

Surfaces: Smooth and Discrete

Justin Solomon

6.8410: Shape Analysis

Fall 2026



What's Next?

Step up
one dimension
from curves to surfaces.

- Theoretical definition
- Discrete representations
- Higher dimensionality

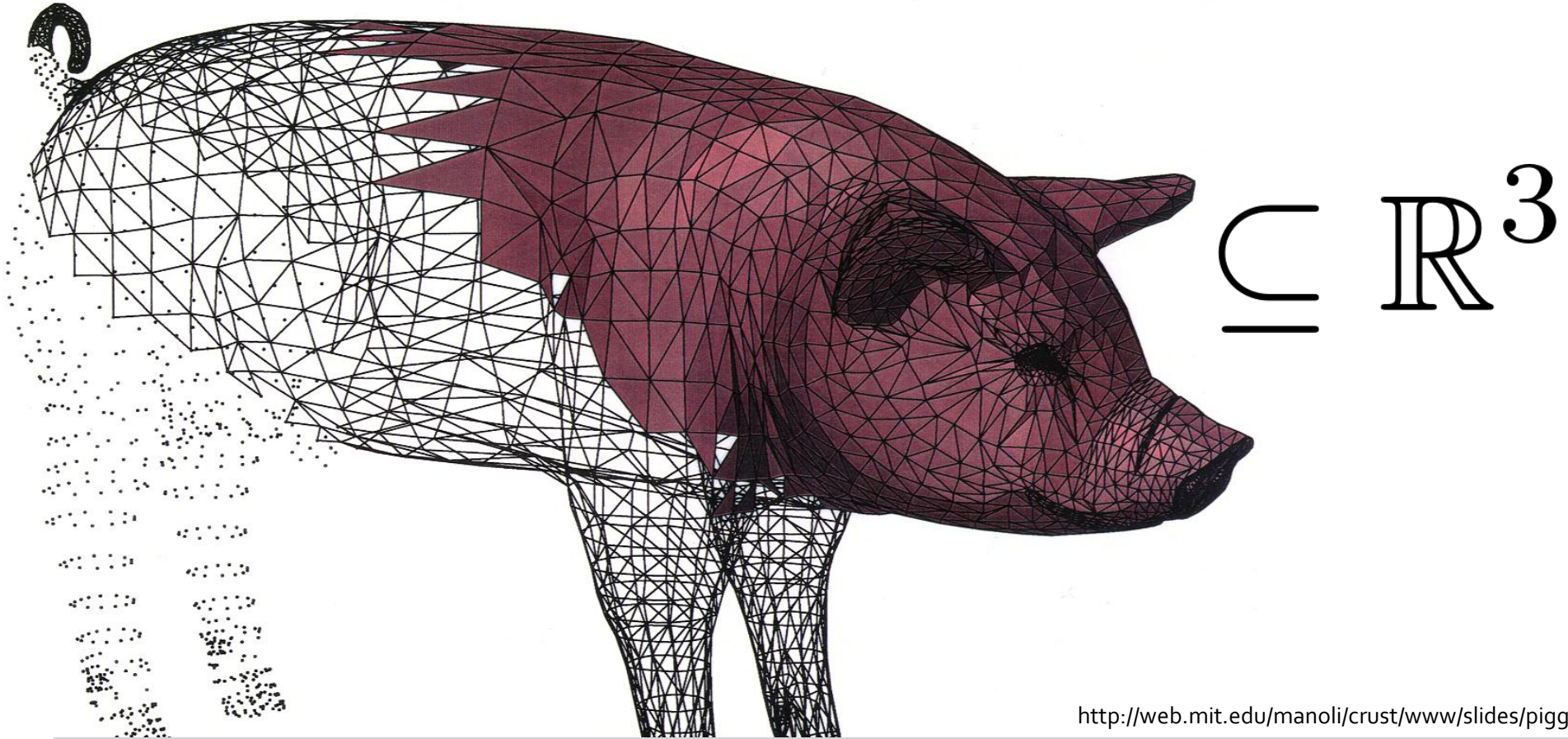
Briefly Will Mention

Step up
 n dimensions
from surfaces to (sub)manifolds.

Easier transition.

*Not entirely true:
e.g. topology of 3-manifolds*

Our Focus



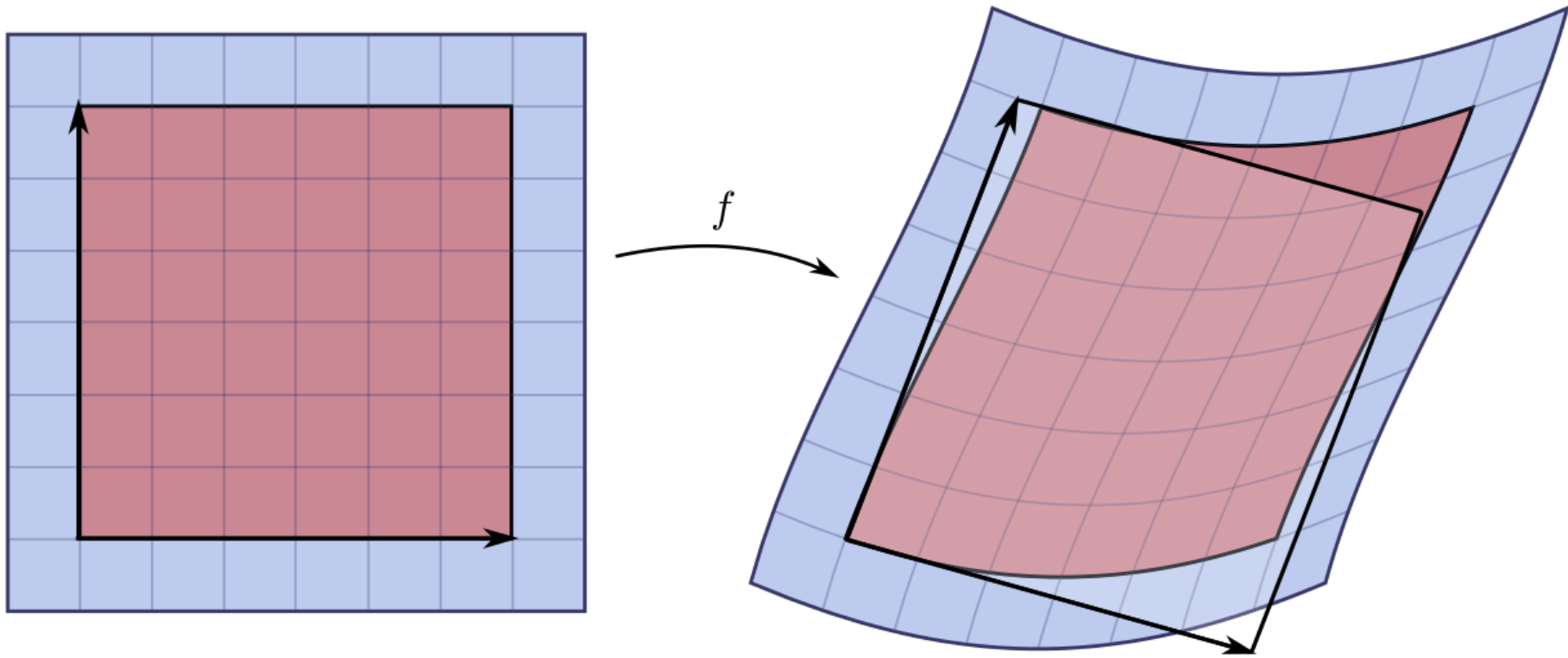
<http://web.mit.edu/manoli/crust/www/slides/piggy.jpg>

Embedded geometry



What is an
embedded surface?

Warm Up: Parametric Surface

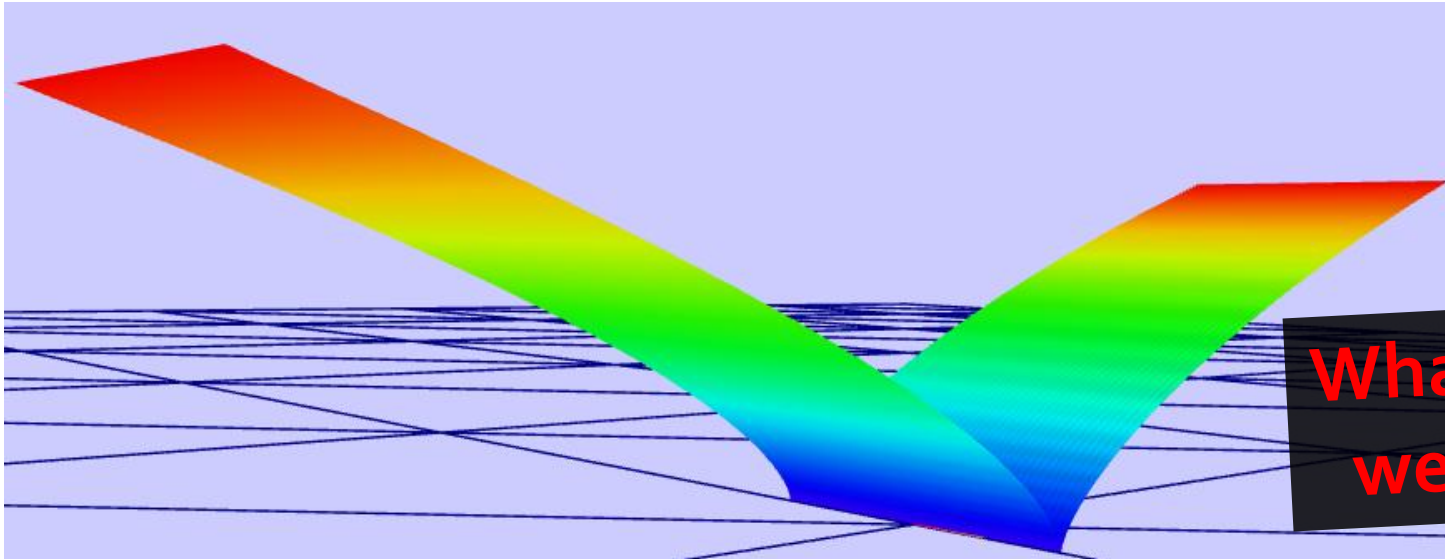


Pathological Cases

$$f(u, v) = (u, u^2, \cos u)$$

$$f(u, v) = (0, 0, 0)$$

$$f(u, v) = (u, v^3, v^2)$$



What condition do we need to add?

Review: Jacobian Matrix

$$f : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

Jacobian matrix:

$$(Df)_j^i = \left(\frac{\partial f^i}{\partial x^j} \right)$$

Regularity (Injectivity/One-to-One) Condition

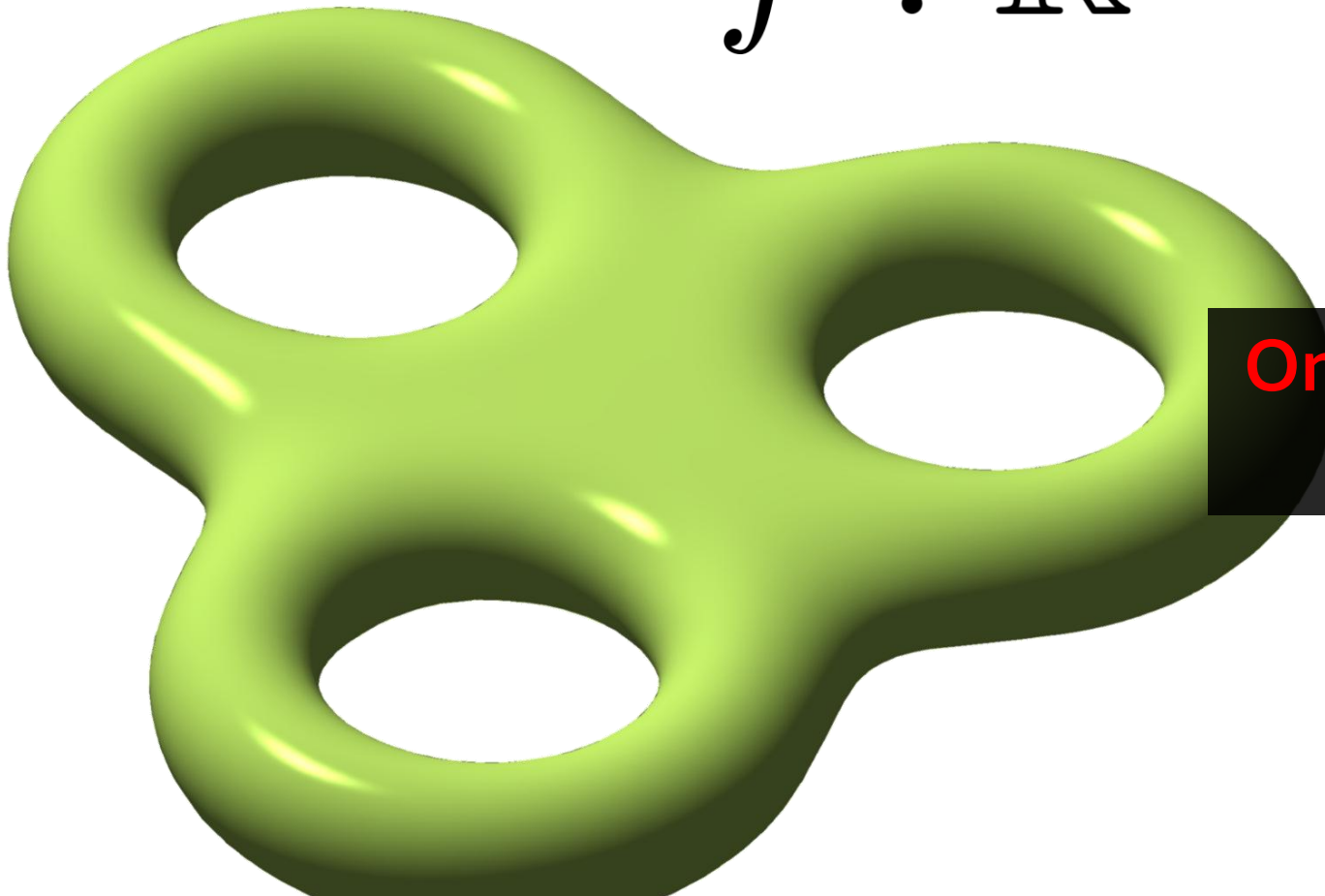
$$f : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

