Continuous Curves

Justin Solomon

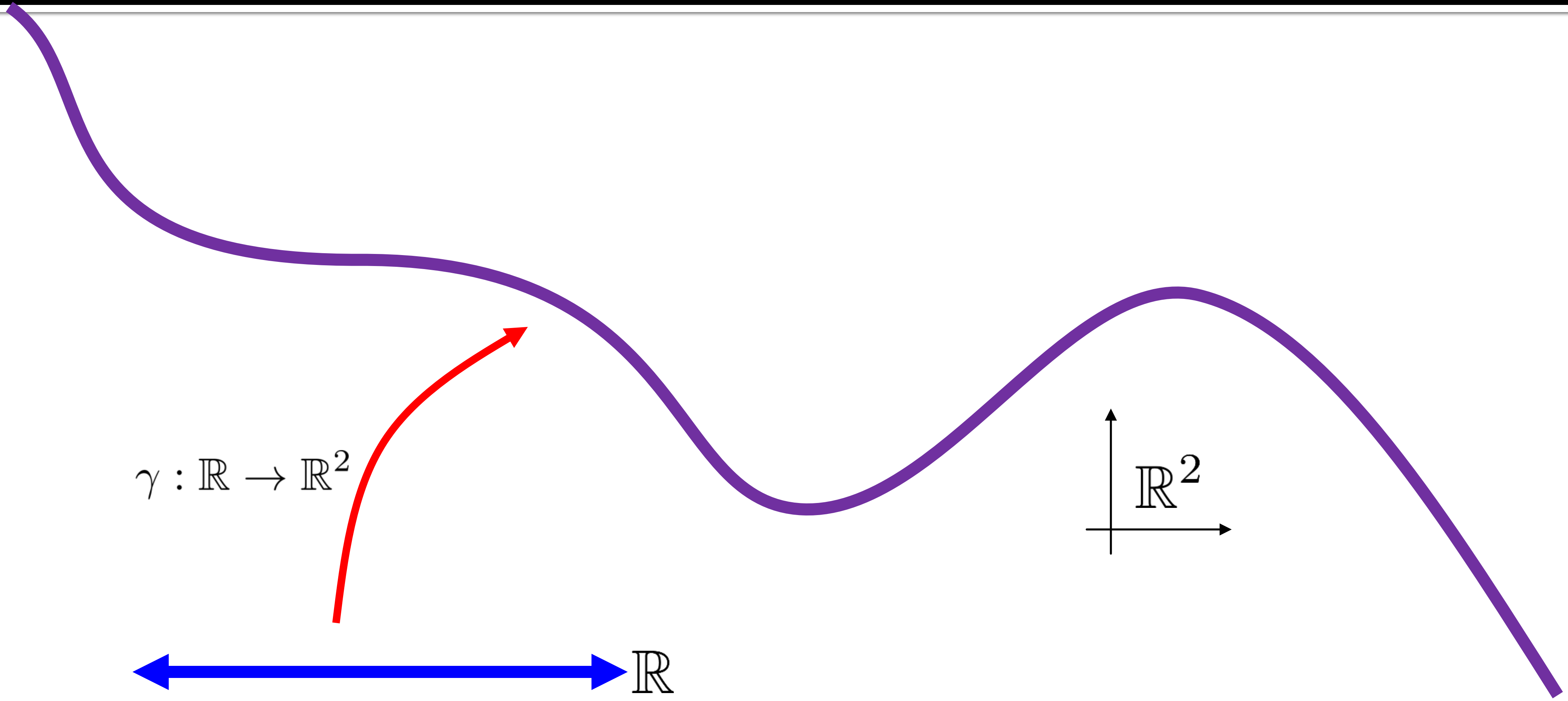
6.8410: Shape Analysis

Spring 2023

Q:

What is a **curve**?

Defining "Curve"



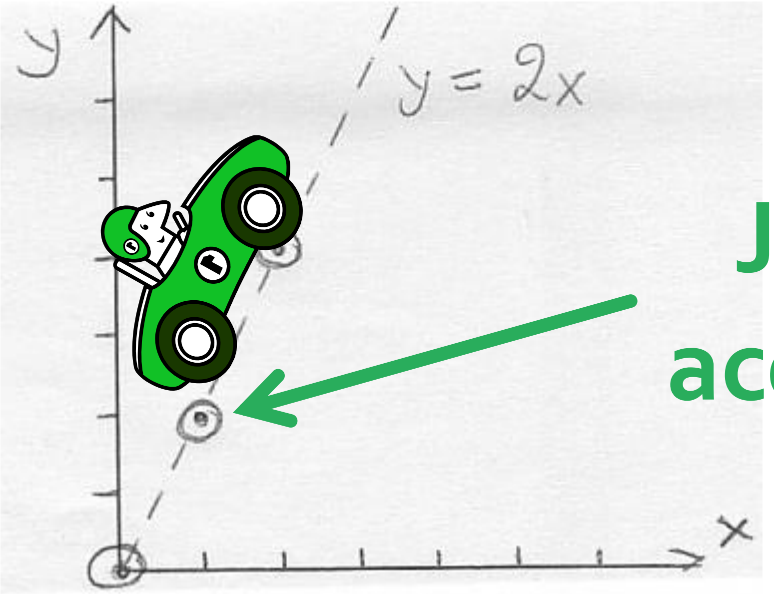
A function?

Subtlety

$$\gamma(t) \equiv (0,0)$$

Not a curve

Different from Calculus



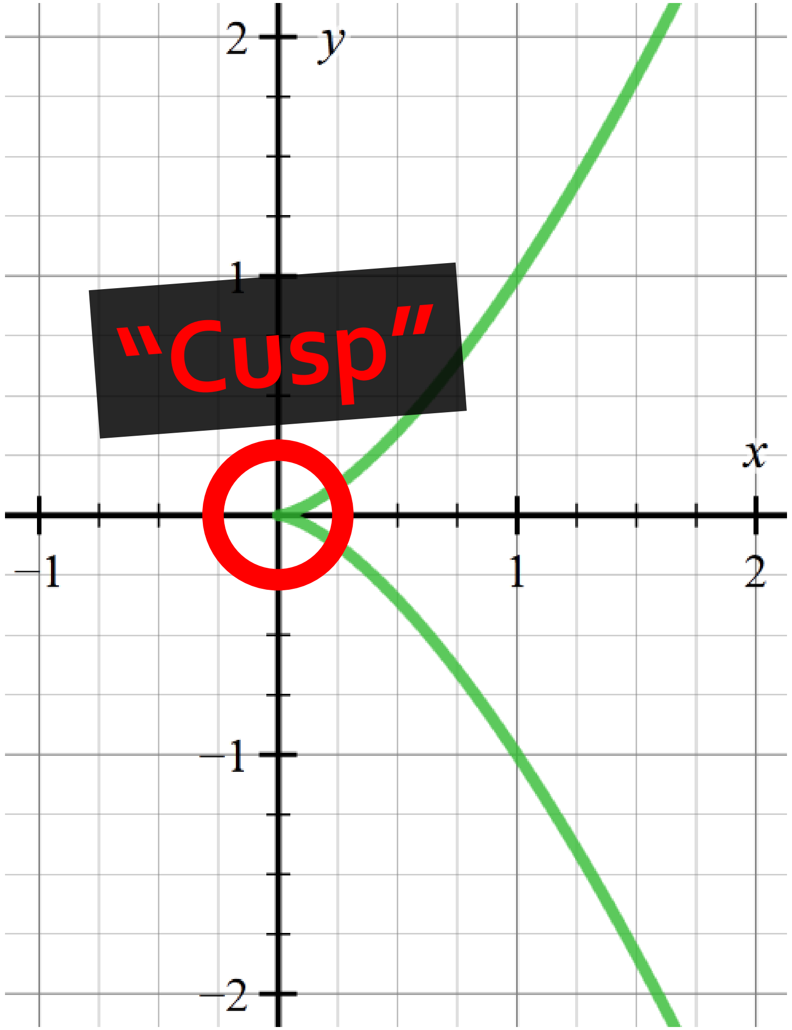
Jams on accelerator

$$\gamma_1(t) = (t, 2t)$$

$$\gamma_2(t) = \begin{cases} (t, 2t) & t \leq 1 \\ (2(t - \frac{1}{2}), 4(t - \frac{1}{2})) & t > 1 \end{cases}$$

http://sd271.k12.id.us/lchs/faculty/sjacobson/ibphysics/compendium/12_files/image003.jpg

Graphs of Smooth Functions



$$\gamma(t) = (t^2, t^3)$$

http://en.wikipedia.org/wiki/Singular_point_of_a_curve

Geometry of a Curve

A curve is a

set of points

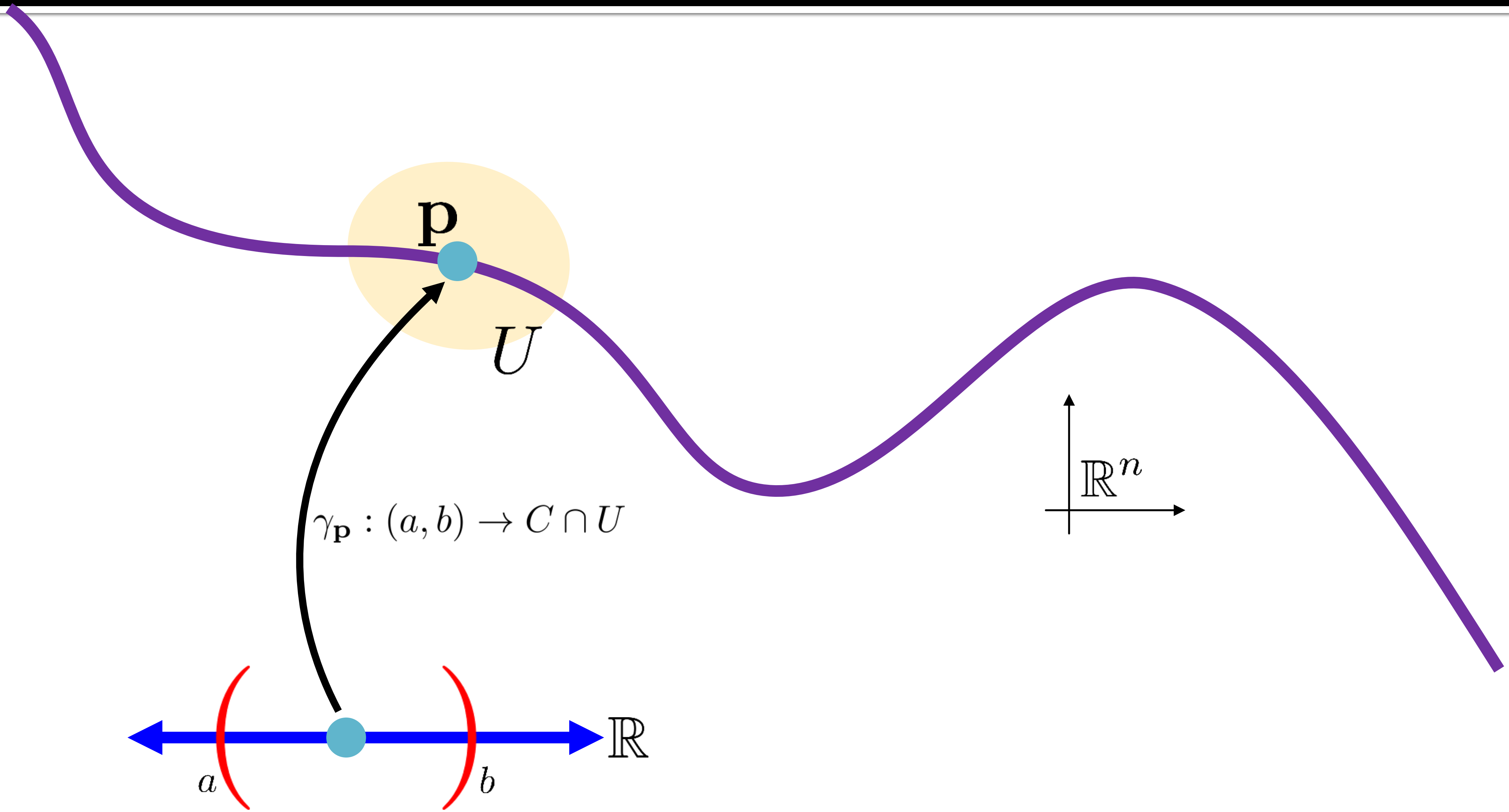
with certain properties.

It is not a function.

Geometric Definition

Set of points that locally looks like a line.

Differential Geometry Definition

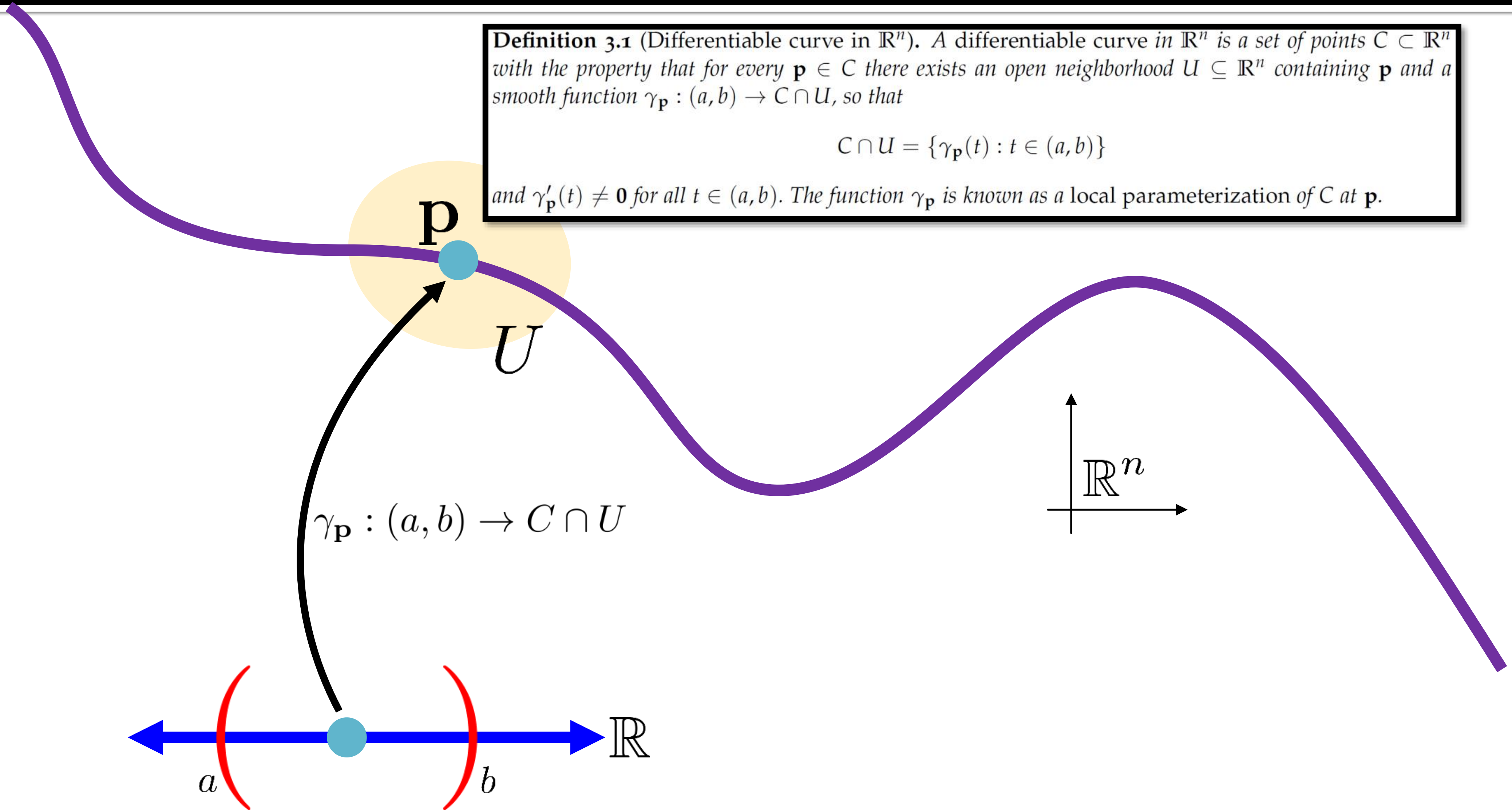


Formal Statement

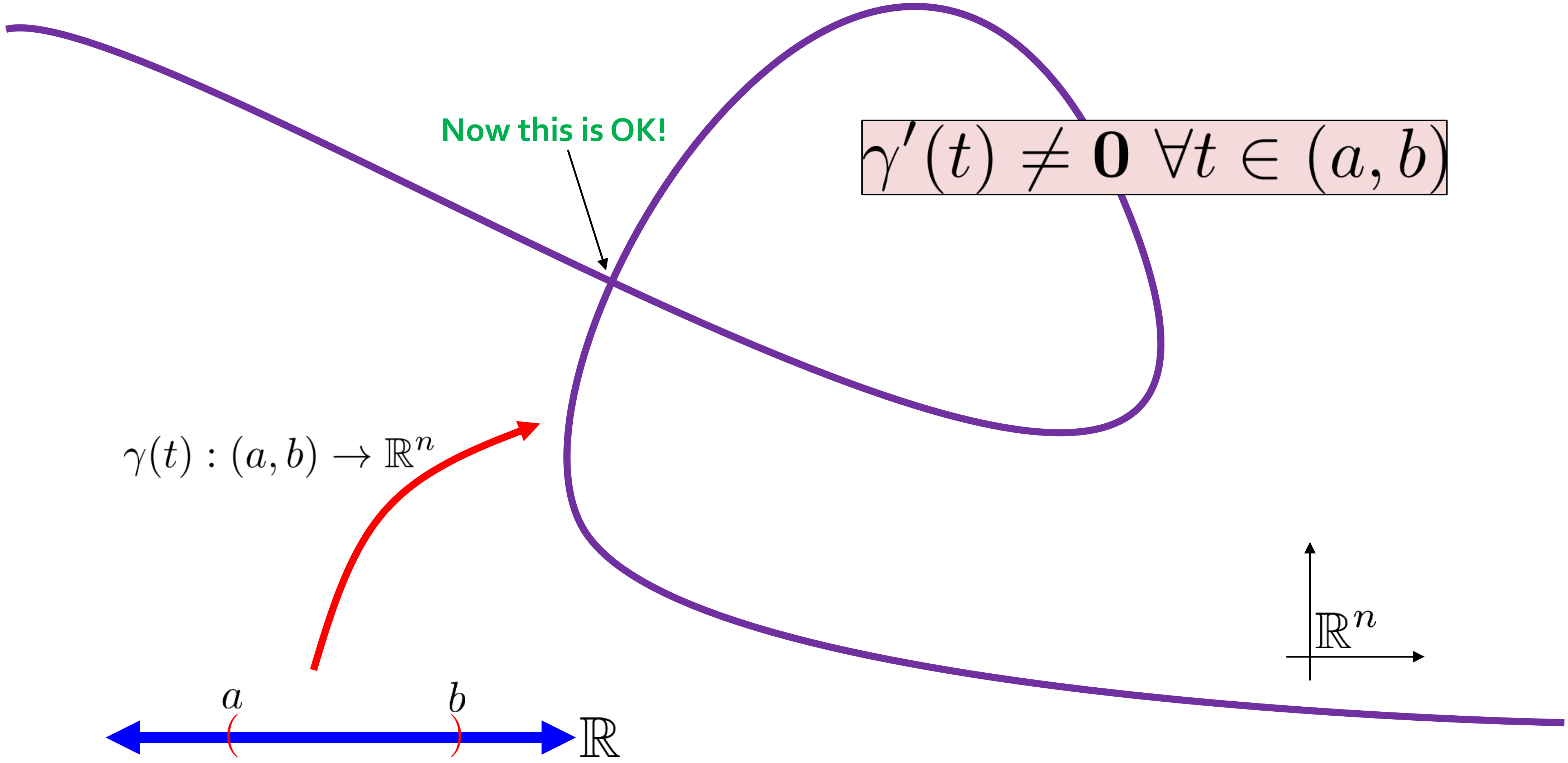
Definition 3.1 (Differentiable curve in $\mathbb{R}^n$). *A* differentiable curve *in $\mathbb{R}^n$ is a set of points $C \subset \mathbb{R}^n$ with the property that for every $\mathbf{p} \in C$ there exists an open neighborhood $U \subseteq \mathbb{R}^n$ containing $\mathbf{p}$ and a smooth function $\gamma_{\mathbf{p}} : (a,b) \to C \cap U$, so that*

$$C \cap U = \{\gamma_{\mathbf{p}}(t) : t \in (a,b)\}$$

and $\gamma'_{\mathbf{p}}(t) \neq \mathbf{0}$ for all $t \in (a,b)$. The function $\gamma_{\mathbf{p}}$ is known as a local parameterization *of C at $\mathbf{p}$.*



Parameterized Curve



Some Vocabulary

- **Trace** of parameterized curve

$$\{\gamma(t) : t \in (a, b)\} \subseteq \mathbb{R}^n$$

- **Component** functions

$$\gamma(t) = (x(t), y(t), z(t))$$

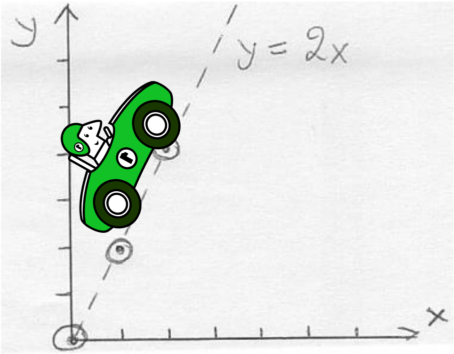
Change of Parameter

$$t \mapsto \gamma \circ \phi(t)$$

Geometric measurements should be

invariant

to changes of parameter.



