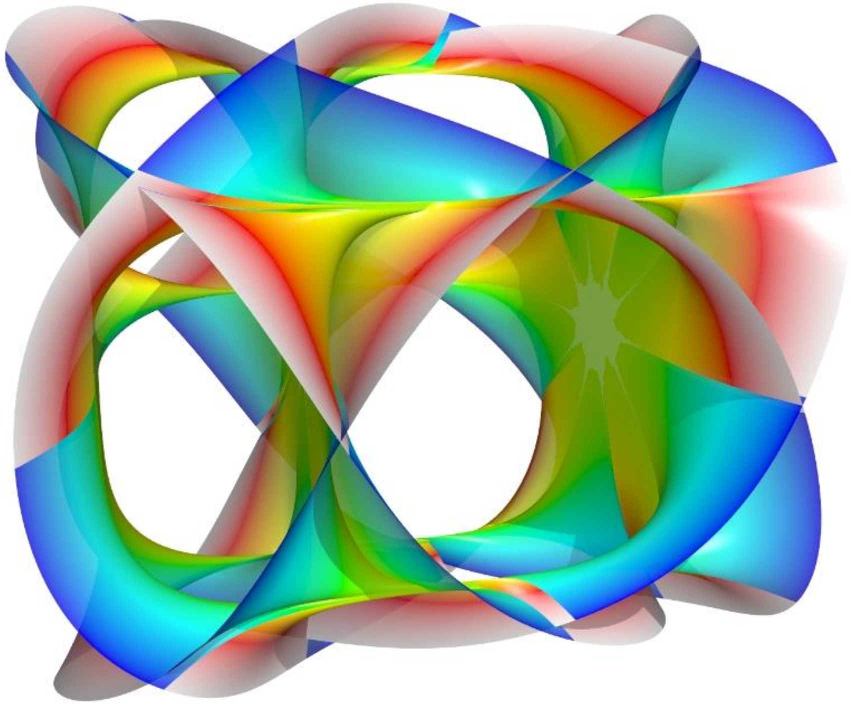




Not Orientable

More General Definition: Manifold

Definition 4.2 (Manifold). *An m-dimensional (topological)* manifold $\mathcal{M}$ *is a Hausdorff space for which each* $\mathbf{p} \in \mathcal{M}$ *admits open sets* $U \subseteq \mathbb{R}^m, W \subseteq \mathcal{M}$ *and a homeomorphism (continuous map with continuous inverse)* $g : U \to W$.



To think about:

No notion of normal!

Tangent vectors exist but have no length!

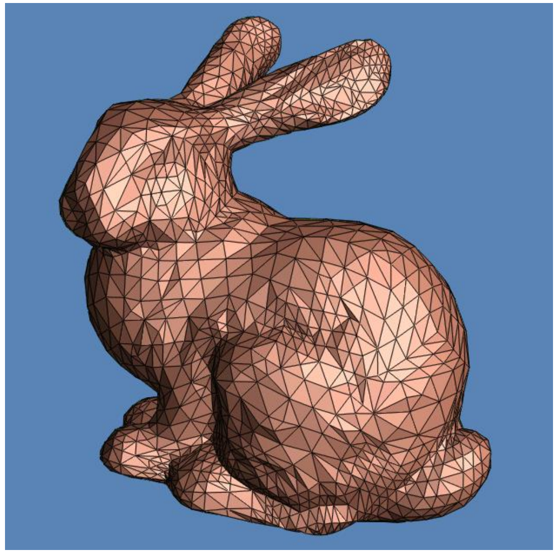
How do you detect orientability?

http://www.math.sjsu.edu/~simic/Pics/Calabi-Yau.jpg

No Euclidean embedding

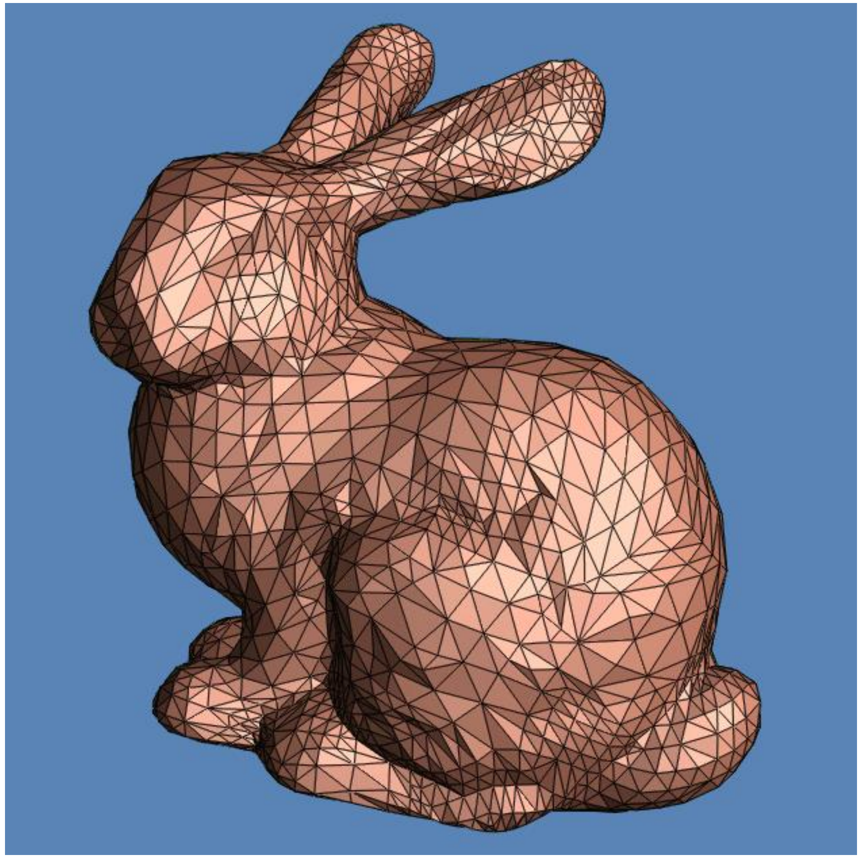
Discrete Problem

What is a discrete surface?
How do you store it?

http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg
http://www.stat.washington.edu/wxs/images/BUNMID.gif

Common Representation



http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg
http://www.stat.washington.edu/wxs/images/BUNMID.gif

Triangle mesh

Triangle Mesh

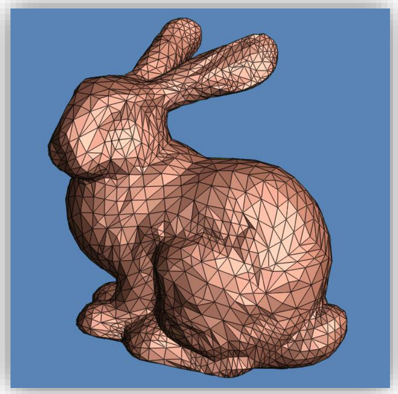
$$V = (v_1, v_2, \ldots, v_n) \subset \mathbb{R}^3$$

$$E = (e_1, e_2, \ldots, e_k) \subseteq V \times V$$

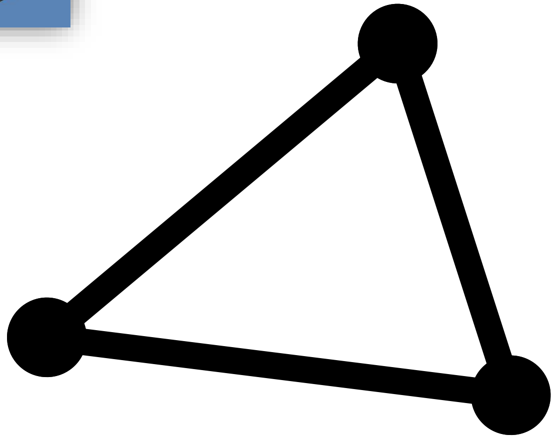
$$F = (f_1, f_2, \ldots, f_m) \subseteq V \times V \times V$$

Plus manifold topological conditions

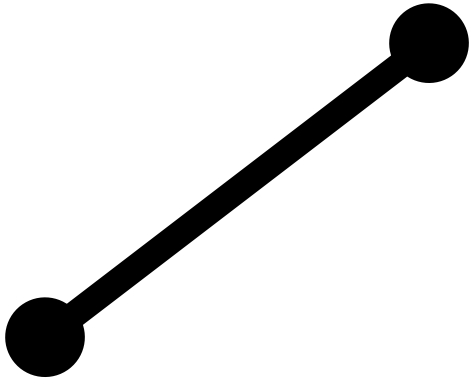
Dimensionality Structure



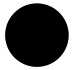
Simplicial complex



Face
Dimension 2



Edge
Dimension 1



Vertex
Dimension 0

Is This a Discrete Surface?

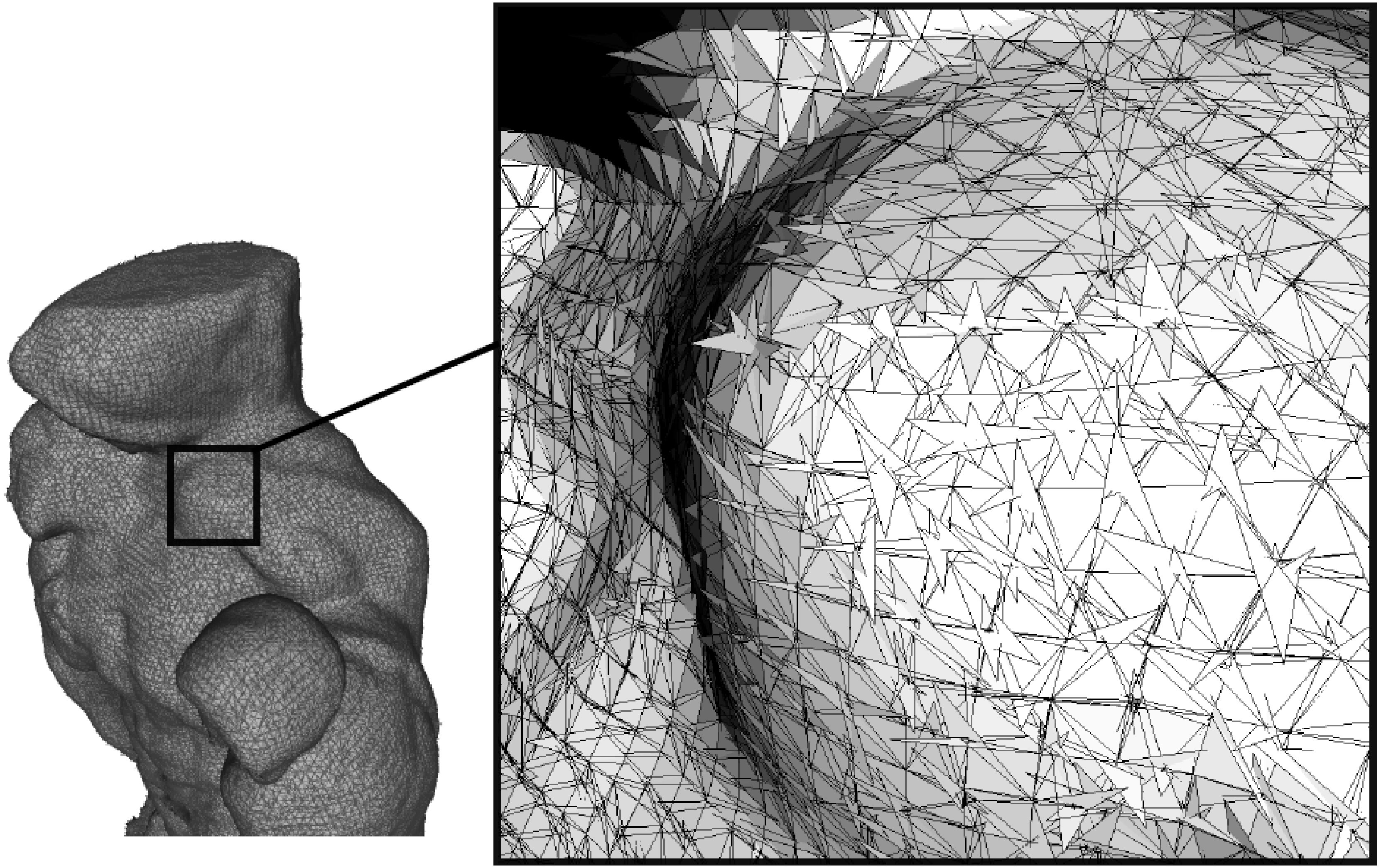


Image from "Global Parametrization of Range Image Sets" (Pietroni et al.)

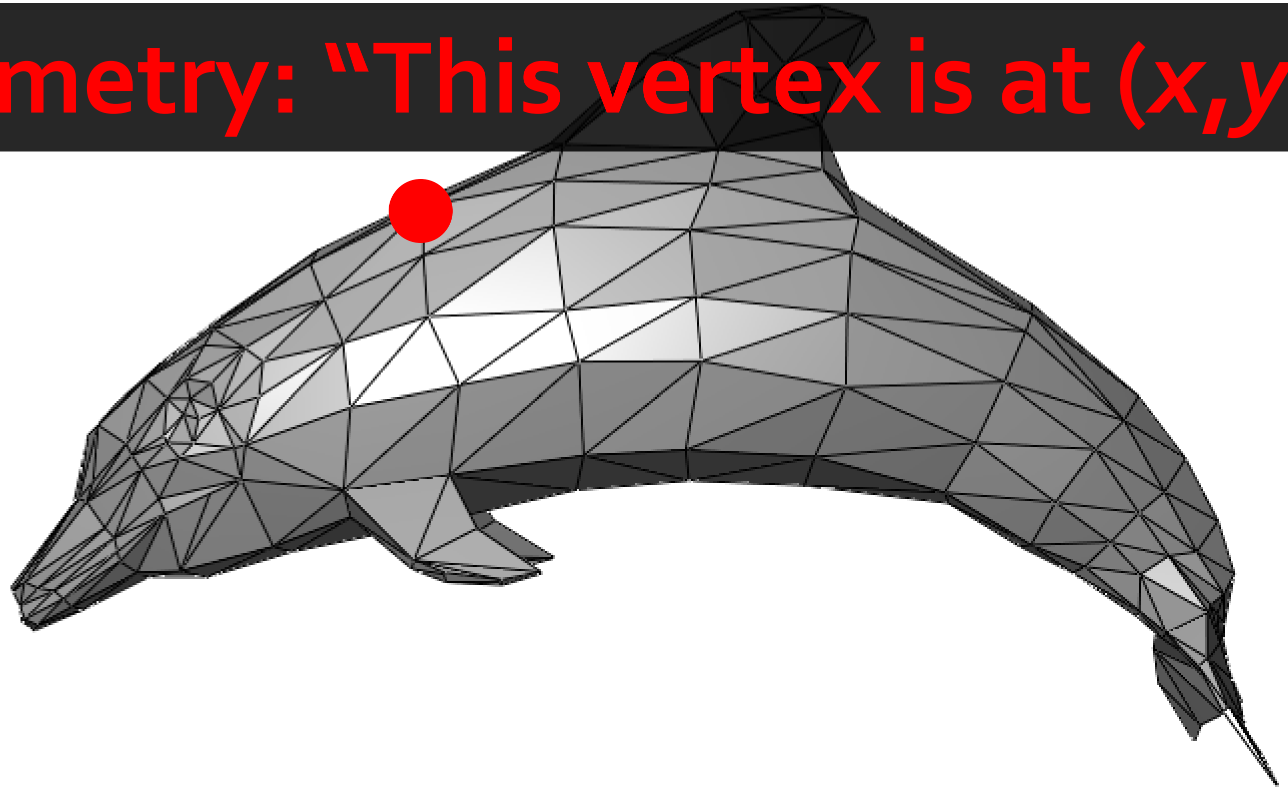
Topology [t*uh*-**pol**-*uh*-jee]:

The study of geometric properties that remain invariant under certain transformations

http://dictionary.reference.com/browse/topology?s=t

Mesh Topology vs. Geometry

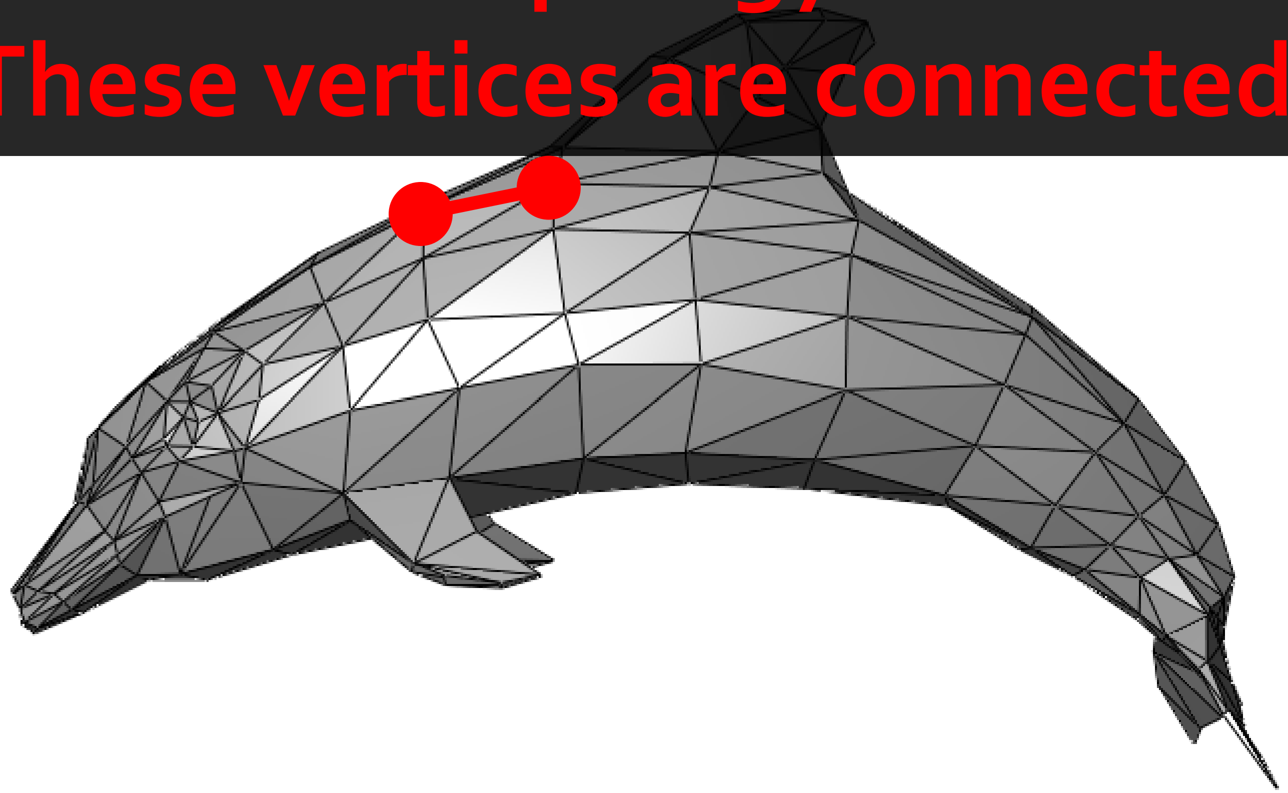
Geometry: "This vertex is at (*x*,*y*,*z*)."



http://upload.wikimedia.org/wikipedia/commons/f/fb/Dolphin_triangle_mesh.png

Mesh Topology vs. Geometry

Topology:
"These vertices are connected."