Matrix condition:
 Df full rank



Moving Away from Parametric Surfaces

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad ?$$

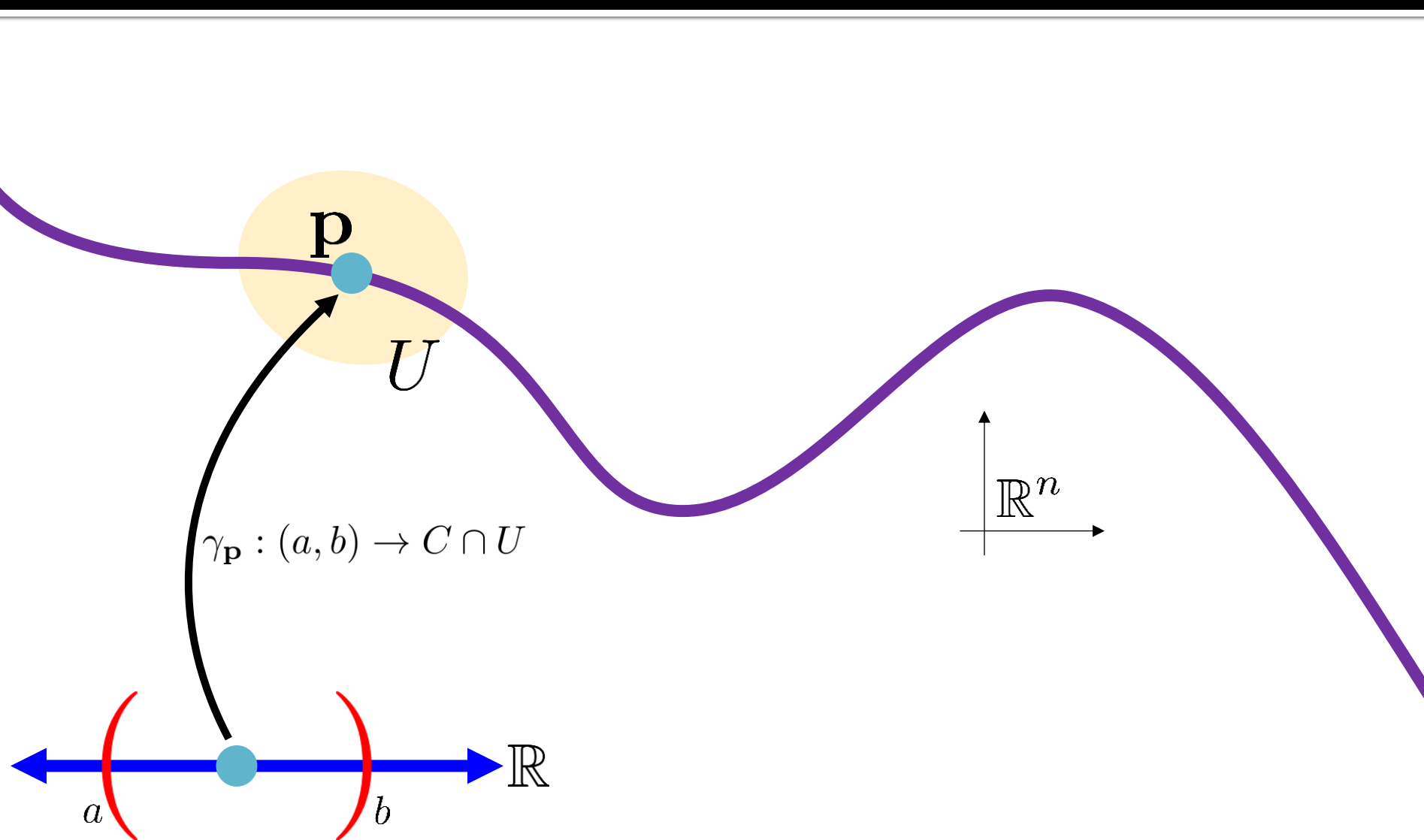


**One function isn't
enough!**

Major difference from curves!

Recall:

Differential Geometry Definition

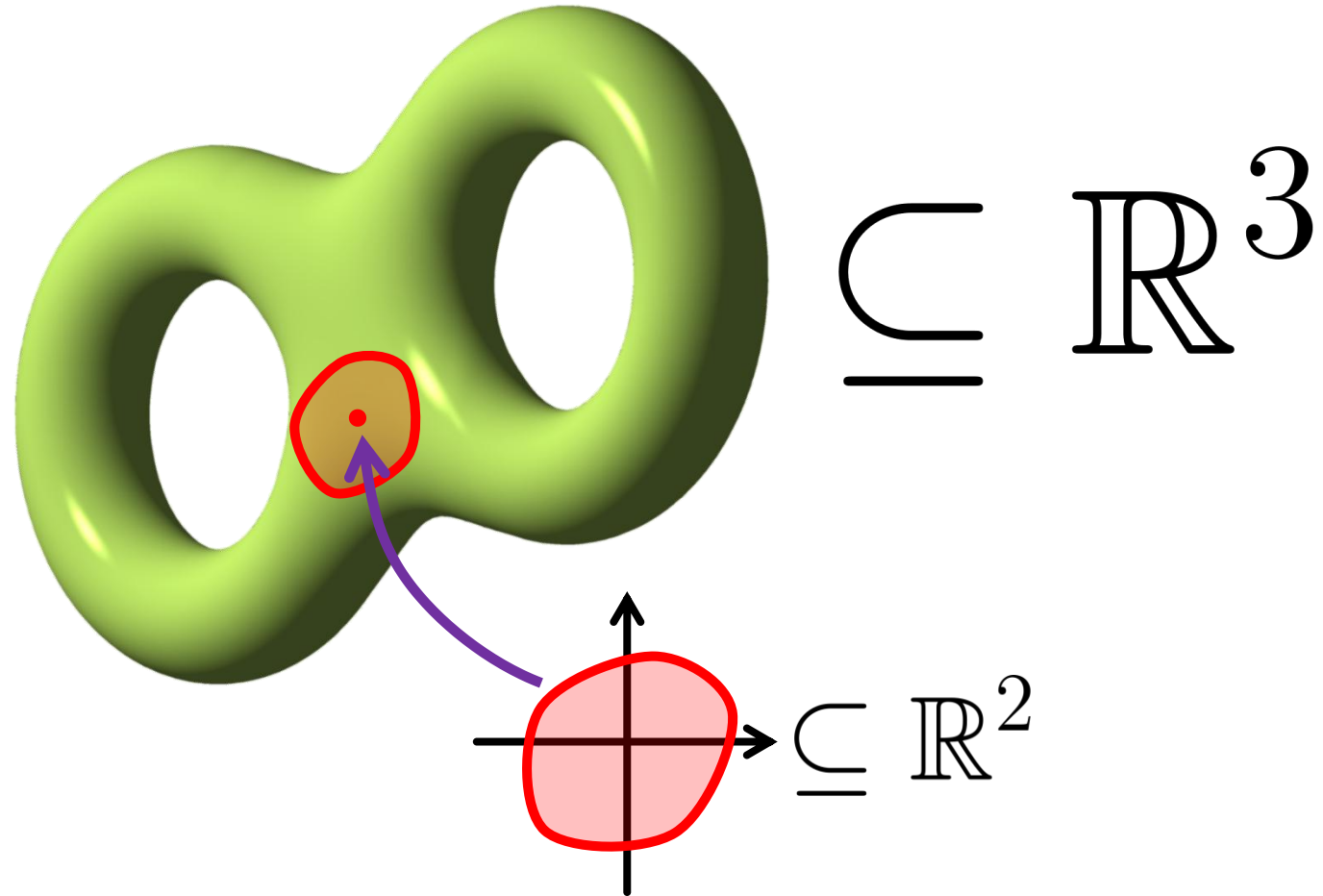


Just Like Curves

A surface is a
set of points
with certain properties.

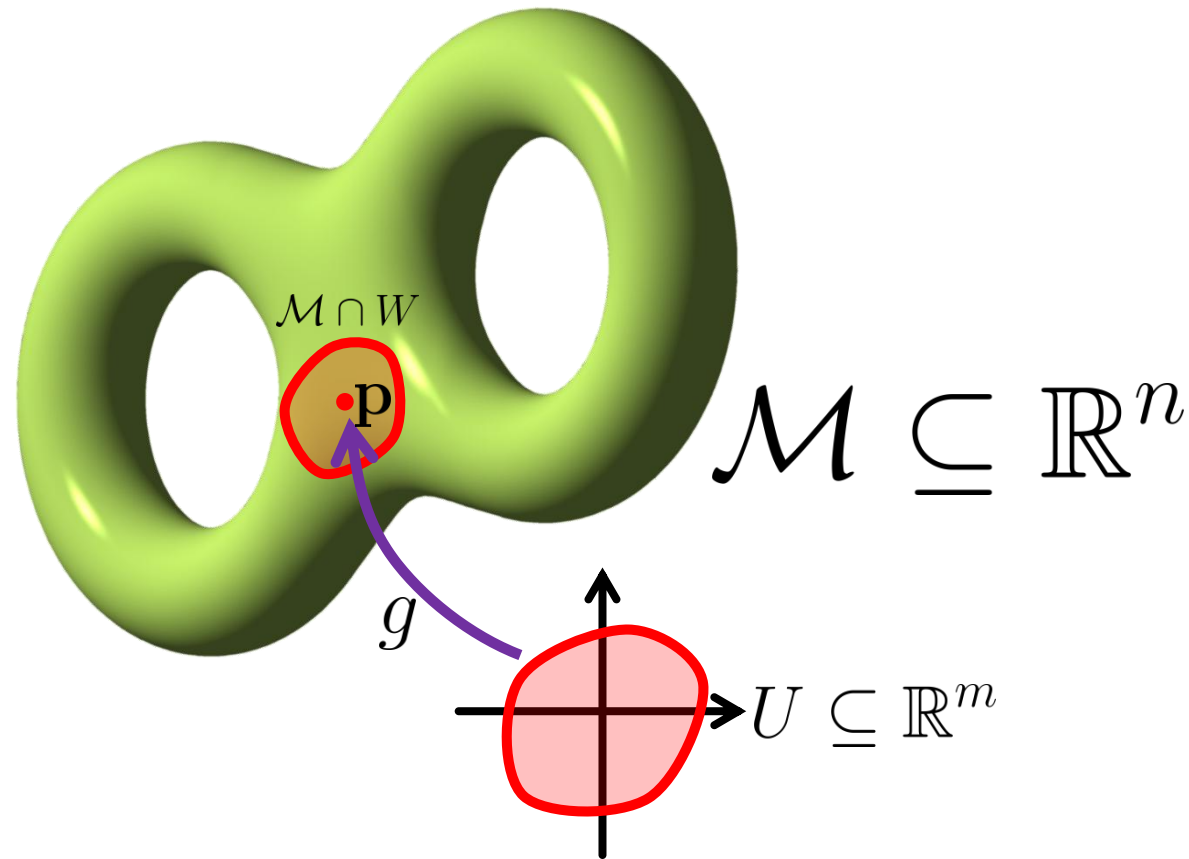
It is not a function.