Dependence of Velocity

$$\tilde{\gamma}(t) := \gamma(\phi(t))$$

Effect on velocity and acceleration?

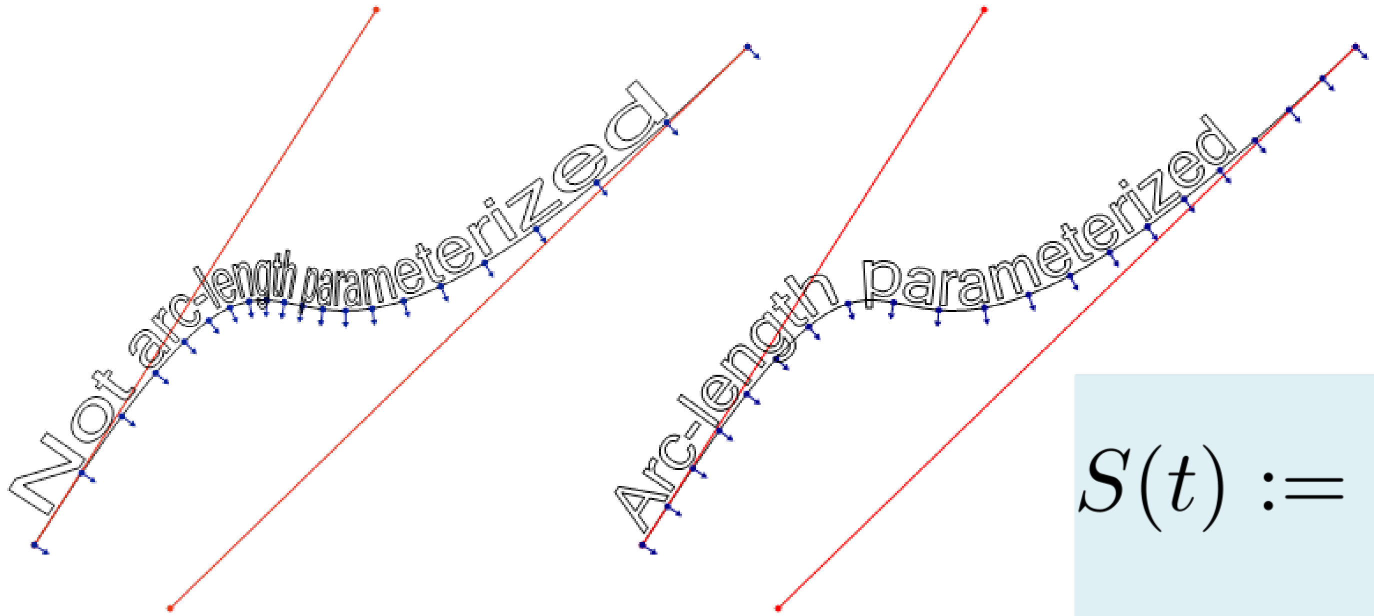
Arc Length

$$\int_a^b \|\gamma'(t)\|_2 \, dt$$

Independent of parameter!

Parameterization by Arc Length

http://www.planetclegg.com/projects/WarpingTextToSplines.html



$$S(t) := \int_{t_0}^{t} \|\gamma'(t)\|_2 \, dt$$

$$t = \phi \circ S(t)$$

Inverse functions

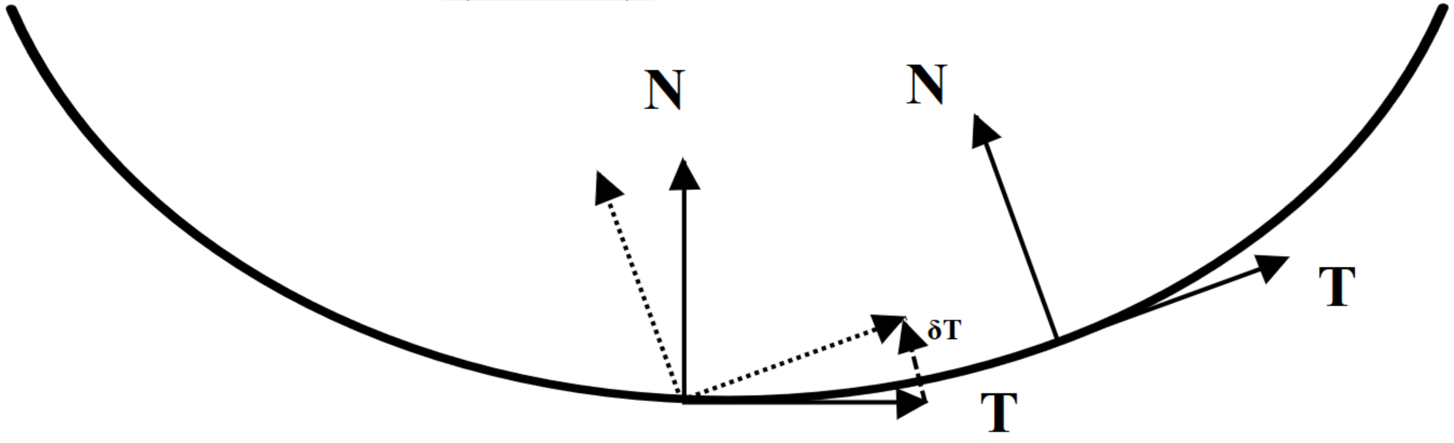
$$\tilde{\gamma}(s) := \gamma \circ \phi(s)$$

Constant-speed parameterization

Moving Frame in 2D

$$\mathbf{T}(s) := \gamma'(s)$$
$$\implies \|\mathbf{T}(s)\|_2 \equiv 1$$
$$\mathbf{N}(s) := J\mathbf{T}(s) = \kappa(s)^{-1}\mathbf{T}'(s)$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

Philosophical Point

Differential geometry "should" be

coordinate-invariant.

Referring to *x* and *y* is a hack!

(but sometimes convenient…)

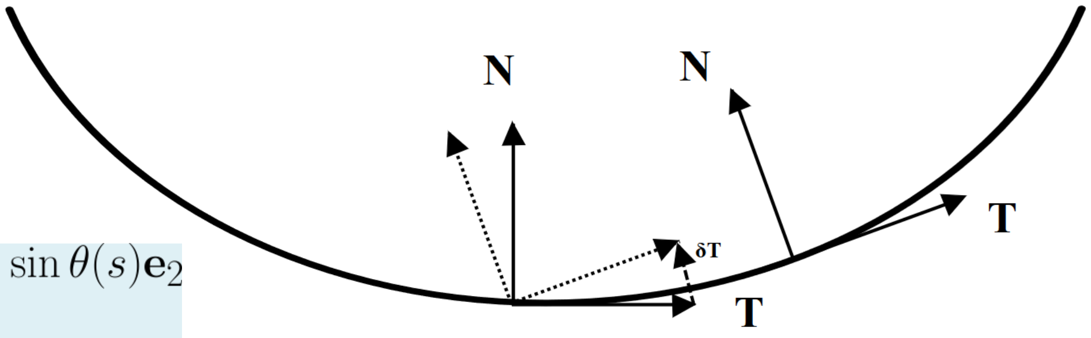
Q:

How do you describe a curve without coordinates?

Turtles All The Way Down

$$\frac{d}{ds}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\end{pmatrix} := \begin{pmatrix}0 & \kappa(s)\\ -\kappa(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\end{pmatrix}$$

Signed curvature κ is rate of change of turning angle θ.



$$\mathbf{T}(s) = \cos\theta(s)\mathbf{e}_1 + \sin\theta(s)\mathbf{e}_2$$
$$\kappa(s) := \theta'(s)$$

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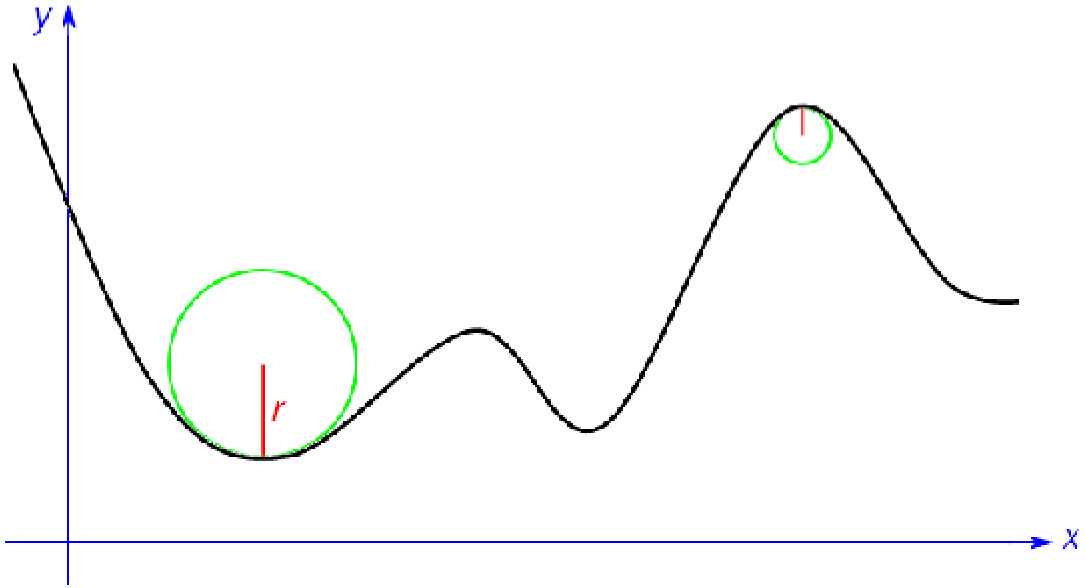
Use coordinates *from* the curve to express its shape!

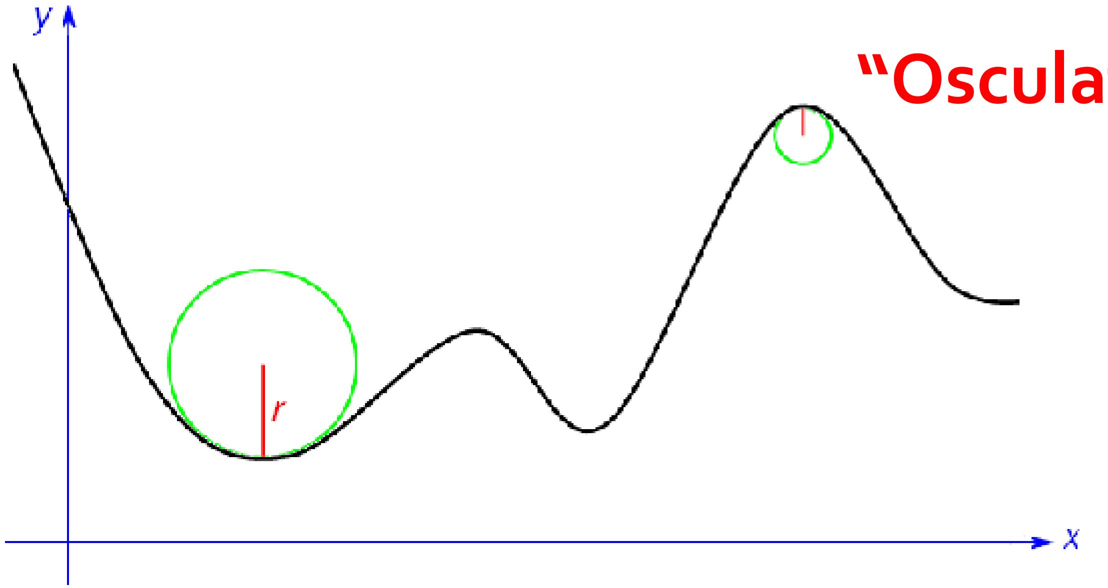
Radius of Curvature



$$r(s) := \frac{1}{\kappa(s)}$$

https://www.quora.com/What-is-the-base-difference-between-radius-of-curvature-and-radius-of-gyration

Radius of Curvature



"Osculating circle"

$$r(s) := \frac{1}{\kappa(s)}$$

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Fundamental theorem of the local theory of plane curves:

$\kappa(s)$ distinguishes a planar curve up to rigid motion.

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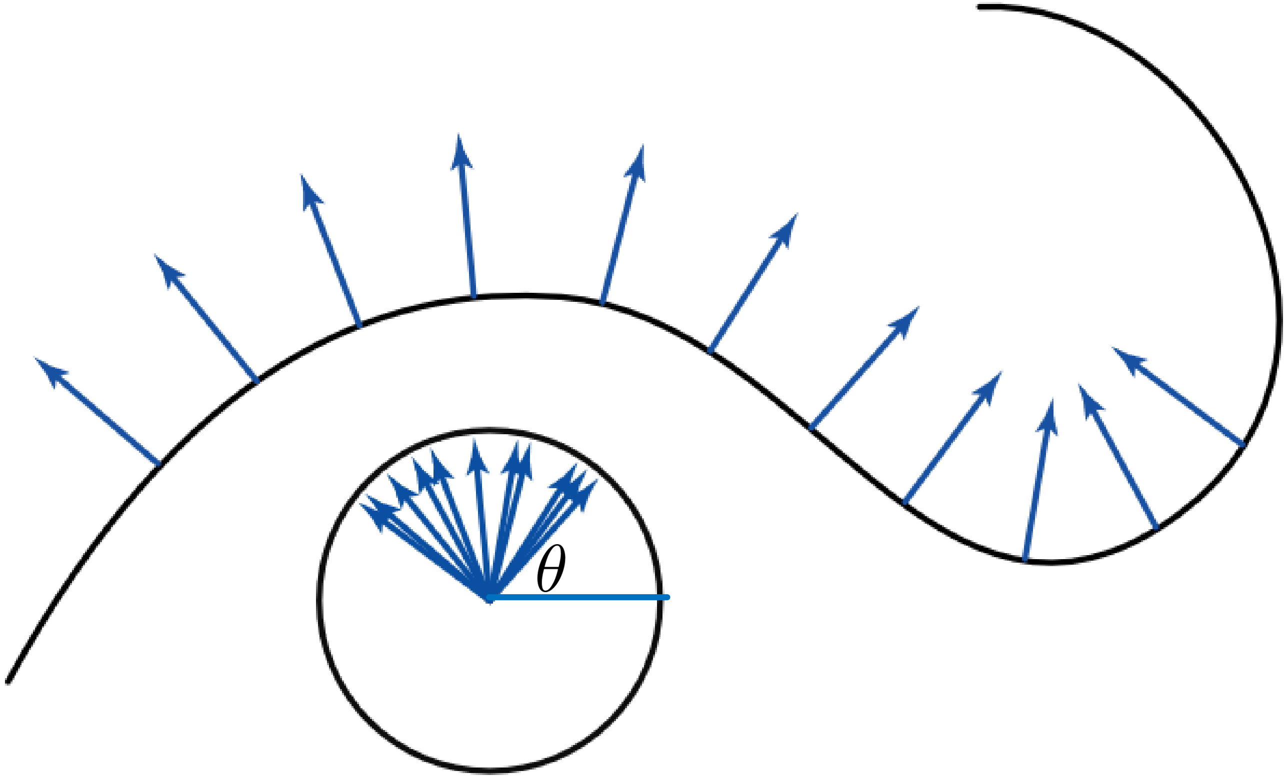
$\kappa(s)$ distinguishes a planar curve up to rigid motion.

Statement shorter than the name!

Idea of Proof



$$\mathbf{T}(s) := (\cos\theta(s), \sin\theta(s))$$

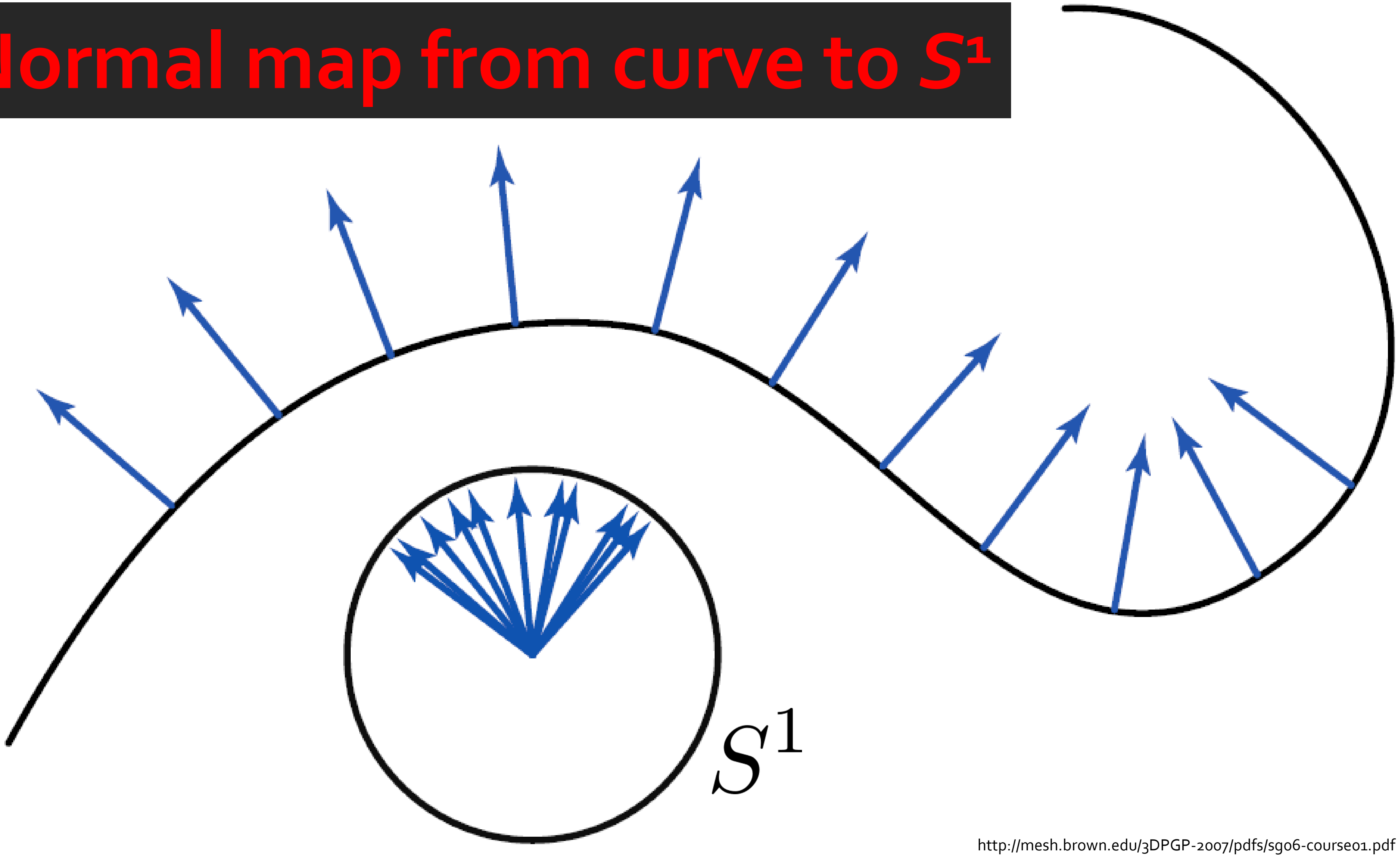
$$\Longrightarrow \kappa(s) := \theta'(s)$$

Image from DDG course notes by E. Grinspun

Provides intuition for curvature

Gauss Map

Normal map from curve to S^1



http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

Winding Number

$$W[\gamma] := \frac{1}{2\pi}\int_a^b \kappa(s)\,ds \in \mathbb{Z}$$

$W[\gamma]$ is an integer, and smoothly deforming γ does not affect $W[\gamma]$.