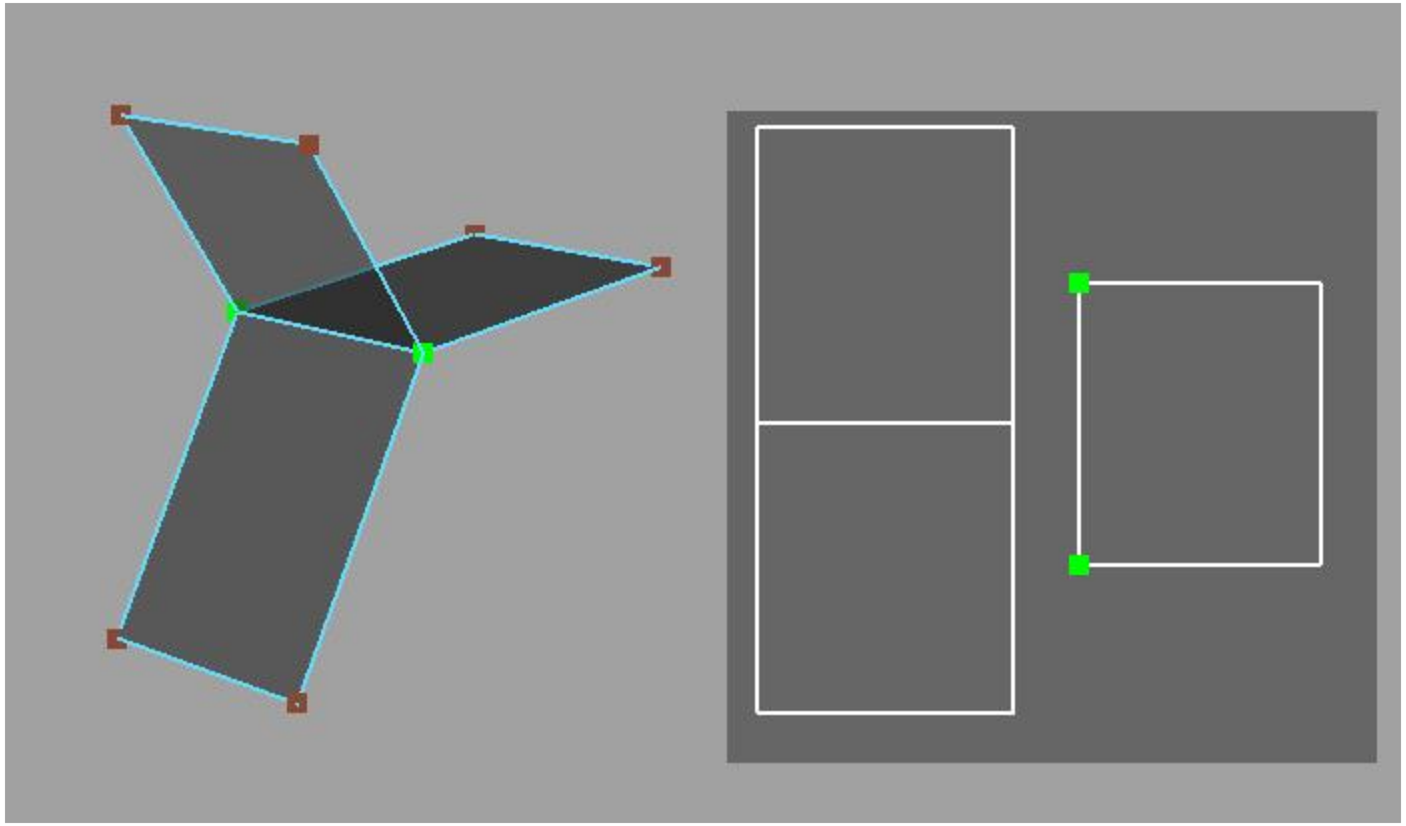


http://upload.wikimedia.org/wikipedia/commons/f/fb/Dolphin_triangle_mesh.png

To read: More general story

“Orientable combinatorial manifold”

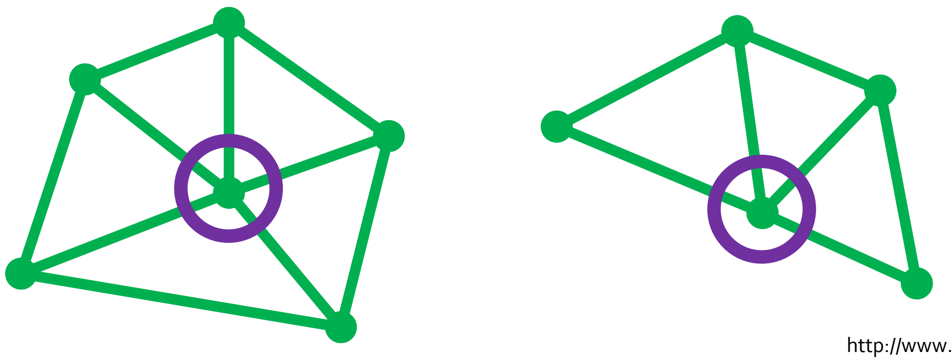
Nonmanifold Edge



http://blog.mixamo.com/wp-content/uploads/2011/01/nonmanifold.jpg

Manifold Triangle Mesh

1. Each **edge** is incident to one or two faces
2. **Faces** incident to a vertex form a closed or open fan



http://www.cs.mtu.edu/~shene/COURSES/cs3621/SLIDES/Mesh.pdf

Manifold Triangle Mesh

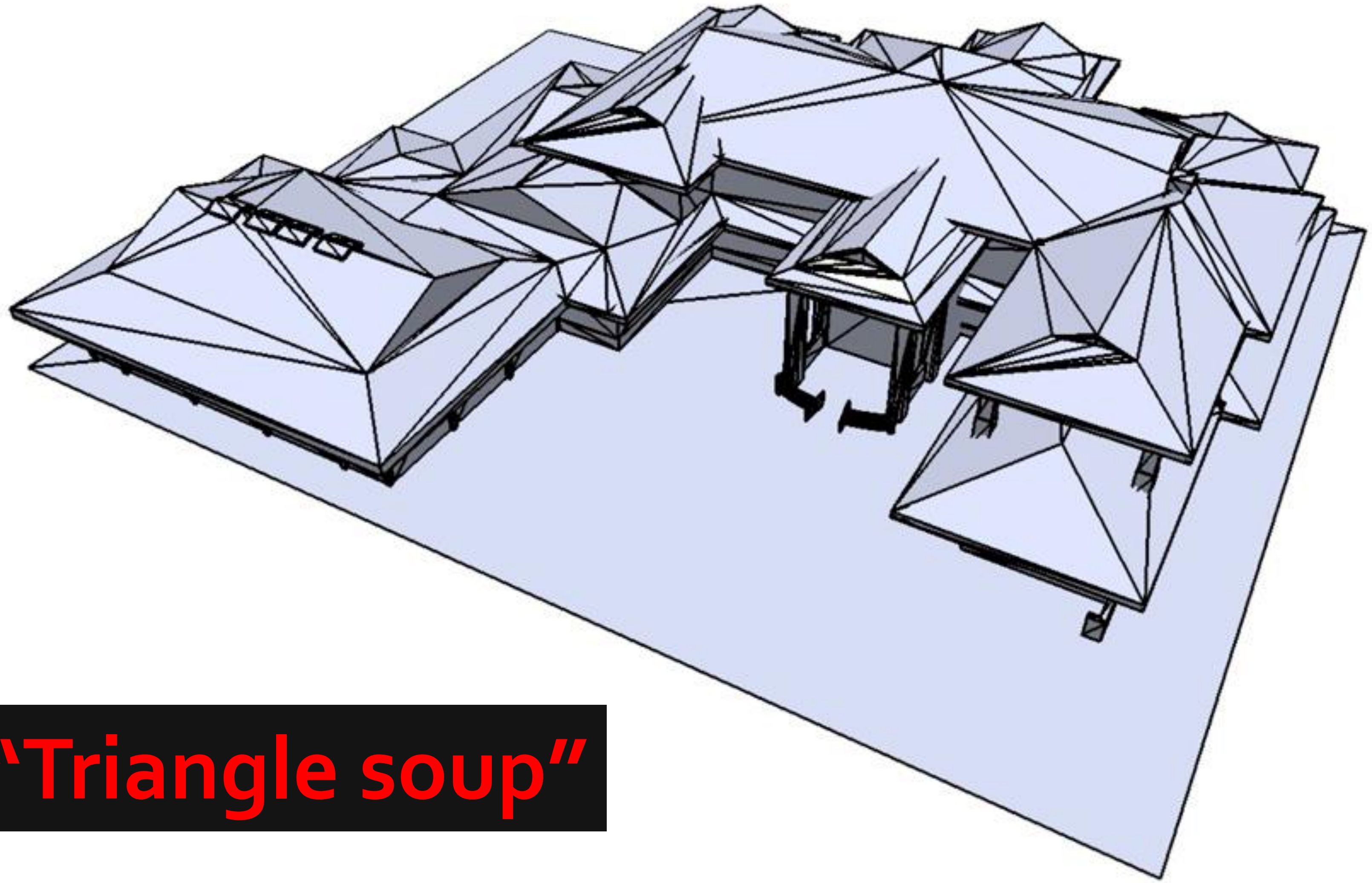
1. Each edge is incident to one or two faces

2. Faces incident to a vertex form a closed or open fan

Assume meshes are manifold (for now)

http://www.cs.mtu.edu/~shene/COURSES/cs3621/SLIDES/Mesh.pdf

Easy-to-Violate Assumption



"Triangle soup"

http://www.pmp-book.org/download/slides/Representations.pdf

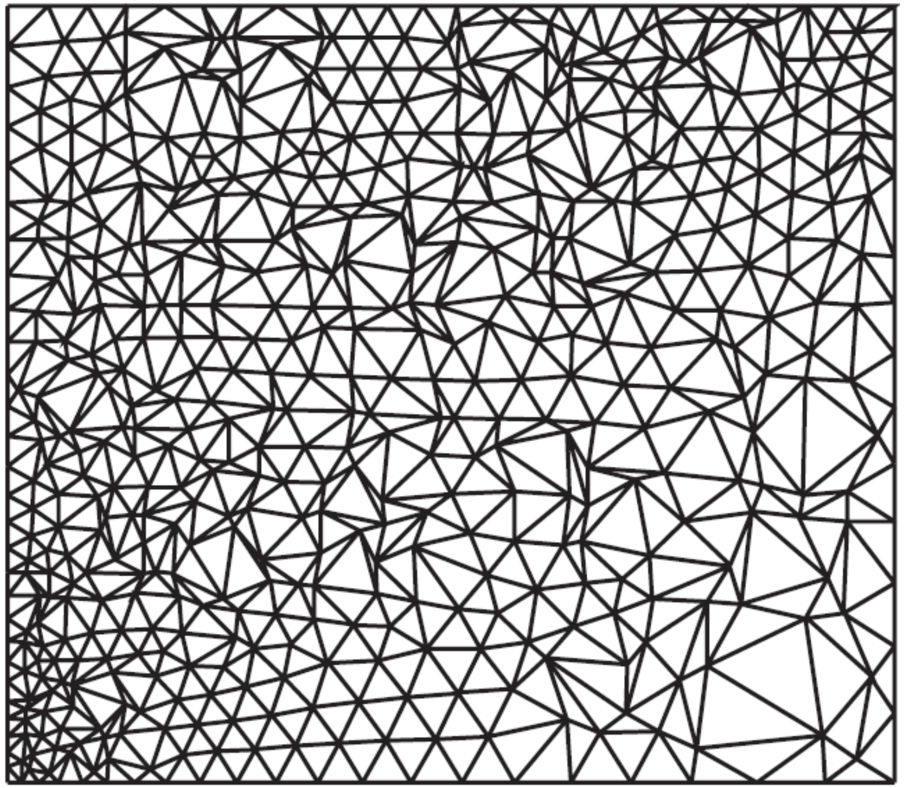
Basic Observation

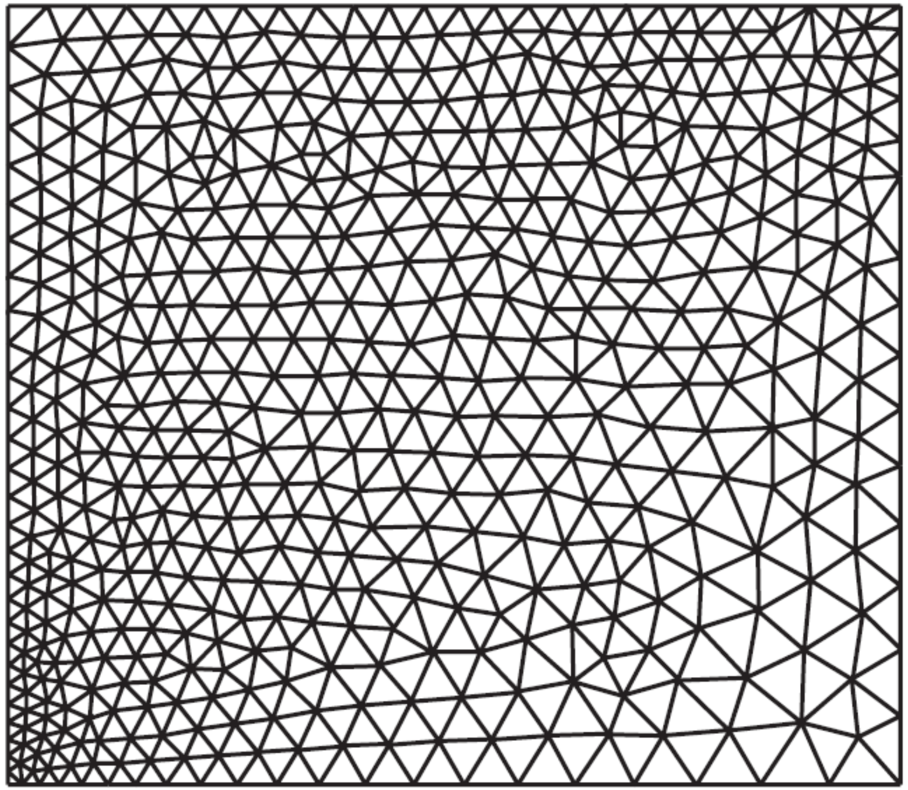
Piecewise linear faces are reasonable building blocks.

Additional Advantages

- Simple to render
- Arbitrary topology possible
- Basis for subdivision, refinement

Invalid Meshes vs. Bad Meshes

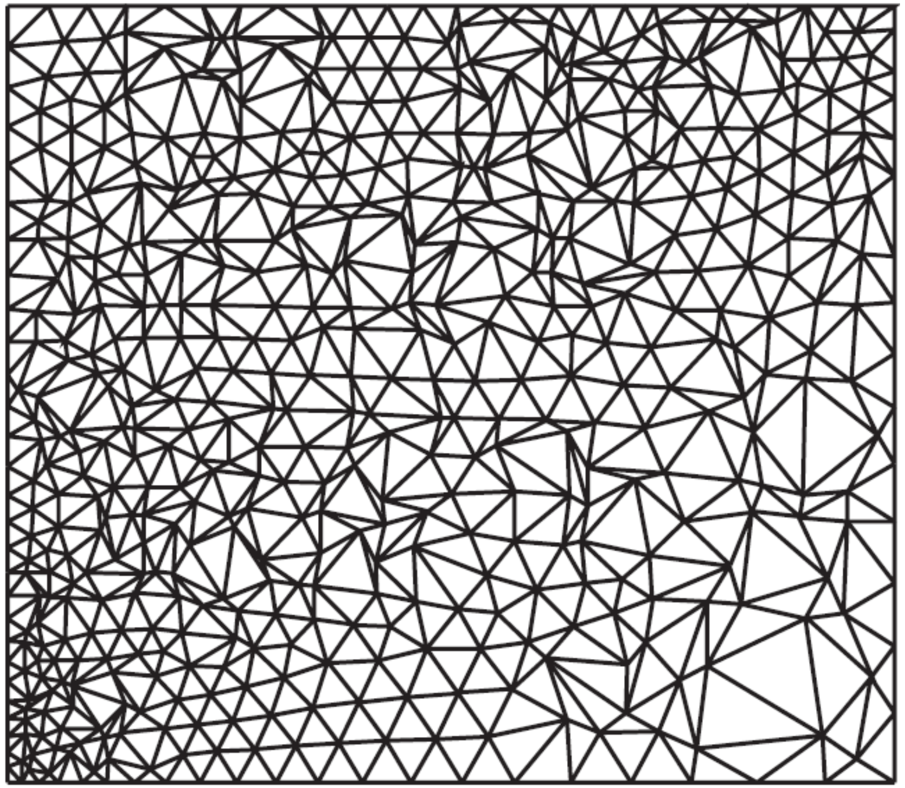




Nonuniform
areas and angles

http://www.sciencedirect.com/science/article/pii/S016887406000795

Why is Meshing an Issue?