Theoretical Definition of Surface



Theoretical Definition: (Sub)Manifold

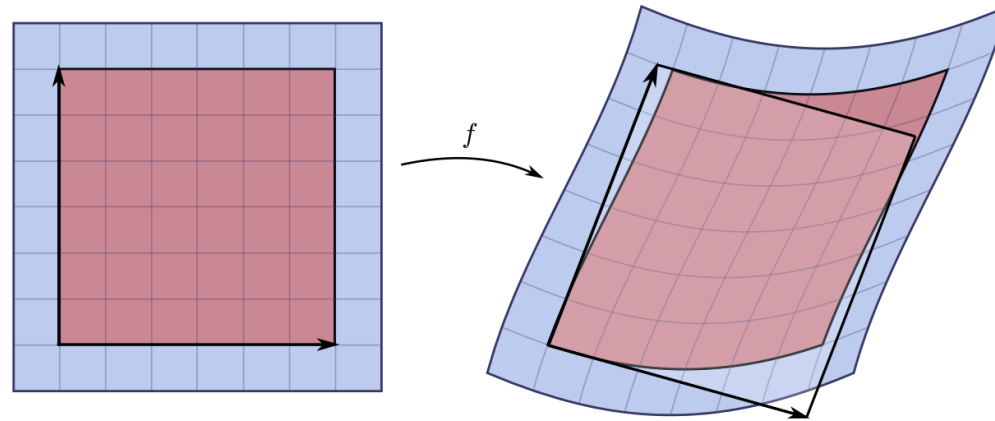
Definition (Submanifold of $\mathbb{R}^n$, with and without boundary). A set $\mathcal{M} \subseteq \mathbb{R}^n$ is an m -dimensional submanifold of $\mathbb{R}^n$ if for each $\mathbf{p} \in \mathcal{M}$ there exist open sets $U \subseteq \mathbb{R}^m$, $W \subseteq \mathbb{R}^n$ and a function $g : U \cap \mathcal{H}_m \rightarrow \mathcal{M} \cap W$ such that $\mathbf{p} \in W$ and g is a one-to-one and smooth map whose Jacobian is rank- m and admitting a continuous inverse $g^{-1} : W \cap \mathcal{M} \rightarrow U$.



$$\mathcal{H}_m := \{\mathbf{x} \in \mathbb{R}^m : x^m \geq 0\}$$

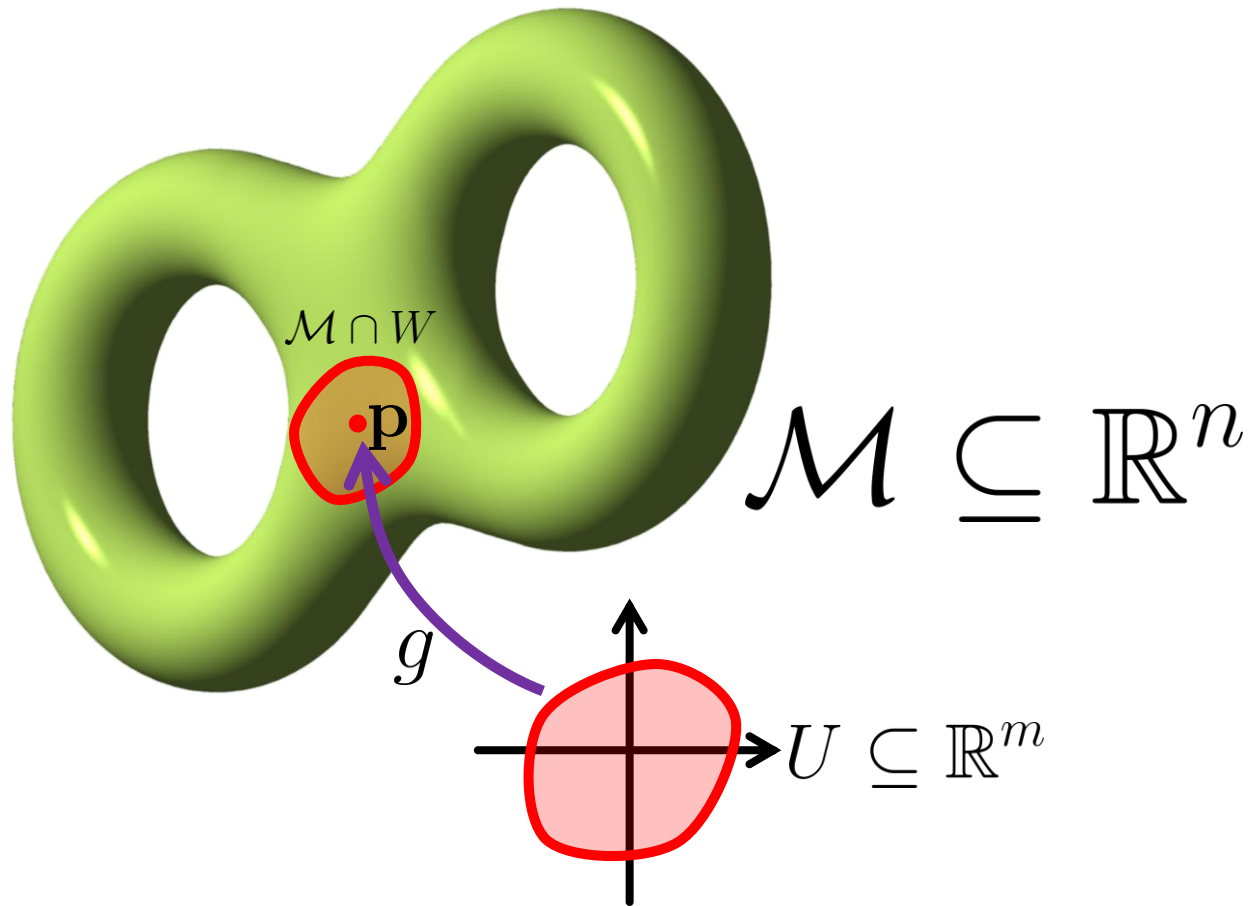
Differential Geometer's Mantra

A surface is
locally planar.



Tangent Space

$$T_{\mathbf{p}}\mathcal{M} = \gamma'(0), \text{ where } \gamma(0) = \mathbf{p}$$



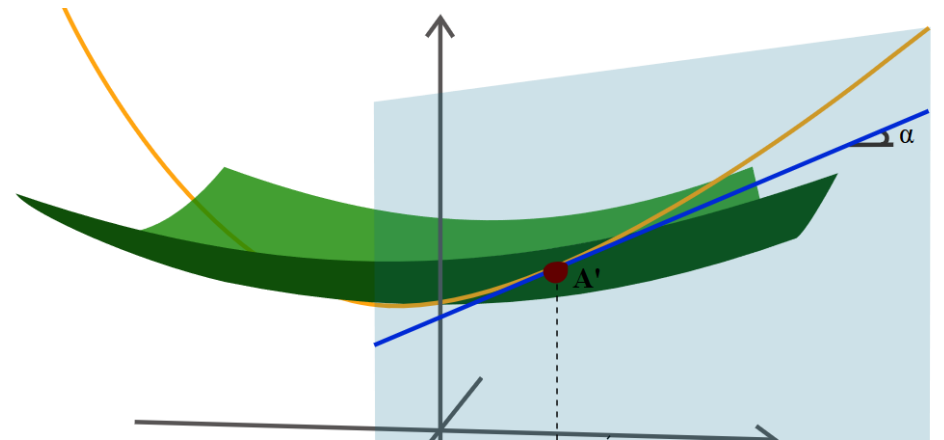
Recall: Differential

$$df_{\mathbf{x}_0}(\mathbf{v}) := \lim_{h \rightarrow 0} \frac{f(\mathbf{x}_0 + h\mathbf{v}) - f(\mathbf{x}_0)}{h}$$

Proposition. df_{x_0} is a linear operator.

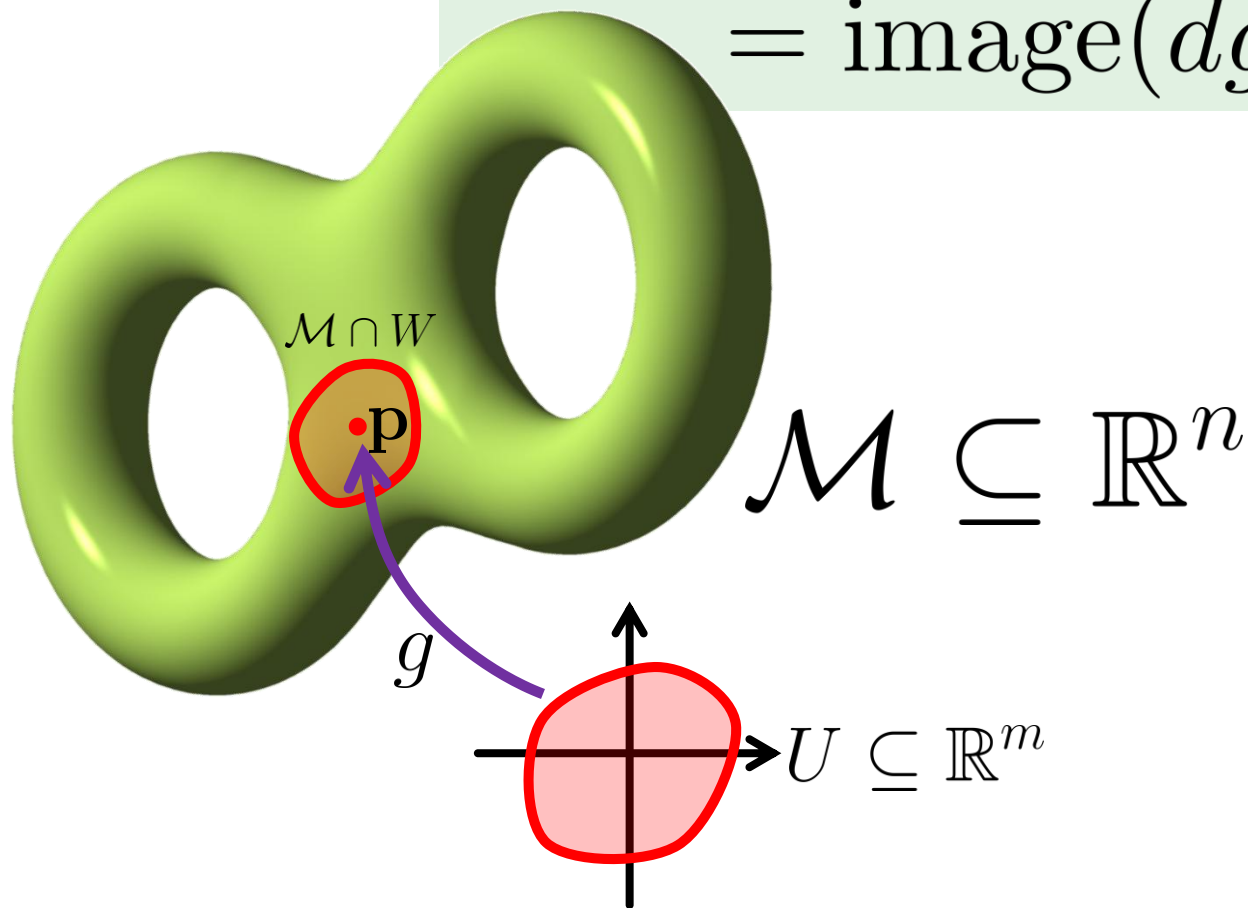
$$df_{\mathbf{x}_0}(\mathbf{v}) = Df(\mathbf{x}_0) \cdot \mathbf{v}$$

Note: Technically we derived the 1D version. Nothing changes!



