

Homework 1: Discrete and Smooth Curves

Due September 22, 2026

This is the first homework assignment for 6.8410. Check the course website for additional materials, the AI policy, and the late policy. You may work on assignments in groups, but every student must submit their own write up; please note your collaborators, if any, on your write up.

Some of the notation used in this homework may be unfamiliar or rusty for computer science students, but undergrad calculus should be sufficient to answer all the problems. Get started early, and reach out for help during office hours and/or on Piazza—we will be generous!

Problem 1 (30 points). In this problem, you'll practice using variational calculus, one of the main tools of the applied geometry toolbox.

- (a) Suppose you are given a regular plane curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2$, and take $\mathbf{v} : [0, 1] \rightarrow \mathbb{R}^2$ to be a vector field along γ . Recall that the arc length of γ is given by

$$s[\gamma] = \int_0^1 \|\gamma'(t)\|_2 dt.$$

We can think of $\gamma_h(t) := \gamma(t) + h\mathbf{v}(t)$ to be a displacement of γ along $\mathbf{v}$. Differentiate $f(h) := s[\gamma + h\mathbf{v}]$ with respect to h at $h = 0$ to yield an expression for $\frac{d}{dh}s[\gamma + h\mathbf{v}]|_{h=0}$.

Hint: The “differentiation under the integral sign” rule shows $\frac{d}{dh} \int_a^b g(t, h) dt = \int_a^b \frac{\partial g}{\partial h}(t, h) dt$.

- (b) You can think of each $\mathbf{v}$ as an infinitesimal displacement (a “variation”) of the entire curve γ at once. Explain how the derivative you took in (a) can be thought of as a directional derivative of arc length in the $\mathbf{v}$ “direction.” In variational calculus, this derivative is known as the Gâteaux or variational derivative of $s[\cdot]$.

- (c) Suppose $\mathbf{v}(0) = \mathbf{v}(1) = \mathbf{0}$. Define a vector-valued function $\mathbf{w}(s)$ so that

$$\left. \frac{d}{dh} s[\gamma + h\mathbf{v}] \right|_{h=0} = \int_0^{s(1)} \mathbf{v}(s^{-1}(\bar{s})) \cdot \mathbf{w}(\bar{s}) d\bar{s},$$

where $s(t) = \int_0^t \|\gamma'(\bar{t})\|_2 d\bar{t}$ on the right-hand side is the arc length function and $\mathbf{w}$ can be written in terms of the curvature and Frenet frame of γ .

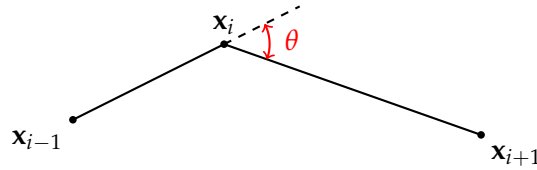
Note: Your formula for $\mathbf{w}$ should *not* include any term involving $\mathbf{v}$.

Hint: Simplify the left-hand side using your answer to (a). Use integration by parts. Also recall that the Frenet formulas involve *arc length*-derivatives.

For coding assignments, please use MATLAB as your programming language. The starter code handles visualization/problem setup and indicates where you should fill in your solutions.

Problem 2 (35 points). In this problem, you will develop a notion of discrete curvature of a plane curve.

- (a) Suppose we have a discrete curve given by a series of points $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^2$. You can think of the vertex positions as parameterized by a vector $\mathbf{x} \in \mathbb{R}^{2n}$. Define an arc length functional $s(\mathbf{x}) : \mathbb{R}^{2n} \rightarrow \mathbb{R}_+$; for convenience, it is acceptable to notate $s(\mathbf{x}) = s(\mathbf{x}_1, \dots, \mathbf{x}_n)$.
- (b) Suppose $1 < i < n$. Write an expression for the gradient $\nabla_{\mathbf{x}_i} s$ of $s(\cdot)$ with respect to $\mathbf{x}_i$ and show that its norm is $2 \sin \frac{\theta}{2}$, where θ is the (unsigned) turning angle between the two segments adjacent to $\mathbf{x}_i$ (see figure).