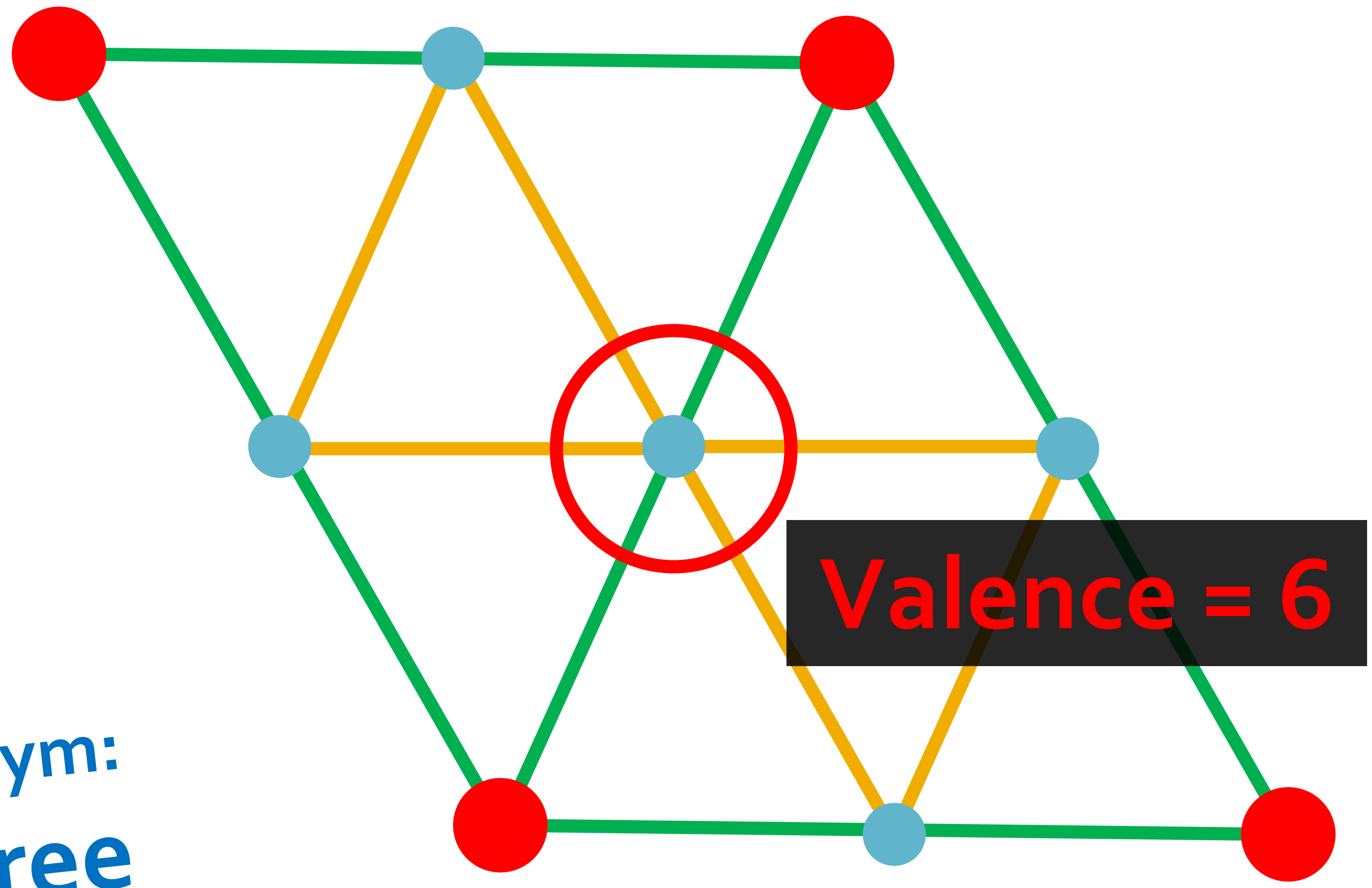


How to you interpret

one value per vertex?

http://www.sciencedirect.com/science/article/pii/S0168874X06000795

Returning to Topology: Valence



Synonym:
Degree

Euler Characteristic for Meshes

$$V - E + F := \chi$$

$$\chi = 2 - 2g$$

$g = 0$

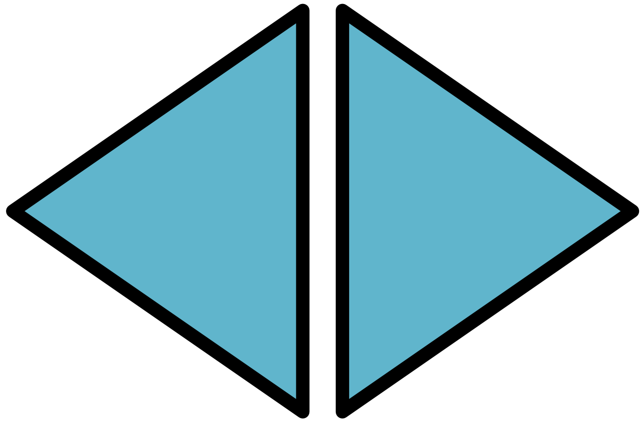
$g = 1$

$g = 2$

Consequences for Triangle Meshes

$$V - E + F := \chi$$

"Each edge is adjacent to two faces. Each face has three edges."



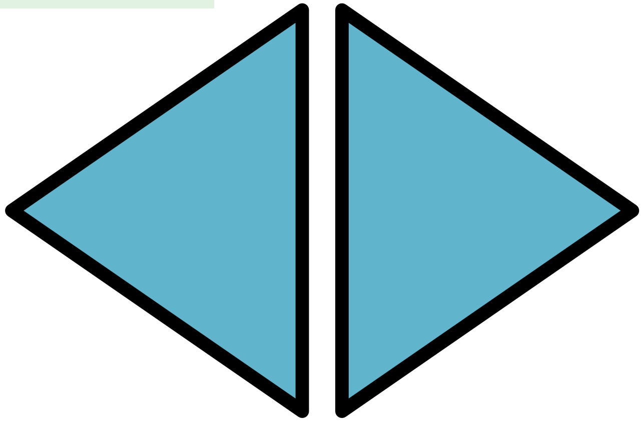
$$2E = 3F$$

Closed mesh: Easy estimates!

Consequences for Triangle Meshes

$$V - \frac{1}{2}F := \chi$$

"Each edge is adjacent to two faces. Each face has three edges."



$$2E = 3F$$

Closed mesh: Easy estimates!

Consequences for Triangle Meshes

Big number

$$V - \frac{1}{2}F := \chi$$

Small number

$$F \approx 2V$$

Closed mesh: Easy estimates!

Consequences for Triangle Meshes

$$E \approx 3V$$

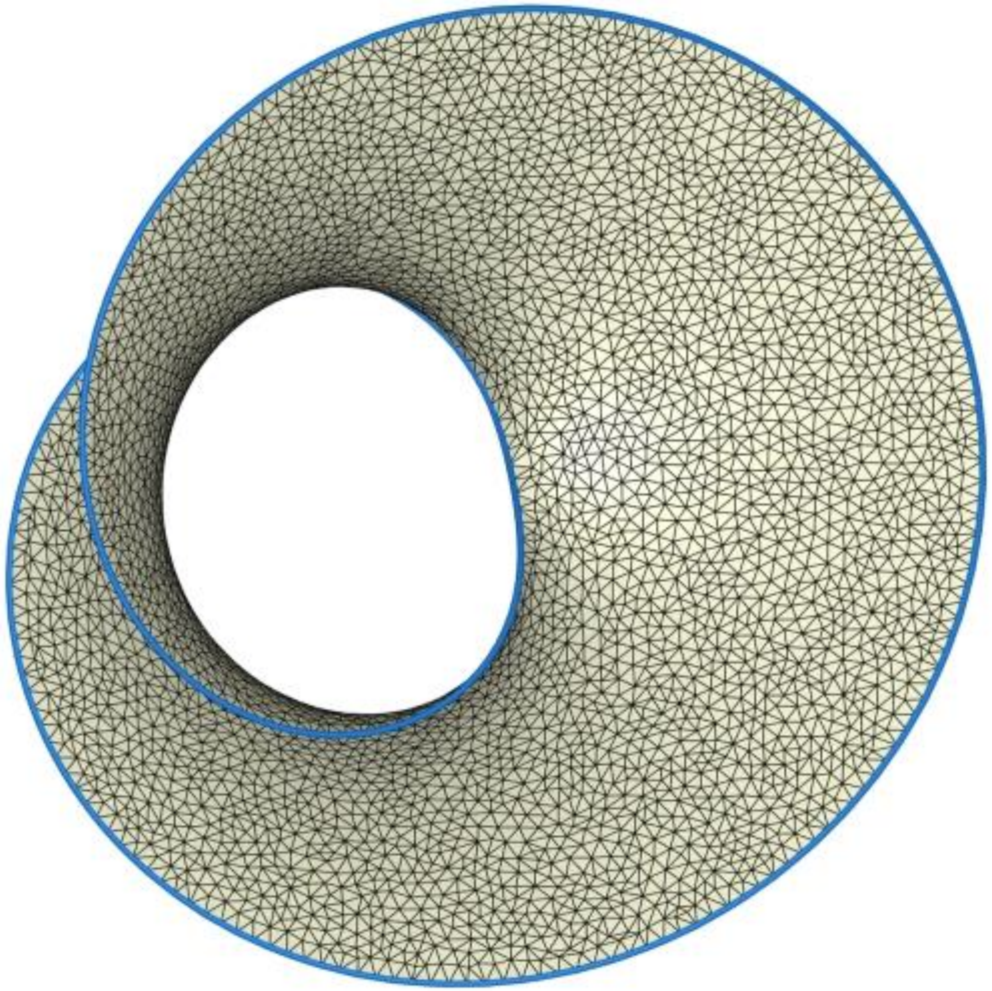
$$F \approx 2V$$

$$\text{average valence} \approx 6$$

Why?!

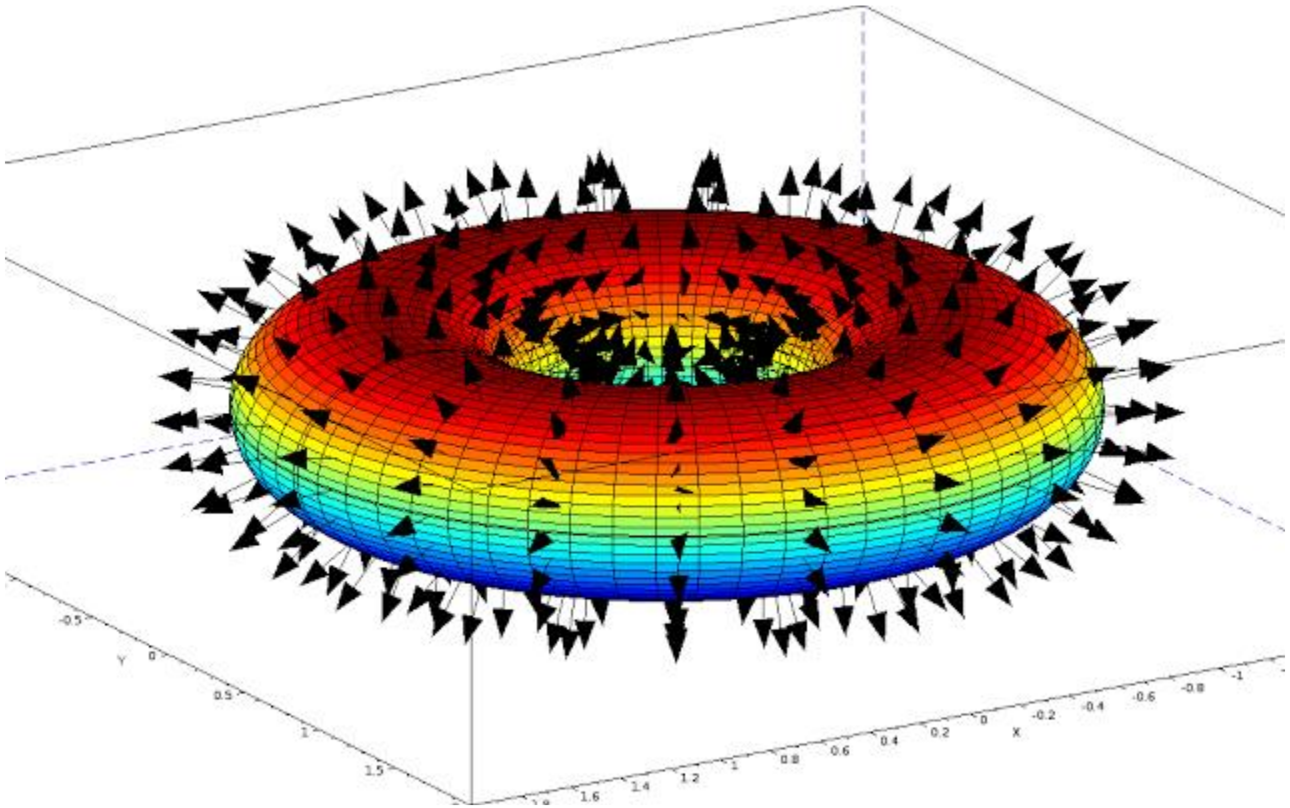
General estimates

Orientability



http://www.cse.ohio-state.edu/~tamaldey/isotopic.html

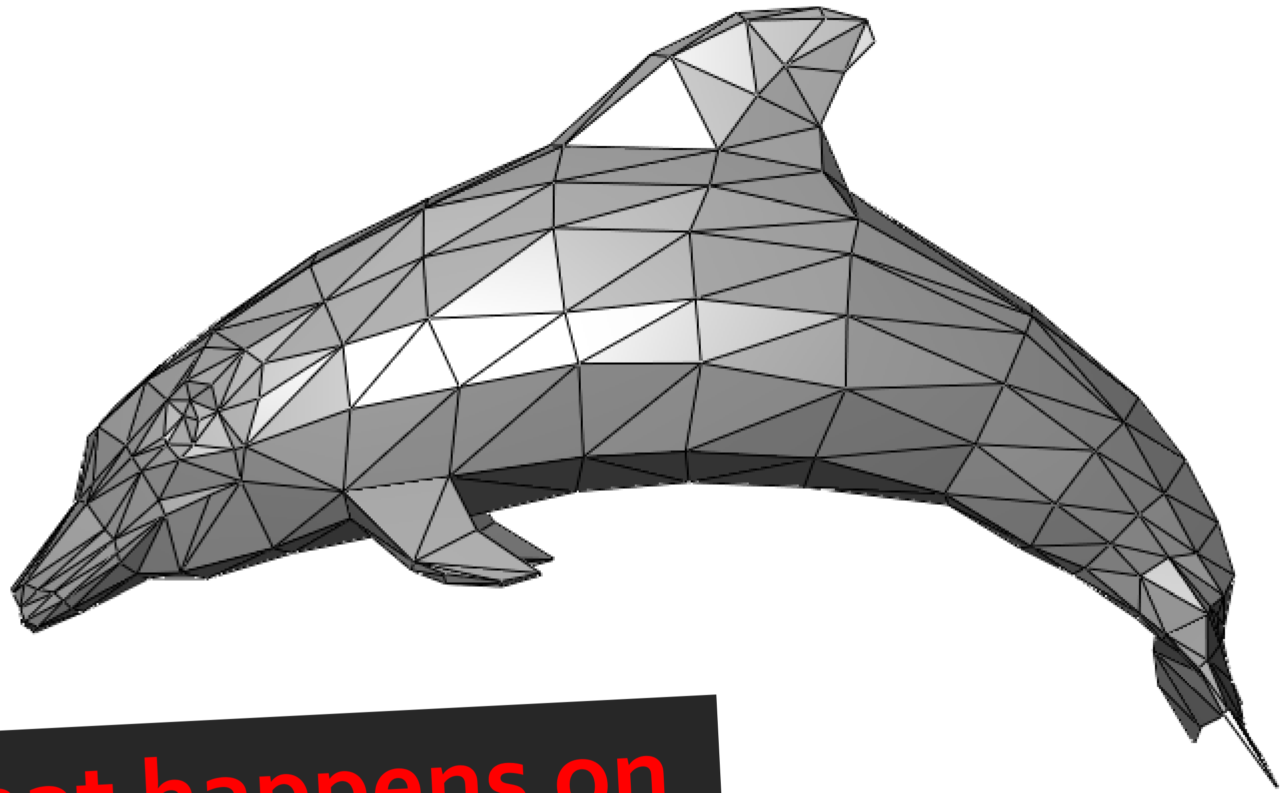
Smooth Surface Definition



https://lh3.googleusercontent.com/-njXPH7NSX5c/VV4PXu54n9I/AAAAAAAAJjM/m6TGg3ZVKGE/w640-h400-p-k/normal_tore.png