Tangent Space

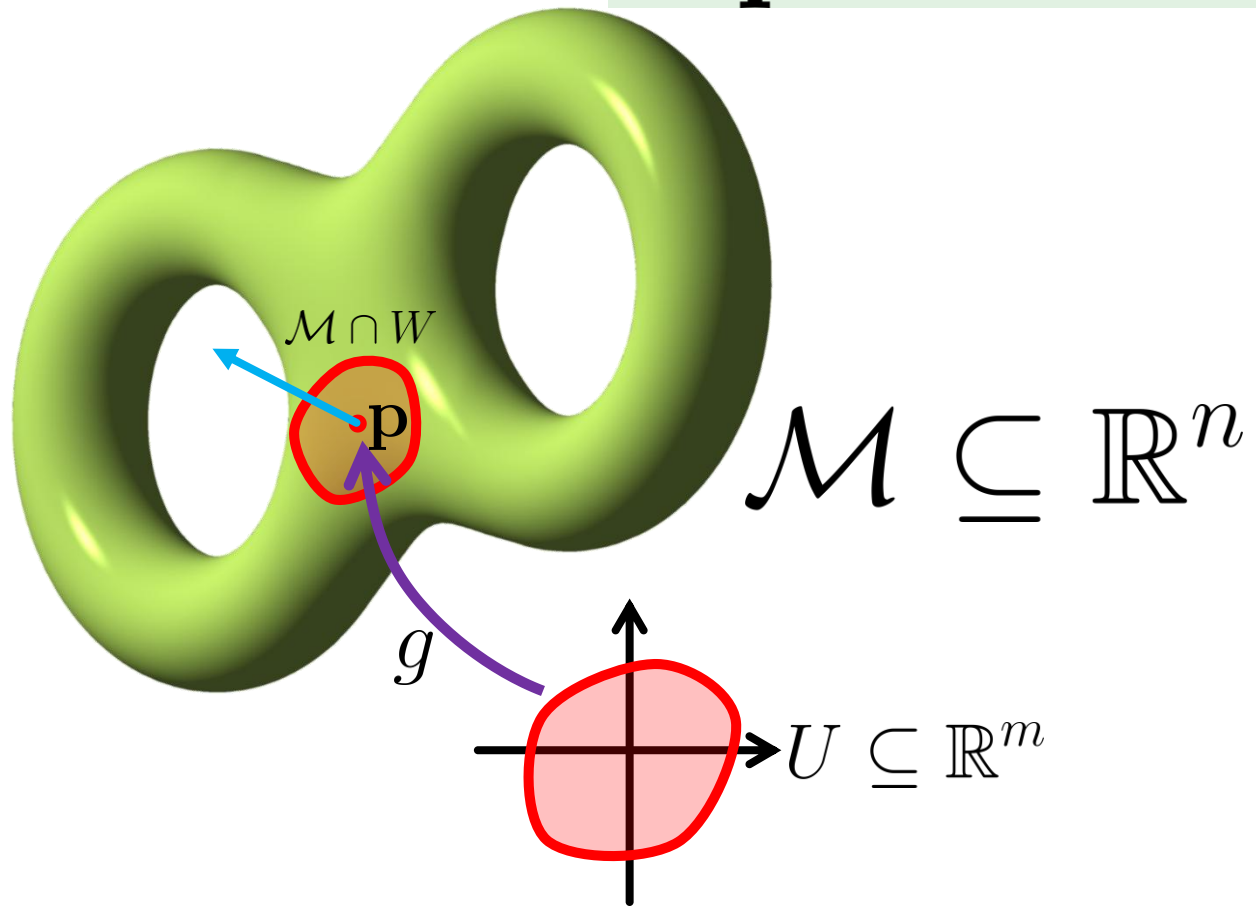
$$T_{\mathbf{p}}\mathcal{M} = \gamma'(0), \text{ where } \gamma(0) = \mathbf{p} \\ = \text{image}(dg_{g^{-1}(\mathbf{p})})$$



Skipping:
Independence of choice of g .

Normal Space

$$N_{\mathbf{p}}\mathcal{M} := (T_{\mathbf{p}}\mathcal{M})^{\perp}$$



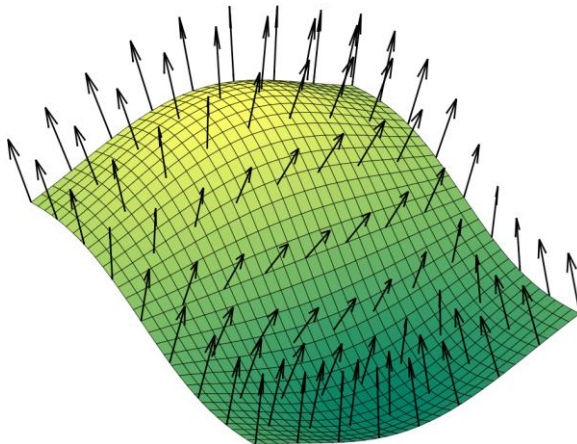
Orientable Submanifold

Admits a continuous map

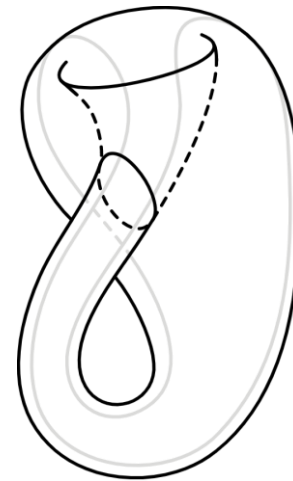
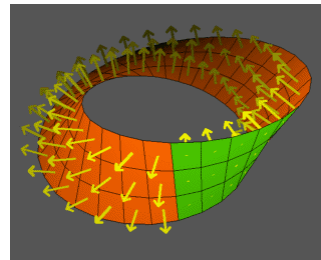
$$\mathbf{n}(\mathbf{p}) : \mathcal{M} \setminus \partial\mathcal{M} \rightarrow \mathcal{S}^{n-1}$$

with

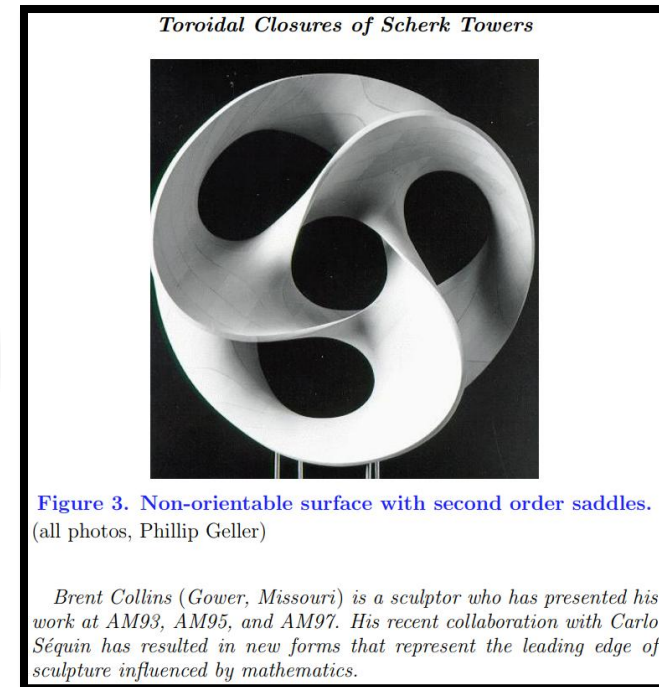
$$\mathbf{n}(\mathbf{p}) \in N_{\mathbf{p}}\mathcal{M}$$



Orientable

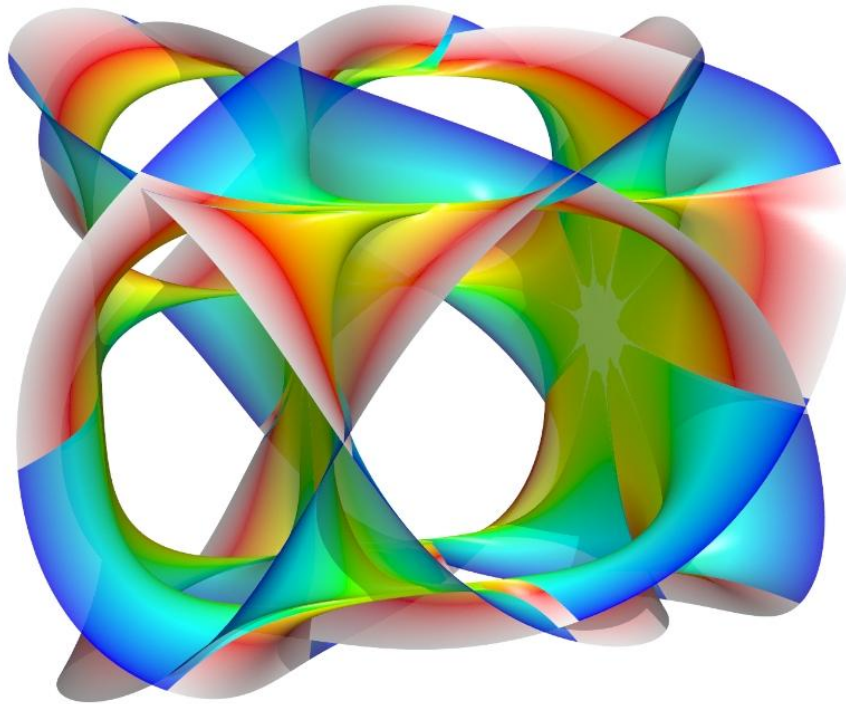


Not Orientable



More General Definition: Manifold

Definition 4.2 (Manifold). *An m -dimensional (topological) manifold $\mathcal{M}$ is a Hausdorff space for which each $\mathbf{p} \in \mathcal{M}$ admits open sets $U \subseteq \mathbb{R}^m, W \subseteq \mathcal{M}$ and a homeomorphism (continuous map with continuous inverse) $g : U \rightarrow W$.*



To think about:

No notion of normal!

Tangent vectors exist but have no length!

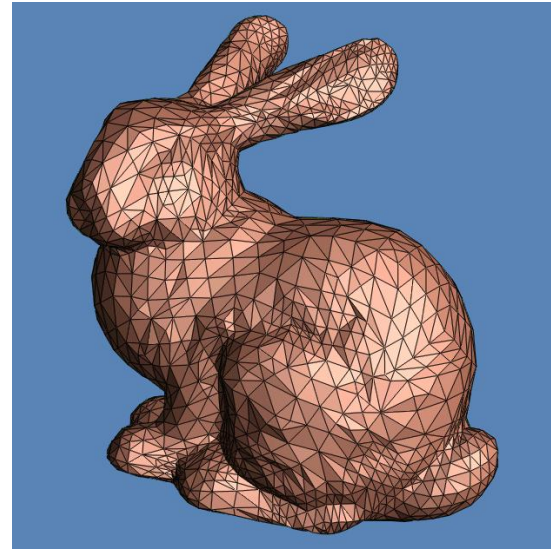
How do you detect orientability?

<http://www.math.sjsu.edu/~simic/Pics/Calabi-Yau.jpg>

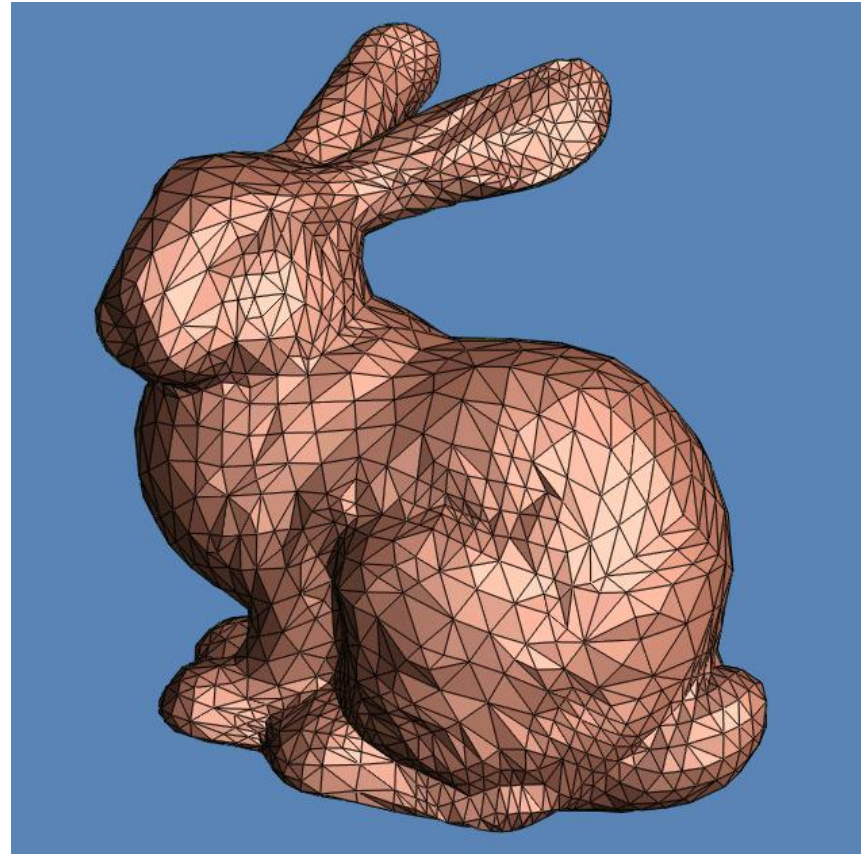
No Euclidean embedding

Discrete Problem

What is a discrete surface?
How do you store it?