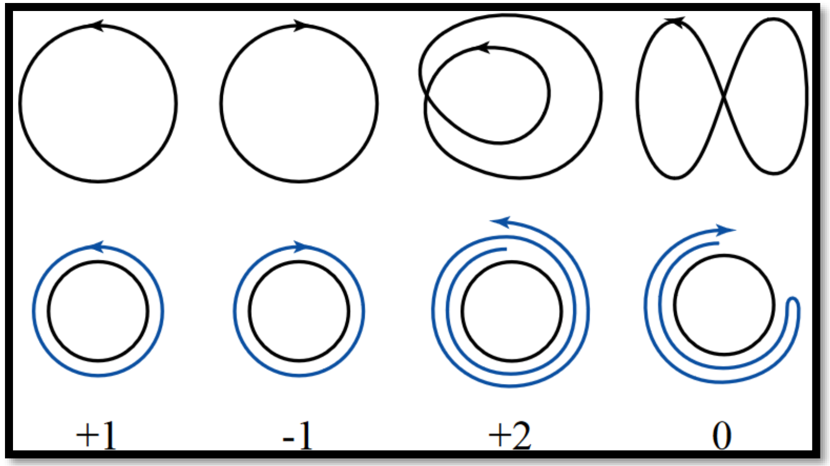
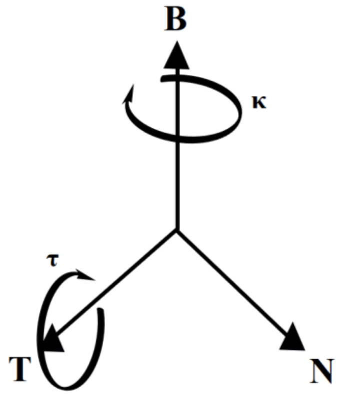


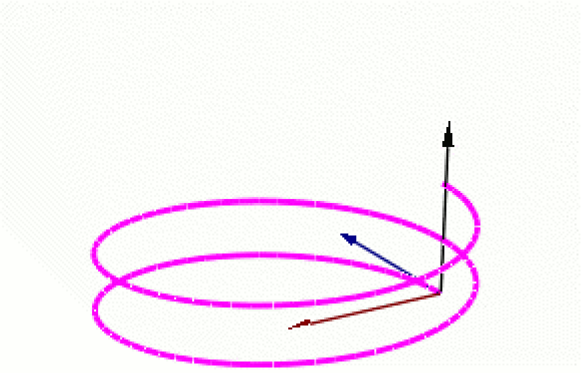
Image from: Grinspun and Secord, "The Geometry of Plane Curves" (SIGGRAPH 2006)

Frenet Frame: Curves in $\mathbb{R}^3$

$$\frac{d}{ds}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix} = \begin{pmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix}$$

- **Binormal:** $\boldsymbol{T} \times \boldsymbol{N}$
- **Curvature:** In-plane motion
- **Torsion:** Out-of-plane motion





Fundamental theorem of the local theory of space curves:

Curvature and torsion distinguish a 3D curve up to rigid motion.

Aside: Generalized Frenet Frame

$$\gamma(s) : \mathbb{R} \to \mathbb{R}^n$$

$$\frac{d}{ds}\begin{pmatrix} \mathbf{e}_1(s) \\ \mathbf{e}_2(s) \\ \vdots \\ \mathbf{e}_n(s) \end{pmatrix} = \begin{pmatrix} 0 & \chi_1(s) & & \\ -\chi_1(s) & \ddots & \ddots & \\ & \ddots & 0 & \chi_{n-1}(s) \\ & & -\chi_{n-1}(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1(s) \\ \mathbf{e}_2(s) \\ \vdots \\ \mathbf{e}_n(s) \end{pmatrix}$$

Suspicion: Application to time series analysis? ML?

C. Jordan, 1874

Gram-Schmidt on first *n* derivatives

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Discrete Curves

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6.8410: Shape Analysis

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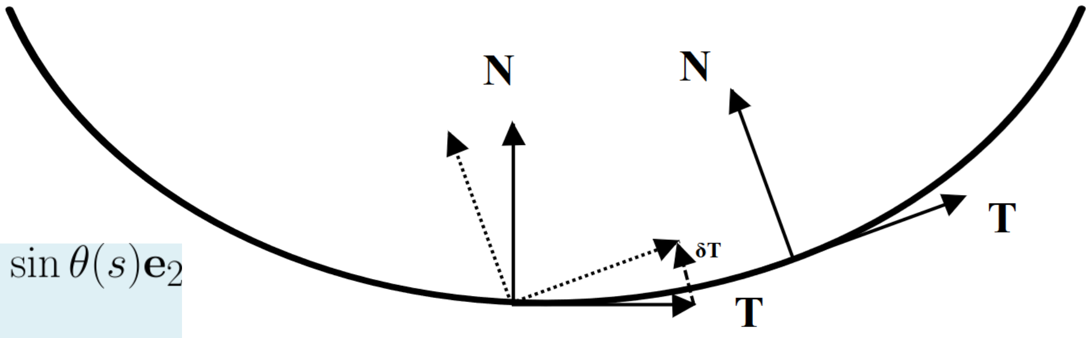

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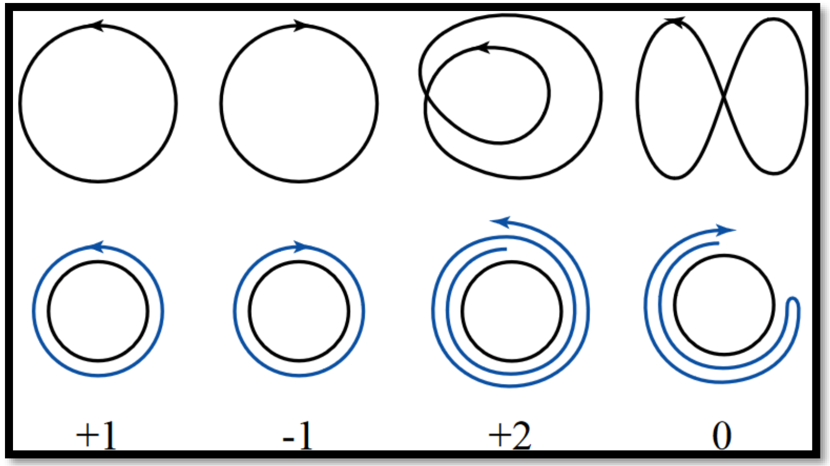

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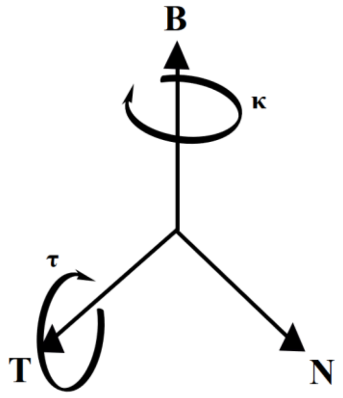
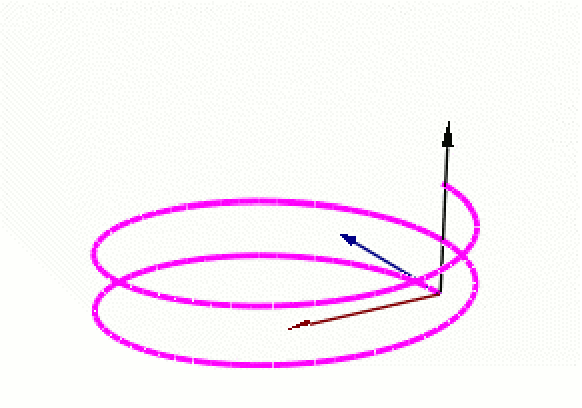
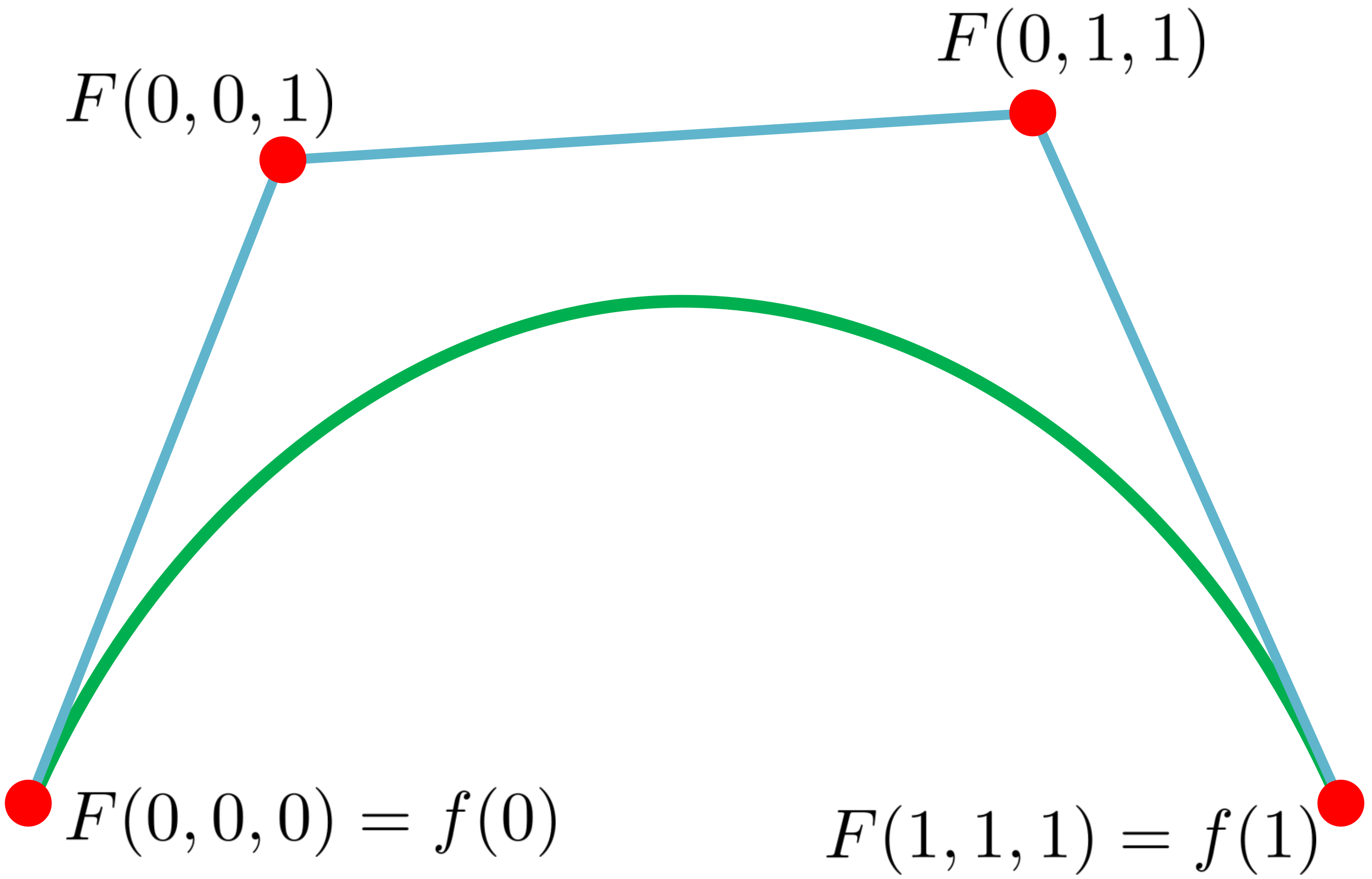




Q:

What do these calculations look like in software?

Old-School Approach



Piecewise smooth approximations

Question

What is the arc length of a cubic Bézier curve?

$$\int_a^b \|\gamma'(t)\|_2 \, dt$$

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What is the arc length of a cubic Bézier curve?

$$\int_a^b \|\gamma'(t)\|_2 \, dt$$

Not known in closed form.

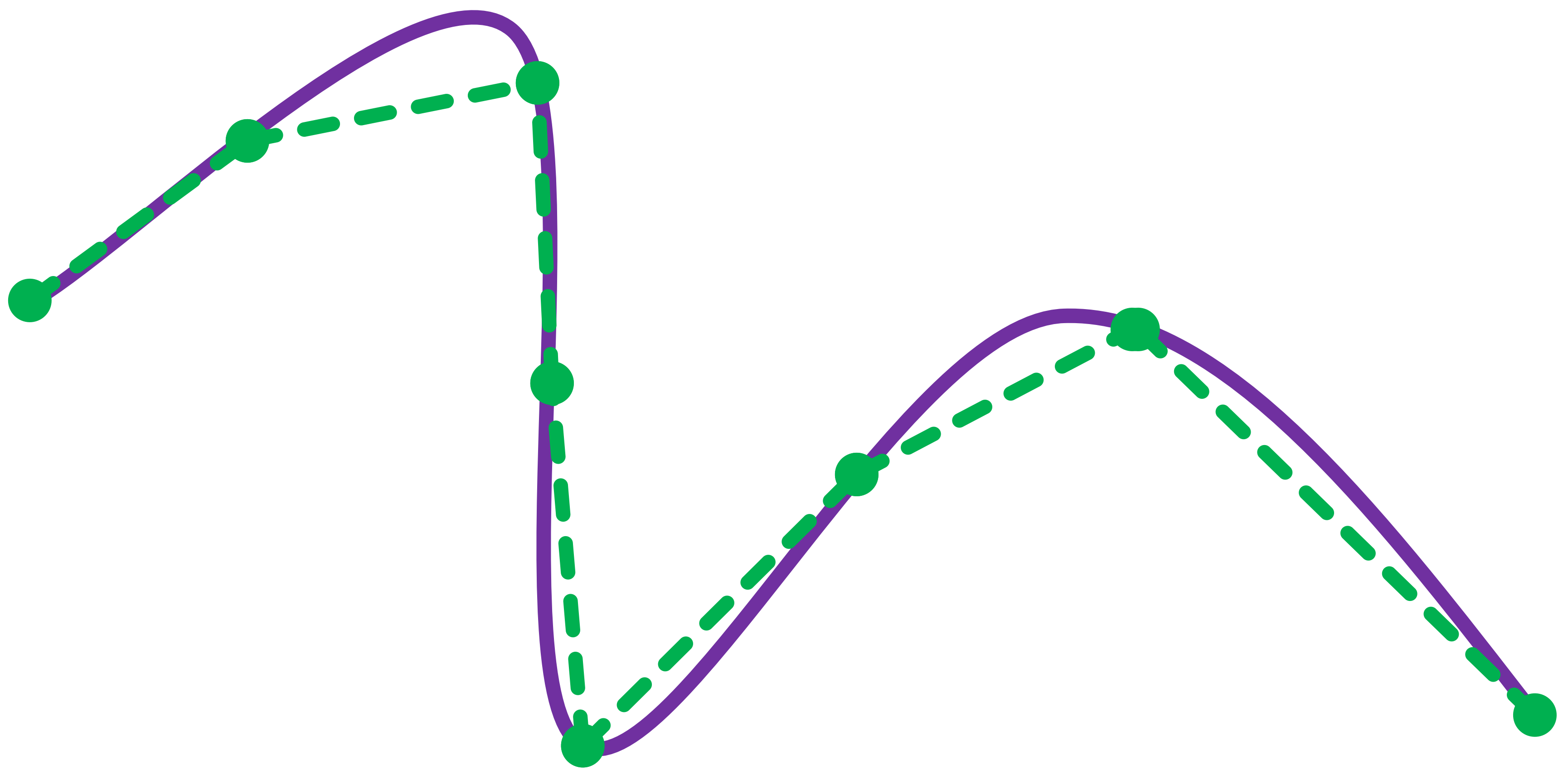
Sad fact:

Closed-form expressions rarely exist. When they do exist, they usually are messy.

Only Approximations Anyway

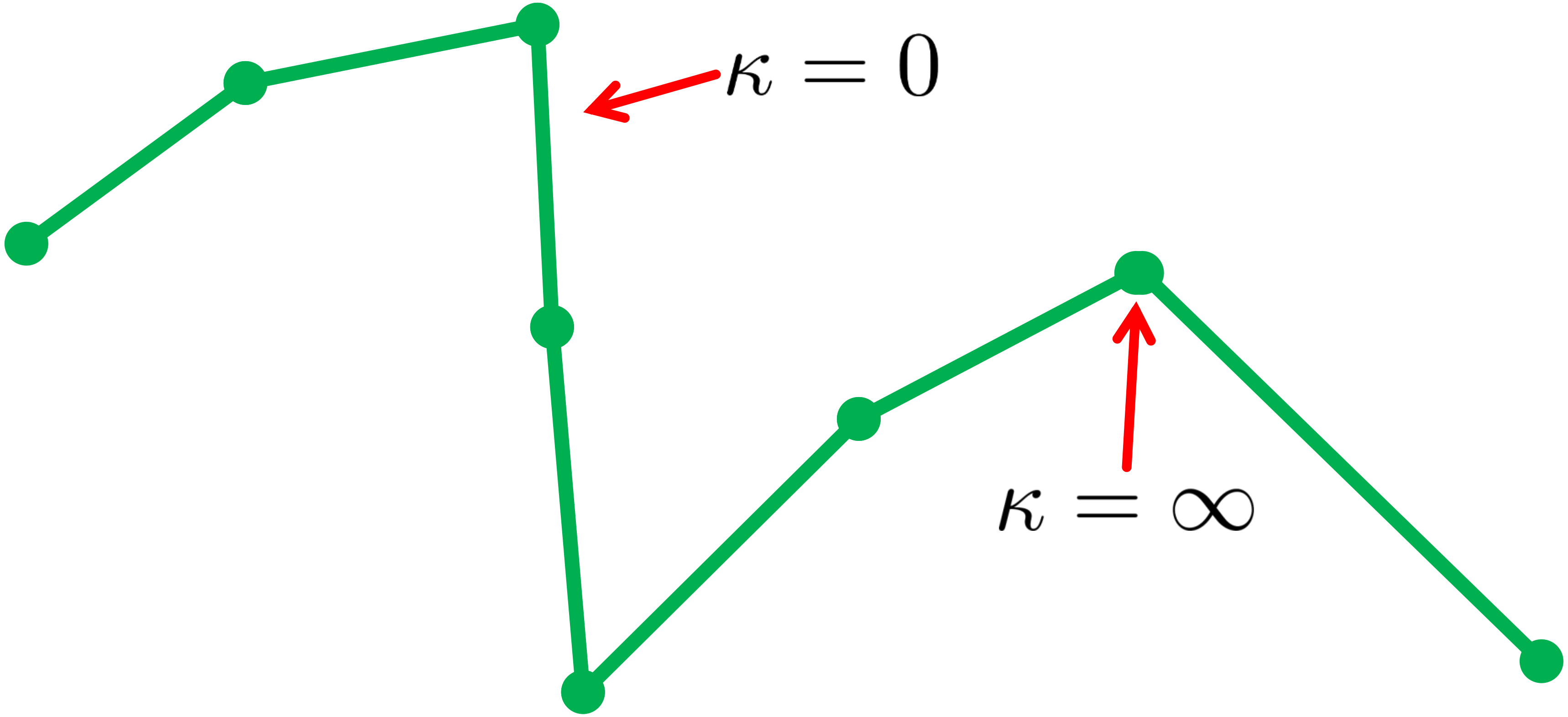
$$\{\text{Bézier curves}\} \subsetneq \{\gamma : \mathbb{R} \to \mathbb{R}^3\}$$

Simpler Approximation



Piecewise linear: Poly-line

Big Problem



Boring differential structure

Finite Difference Approach

$$f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$$

THEOREM: As $\Delta h \to 0$, [insert statement].

Reality Check

$$f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$$

THEOREM $\boxed{h > 0}$ tatement].

Two Key Considerations

- Convergence to continuous theory
- Discrete behavior

Goal

Examine discrete theories of differentiable curves.