Continuous field of normal vectors

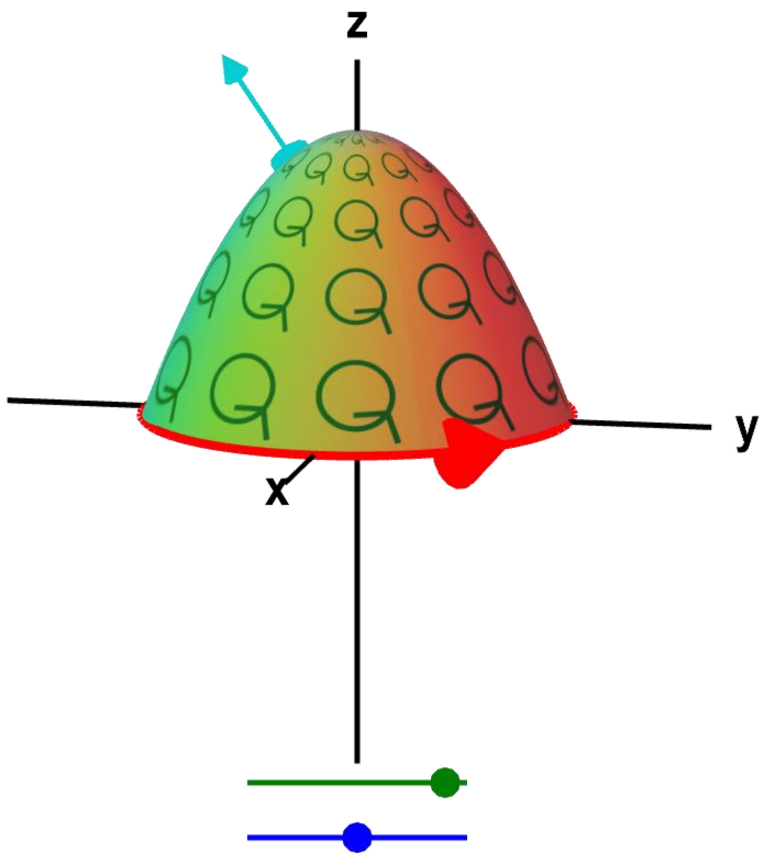
Issue on Triangle Mesh

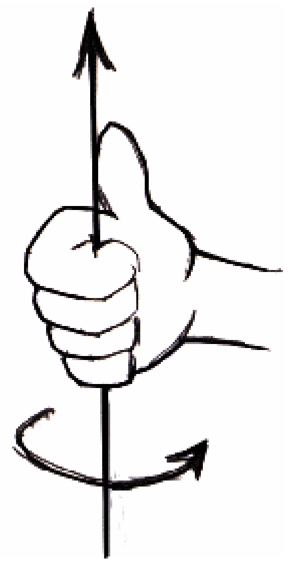


What happens on edges/vertices?

http://upload.wikimedia.org/wikipedia/commons/f/fb/Dolphin_triangle_mesh.png

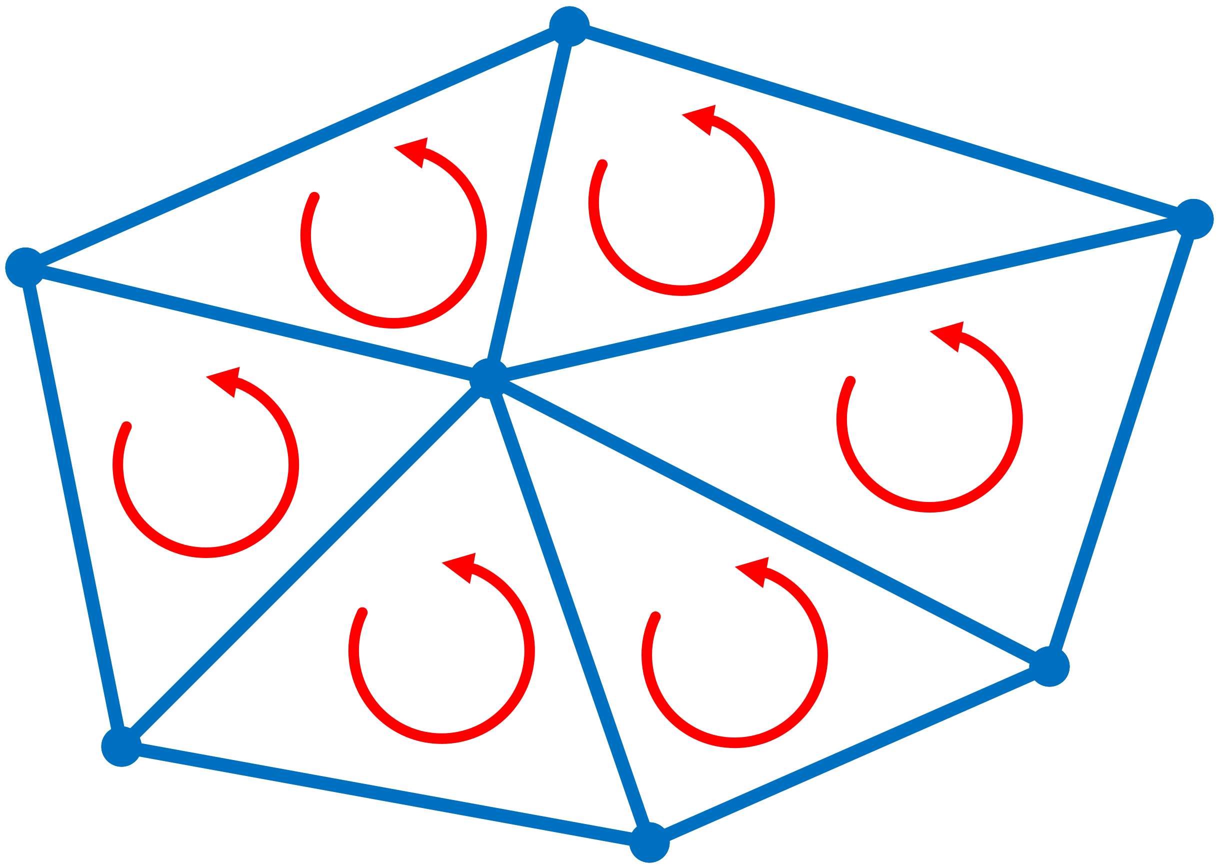
Right-Hand Rule





http://viz.aset.psu.edu/gho/sem_notes/3d_fundamentals/html/3d_coordinates.html
http://mathinsight.org/stokes_theorem_orientation

Discrete Orientability



Normal field isn't continuous

Data Structures for Surfaces

Must represent geometry <u>and</u> topology.

Simplest Format

```
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
```

No topology!

`glBegin(GL_TRIANGLES)`

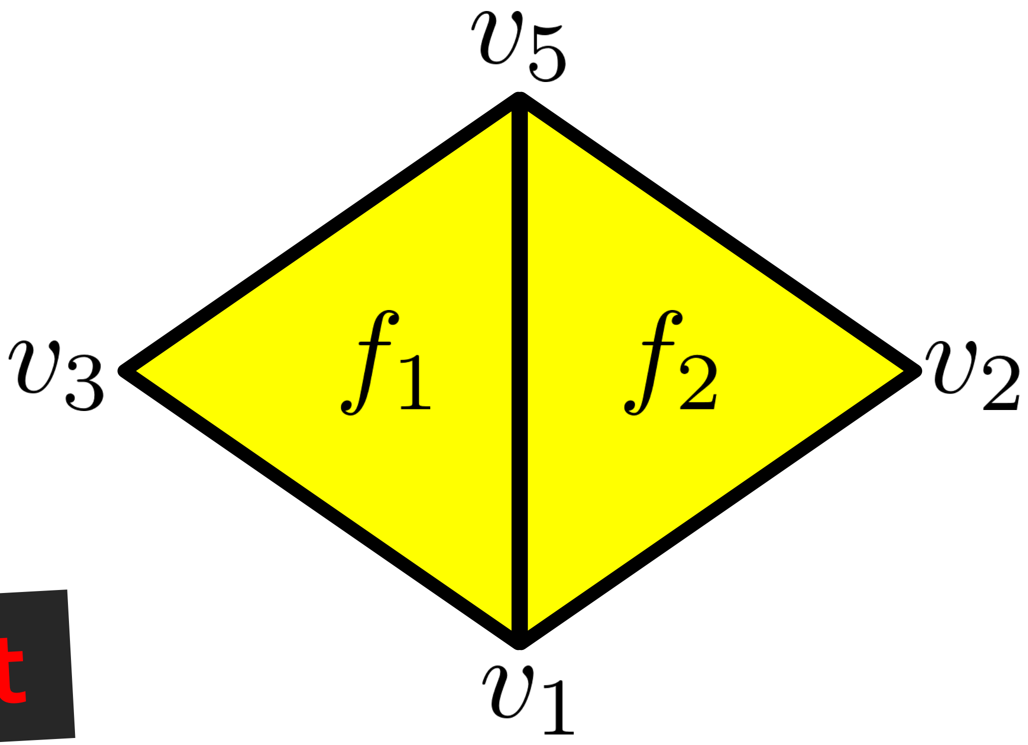
CS 468 2011 (M. Ben-Chen), other slides

Triangle soup

Factor Out Vertices

```
f 1 5 3
f 5 1 2
…
v 0.2 1.5 3.2
v 5.2 4.1 8.9
…
```

.obj format



CS 468 2011 (M. Ben-Chen), other slides

Shared vertex structure

Simple Mesh Smoothing

```
for i=1 to n
   for each vertex v
      v = .5*v +
        .5*(average of neighbors);
```

Typical Queries

- Neighboring vertices to a vertex
- Neighboring faces to an edge
- Edges adjacent to a face
- Edges adjacent to a vertex
- …

Mostly localized

Typical Queries

- Neighboring vertices to a vertex
- Neighboring faces to an edge
- Edges adjacent to a face
- Edges adjacent to a vertex
- …

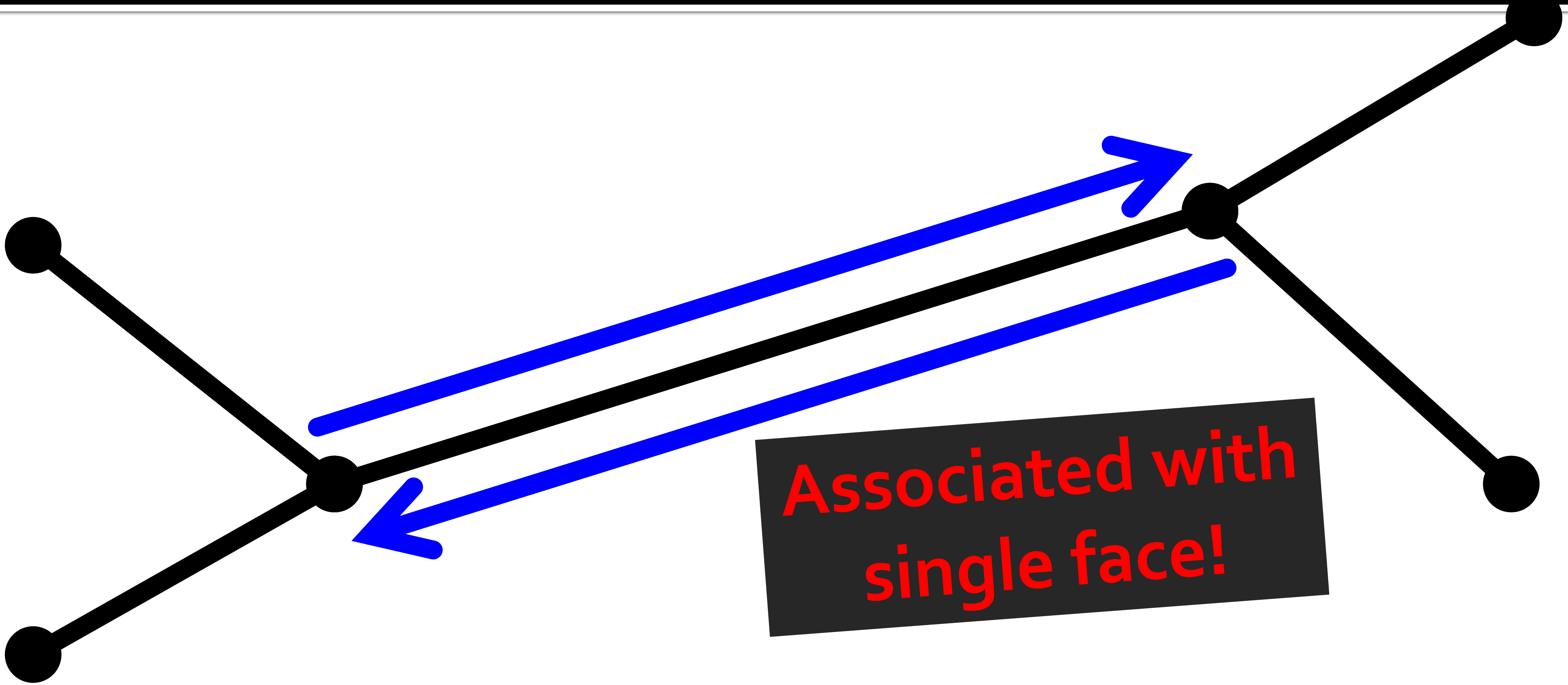
Mostly localized

Pieces of Halfedge Data Structure

- Vertices
- Faces
- Half-edges

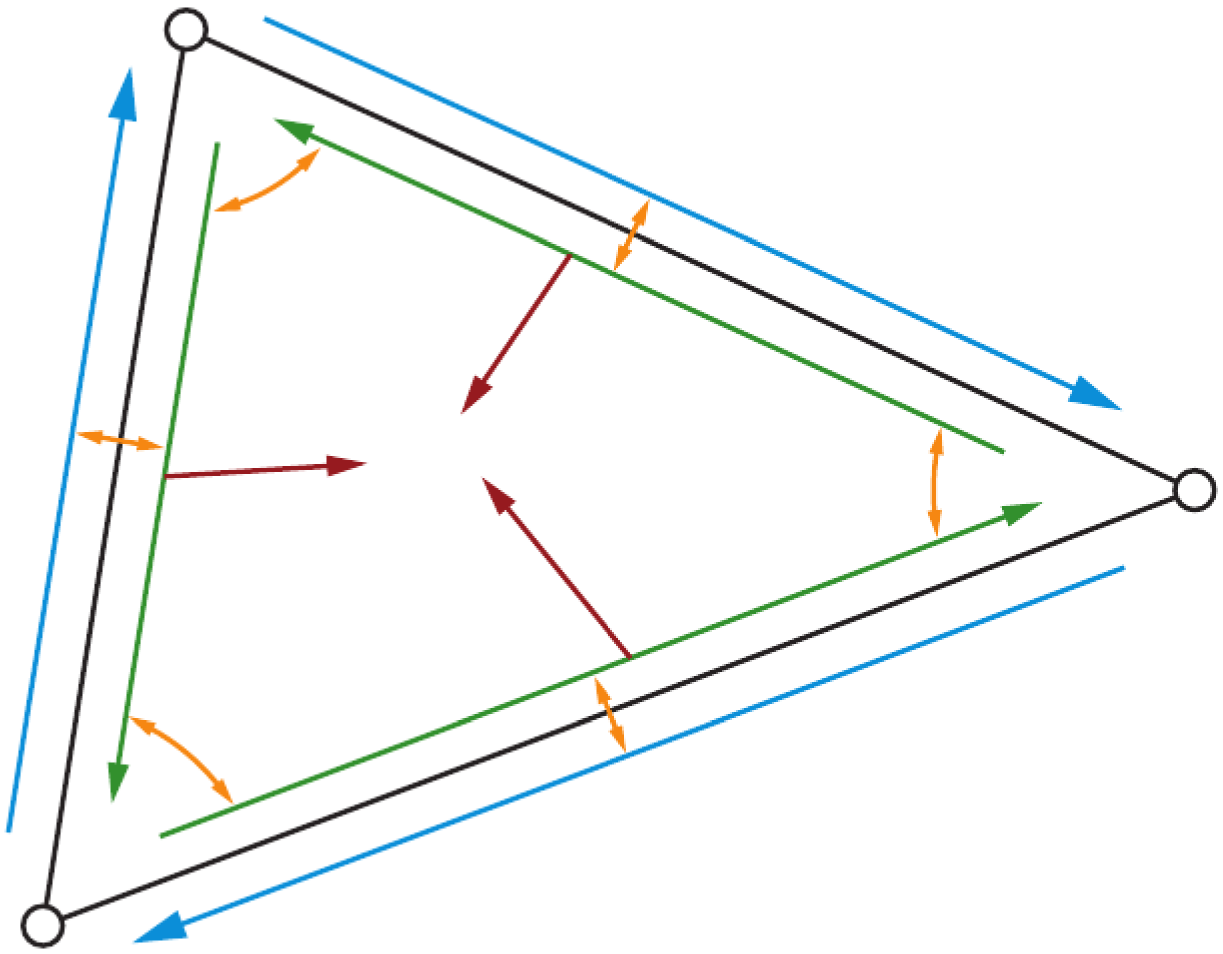
Structure tuned for meshes

Halfedge?