Common Representation



<http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg>
<http://www.stat.washington.edu/wxs/images/BUNMID.gif>

Triangle mesh

Triangle Mesh

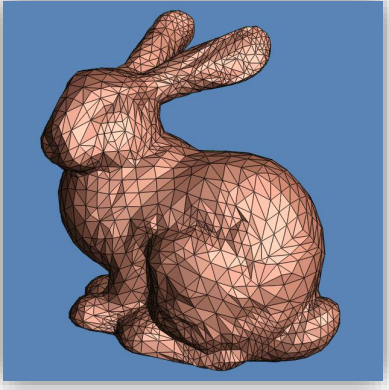
$$V = (v_1, v_2, \dots, v_n) \subset \mathbb{R}^3$$

$$E = (e_1, e_2, \dots, e_k) \subseteq V \times V$$

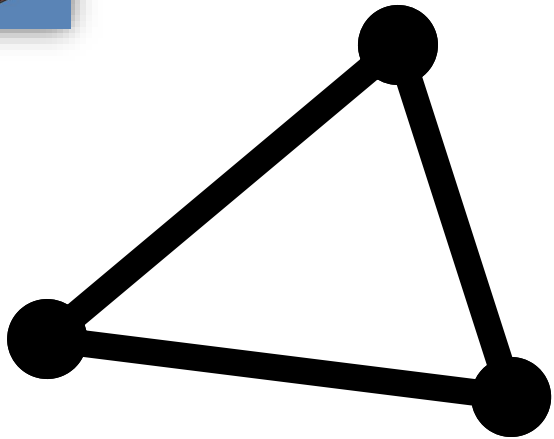
$$F = (f_1, f_2, \dots, f_m) \subseteq V \times V \times V$$

**Plus manifold
topological conditions**

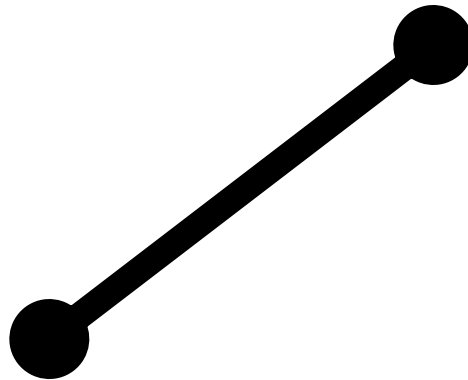
Dimensionality Structure



**Simplicial
complex**



Face
Dimension 2

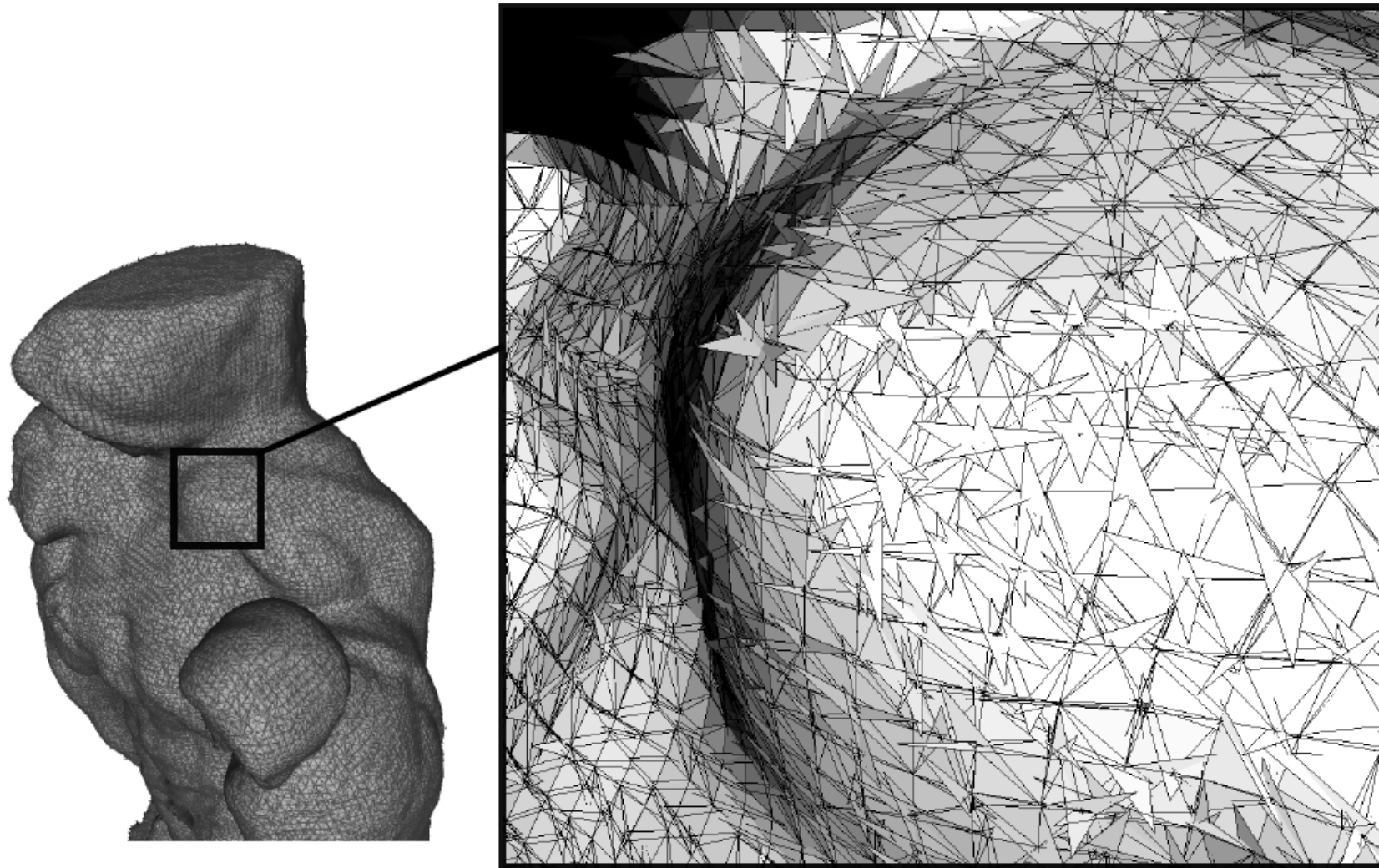


Edge
Dimension 1



Vertex
Dimension 0

Is This a Discrete Surface?

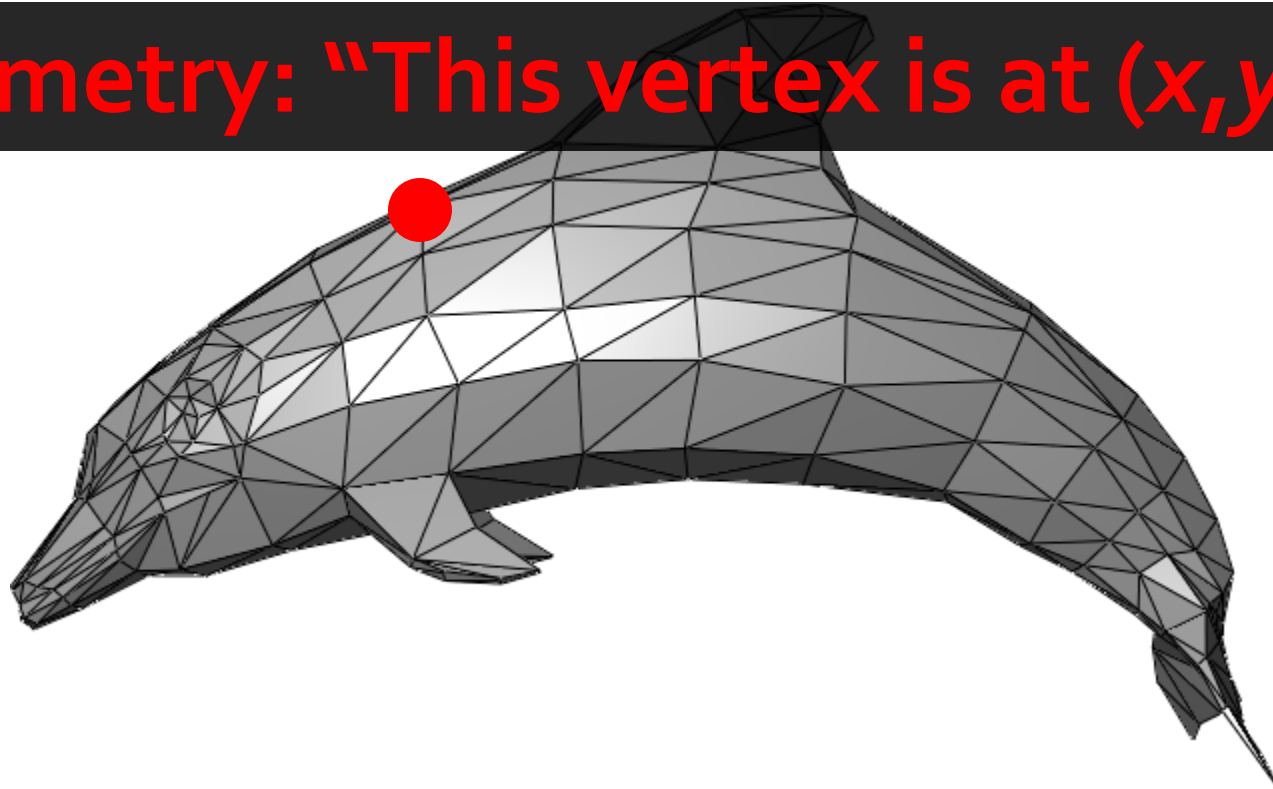


Topology [*tuh-pol-uh-jee*]:
The study of geometric
properties that remain
invariant under certain
transformations



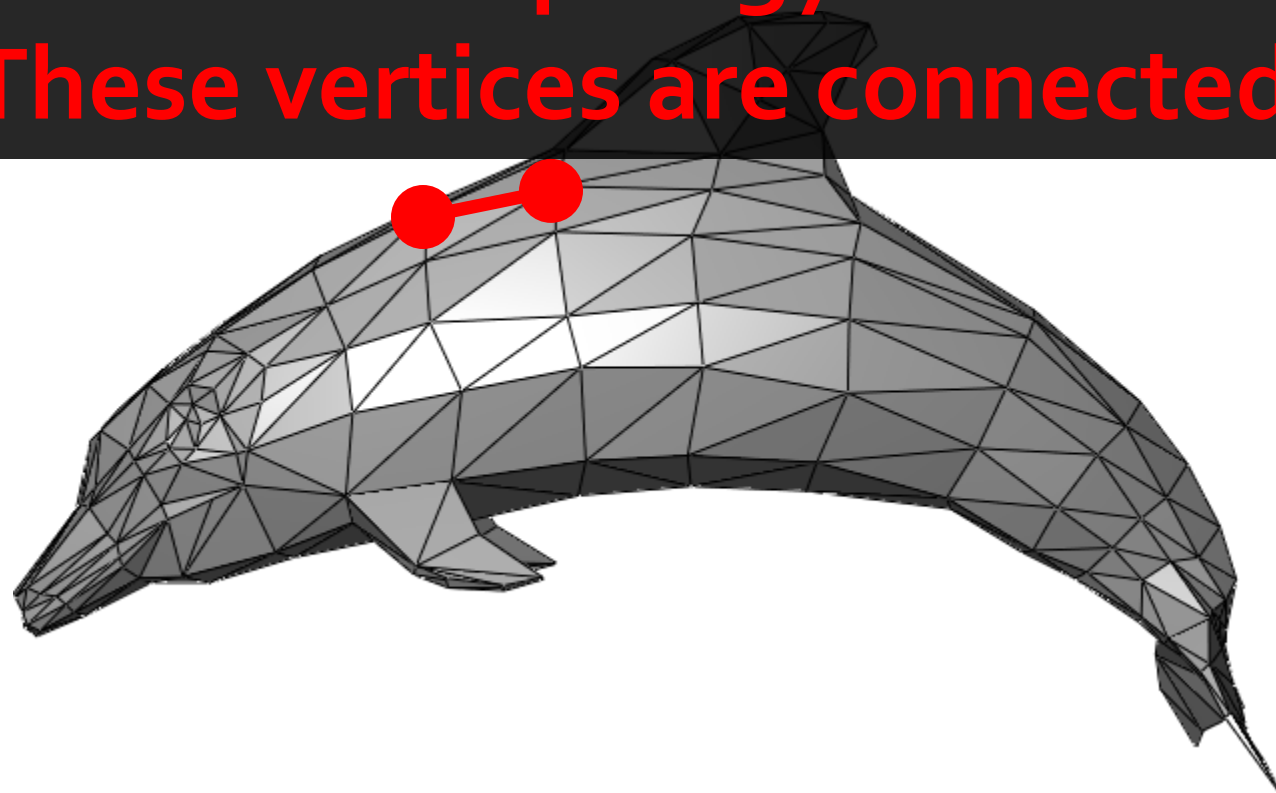
Mesh Topology vs. Geometry

Geometry: "This vertex is at (x,y,z) ."