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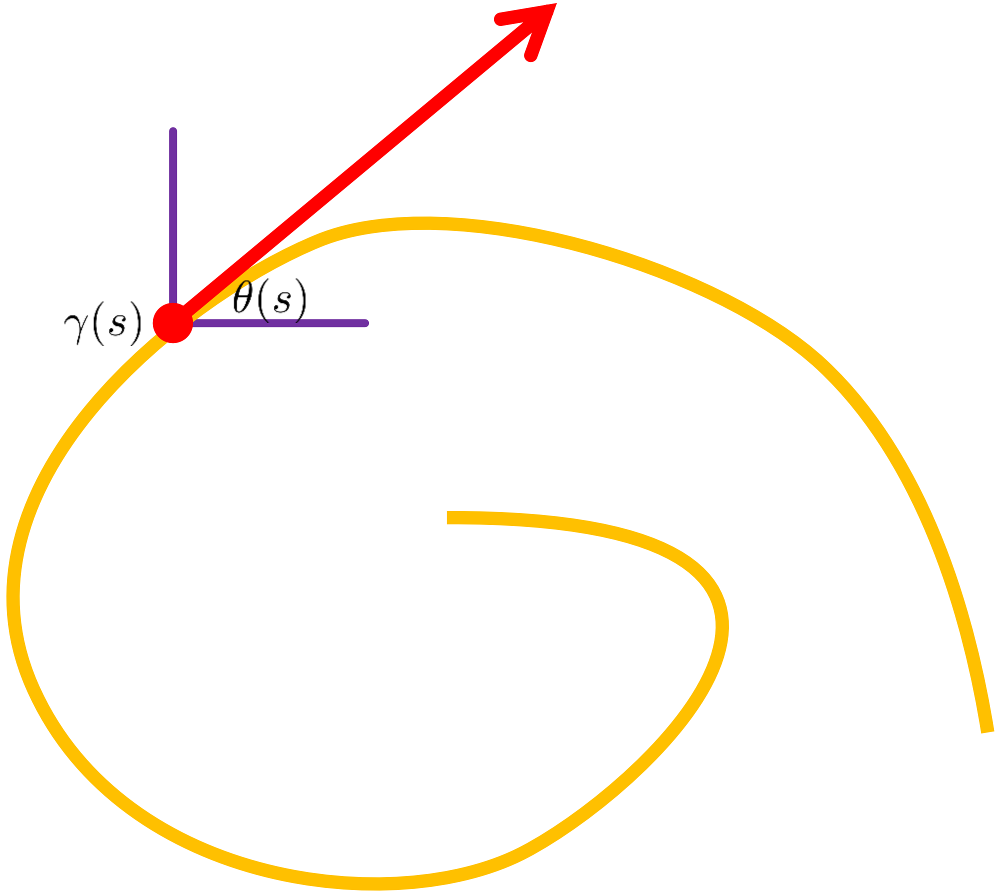
Examine discrete theor<u>ies</u> of differentiable curves.

Recall:

Signed Curvature on Plane Curves

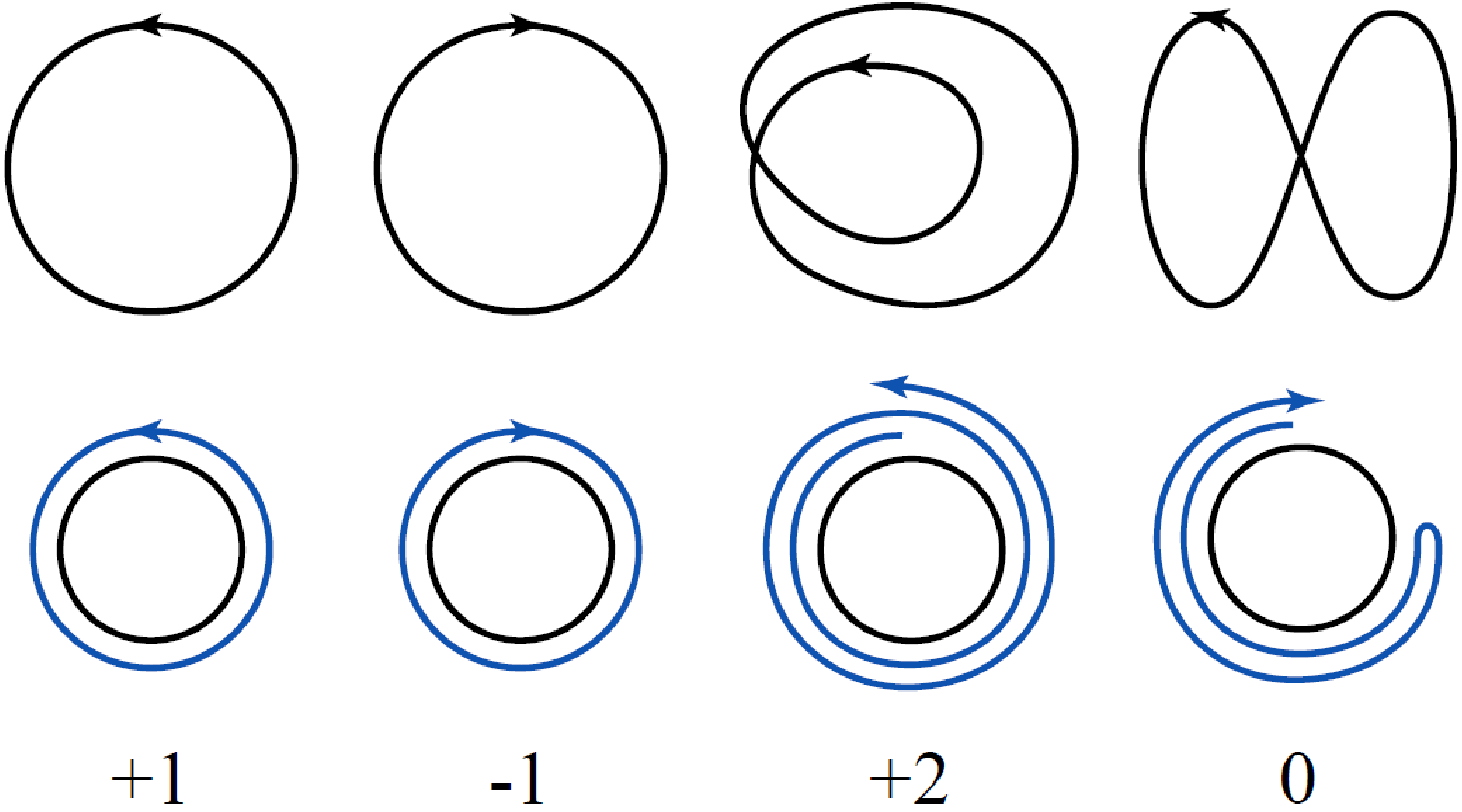
$$\mathbf{T}(s) = (\cos\theta(s), \sin\theta(s))$$



Gauss map:
Map from curve to its normals.

$$\mathbf{T}'(s) = \theta'(s)(-\sin\theta(s), \cos\theta(s))$$
$$:= \kappa(s)\mathbf{N}(s)$$

Turning Numbers



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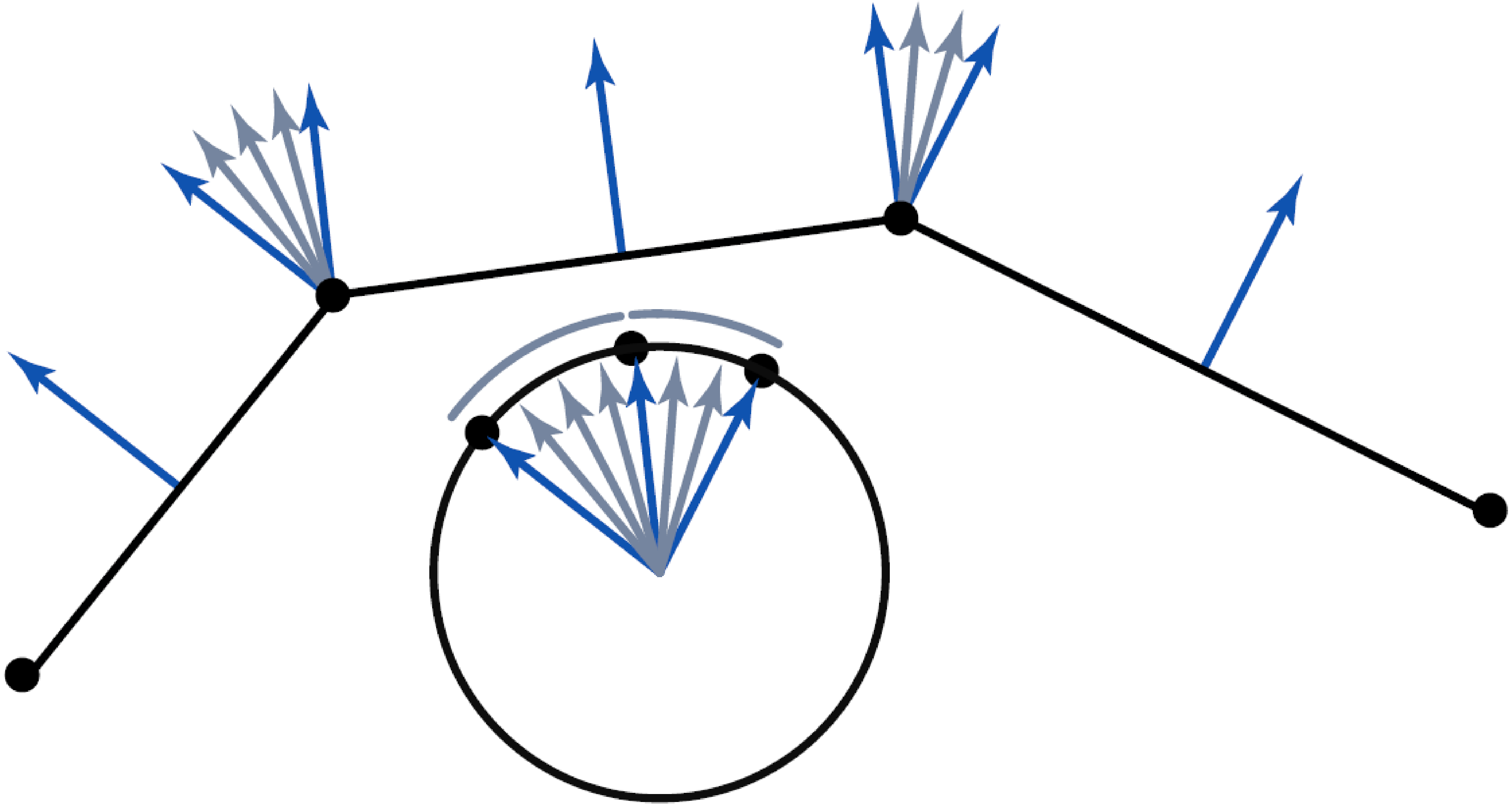
Discrete Gauss Map



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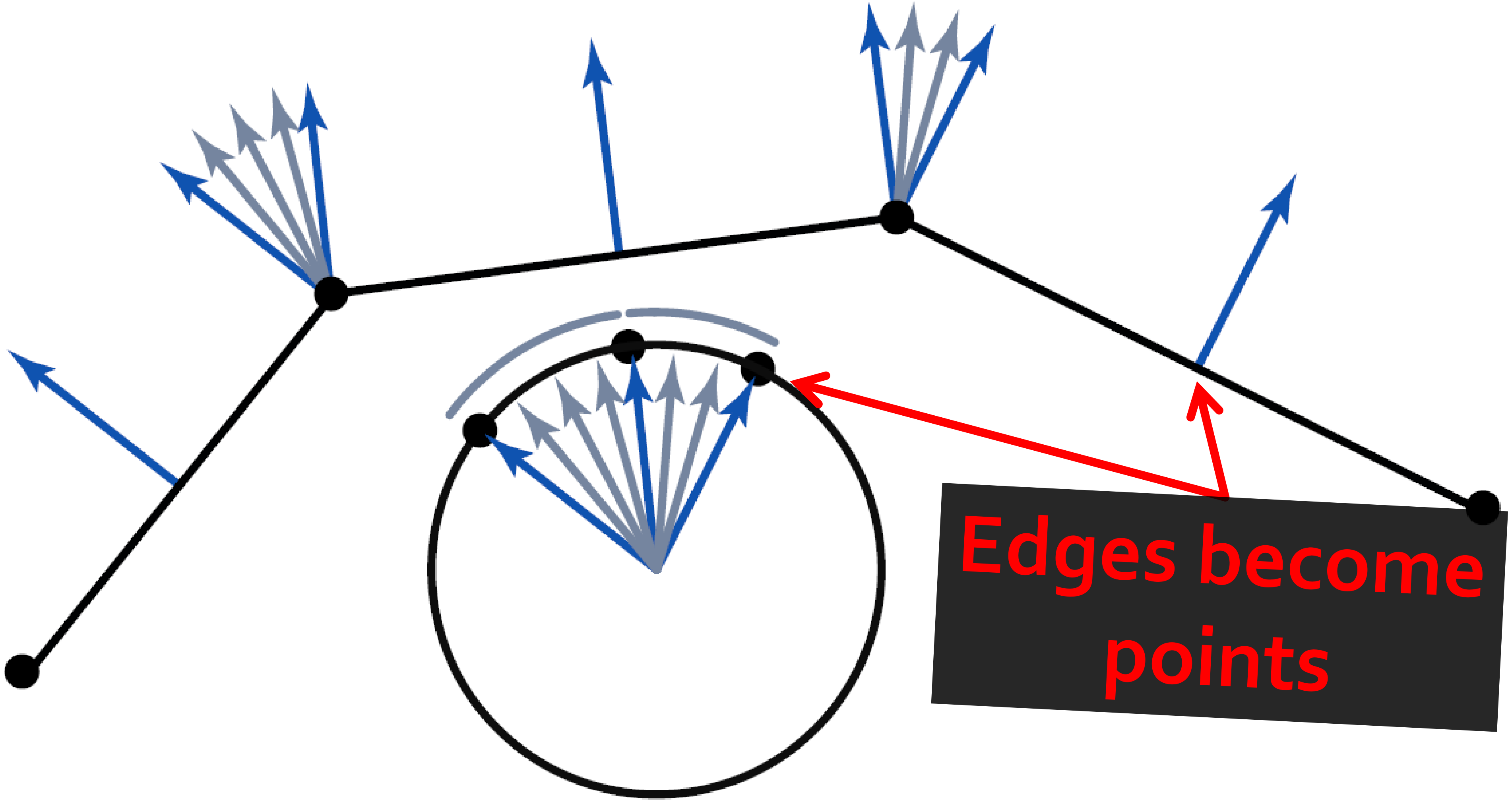

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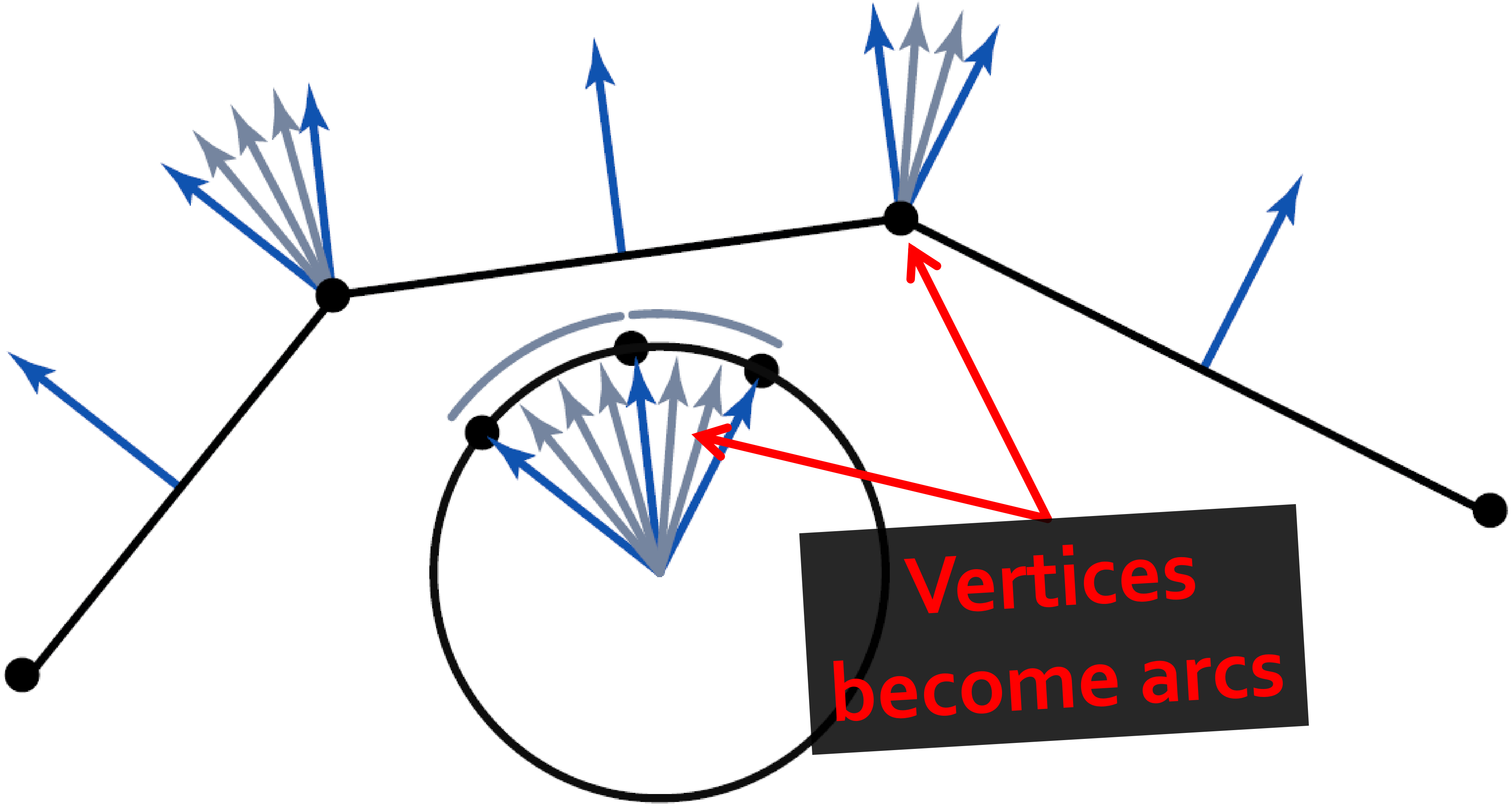