- (c) Take a look at `discreteCurve.m` (or `discreteCurve.jl`). The curve generates an $n \times 2$ array representing n points on a discrete two-dimensional curve. Modify the code to plot the derivative you computed in part (b).
- (d) Propose a measure of discrete (unsigned) per-vertex curvature of a 2D discrete curve based on your answers to 1(c) and 2(b), and draw the curve colored by this value. For this, fill in code to compute `kappa` in `discreteCurve.m` (`discreteCurve.jl`).

Note: Make sure that your notion of curvature does not go to zero as you increase the number n of samples. Multiple answers are possible.

- (e) For sufficiently small $h > 0$, one simple way to decrease the length of the curve would be to replace each point $\mathbf{x}_i$ with a new point $\mathbf{x}'_i := \mathbf{x}_i - (\nabla_{\mathbf{x}_i} s)h$, where $\nabla_{\mathbf{x}_i} s$ is the derivative of s with respect to $\mathbf{x}_i$ (make sure you understand why!). Implement this forward integration scheme with the endpoints fixed, and make sure that if you iterate enough times the curve approximates a straight line. What happens if h is too large?

Note: the gradients with respect to all the points should be computed together before replacing any points.

Problem 3 (35 points). In this problem, you will implement a special case of the “Discrete Elastic Rods” paper discussed in class—a closed loop formed from naturally straight, isotropic rod. Take a look at `elasticRods.m` (or `elasticRods.jl`) for starter code.

- (a) Add code to compute the $3 \times 3 \times n$ array `parallelTransport`. Each 3×3 page of this array is a rotation matrix aligning the Bishop frames between successive edges (denoted P_i in the paper).

(b) The bending energy has the form

$$E_{\text{bend}} = \alpha \sum_i \frac{|(\kappa \mathbf{b})_i|^2}{\bar{l}_i},$$

where $\bar{l}_i$ is the dual length of vertex i (see §4.2.1 of the paper). The bending force at vertex i is thus

$$-\nabla_i E_{\text{bend}} = \sum_j \frac{-2\alpha}{\bar{l}_j} (\nabla_i (\kappa \mathbf{b})_j)^\top (\kappa \mathbf{b})_j.$$

Show that

$$\nabla_i (\kappa \mathbf{b})_j = \frac{1}{|\bar{\mathbf{e}}^{j-1}| |\bar{\mathbf{e}}^j| + \mathbf{e}^{j-1} \cdot \mathbf{e}^j} \begin{cases} 2[\mathbf{e}^j] + (\kappa \mathbf{b})_j (\mathbf{e}^j)^\top & i = j - 1 \\ 2[\mathbf{e}^{j-1}] - (\kappa \mathbf{b})_j (\mathbf{e}^{j-1})^\top & i = j + 1 \\ -(2[\mathbf{e}^j] + (\kappa \mathbf{b})_j (\mathbf{e}^j)^\top + 2[\mathbf{e}^{j-1}] - (\kappa \mathbf{b})_j (\mathbf{e}^{j-1})^\top) & i = j \\ 0 & \text{otherwise,} \end{cases}$$

where $[\mathbf{x}]$ denotes a 3×3 matrix defined so that $[\mathbf{x}]\mathbf{y} = \mathbf{x} \times \mathbf{y}$ (see §7.1 of the paper). *Note:* you may assume as in the paper that the rod is inextensible, i.e., the edge lengths $|\bar{\mathbf{e}}^k|$ are constant over time.

- (c) Implement the functions `computeBendForce` and `computeTwistForce`. Make sure to read the relevant sections in the paper carefully—in particular, note that $\bar{l}_i$ is *twice* the dual cell length $|\mathcal{D}_i|$ (see §4.2.1 in the paper), and pay attention to the transposes in the force equations.

To test your work, run the `elasticRods` function. Plot the total energy, kinetic energy, twist energy, and bending energy over time. The total energy should stay constant to within a close approximation.

Hint: You should not have to form the matrices $[\mathbf{e}^j]$ to compute your forces.

- (d) **Challenge problem** (5 points of the 35). Fixing the material parameters α and β (`bendModulus` and `twistModulus` in the code), plot the maximum displacement over time for a variety of total twist values. Try to determine the critical twist at which the *Michell instability* phase transition occurs. Does it correspond to the theoretical value $\theta = 2\pi\sqrt{3}\alpha/\beta$? Why or why not?