Mesh Topology vs. Geometry

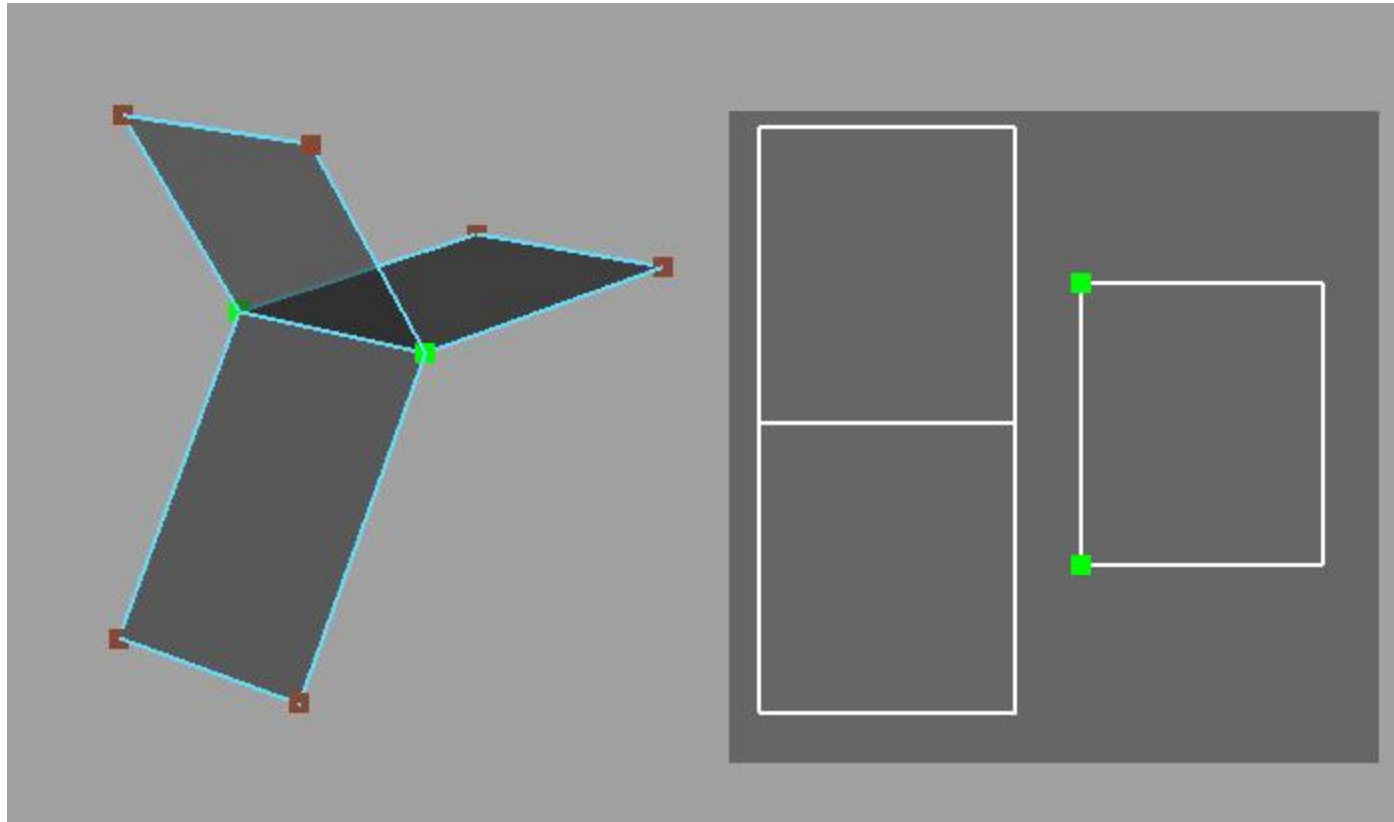
Topology:
“These vertices are connected.”



To read: More general story

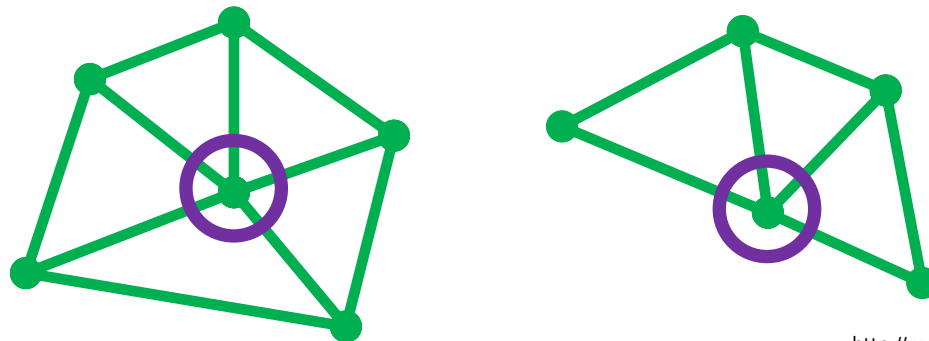
“Orientable combinatorial manifold”

Nonmanifold Edge



Manifold Triangle Mesh

1. Each **edge** is incident to one or two faces
2. **Faces** incident to a vertex form a closed or open fan



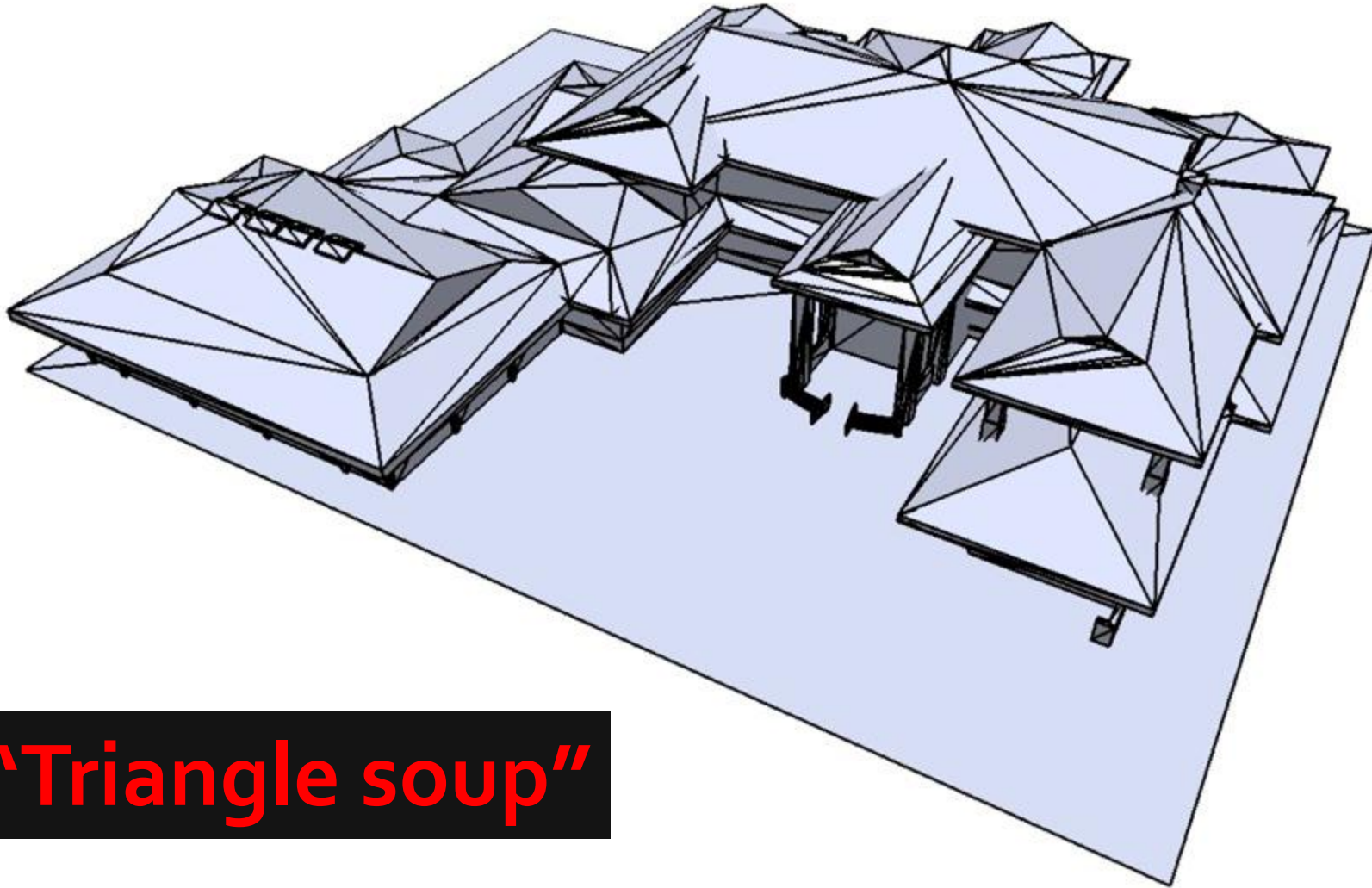
Manifold Triangle Mesh

1. Each **edge** is incident to one or two faces
2. **Faces** incident to a vertex form a closed or open fan

Assume meshes are manifold
(for now)



Easy-to-Violate Assumption



“Triangle soup”

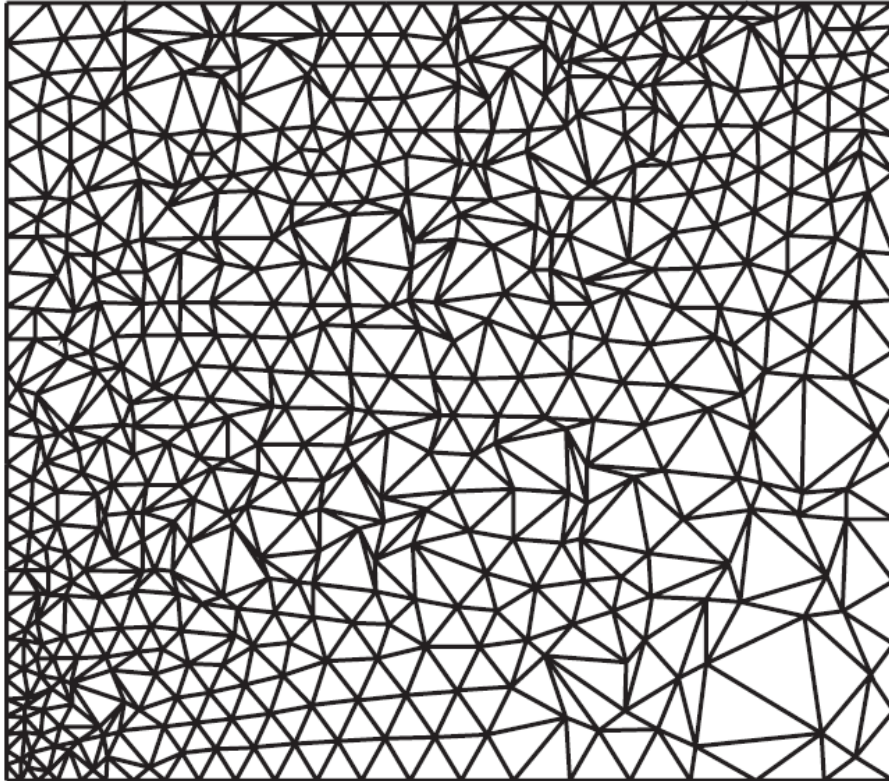
Basic Observation

Piecewise linear faces are
reasonable building blocks.