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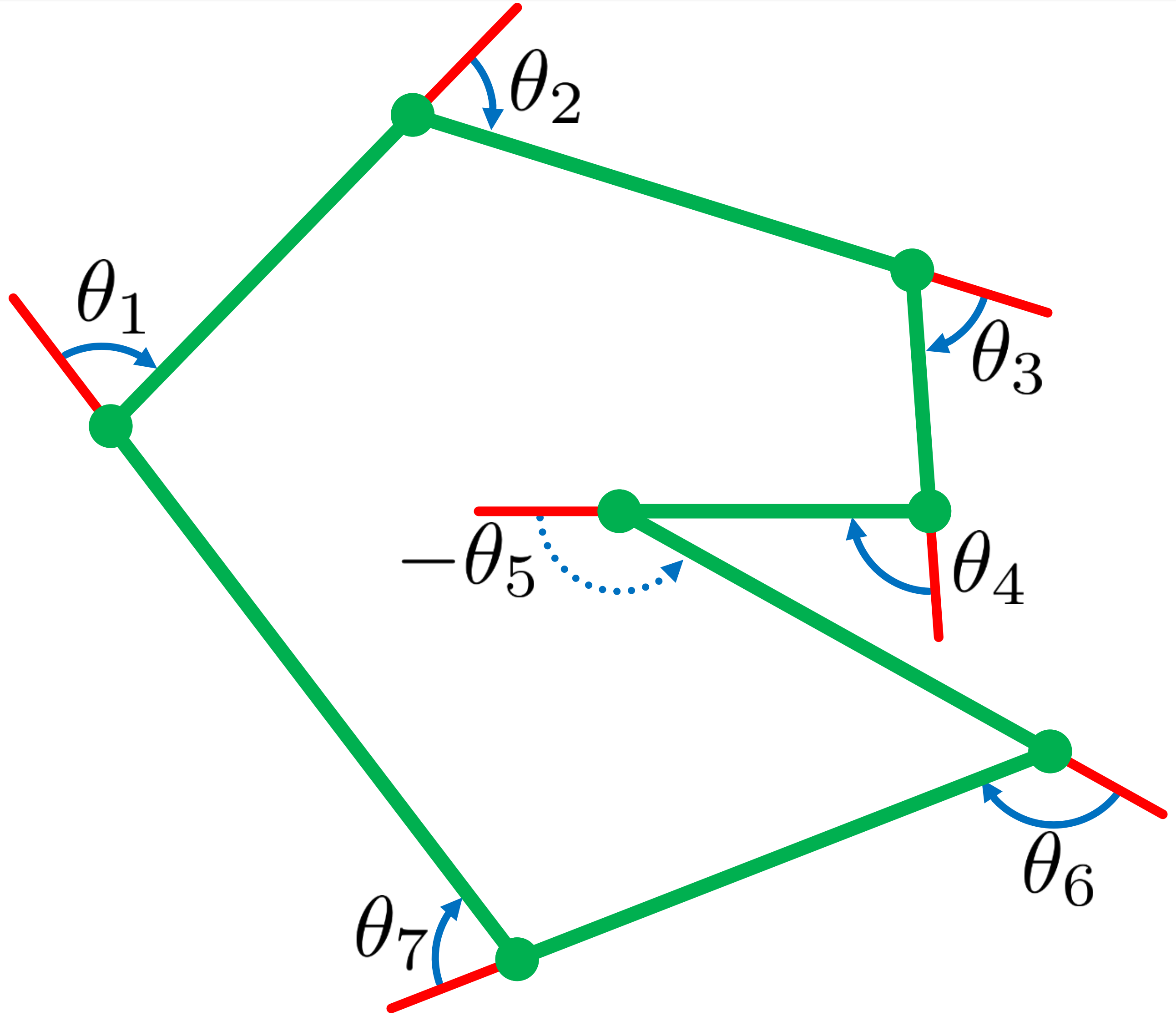
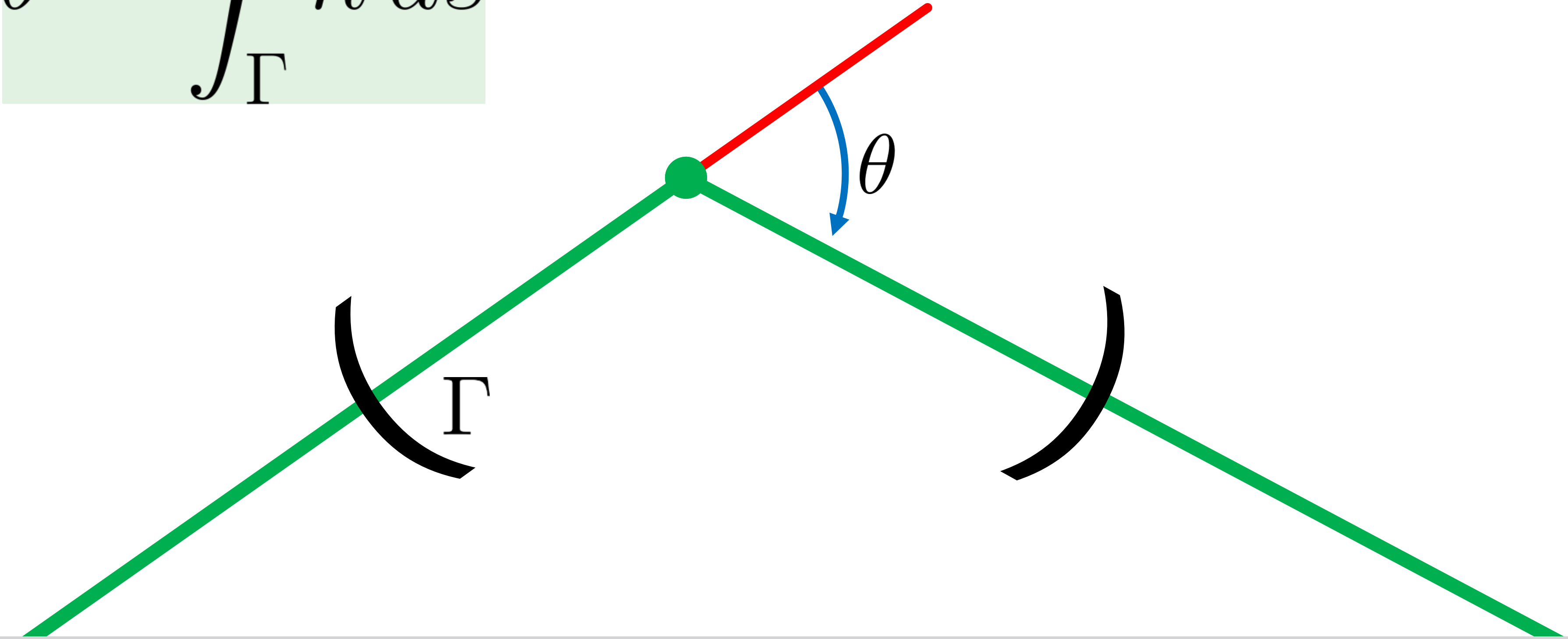
Key Observation



$$\sum_i \theta_i = 2\pi k$$

What's Going On?

$$\theta = \int_\Gamma \kappa \, ds$$

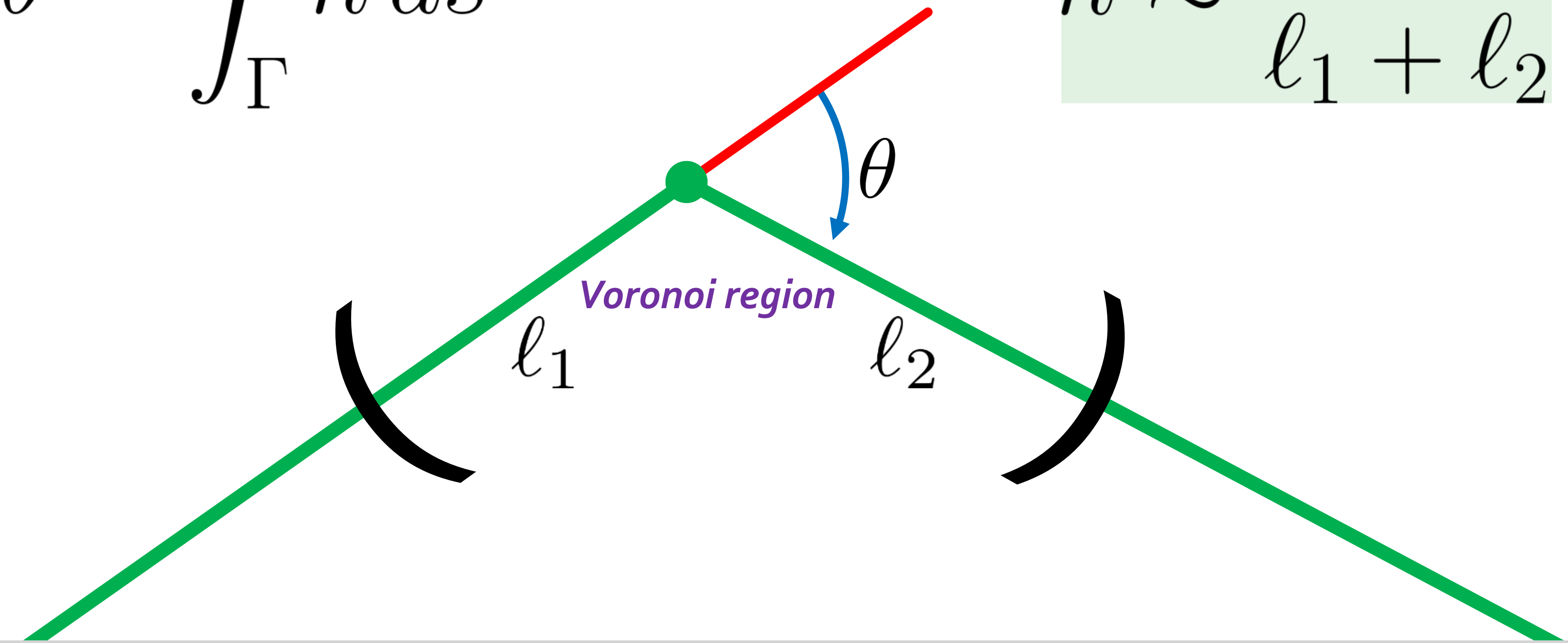


Integrated curvature

What's Going On?

$$\theta = \int_\Gamma \kappa \, ds$$

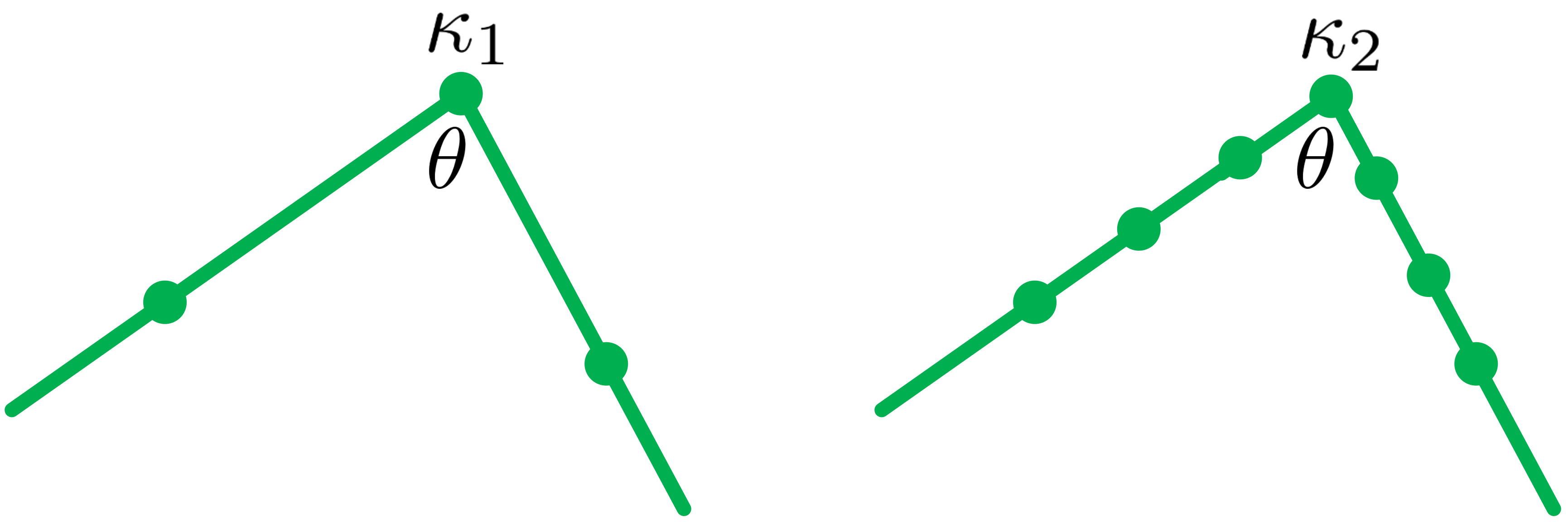
$$\boxed{\kappa \approx \frac{\theta}{\ell_1 + \ell_2}}$$



Total change in curvature

Interesting Distinction

$$\kappa_1 \neq \kappa_2$$



Same integrated curvature

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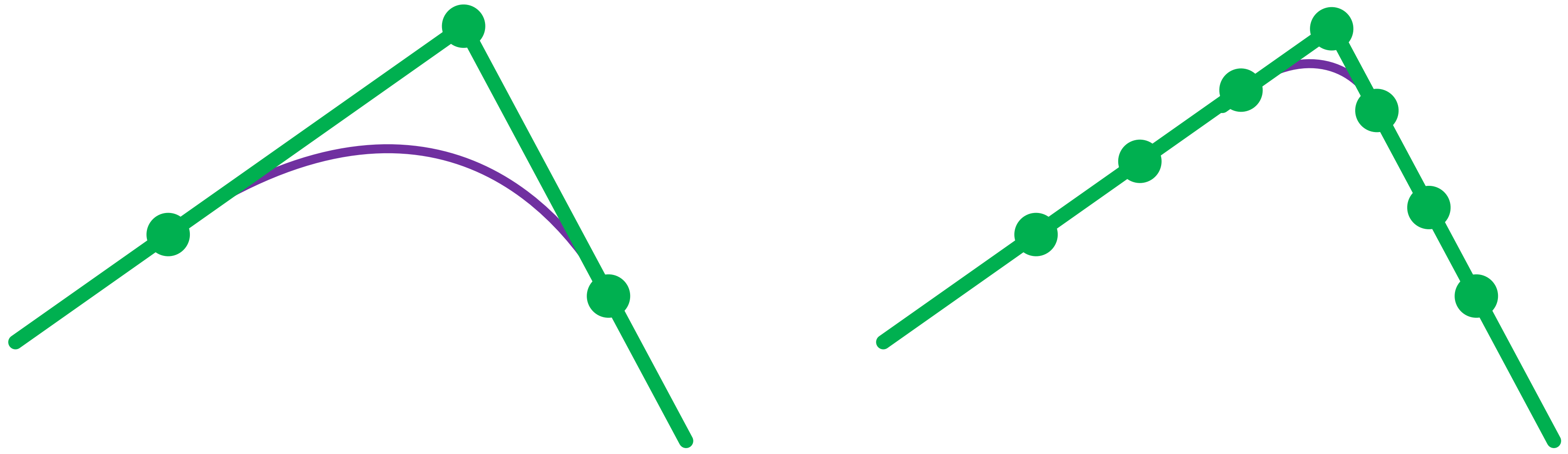

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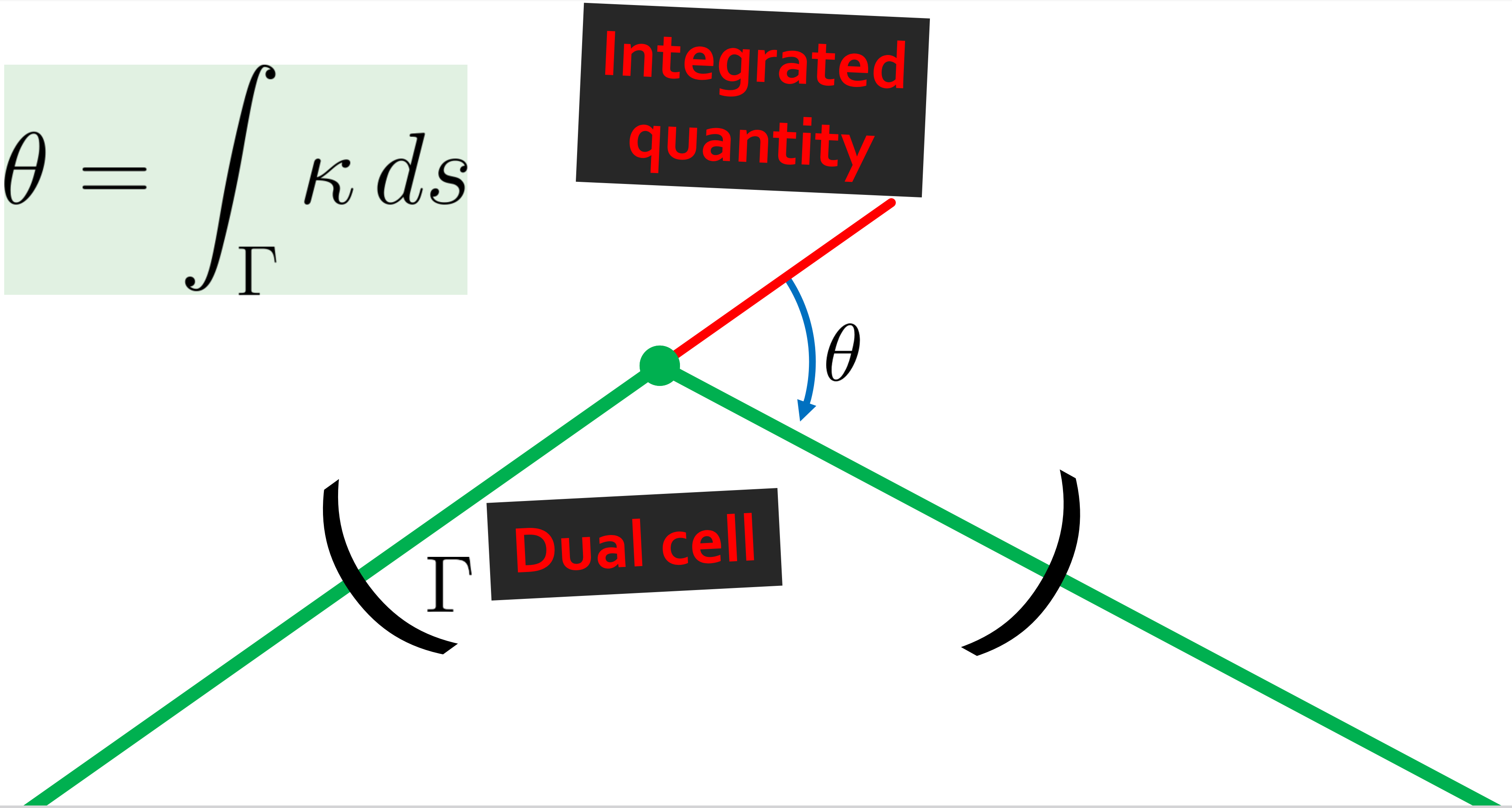
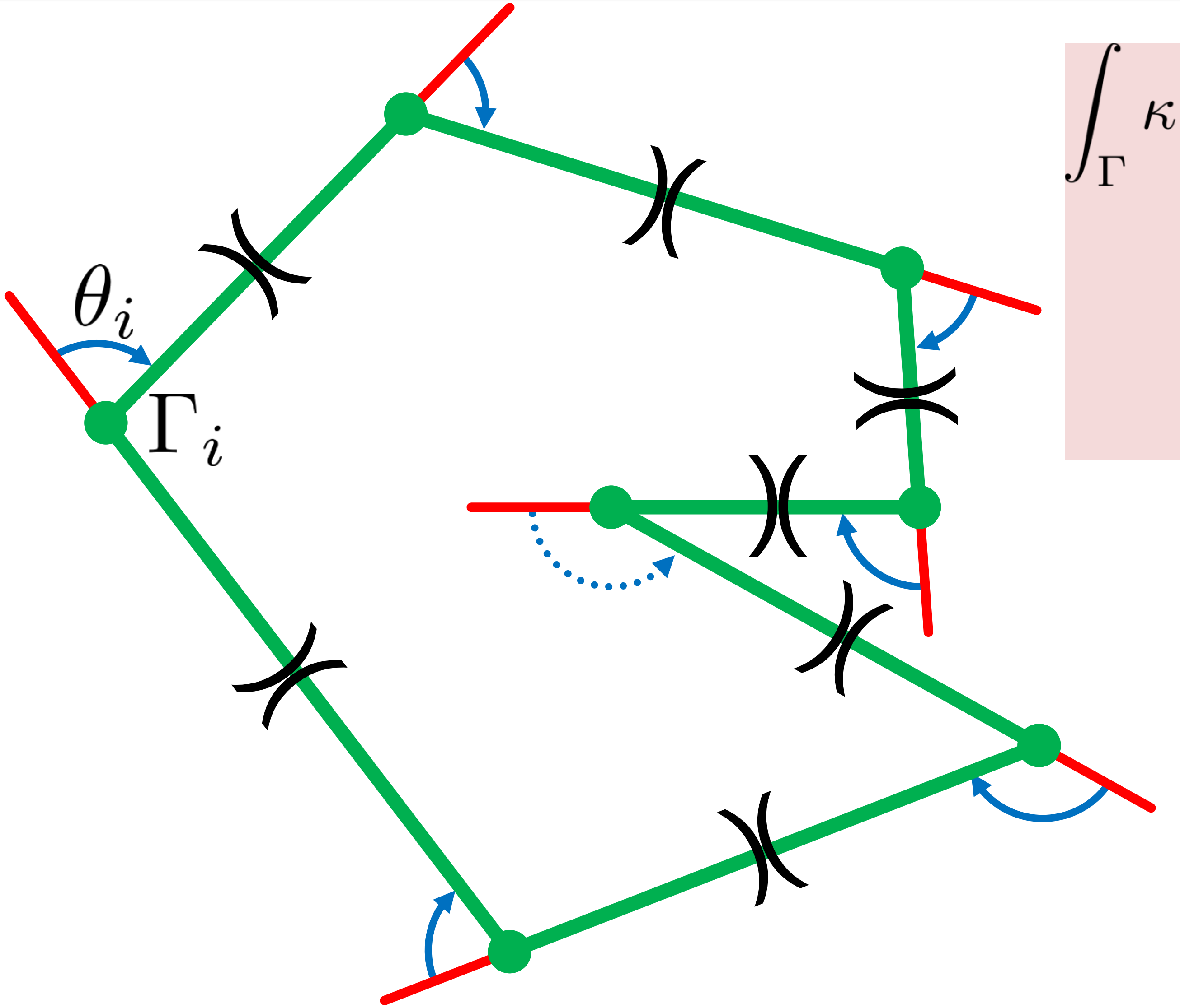