Oriented edge

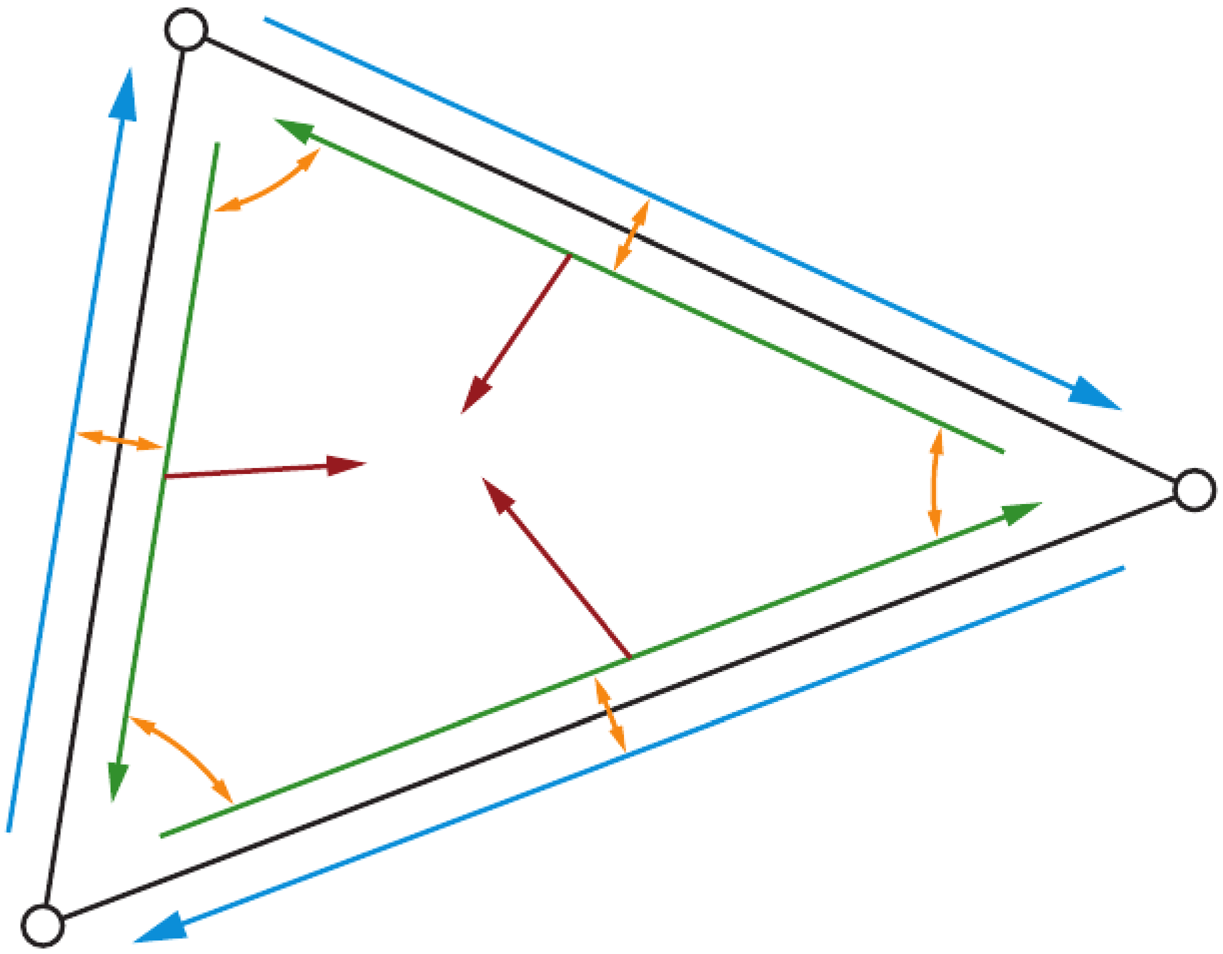
Halfedge Data Types



Vertex stores:

- Arbitrary outgoing halfedge

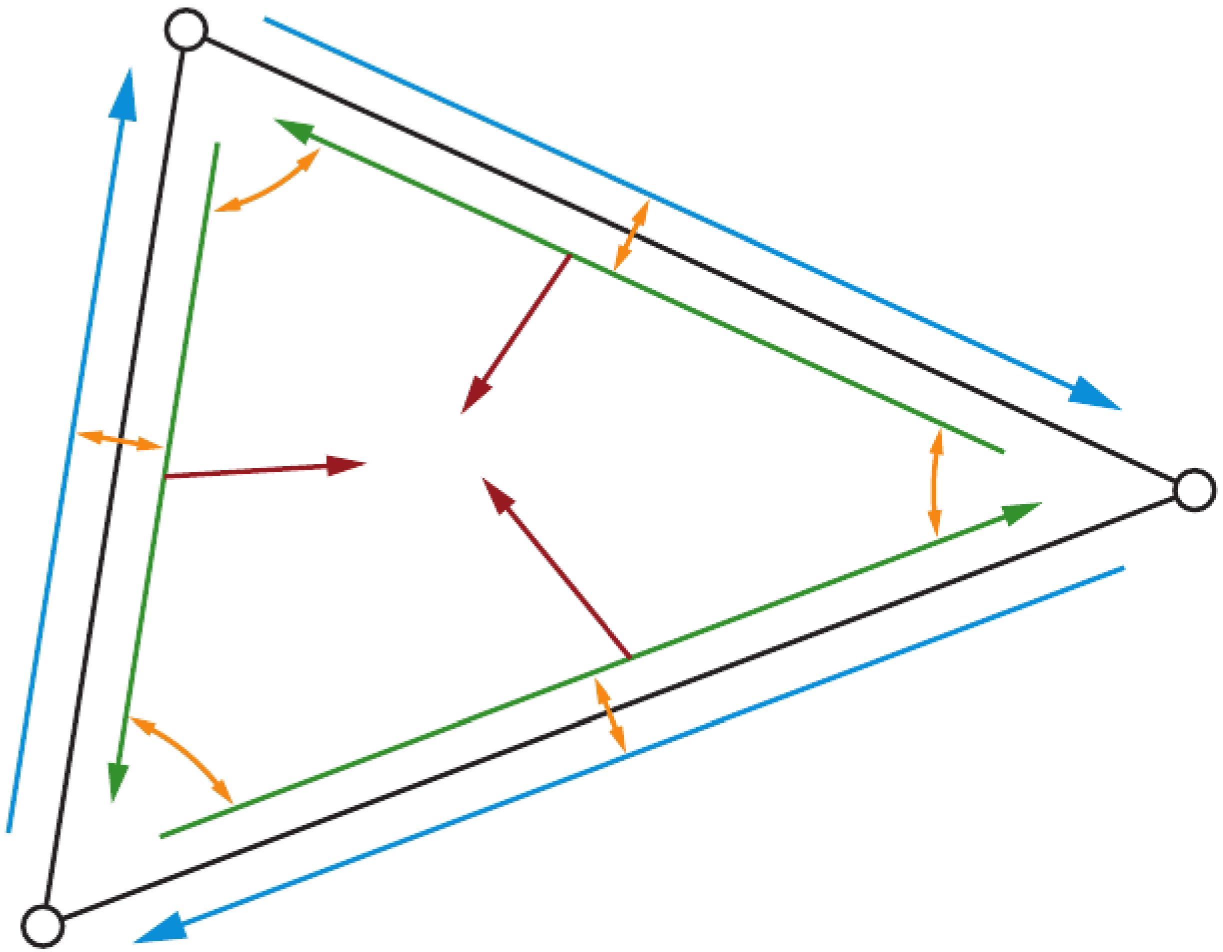
Halfedge Data Types



Face stores:

- Arbitrary adjacent halfedge

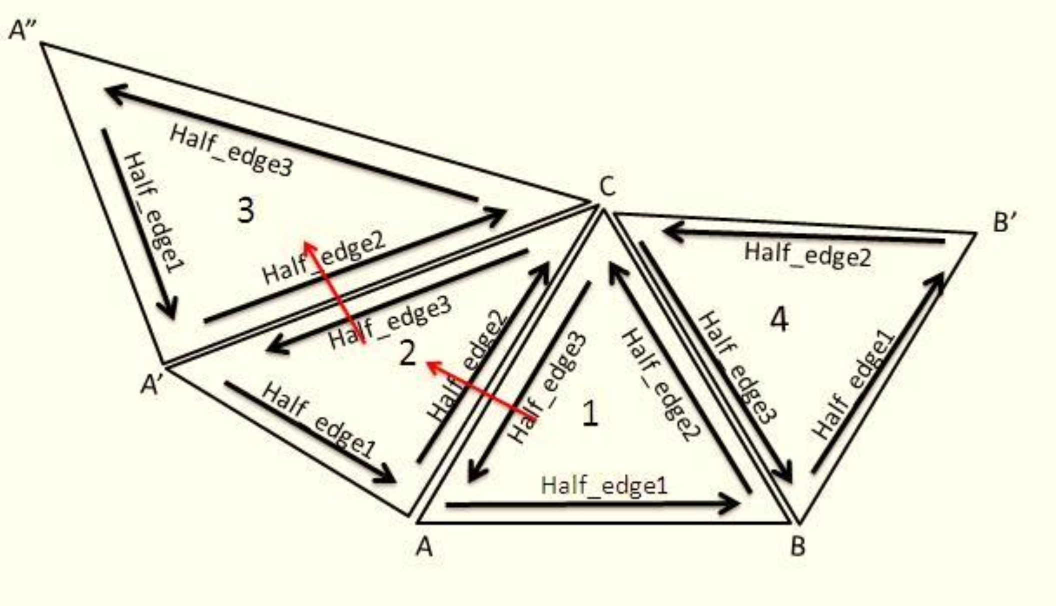
Halfedge Data Types



Halfedge stores:

- Flip
- Next
- Face
- Vertex

Iterating Over Vertex Neighbors




```
Iterate(v):
startEdge = v.out;
e = startEdge;
do
    process(e.flip.from)
    e = e.flip.next
while e != startEdge
```

https://sympa.inria.fr/sympa/arc/cgal-discuss/2008-05/msg00054.html

Only Scratching the Surface

Eurographics Symposium on Geometry Processing (2005)
M. Desbrun, H. Pottmann (Editors)

Streaming Compression of Triangle Meshes

Martin Isenburg[1†] Peter Lindstrom[2] Jack Snoeyink[1]

[1] University of North Carolina at Chapel Hill [2] Lawrence Livermore National Labs

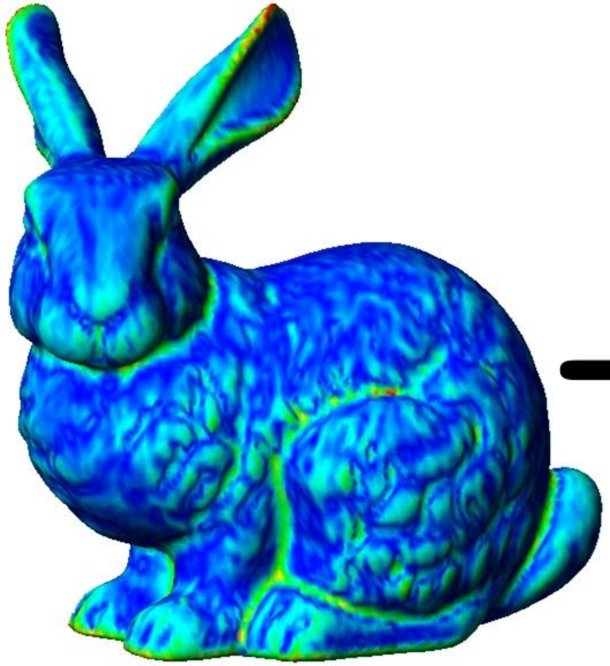
EUROGRAPHICS 2011 / M. Chen and O. Deussen
(Guest Editors) *Volume 30 (2011), Number 2*

SQuad: Compact Representation for Triangle Meshes

Topraj Gurung[1], Daniel Laney[2], Peter Lindstrom[2], Jarek Rossignac[1]

[1]Georgia Institute of Technology
[2]Lawrence Livermore National Laboratory

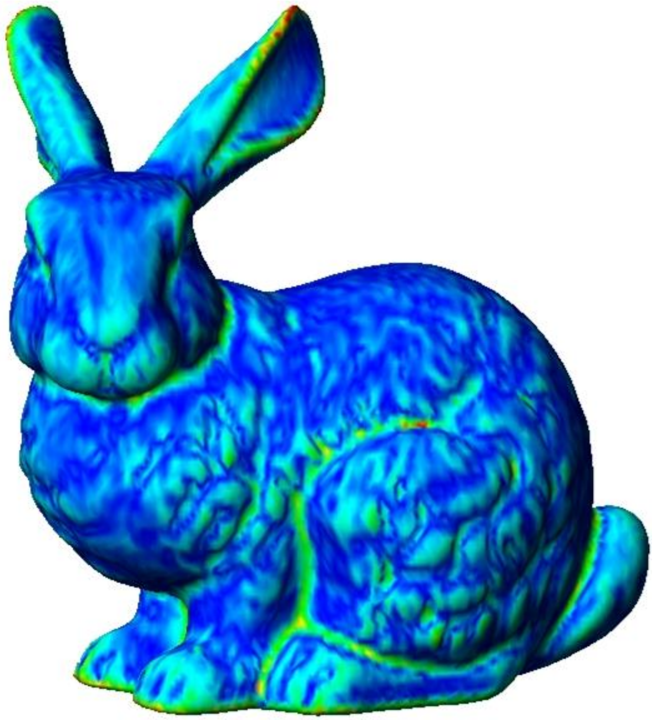
Scalar Functions

$f:$  $\longrightarrow \mathbb{R}$

http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

Map points to real numbers

Discrete Scalar Functions



$$f \in \mathbb{R}^{|V|}$$

http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

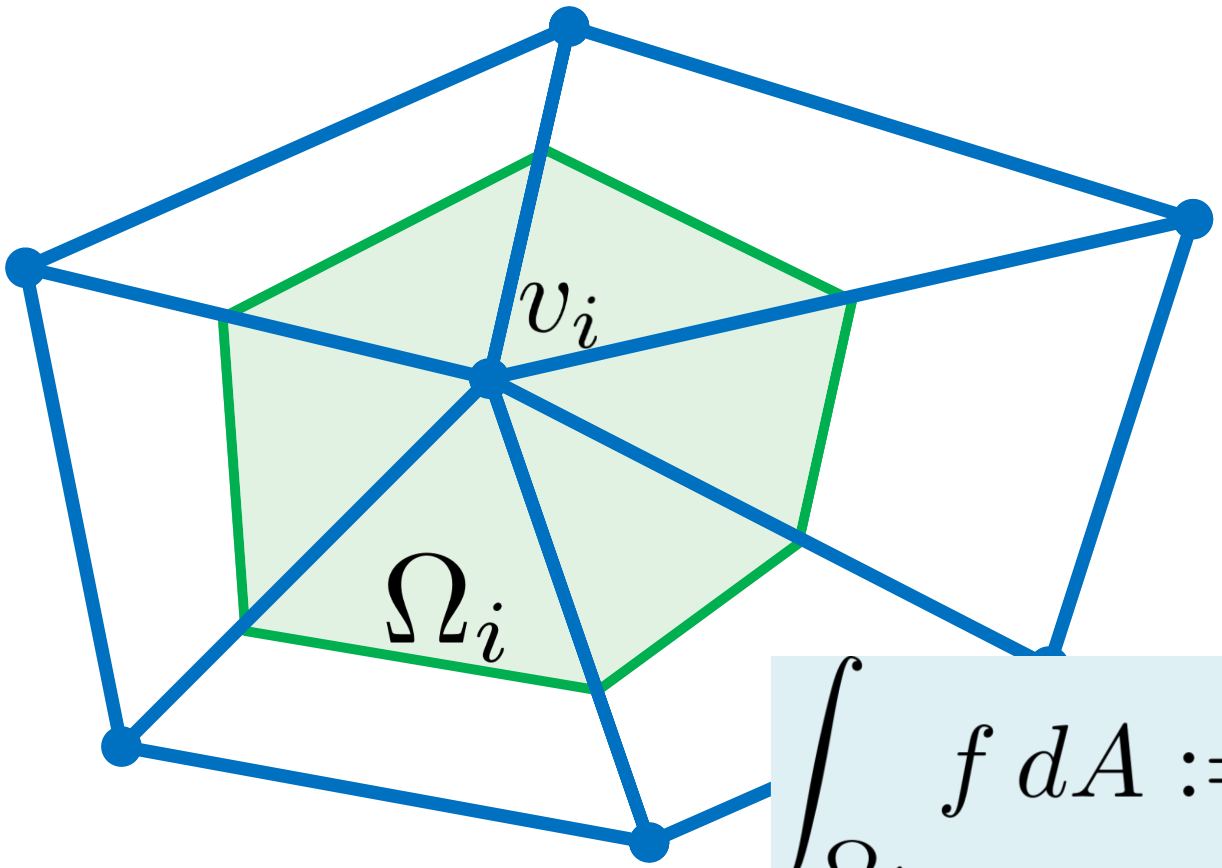
Map vertices to real numbers

Question

What is the integral of *f*?