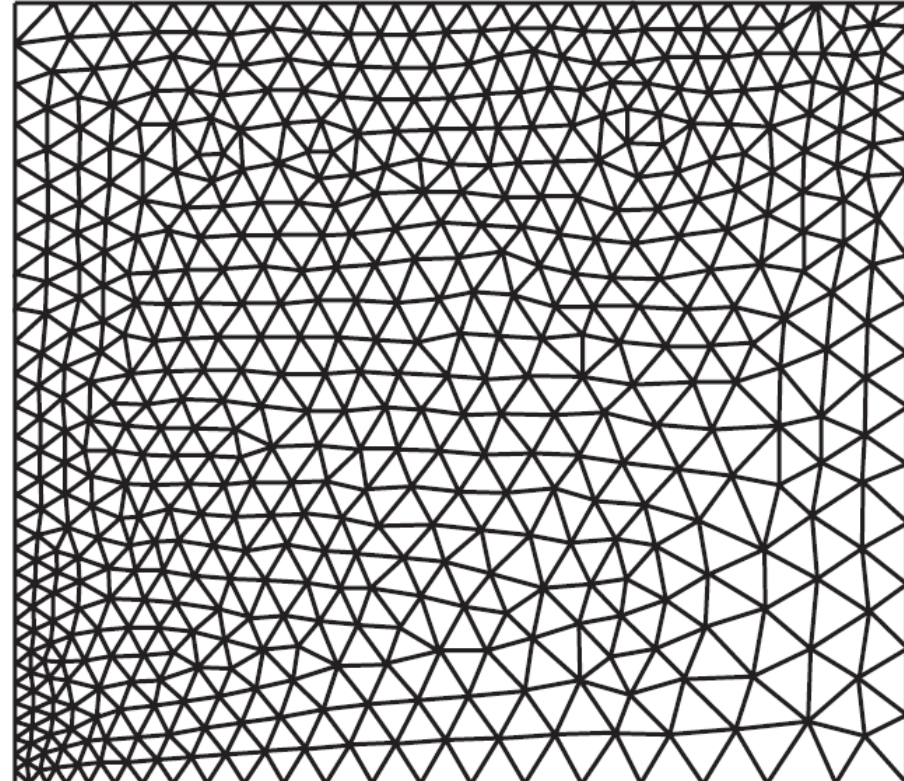
Additional Advantages

- Simple to render
- Arbitrary topology possible
- Basis for subdivision, refinement

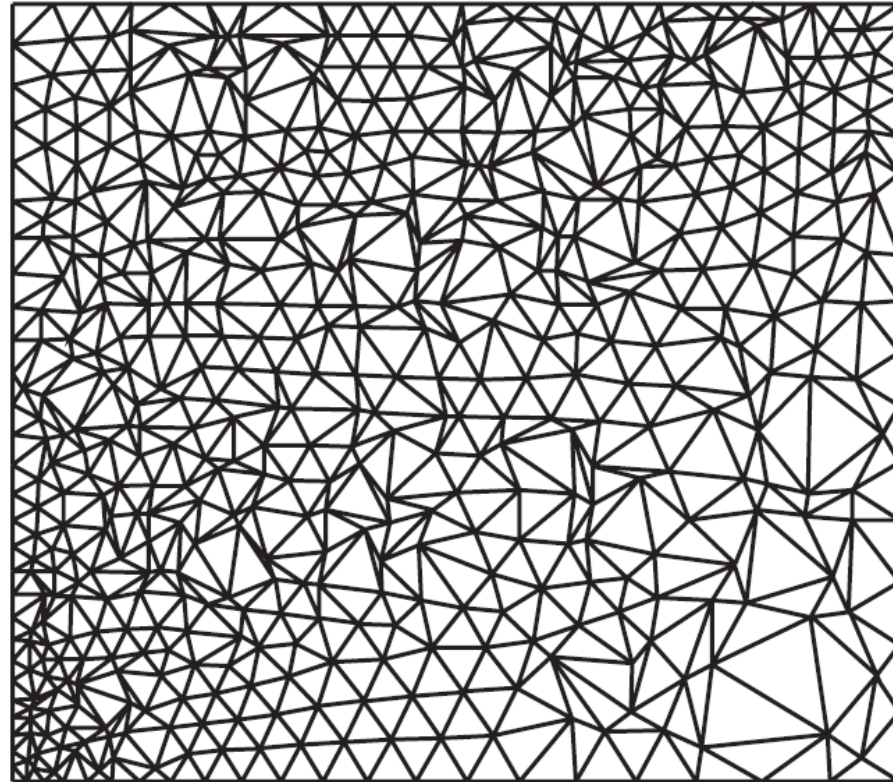
Invalid Meshes vs. Bad Meshes



**Nonuniform
areas and angles**

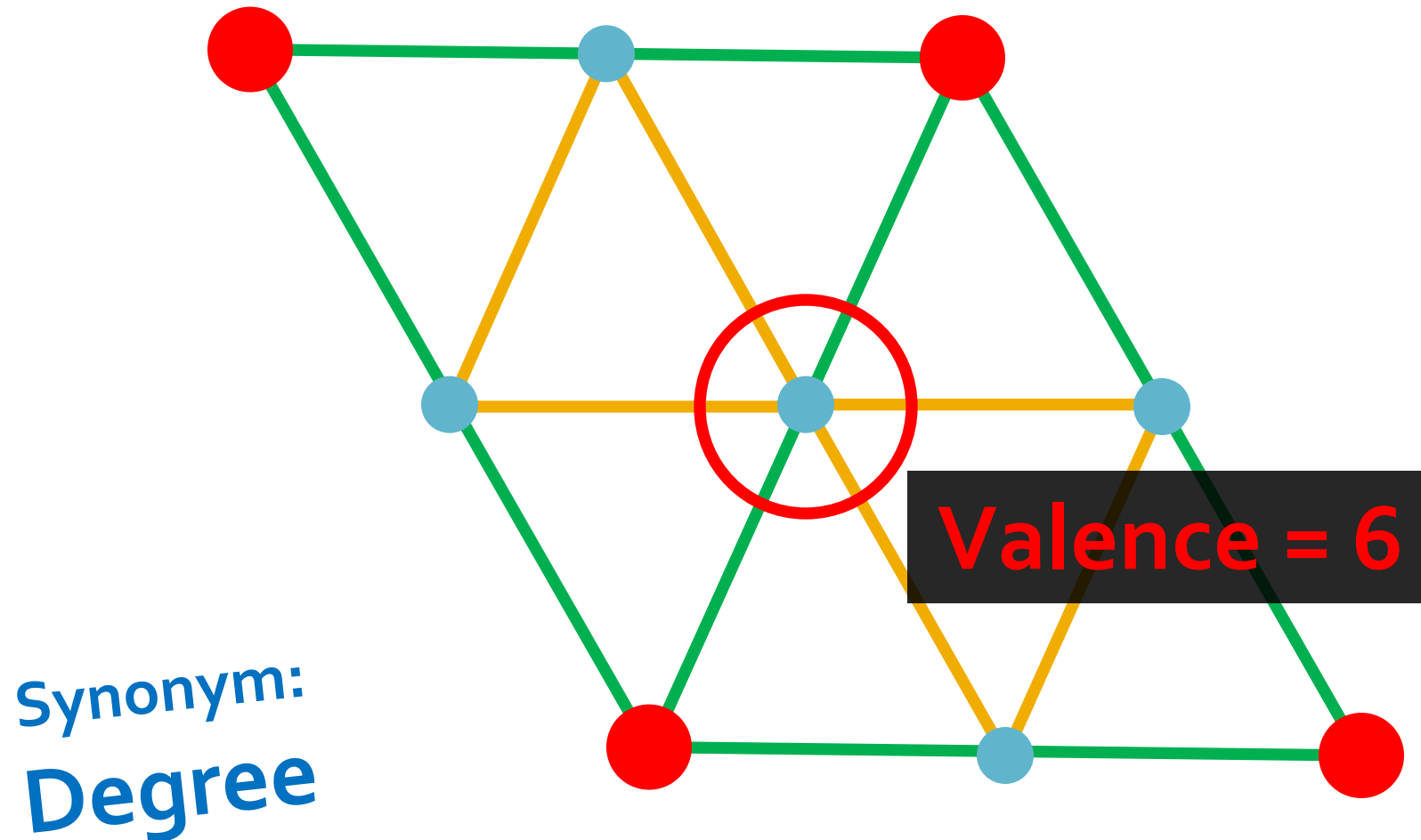


Why is Meshing an Issue?



How to you interpret
one value per vertex?

Returning to Topology: Valence



Euler Characteristic for Meshes

$$V - E + F := \chi$$

$$\chi = 2 - 2g$$



$$g = 0$$



$$g = 1$$

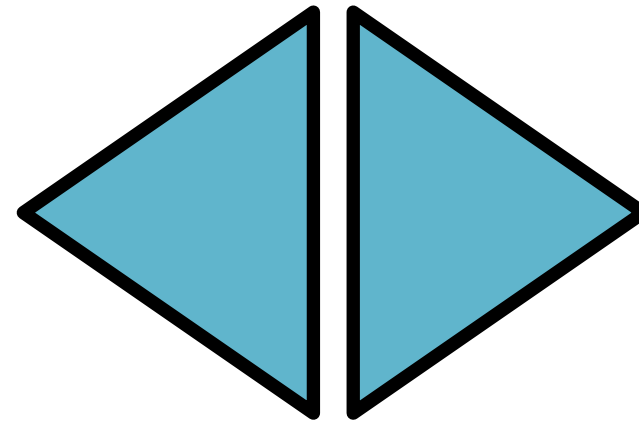


$$g = 2$$

Consequences for Triangle Meshes

$$V - E + F := \chi$$

“Each edge is adjacent to two faces. Each face has three edges.”



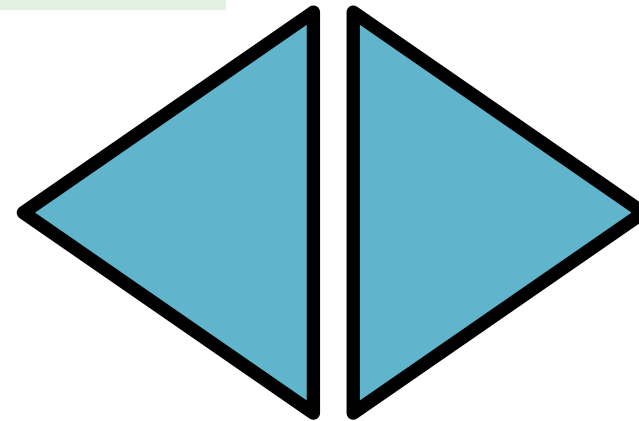
$$2E = 3F$$

Closed mesh: Easy estimates!

Consequences for Triangle Meshes

$$V - \frac{1}{2}F := \chi$$

“Each edge is adjacent to two faces. Each face has three edges.”



$$2E = 3F$$

Closed mesh: Easy estimates!

Consequences for Triangle Meshes

**Big
number**

$$V - \frac{1}{2}F := \chi$$

**Small
number**


$$F \approx 2V$$

Closed mesh: Easy estimates!