Total change in curvature

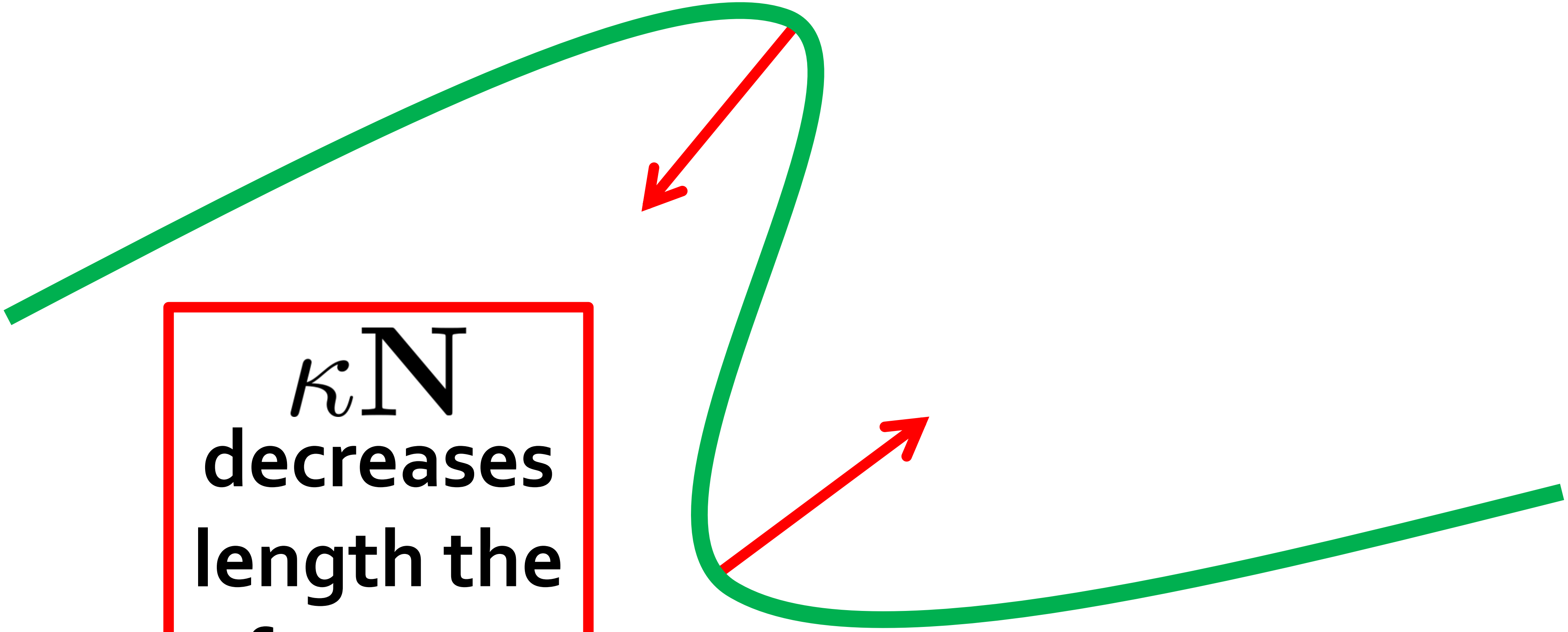
Discrete Turning Angle Theorem



$$\int_\Gamma \kappa\, ds = \sum_i \int_{\Gamma_i} \kappa\, ds$$
$$= \sum_i \theta_i$$
$$= 2\pi k$$

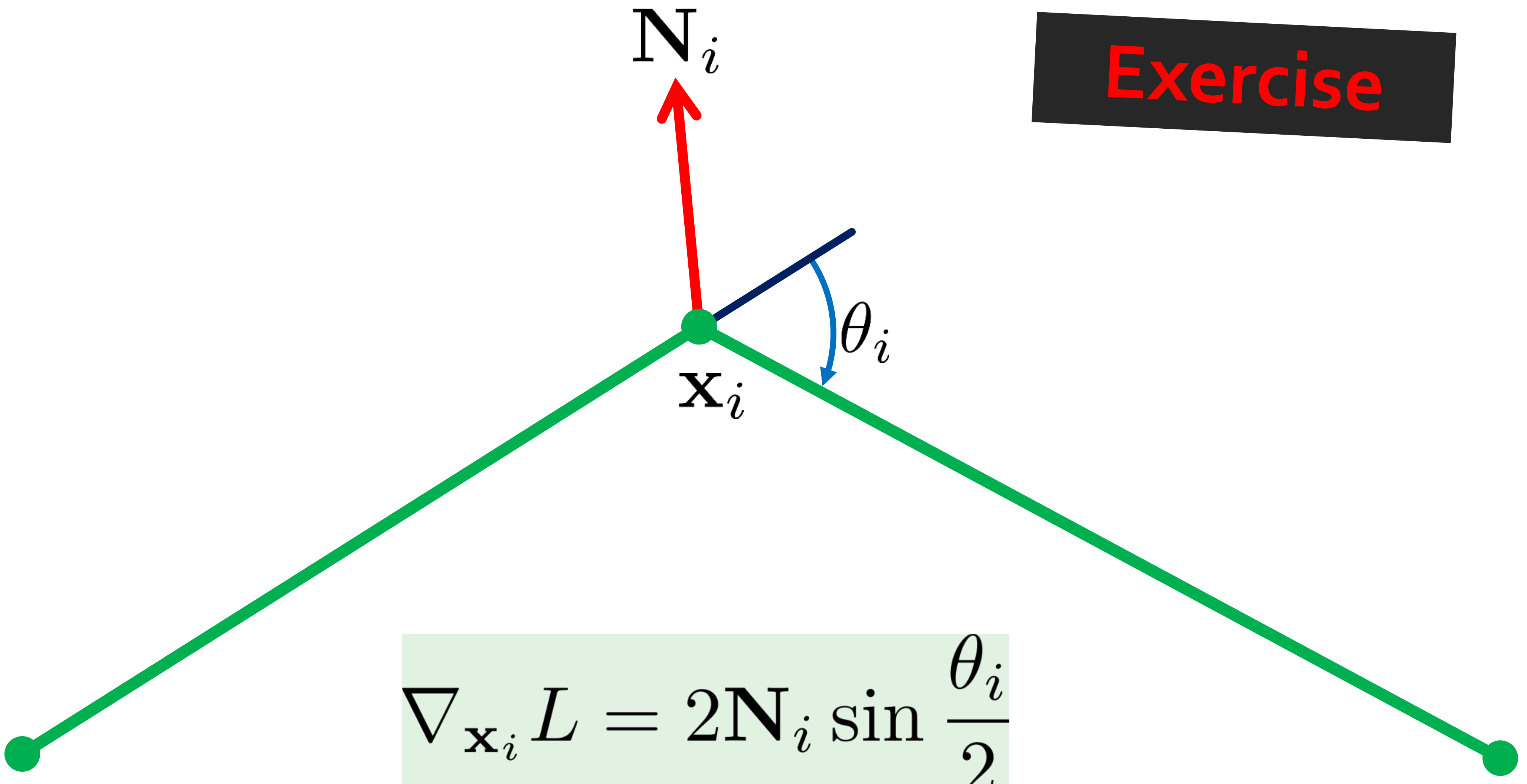
Preserved structure!

First Variation Formula



$\kappa \mathbf{N}$ decreases length the fastest.

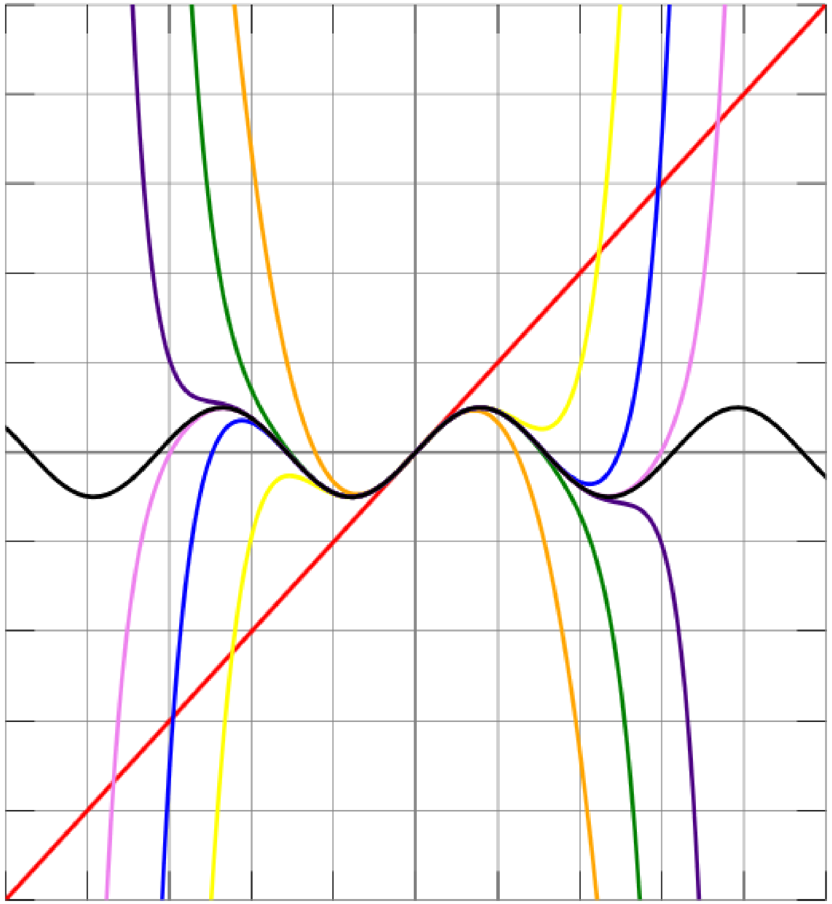
Discrete Case



Exercise

$$\nabla_{\mathbf{x}_i} L = 2\mathbf{N}_i \sin\frac{\theta_i}{2}$$

For Small θ


http://en.wikipedia.org/wiki/Taylor_series

$$2\sin\frac{\theta}{2} \approx 2\cdot\frac{\theta}{2}$$
$$= \theta$$

Same behavior in the limit

No Free Lunch

Choose one:

- Discrete curvature with
  turning angle theorem
- Discrete curvature from
  gradient of arc length

Remaining Question

Does discrete curvature converge in limit?

Yes!

Under some assumptions!

Remaining Question

Does discrete curvature converge in limit?

Yes!

Under some assumptions!

Questions:

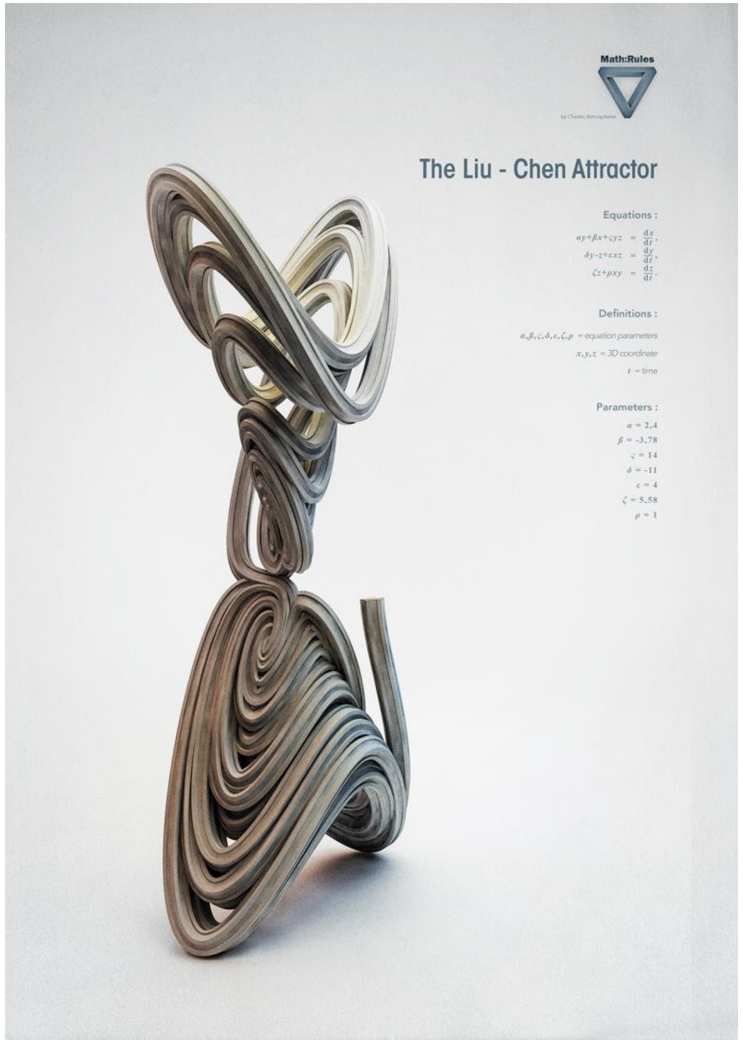
- Type of convergence?
- Sampling?
- Class of curves?

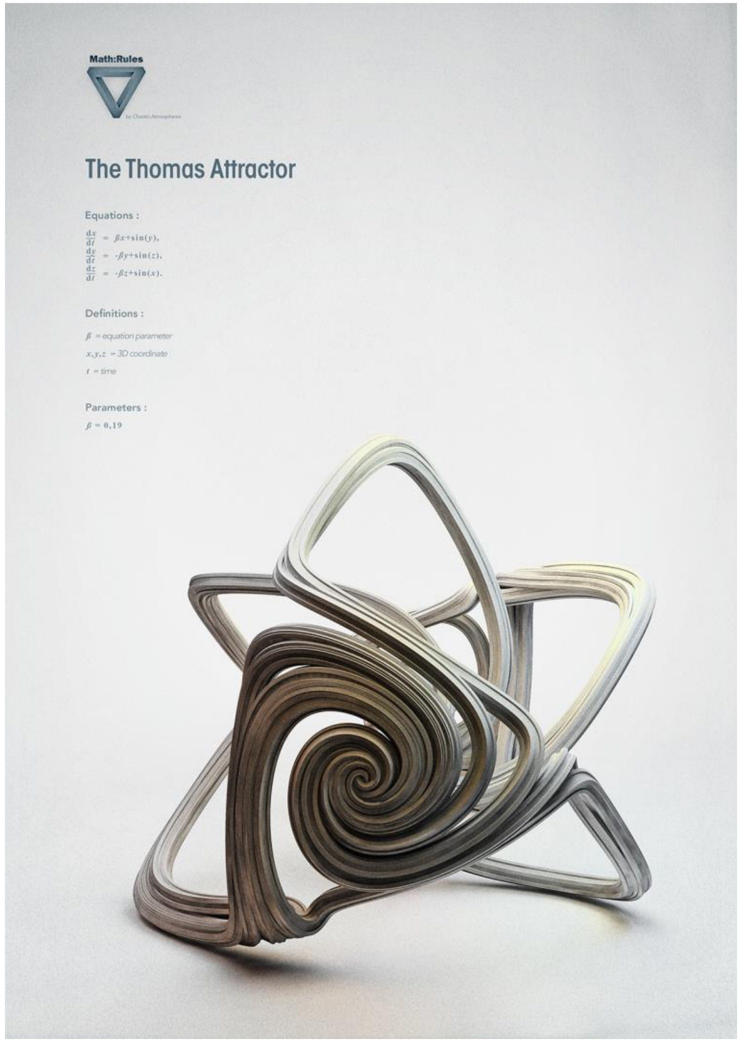
Discrete Differential Geometry

- **Different** discrete behavior

- **Same** convergence

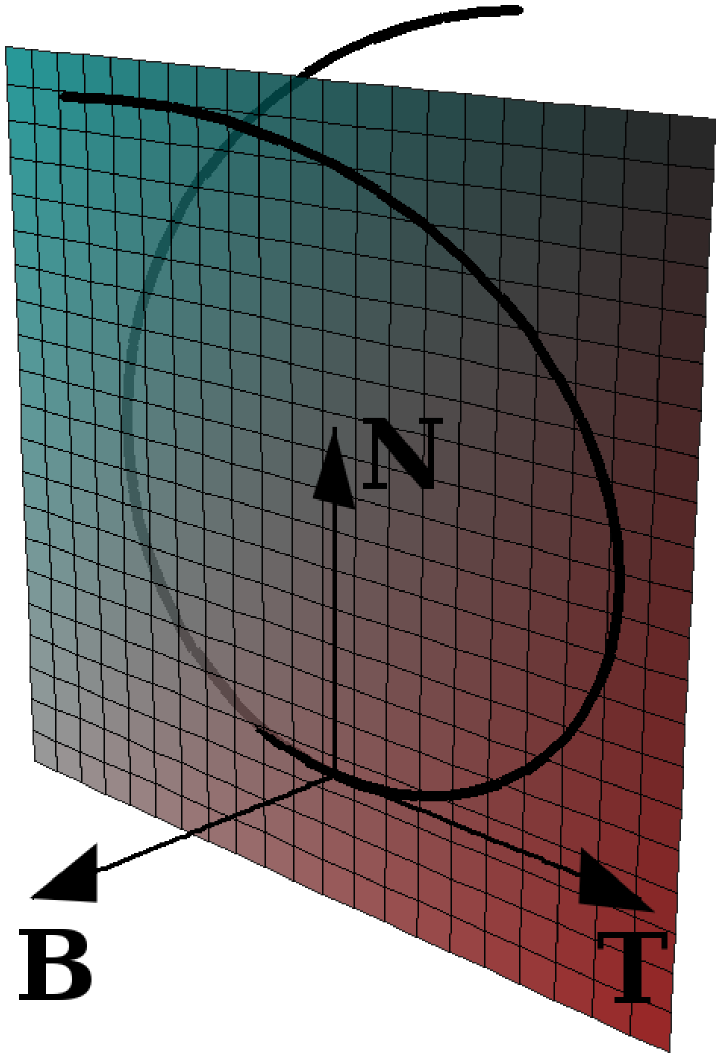
Curves in 3D?





https://www.behance.net/gallery/7618879/Strange-Attractors

Frenet Frame



http://upload.wikimedia.org/wikipedia/commons/6/6f/Frenet.png

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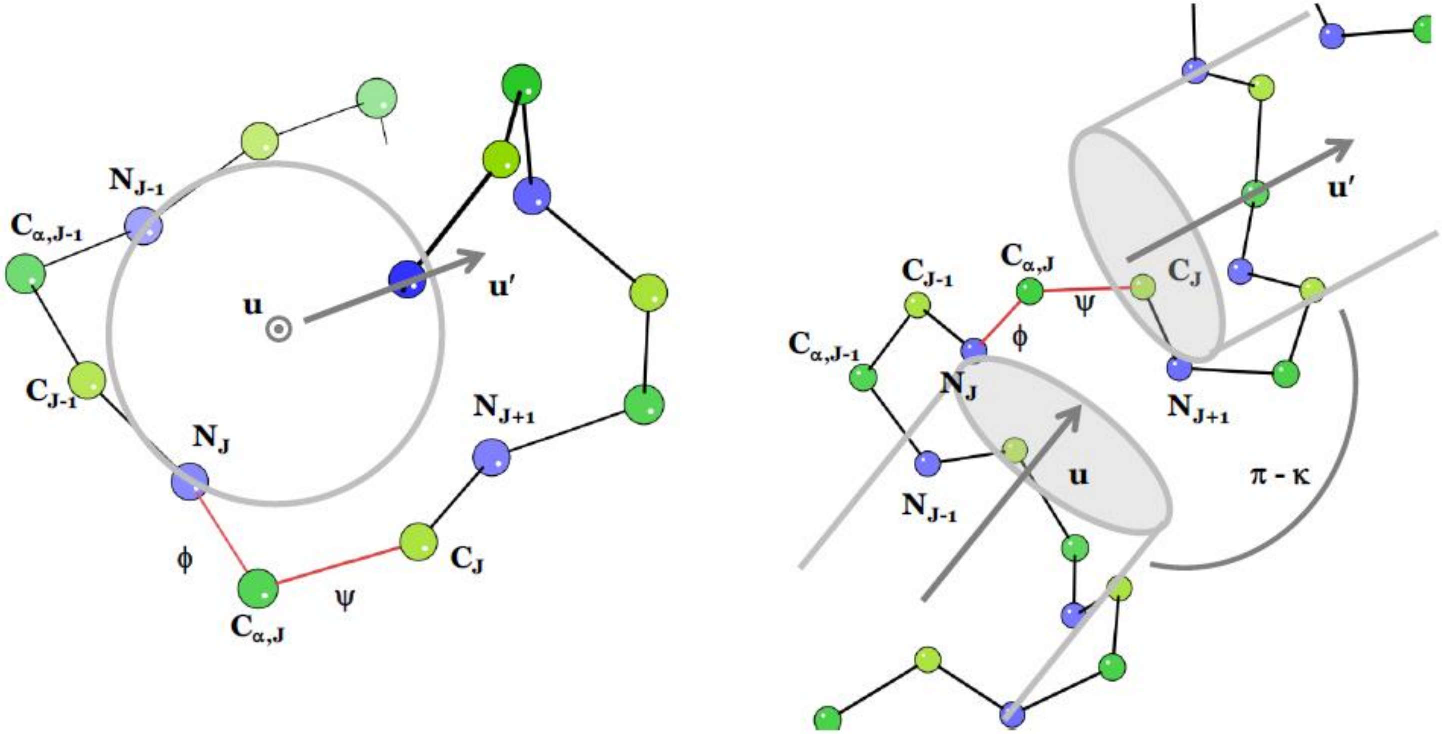
Application

NMR scanner



Kinked alpha helix

Structure Determination of Membrane Proteins Using Discrete Frenet Frame and Solid State NMR Restraints
Achuthan and Quine
Discrete Mathematics and its Applications, ed. M. Sethumadhavan (2006)

Potential Discretization

$$\mathbf{T}_j = \frac{\mathbf{p}_{j+1} - \mathbf{p}_j}{\|\mathbf{p}_{j+1} - \mathbf{p}_j\|_2}$$

$$\mathbf{B}_j = \mathbf{T}_{j-1} \times \mathbf{T}_j$$

$$\mathbf{N}_j = \mathbf{B}_j \times \mathbf{T}_j$$

Discrete Frenet frame

$$\mathbf{T}_k = R(\mathbf{B}_k, \theta_k)\mathbf{T}_{k-1}$$

$$\mathbf{B}_{k+1} = R(\mathbf{T}_k, \phi_k)\mathbf{B}_k$$

"Bond and torsion angles"

(derivatives converge to κ and τ, resp.)