$$\int_M f \, dA$$

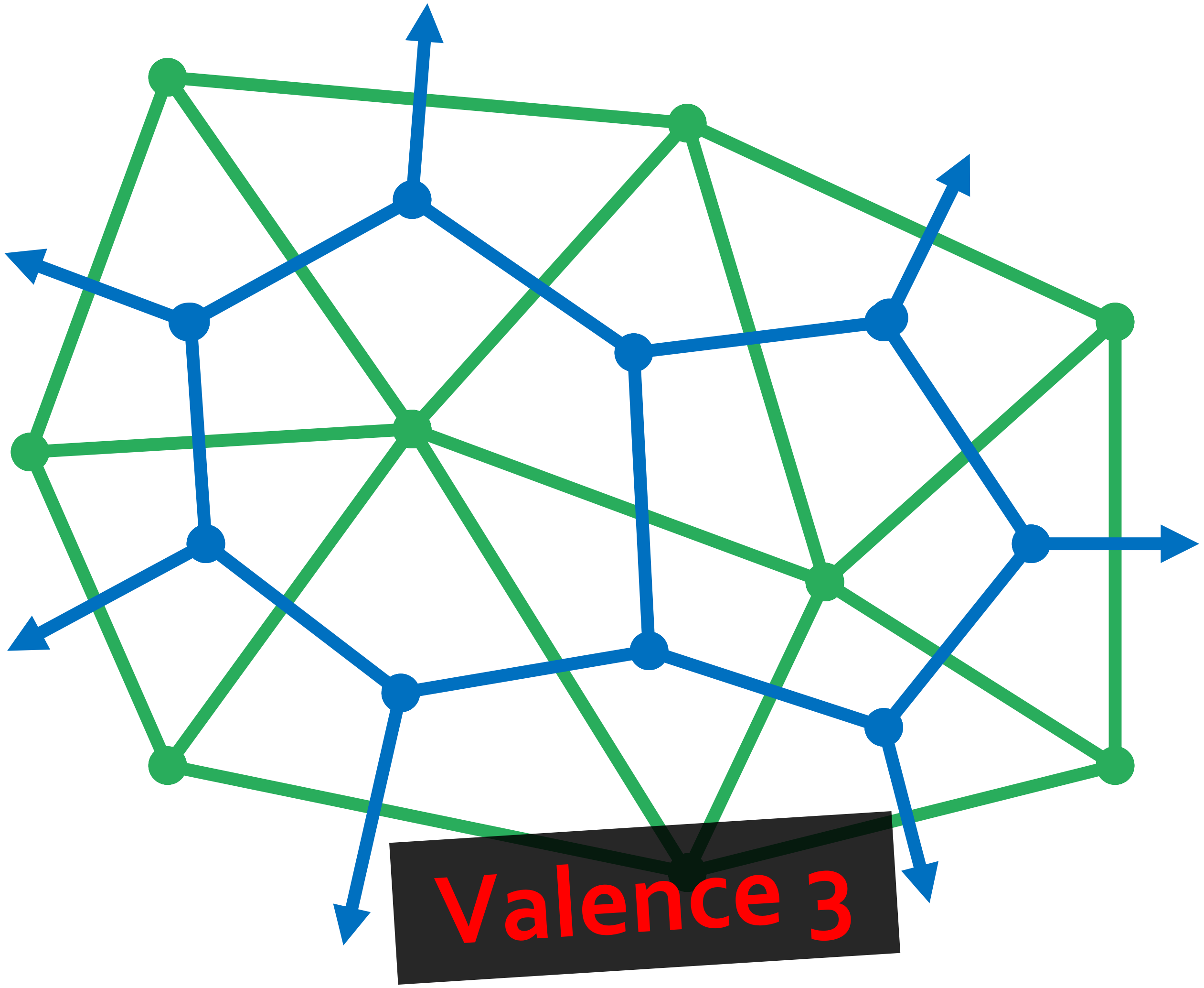
Dual Cell



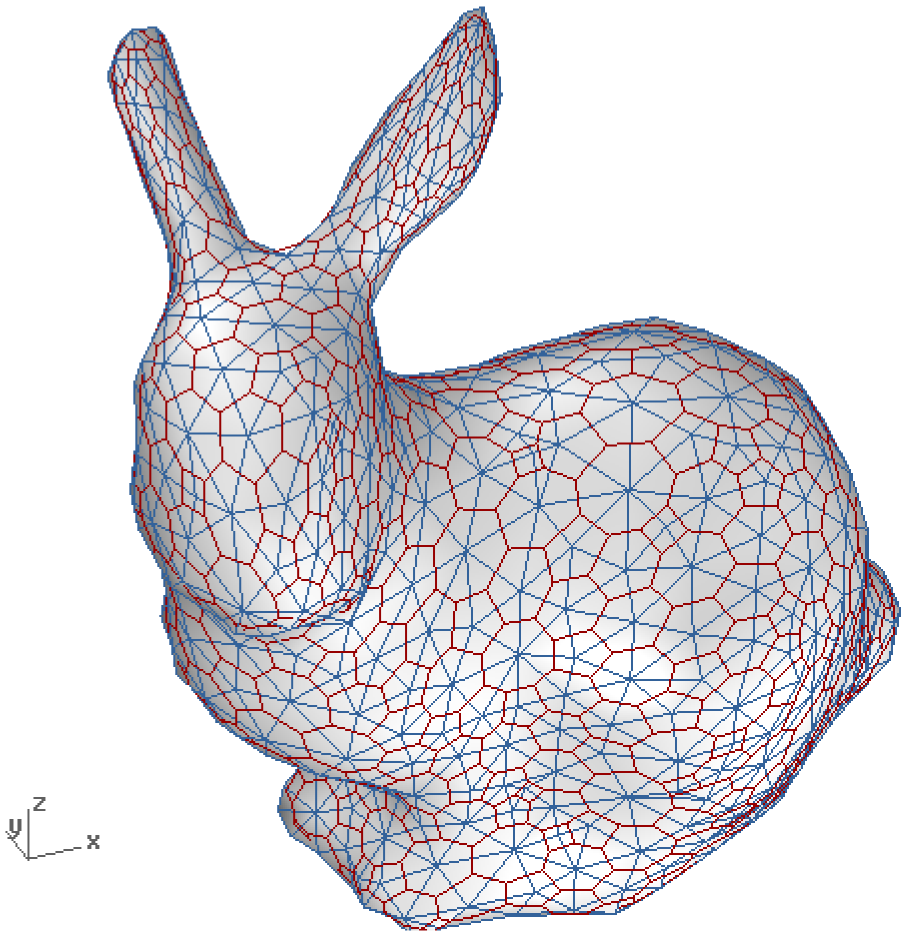
$$\int_{\Omega_i} f\, dA := f_i |\Omega_i|$$

Discrete version of dA

Dual Complex

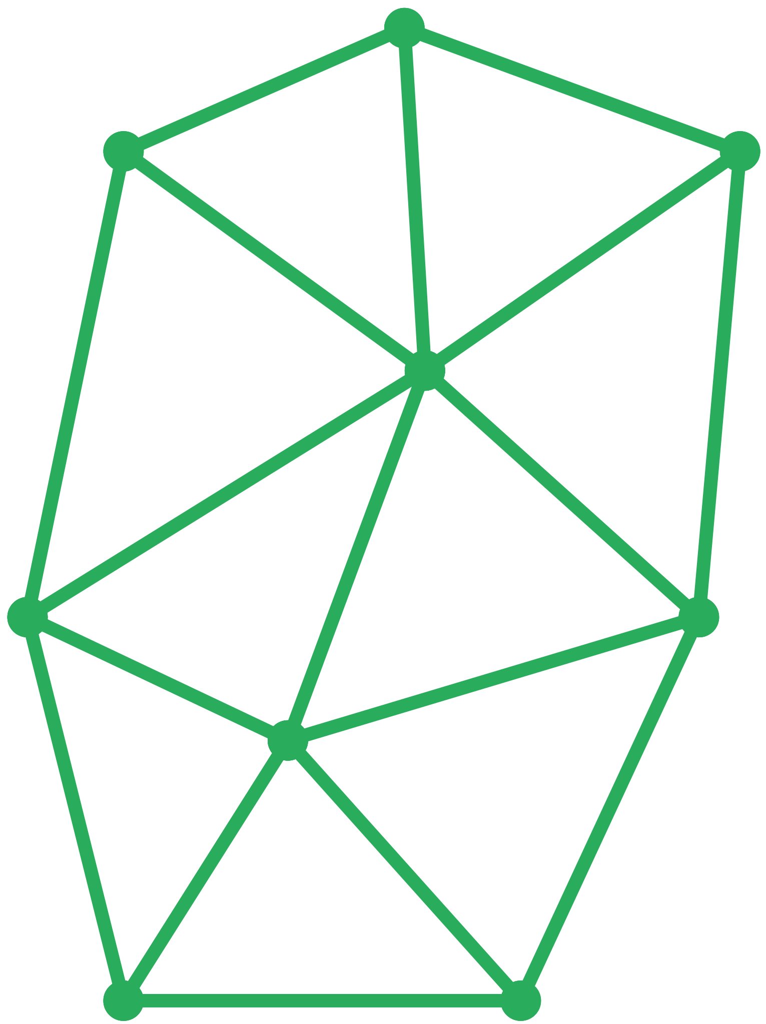


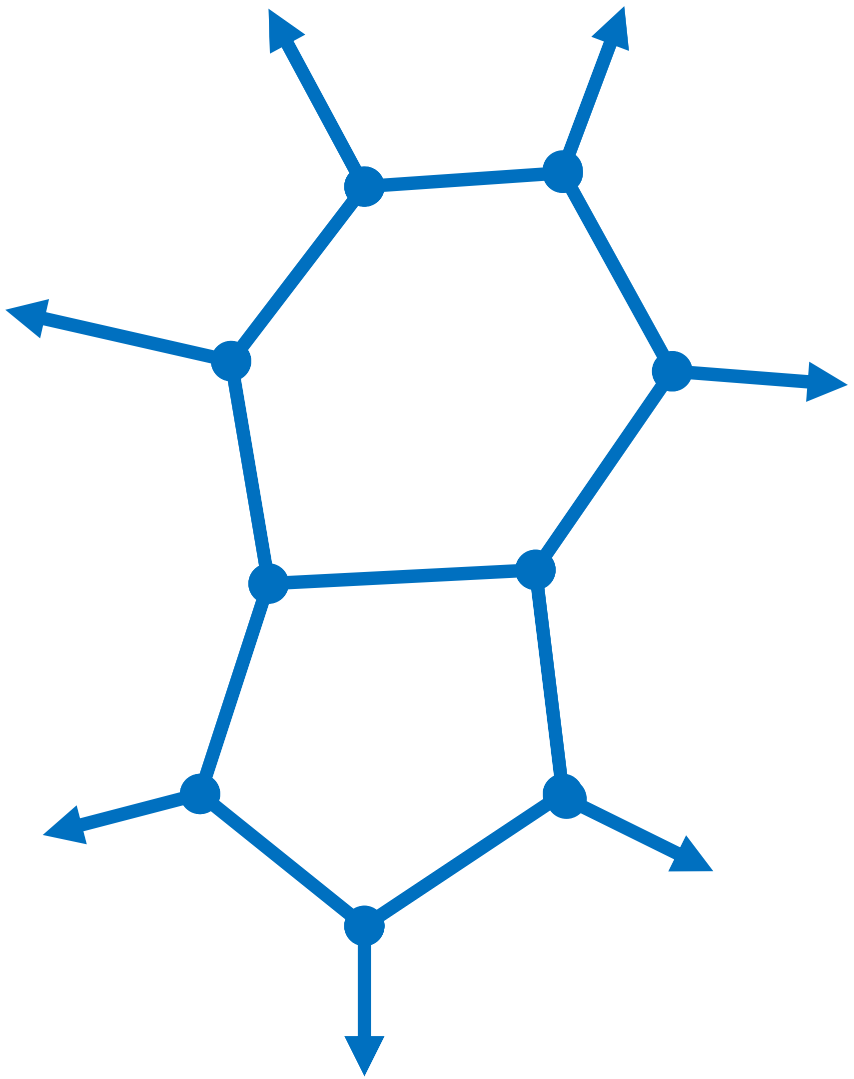
One Surface, Two Meshes



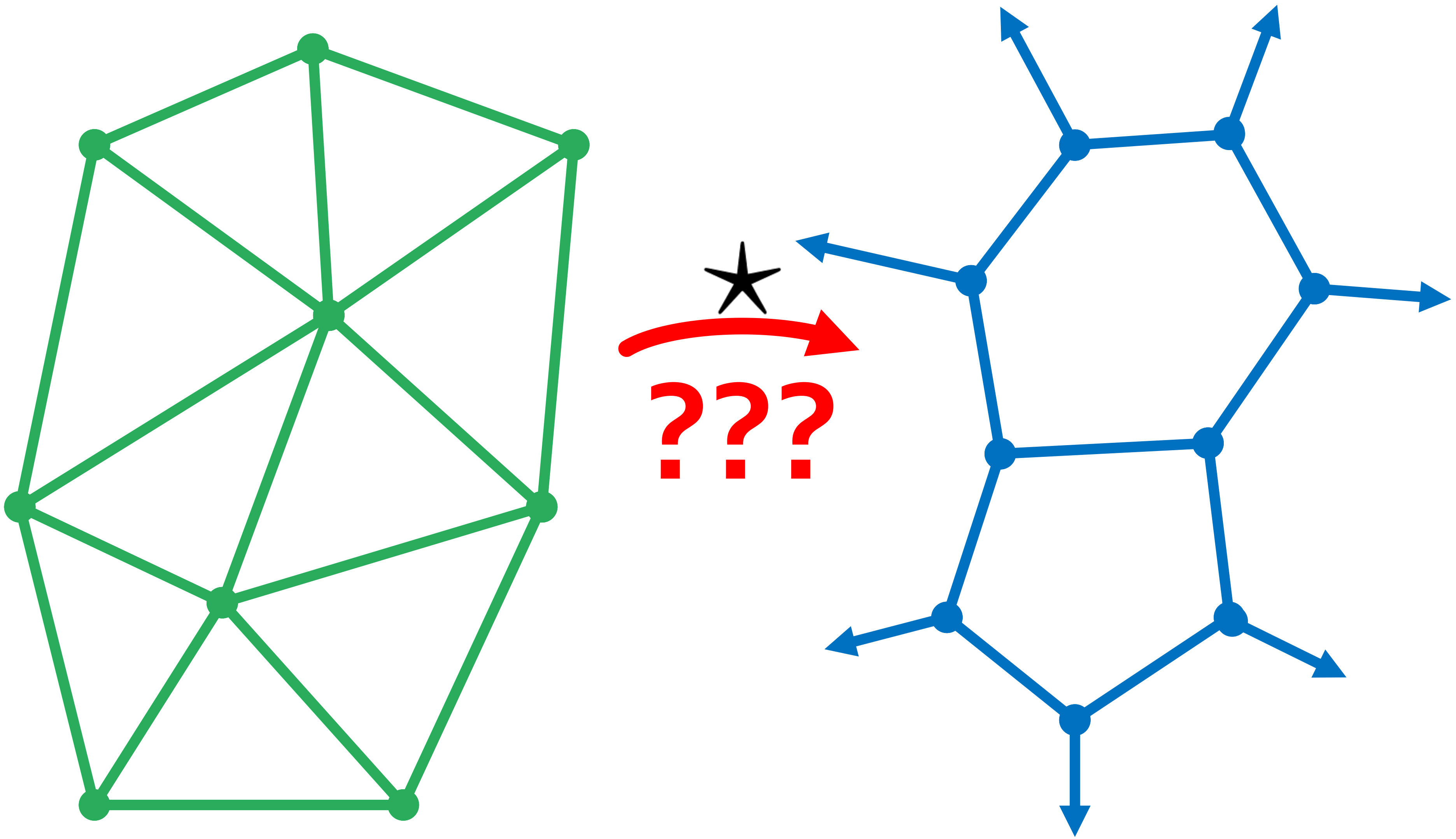
http://www.grasshopper3d.com/group/kangaroo/forum/topics/isosurface-dynamic-remeshing

One Surface, Two Halfedges

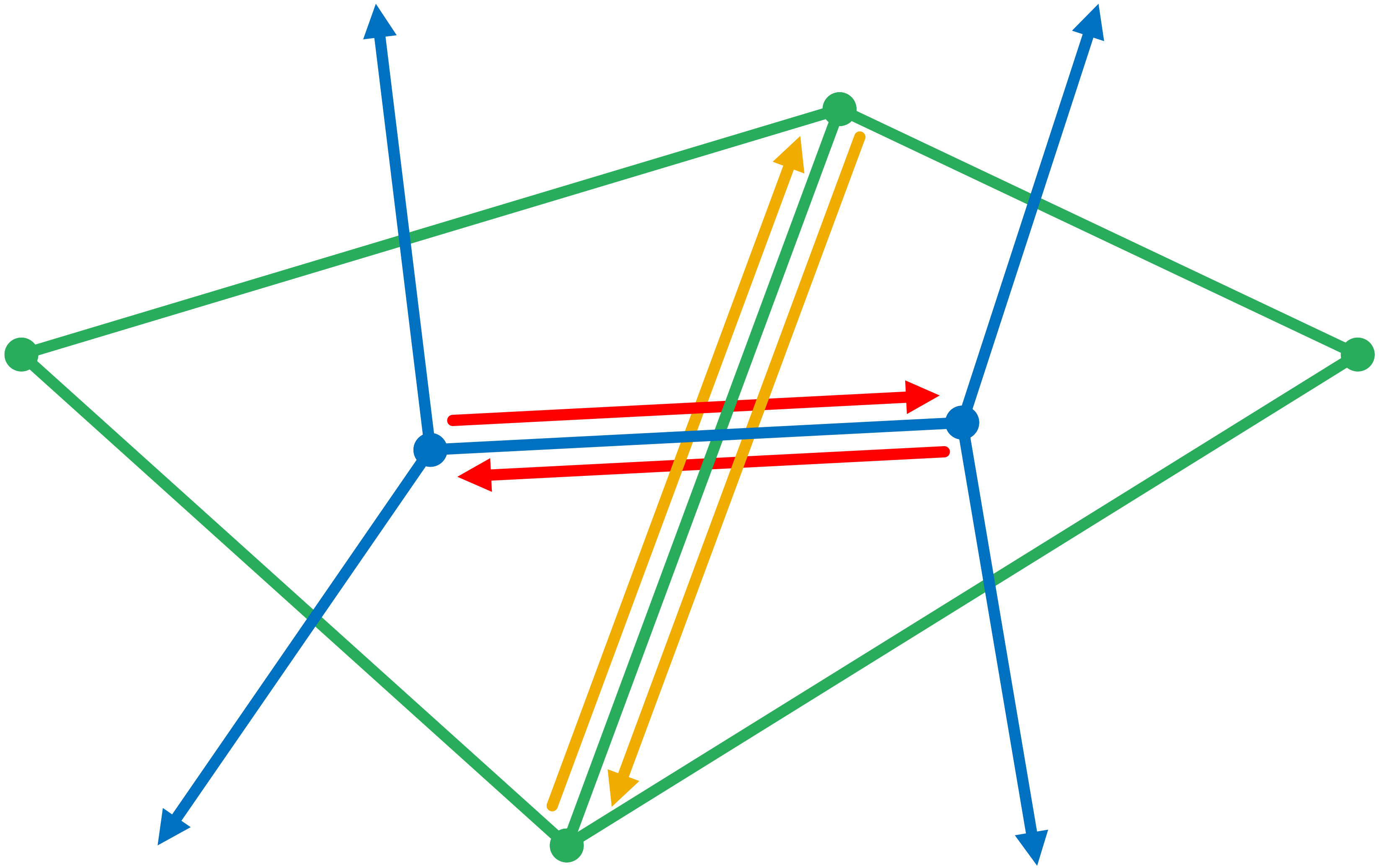




Missing Operation

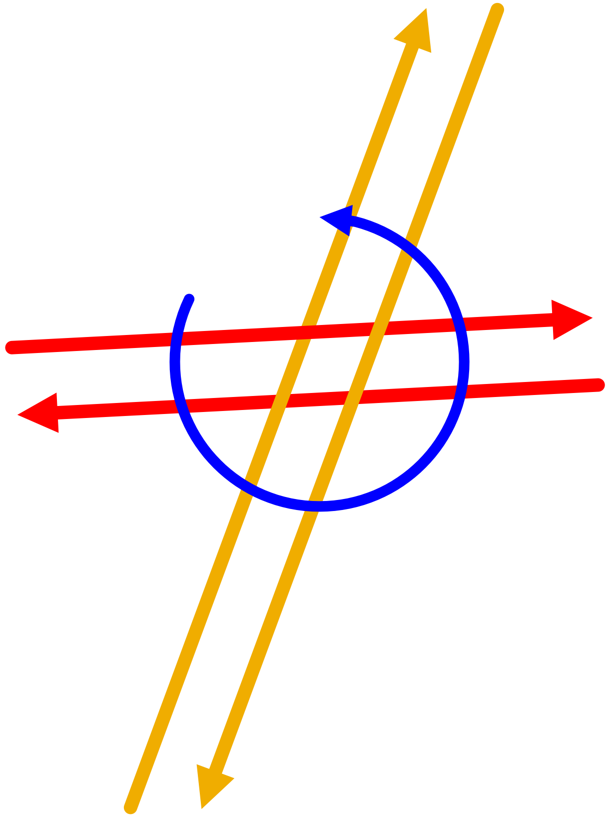


Quad Edge

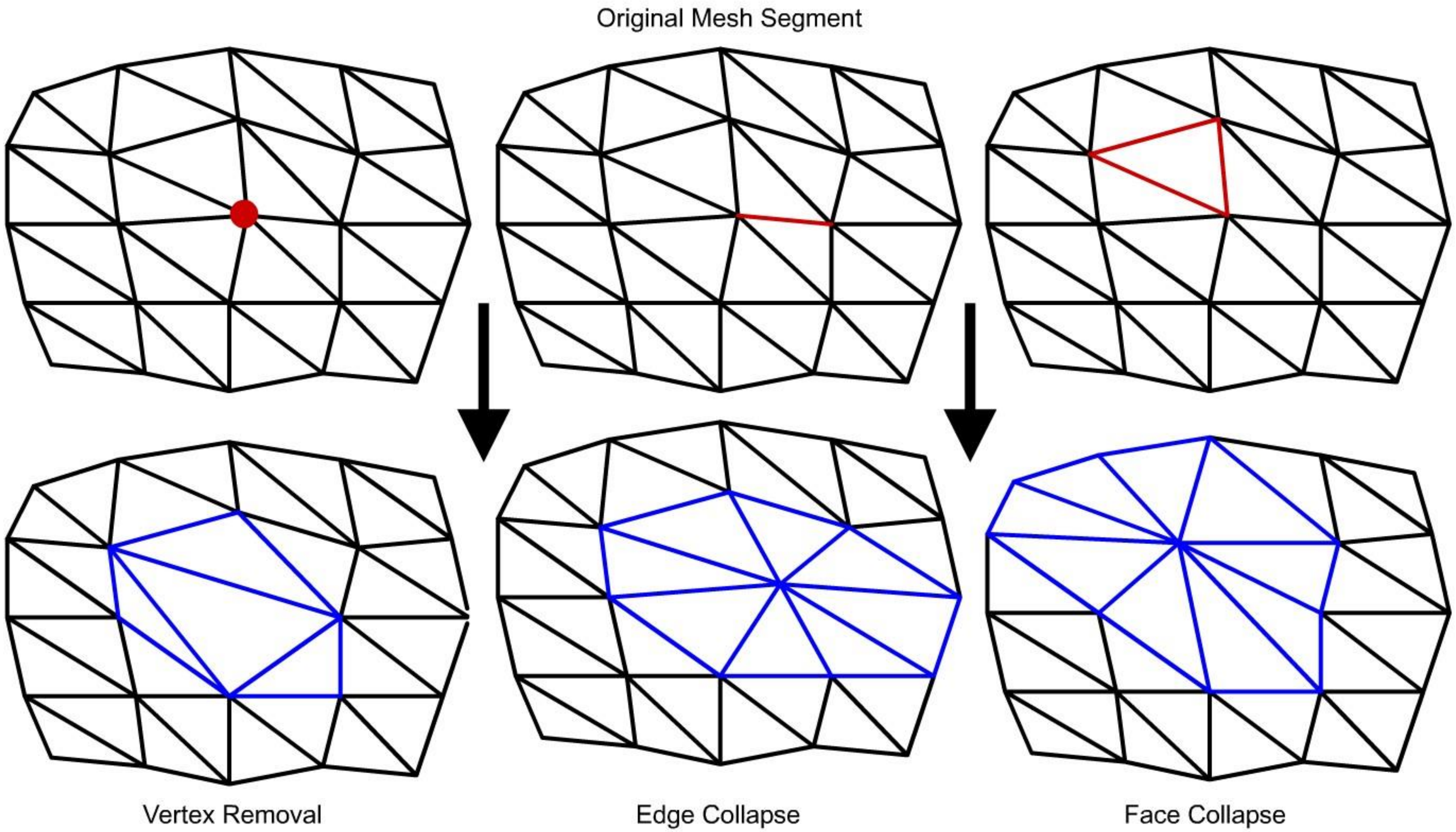


Rotation Operation

$$e \to \text{Rot} \to \text{Rot} = e \to \text{Flip}$$

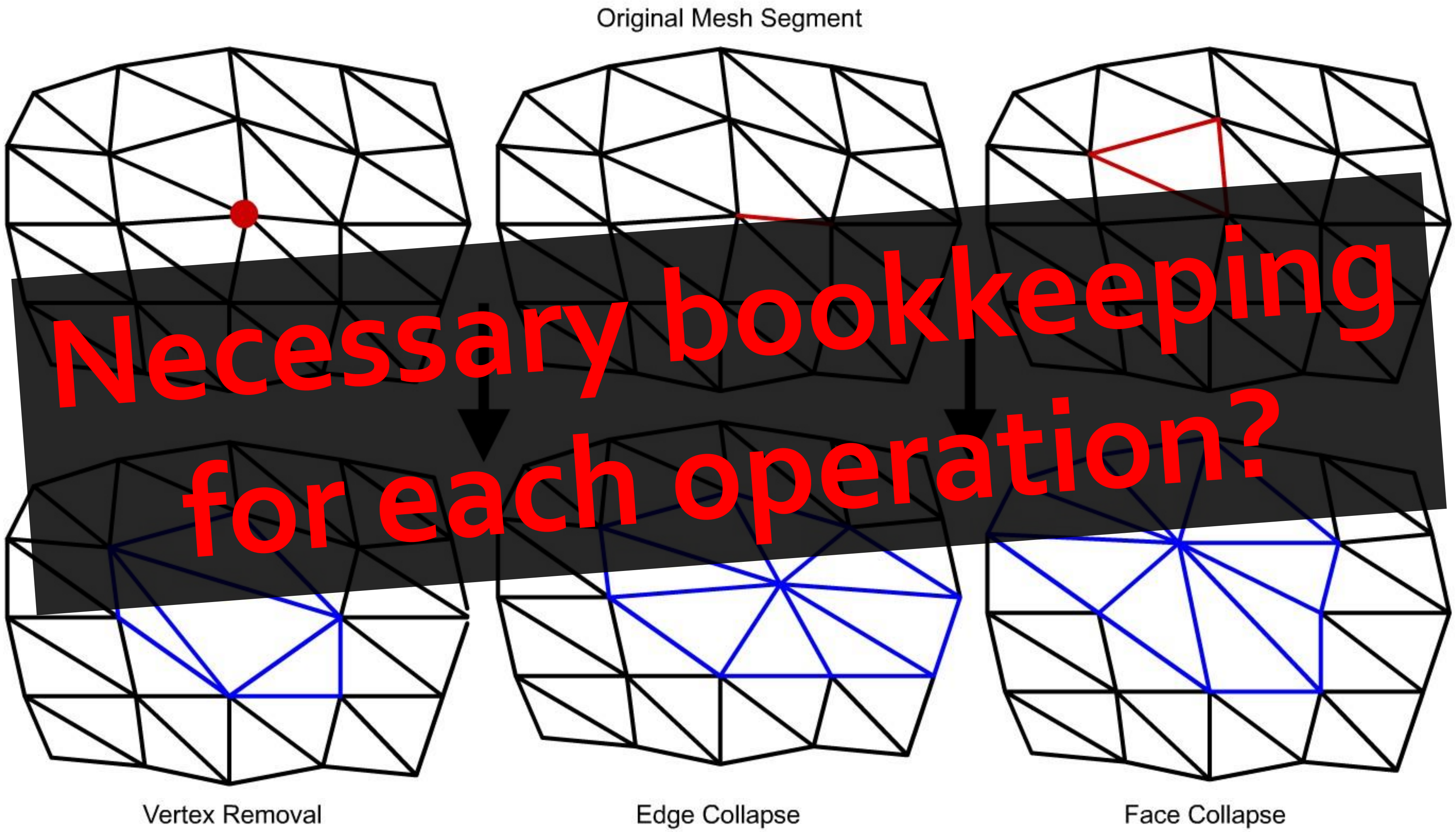


Topological Operations



http://johnrichie.com/V2/richie/simp/images/types1.jpg

Topological Operations



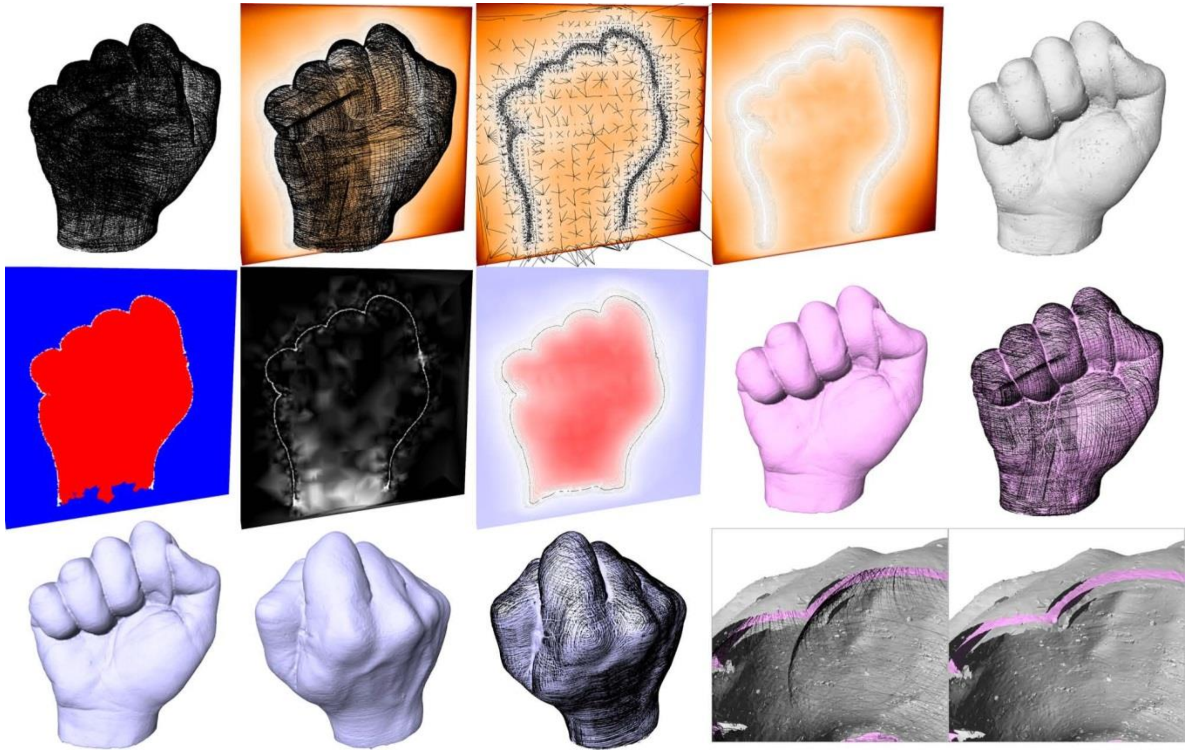
Necessary bookkeeping for each operation?

http://johnrichie.com/V2/richie/simp/images/types1.jpg

Take-Away

Complex data structures enable simpler traversal at cost of more bookkeeping.

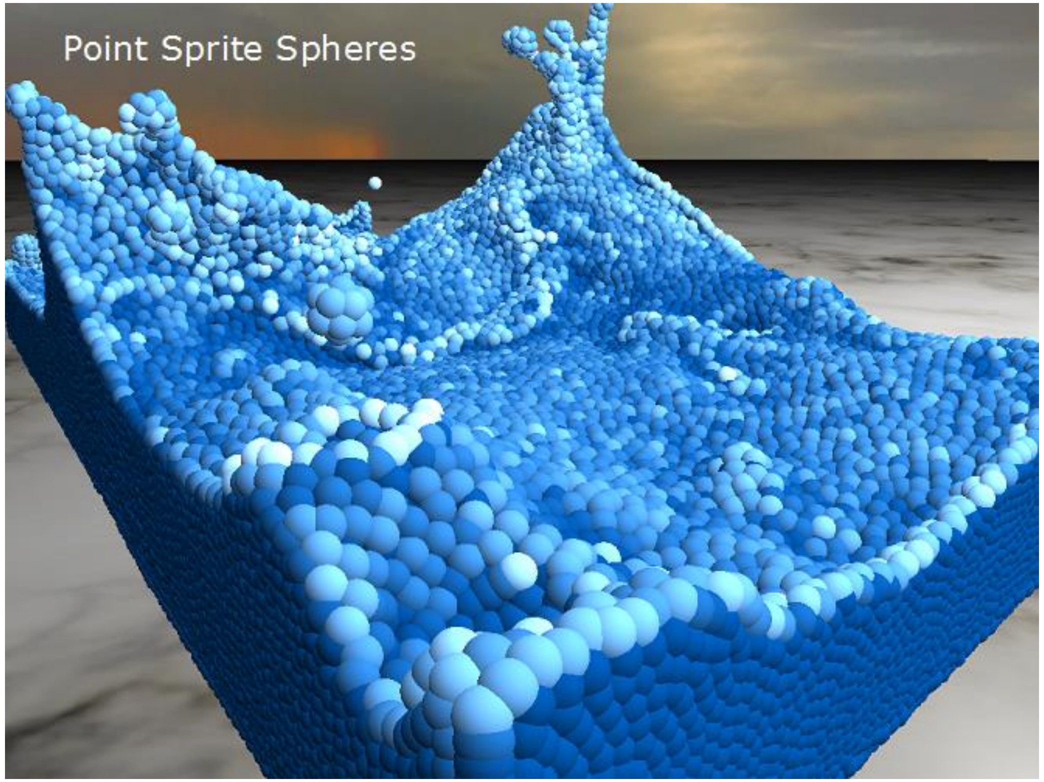
Not the Only Geometric Representation



http://www.cs.umd.edu/class/spring2005/cmsc828v/papers/mpu_implicits.pdf ftp://ftp-sop.inria.fr/geometrica/alliez/signing.pdf

Implicit surfaces

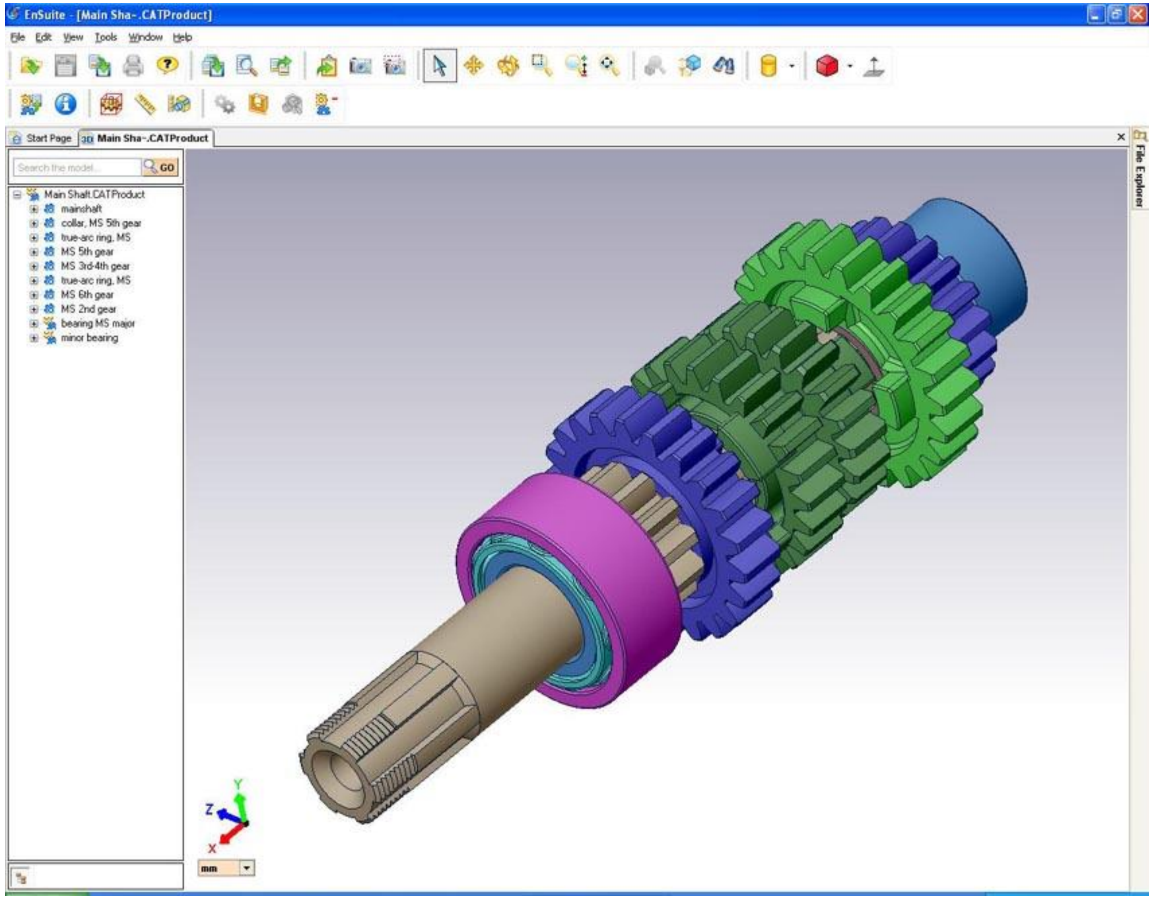
Application of Implicit Surfaces



http://www.itsartmag.com/features/cgfluids/
https://developer.nvidia.com/content/fluid-simulation-alice-madness-returns