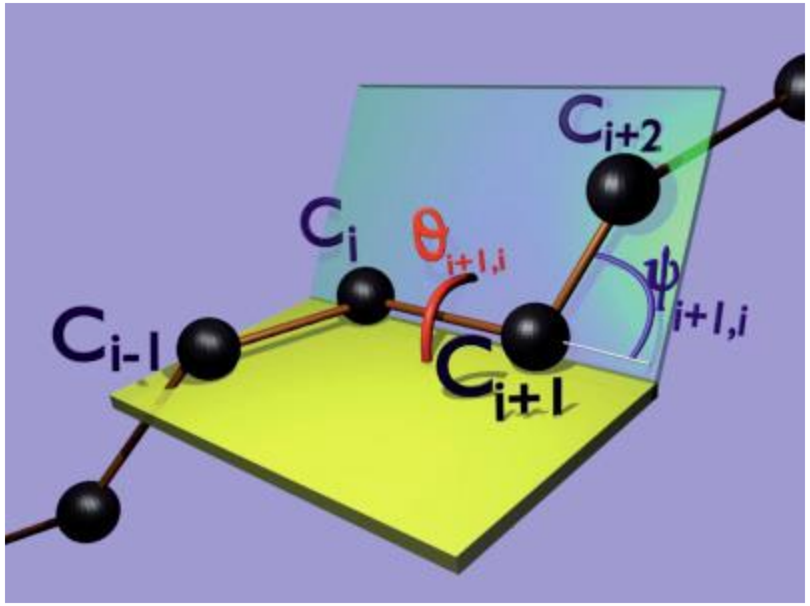
Discrete frame introduced in:

The resultant electric moment of complex molecules

Eyring, Physical Review, 39(4):746—748, 1932.

Transfer Matrix

$$\begin{pmatrix} \mathbf{T}_{i+1} \\ \mathbf{N}_{i+1} \\ \mathbf{B}_{i+1} \end{pmatrix} = R_{i+1,i} \begin{pmatrix} \mathbf{T}_{i} \\ \mathbf{N}_{i} \\ \mathbf{B}_{i} \end{pmatrix}$$

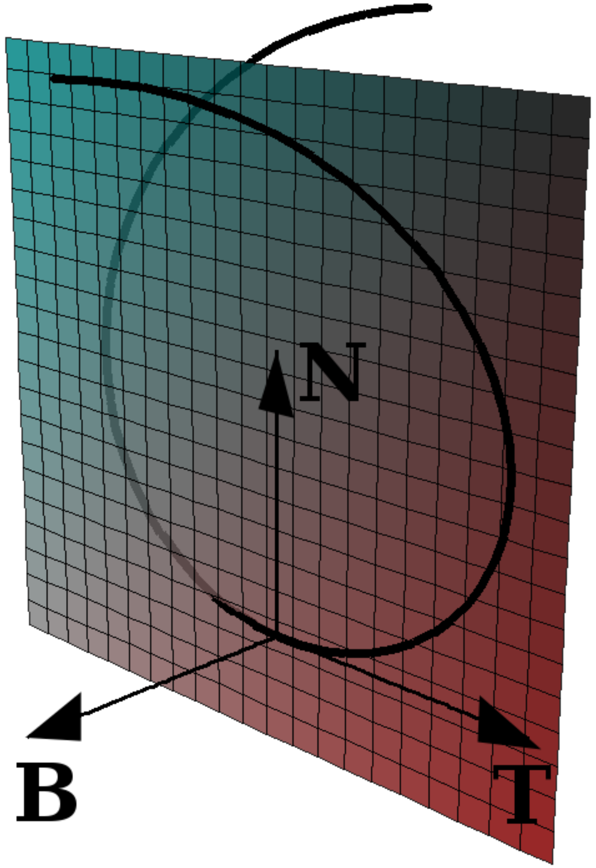


<u>Discrete</u> construction that works for fractal curves and converges in continuum limit.

Discrete Frenet Frame, Inflection Point Solitons, and Curve Visualization with Applications to Folded Proteins
Hu, Lundgren, and Niemi
Physical Review E 83 (2011)

Frenet Frame: Issue

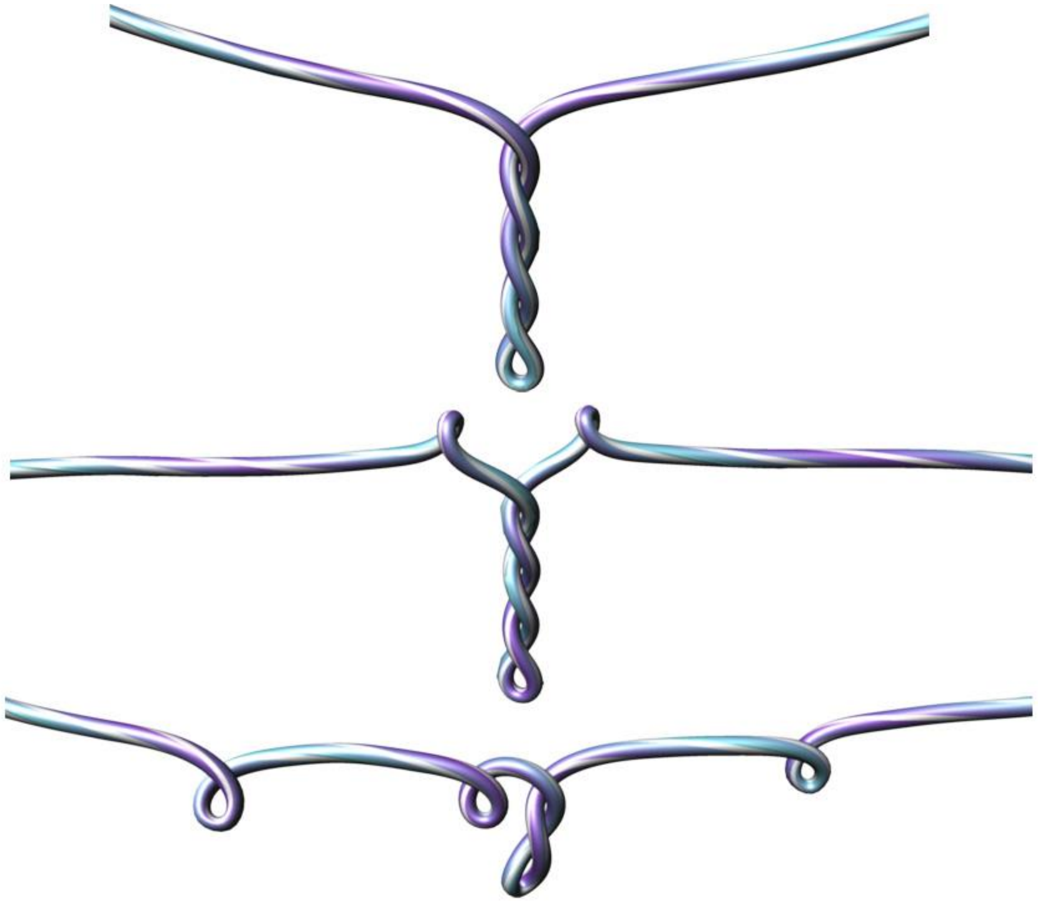
$\kappa = 0?$



$$\frac{d}{ds}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix} = \begin{pmatrix}0 & \boxed{\kappa(s)} & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix}$$

http://upload.wikimedia.org/wikipedia/commons/6/6f/Frenet.png

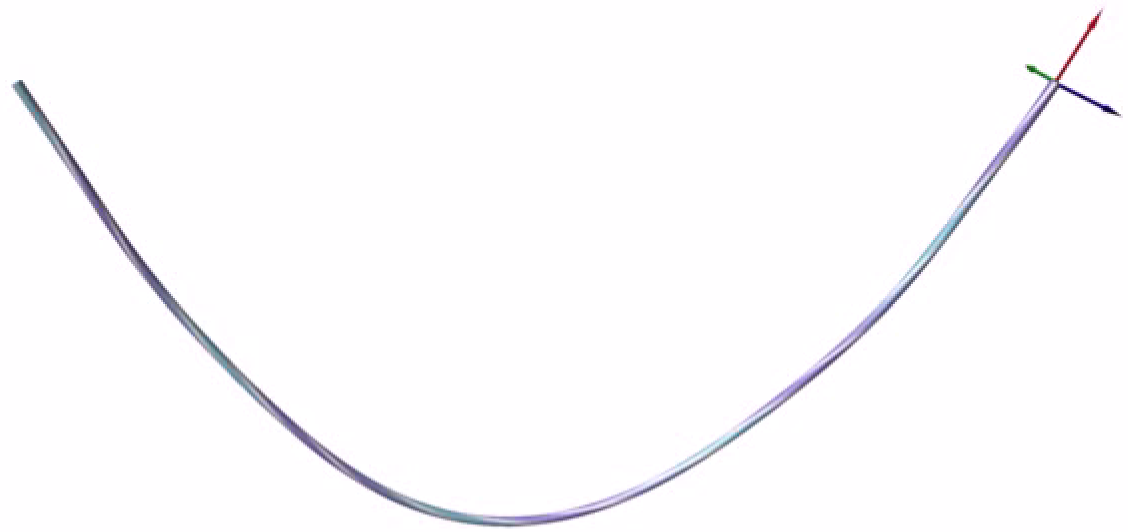
Segments Not Always Enough



Discrete Elastic Rods
Bergou, Wardetzky, Robinson, Audoly, and Grinspun
SIGGRAPH 2008

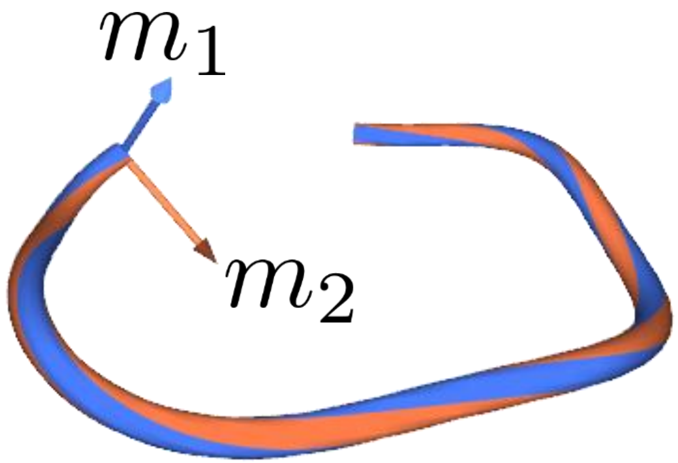
http://www.cs.columbia.edu/cg/rods/

Simulation Goal



http://www.cs.columbia.edu/cg/rods/

Adapted Framed Curve



$$\Gamma = \{\gamma(s); \mathbf{T}, \mathbf{m}_1, \mathbf{m}_2\}$$

Material frame

http://www.cs.columbia.edu/cg/rods/

Normal part encodes twist

Bending Energy

$$E_{\text{bend}}(\Gamma) := \frac{1}{2}\int_{\Gamma} \alpha\kappa^2\, ds$$

Penalize turning the steering wheel

$$\begin{aligned}
\kappa\mathbf{N} &= \mathbf{T}' \\
&= (\mathbf{T}'\cdot\mathbf{T})\mathbf{T} + (\mathbf{T}'\cdot\mathbf{m}_1)\mathbf{m}_1 + (\mathbf{T}'\cdot\mathbf{m}_2)\mathbf{m}_2 \\
&= (\mathbf{T}'\cdot\mathbf{m}_1)\mathbf{m}_1 + (\mathbf{T}'\cdot\mathbf{m}_2)\mathbf{m}_2 \\
&:= \omega_1\mathbf{m}_1 + \omega_2\mathbf{m}_2
\end{aligned}$$

Bending Energy

$$E_{\text{bend}}(\Gamma) := \frac{1}{2}\int_{\Gamma} \alpha(\omega_1^2 + \omega_2^2)\, ds$$

Penalize turning the steering wheel

$$\begin{aligned}
\kappa\mathbf{N} &= \mathbf{T}' \\
&= (\mathbf{T}' \cdot \mathbf{T})\mathbf{T} + (\mathbf{T}' \cdot \mathbf{m}_1)\mathbf{m}_1 + (\mathbf{T}' \cdot \mathbf{m}_2)\mathbf{m}_2 \\
&= (\mathbf{T}' \cdot \mathbf{m}_1)\mathbf{m}_1 + (\mathbf{T}' \cdot \mathbf{m}_2)\mathbf{m}_2 \\
&:= \omega_1\mathbf{m}_1 + \omega_2\mathbf{m}_2
\end{aligned}$$

Twisting Energy

$$E_{\text{twist}}(\Gamma) := \frac{1}{2}\int_{\Gamma} \beta m^2 \, ds$$

Penalize non-tangent change in material frame

$$\begin{aligned} m &:= \mathbf{m}_1' \cdot \mathbf{m}_2 \\ &= \frac{d}{dt}(\mathbf{m}_1 \cdot \mathbf{m}_2) - \mathbf{m}_1 \cdot \mathbf{m}_2' \\ &= -\mathbf{m}_1 \cdot \mathbf{m}_2' \end{aligned}$$

Swapping m_1 and m_2 does not affect E_{twist}!

Bishop Frame: The Hipster Framed Curve

THERE IS MORE THAN ONE WAY TO FRAME A CURVE

RICHARD L. BISHOP

The Frenet frame of a 3-times continuously differentiable (that is, C^3) non-degenerate curve in euclidean space has long been the standard vehicle for analysing properties of the curve invariant under euclidean motions. For arbitrary moving frames, that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the curve parameter in terms of the frame itself, and due to orthonormality the coefficient matrix is always skew-symmetric. Thus it generally has three nonzero entries. The Frenet frame gains part of its special significance from the fact that one of the three derivatives is always zero. Another feature of the Frenet frame is that it is *adapted* to the curve: the members are either tangent to or perpendicular to the curve. It is the purpose of this paper to show that there are other frames which have these same advantages and to compare them with the Frenet frame.

. **Relatively parallel fields.** We say that a normal vector field M along a cur
atively parallel if its derivative is tangential. Such a field turns only whate
unt is necessary for it to remain normal, so it is as close to being parallel
ble without losing normality. Since its derivative is perpendicular to it, a
atively parallel normal fie … h fields occur classically in

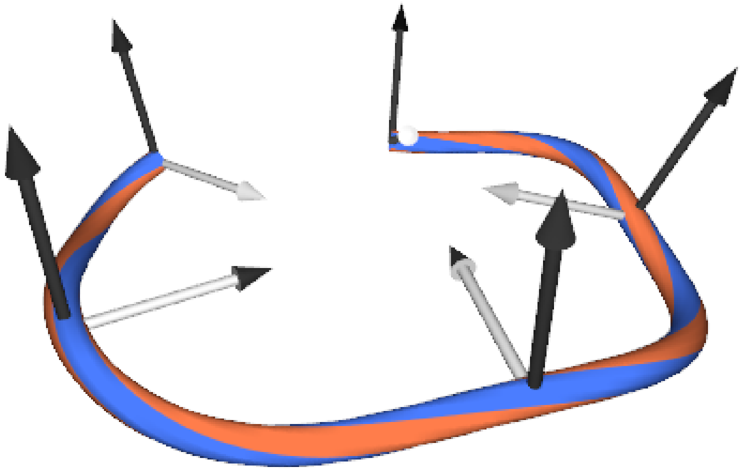
(couldn't decide on a meme)

Bishop Frame

$$\mathbf{T}' = \mathbf{\Omega} \times \mathbf{T}$$

$$\mathbf{u}' = \mathbf{\Omega} \times \mathbf{u}$$

$$\mathbf{v}' = \mathbf{\Omega} \times \mathbf{v}$$



$$\mathbf{\Omega} := \kappa \mathrm{B} \ (\text{“curvature binormal”})$$

Darboux vector

http://www.cs.columbia.edu/cg/rods/

Most relaxed frame

Bishop Frame

$$\mathbf{T}' = \mathbf{\Omega} \times \mathbf{T}$$

$$\mathbf{u}' = \mathbf{\Omega} \times \mathbf{u}$$

$$\mathbf{v}' = \mathbf{\Omega} \times \mathbf{v}$$

$$\mathbf{u}' \cdot \mathbf{v} \equiv 0$$

No twist

("parallel transport")

$$\mathbf{\Omega} := \kappa \mathrm{B} \text{ (“curvature binormal”)}$$

Darboux vector

http://www.cs.columbia.edu/cg/rods/

Most relaxed frame

Curve-Angle Representation

$$\mathbf{m}_1 = \mathbf{u}\cos\theta + \mathbf{v}\sin\theta$$

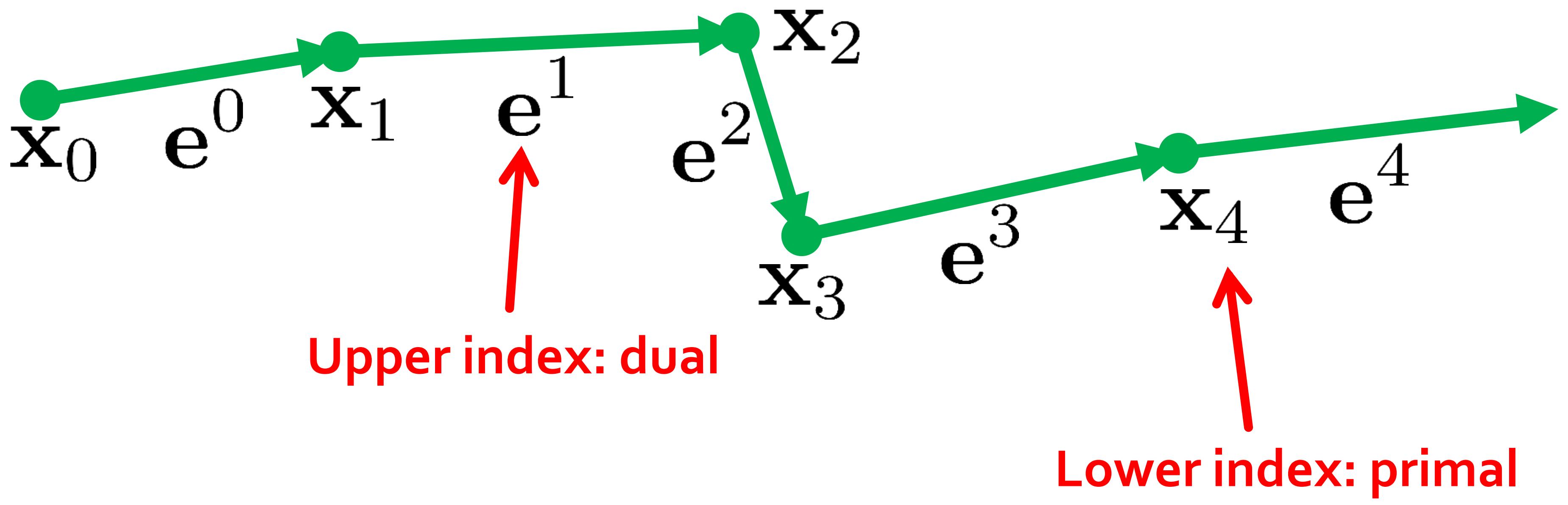
$$\mathbf{m}_2 = -\mathbf{u}\sin\theta + \mathbf{v}\cos\theta$$

$$E_{\mathrm{twist}}(\Gamma) := \frac{1}{2}\int_\Gamma \beta(\theta')^2\, ds$$

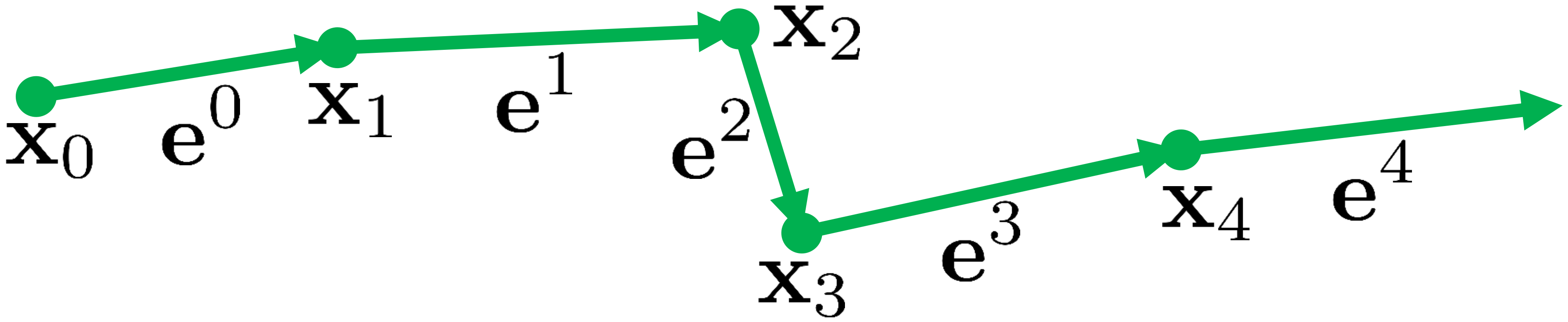
Degrees of freedom for elastic energy:

- Shape of curve
- Twist angle $\boldsymbol{\theta}$

Discrete Kirchoff Rods



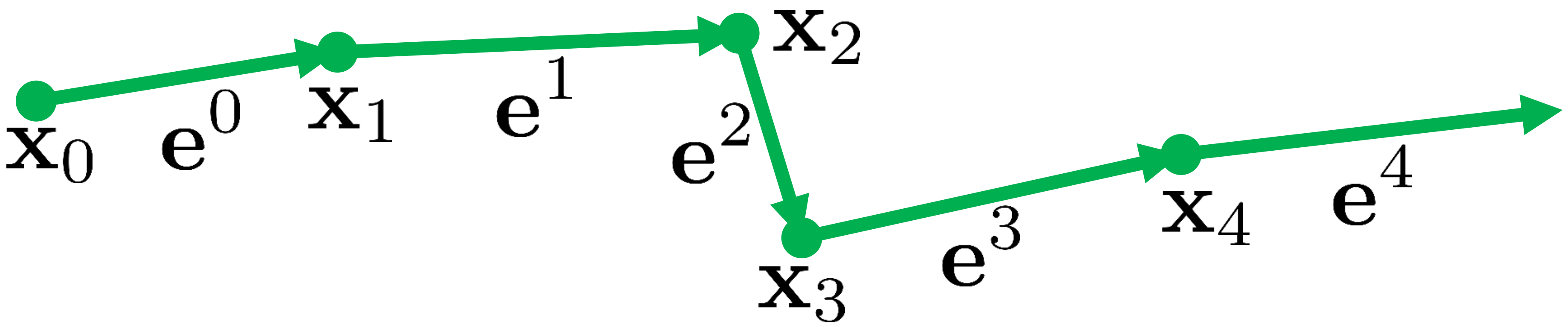
Discrete Kirchoff Rods



$$\mathbf{T}^i := \frac{\mathbf{e}^i}{\|\mathbf{e}^i\|_2}$$

Tangent unambiguous on edge

Discrete Kirchoff Rods



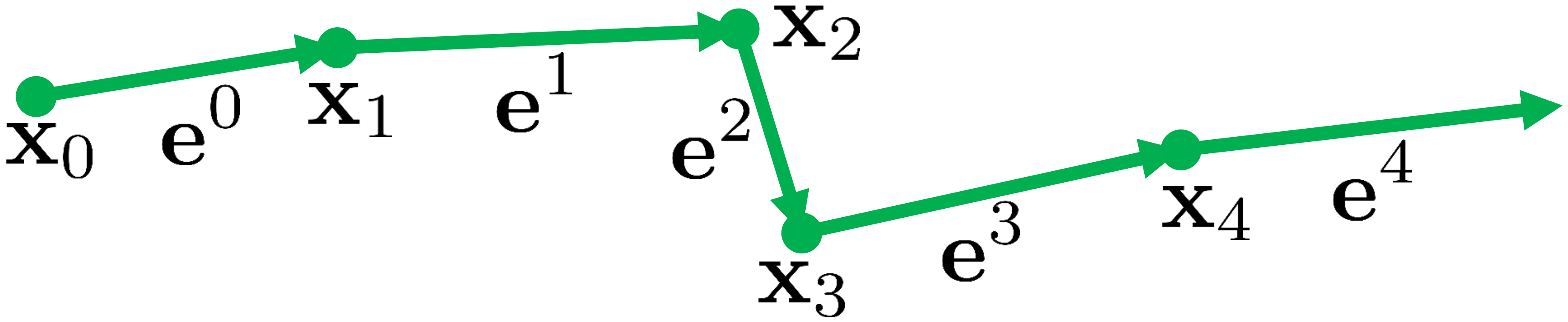
$$\kappa_i := 2\tan\frac{\phi_i}{2}$$

Turning angle

Yet another curvature!

Integrated curvature

Discrete Kirchoff Rods



$$\kappa_i := 2\tan\frac{\phi_i}{2}$$

$$(\kappa \mathbf{B})_i := \frac{2\mathbf{e}^{i-1} \times \mathbf{e}^i}{\|\mathbf{e}^{i-1}\|_2 \|\mathbf{e}^i\|_2 + \mathbf{e}^{i-1} \cdot \mathbf{e}^i}$$

Yet another curvature!

Orthogonal to osculating plane, norm κ_i

Darboux vector

Bending Energy

$$E_{\mathrm{bend}}(\Gamma) := \frac{\alpha}{2}\sum_i \left(\frac{(\kappa \mathbf{B})_i}{\ell_i/2}\right)^2 \frac{\ell_i}{2}$$

$$= \alpha \sum_i \frac{\|(\kappa \mathbf{B})_i\|_2^2}{\ell_i}$$

Convert to pointwise and integrate

Discrete Parallel Transport

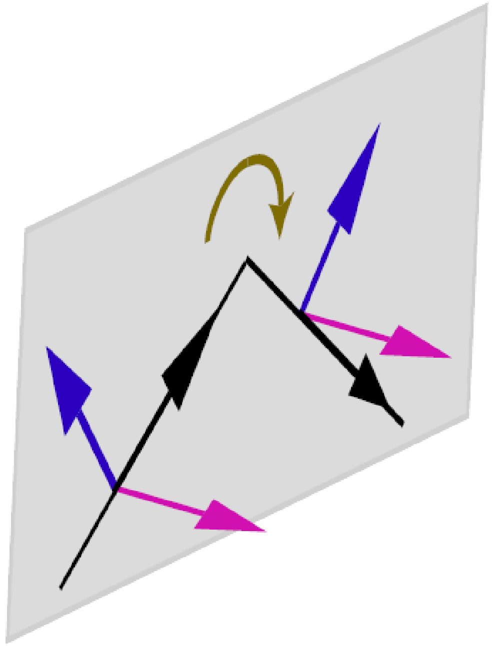
$$P_i(\mathbf{T}^{i-1}) = \mathbf{T}^i$$

$$P_i(\mathbf{T}^{i-1} \times \mathbf{T}^i) = \mathbf{T}^{i-1} \times \mathbf{T}^i$$

- Map tangent to tangent
- Preserve binormal
- Orthogonal

$$\mathbf{u}^i = P_i(\mathbf{u}^{i-1})$$

$$\mathbf{v}^i = \mathbf{T}^i \times \mathbf{u}^i$$

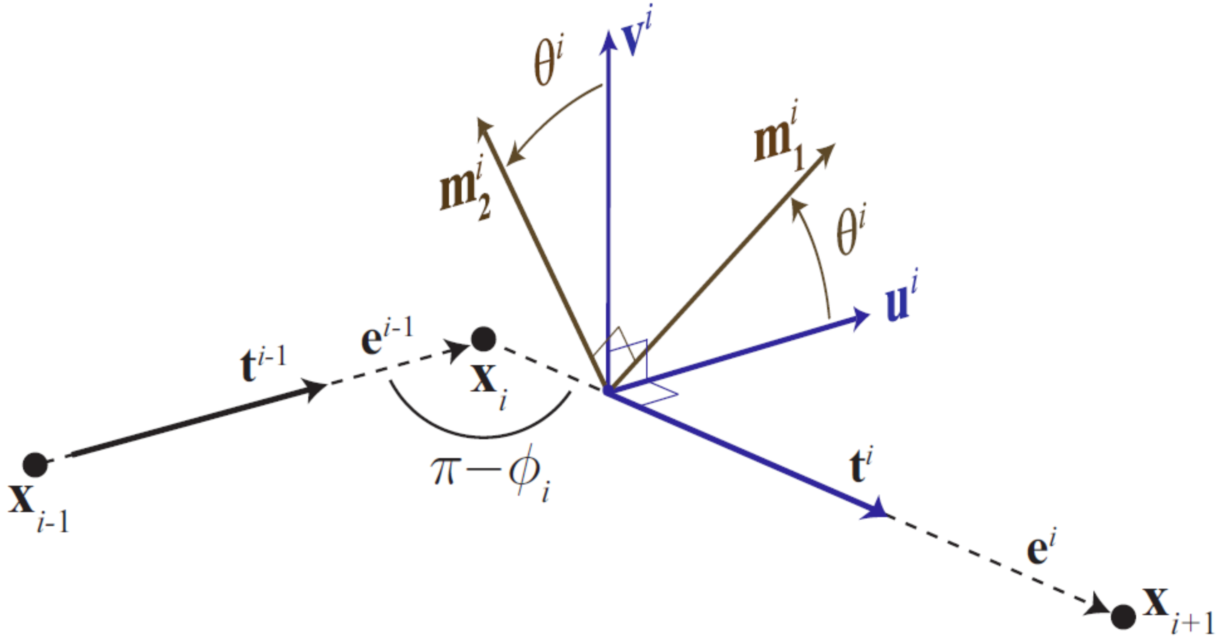


http://www.cs.columbia.edu/cg/rods/

Discrete Material Frame

$$\mathbf{m}_1^i = \mathbf{u}^i \cos\theta^i + \mathbf{v}^i \sin\theta^i$$

$$\mathbf{m}_2^i = -\mathbf{u}^i \sin\theta^i + \mathbf{v}^i \cos\theta^i$$



http://www.cs.columbia.edu/cg/rods/

Discrete Twisting Energy

$$E_{\text{twist}}(\Gamma) := \beta \sum_i \frac{(\theta^i - \theta^{i-1})^2}{\ell_i}$$

Note θ_0 can be arbitrary

Simulation

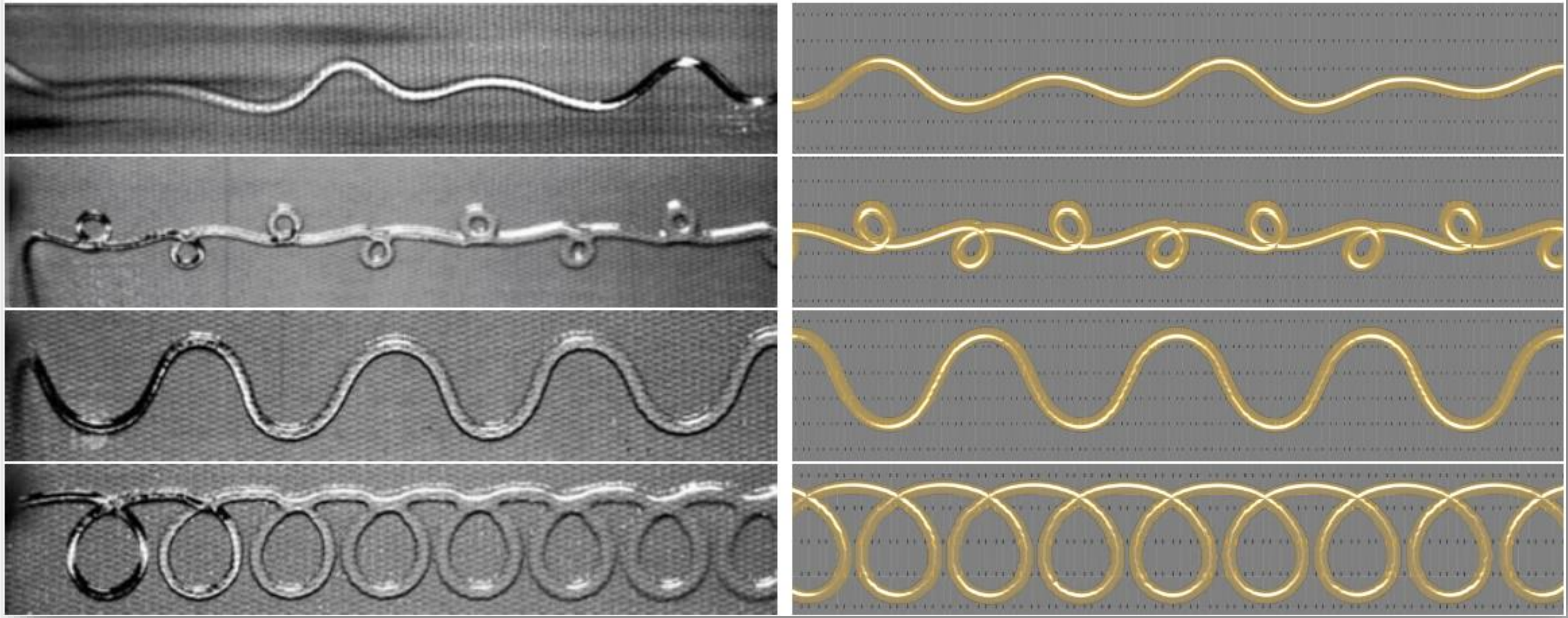
`\omit{physics}`

Worth reading!

Extension and Speedup

Discrete Viscous Threads

Miklós Bergou, Columbia University
Basile Audoly, UPMC Univ. Paris 06 & CNRS
Etienne Vouga, Columbia University
Max Wardetzky, Universität Göttingen
Eitan Grinspun, Columbia University



http://www.cs.columbia.edu/cg/threads/

Extension and Speedup

Discrete Viscous Threads

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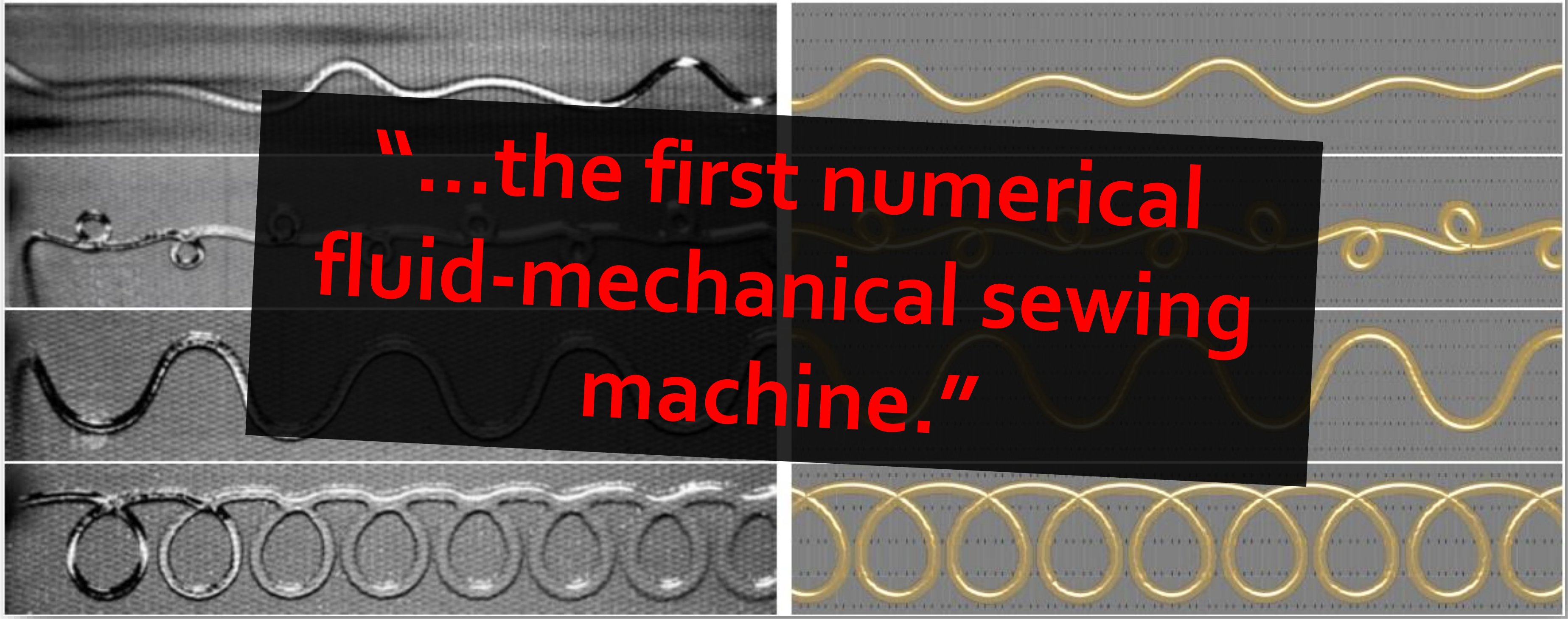


"…the first numerical fluid-mechanical sewing machine."

http://www.cs.columbia.edu/cg/threads/

Morals

One curve,
three curvatures.

$$\theta \qquad\qquad 2\sin\frac{\theta}{2} \qquad\qquad 2\tan\frac{\theta}{2}$$

Morals

Easy theoretical object,
hard to use.

$$\frac{d}{ds}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix}=\begin{pmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix}$$

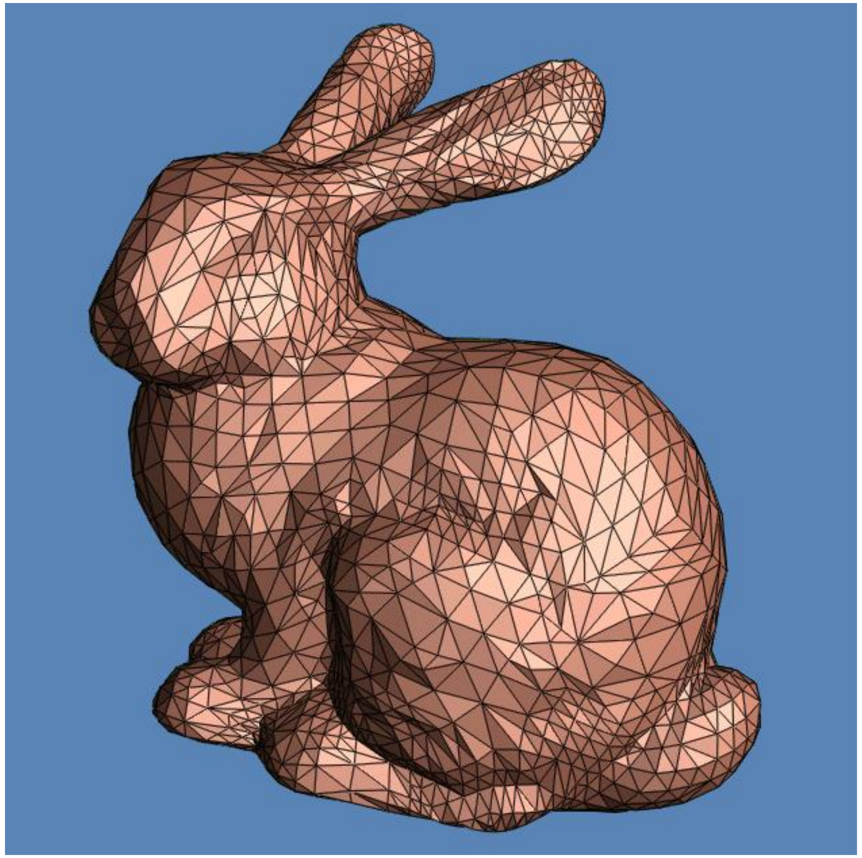
Morals

Proper frames and DOFs go a long way.

$$\mathbf{m}_1^i = \mathbf{u}^i \cos\theta^i + \mathbf{v}^i \sin\theta^i$$

$$\mathbf{m}_2^i = -\mathbf{u}^i \sin\theta^i + \mathbf{v}^i \cos\theta^i$$

Next



http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg
http://www.stat.washington.edu/wxs/images/BUNMID.gif

Surfaces

Discrete Curves

Justin Solomon

6.8410: Shape Analysis

Spring 2023