Smoothed-particle hydrodynamics (SPH)

Not the Only Geometric Representation

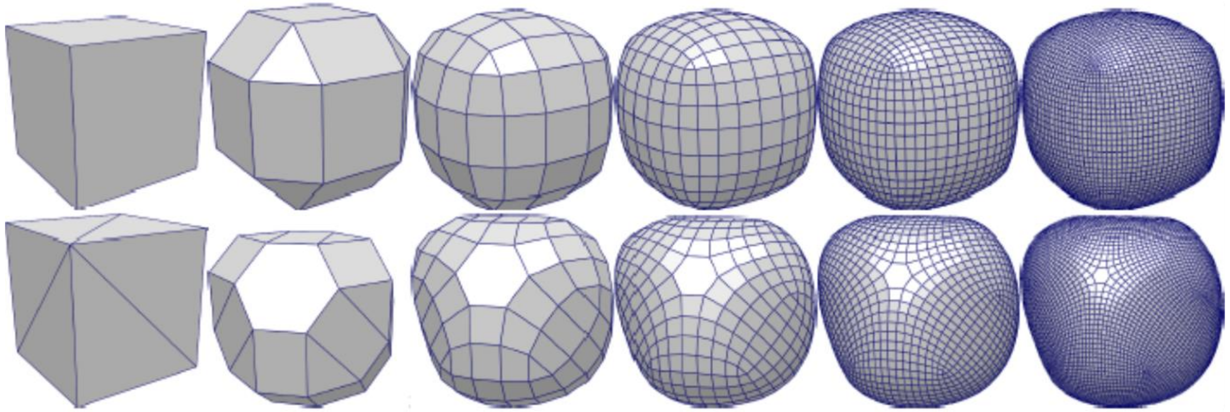


Computer-Aided Design (CAD)

http://www.cad-sourcing.com/wp-content/uploads/2011/12/free-cad-software.jpg

Polynomial/rational patches

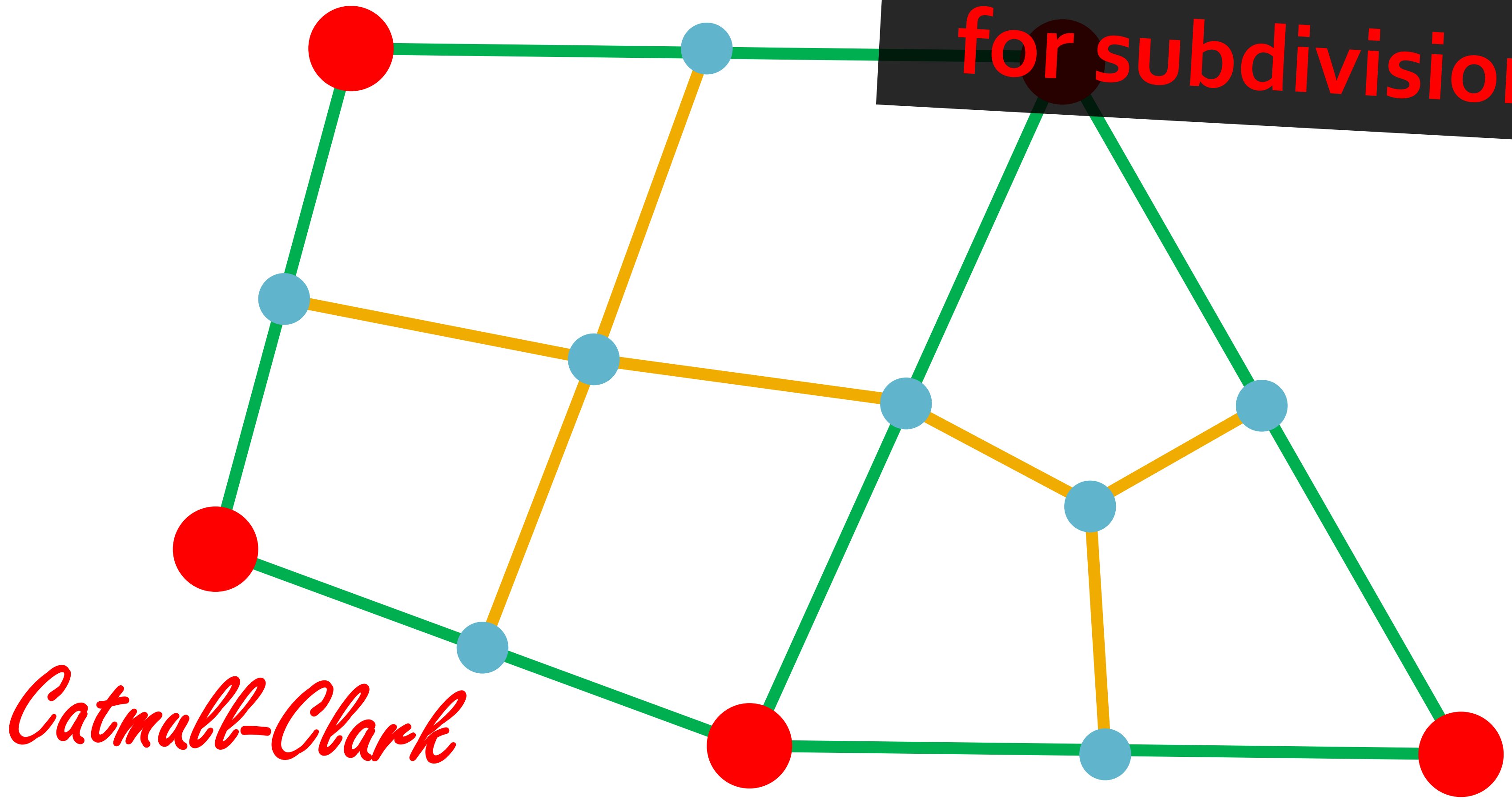
Not the Only Geometric Representation



https://imagecomputing.net/damien.rohmer/teaching/2018_2019/semester_1/m2_mpri_cg_viz/class/01_surface_representation/content/035_subdivision_surfaces/index.html

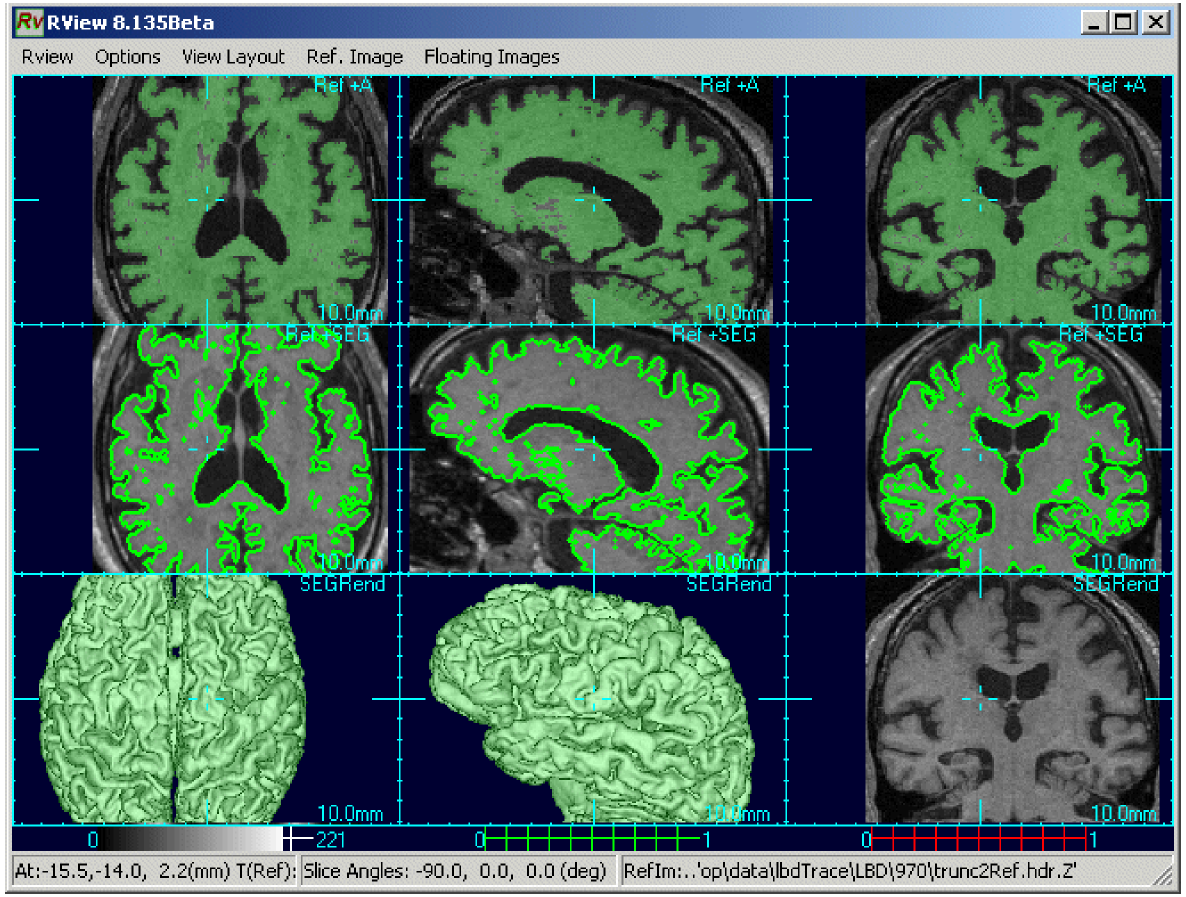
Subdivision Surfaces

Aside



Halfedge suited for subdivision!

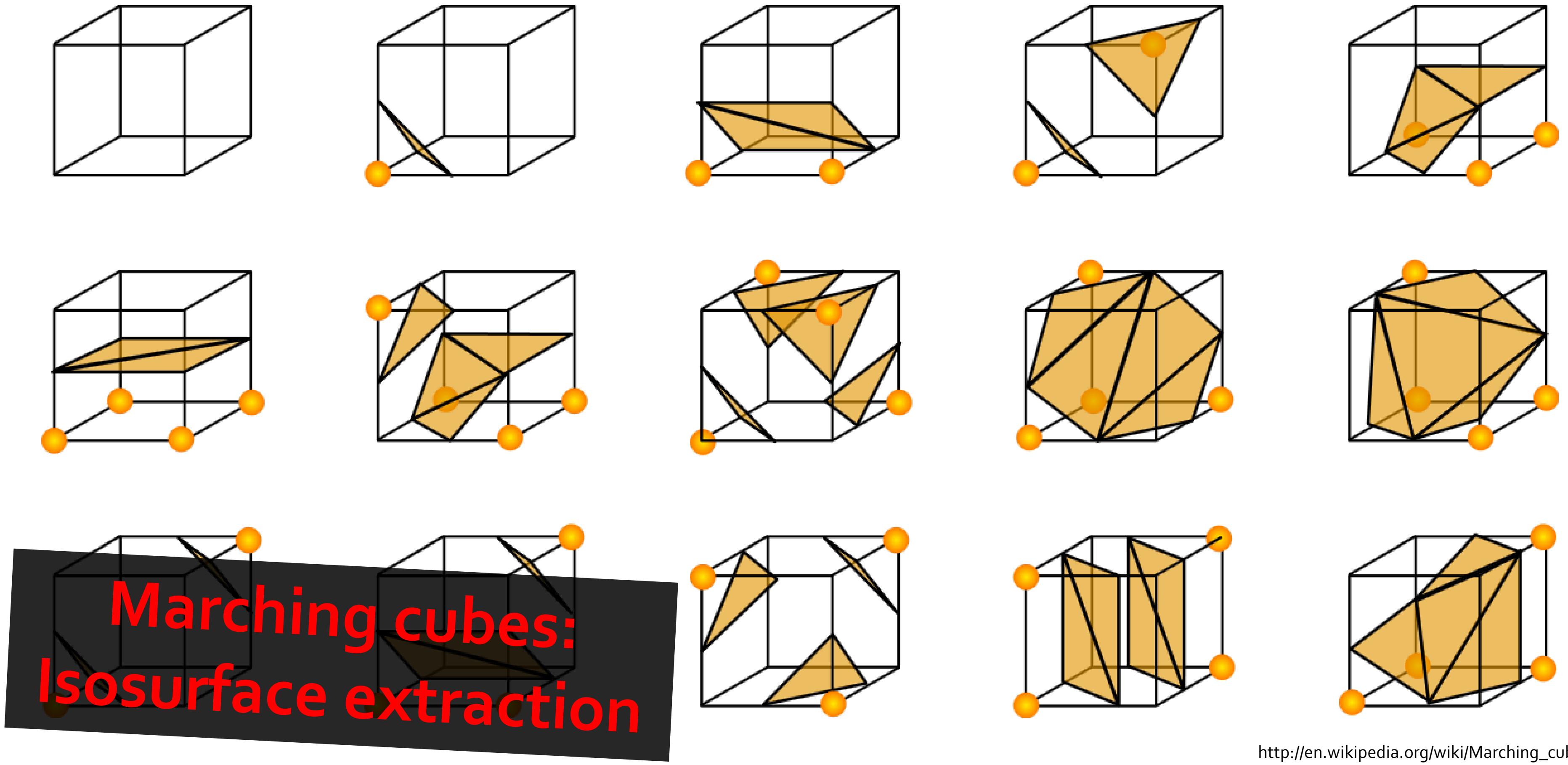
Not the Only Geometric Representation



http://www.colin-studholme.net/software/rview/rvmanual/morphtool5.gif

Volumetric imaging

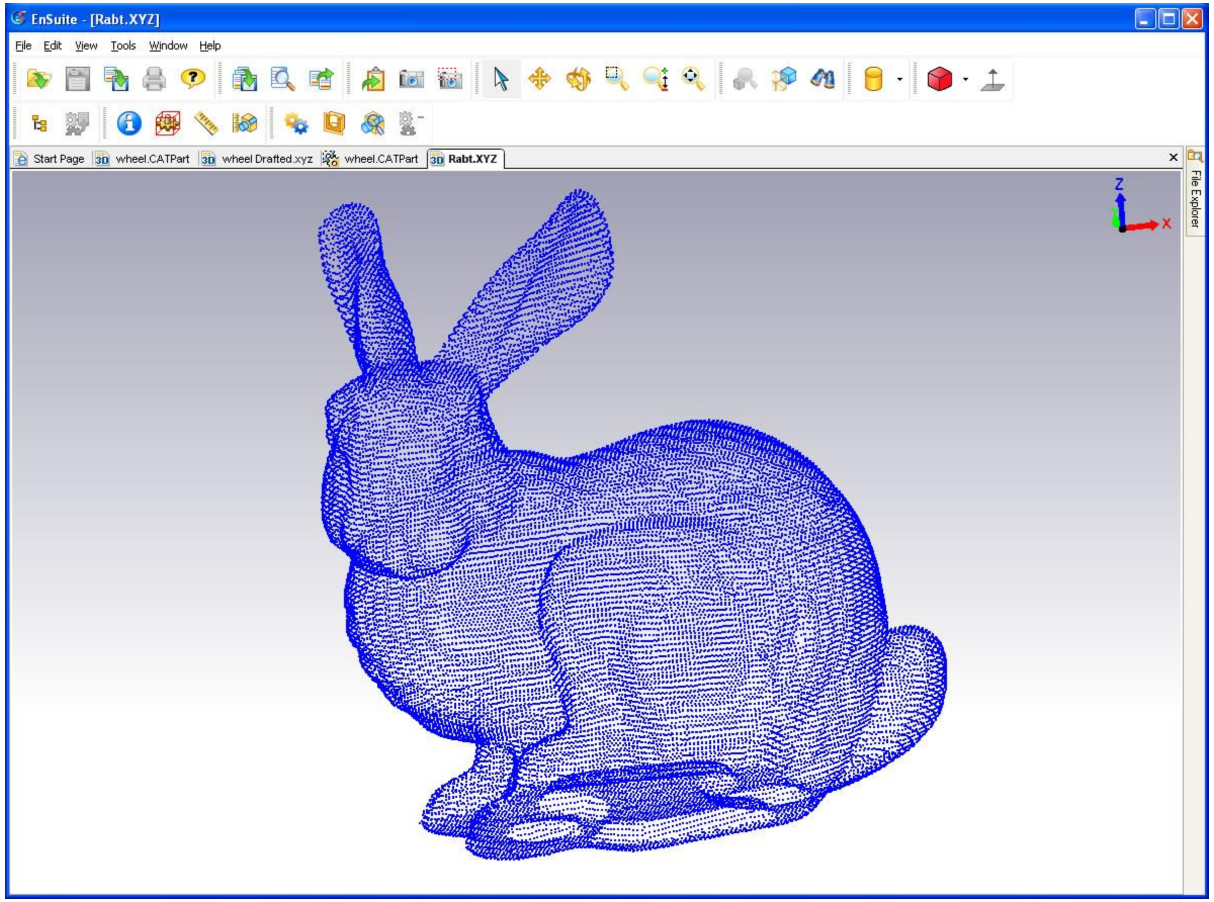
Surfaces from Volumes



http://en.wikipedia.org/wiki/Marching_cubes

Volumetric extraction

Not the Only Geometric Representation



http://www.engineeringspecifier.com/public/primages/pr1200.jpg

Point clouds

Surfaces: Smooth and Discrete

Justin Solomon

6.8410: Shape Analysis

Spring 2023