Consequences for Triangle Meshes

$$E \approx 3V$$

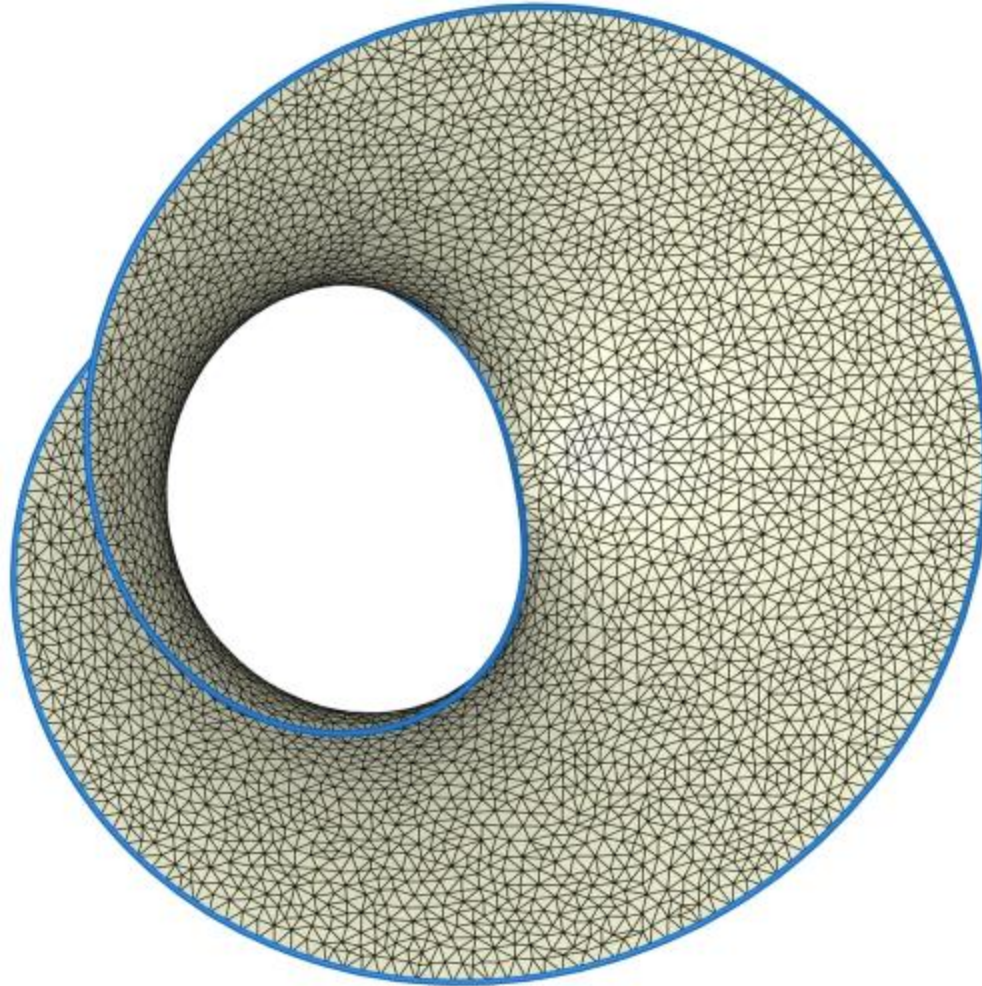
$$F \approx 2V$$

average valence ≈ 6

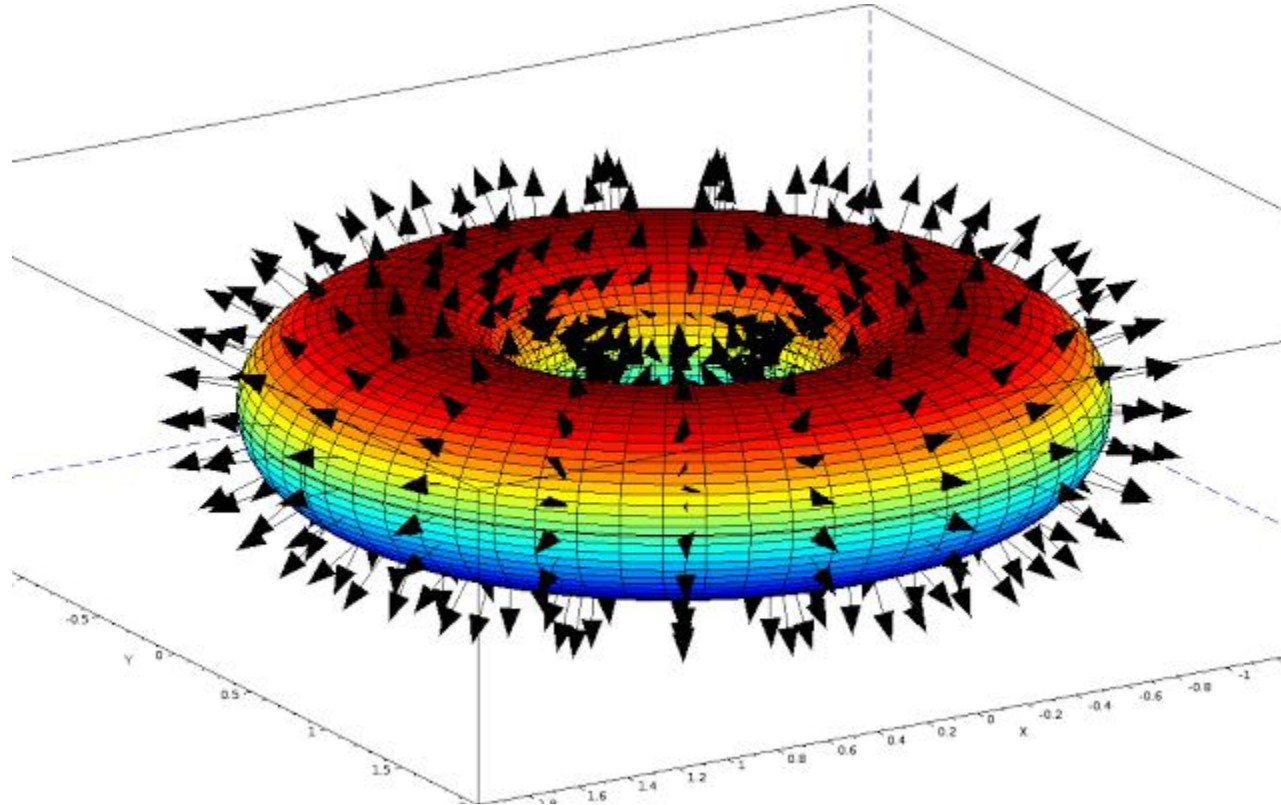
Why?!

General estimates

Orientability



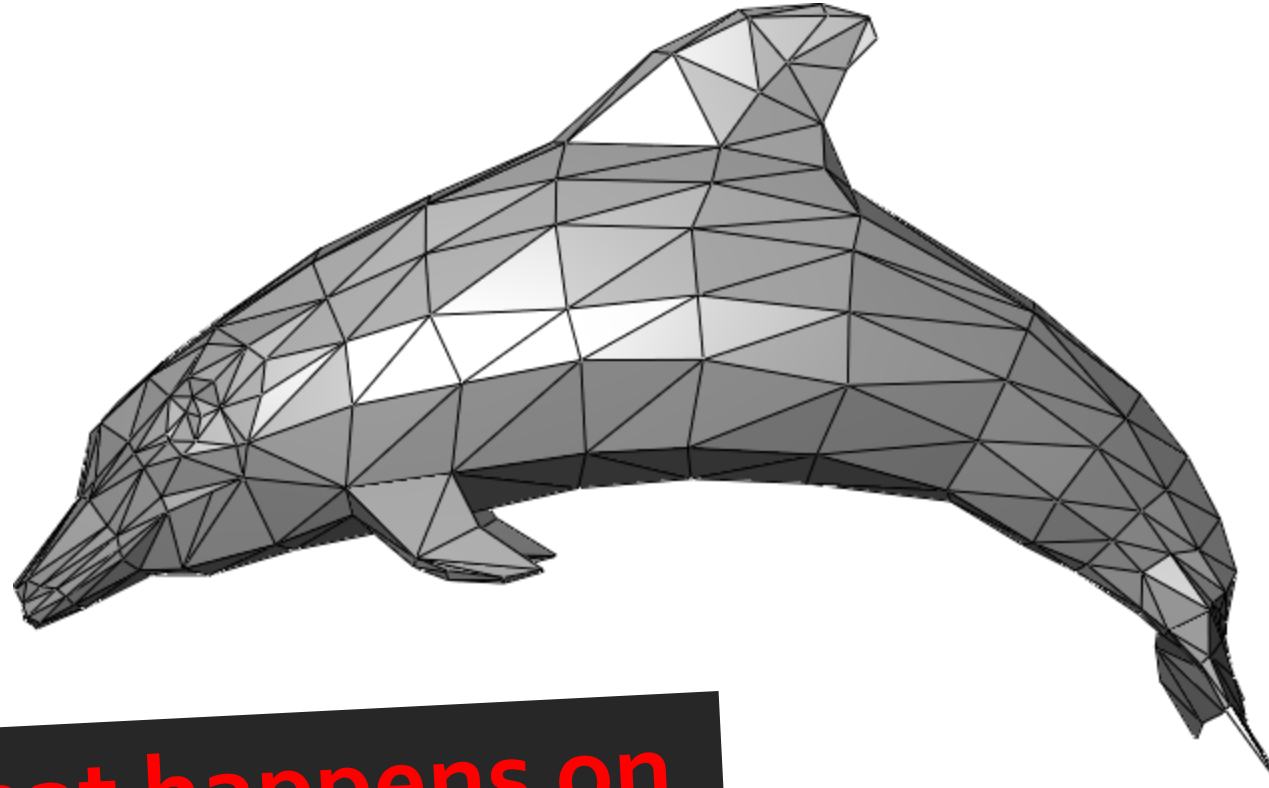
Smooth Surface Definition



https://lh3.googleusercontent.com/-njXPH7NSX5c/VV4PXu54ngI/AAAAAAAAAJM/m6TGg3ZVKGE/w640-h400-p-k/normal_tore.png

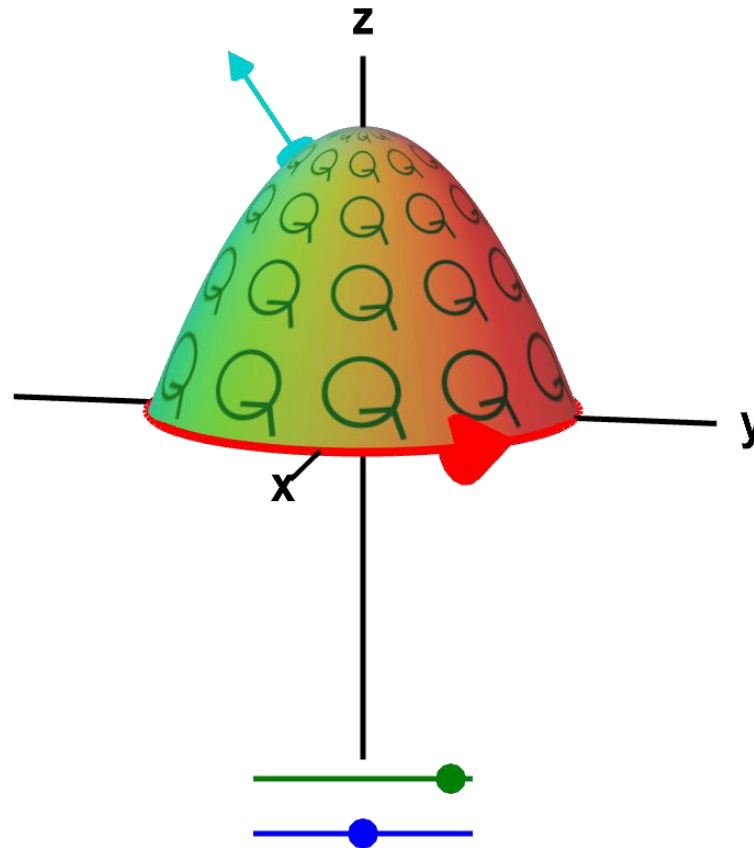
Continuous field of normal vectors

Issue on Triangle Mesh

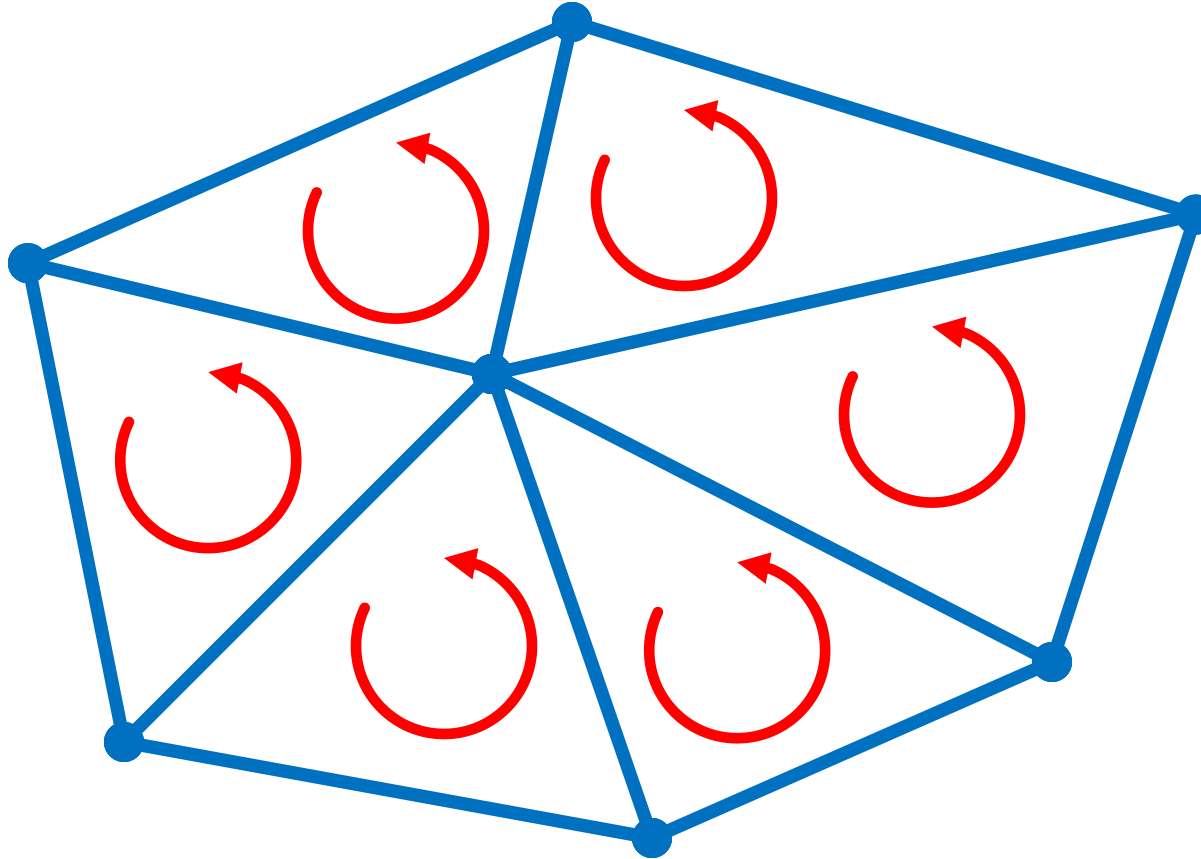


**What happens on
edges/vertices?**

Right-Hand Rule



Discrete Orientability



Normal field isn't continuous

Data Structures for Surfaces

**Must represent geometry
and topology.**

Simplest Format

```
x1 y1 z1 / x2 y2 z2 / x3 y3 z3  
x1 y1 z1 / x2 y2 z2 / x3 y3 z3  
x1 y1 z1 / x2 y2 z2 / x3 y3 z3  
x1 y1 z1 / x2 y2 z2 / x3 y3 z3  
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
```

No topology!

```
glBegin(GL_TRIANGLES)
```

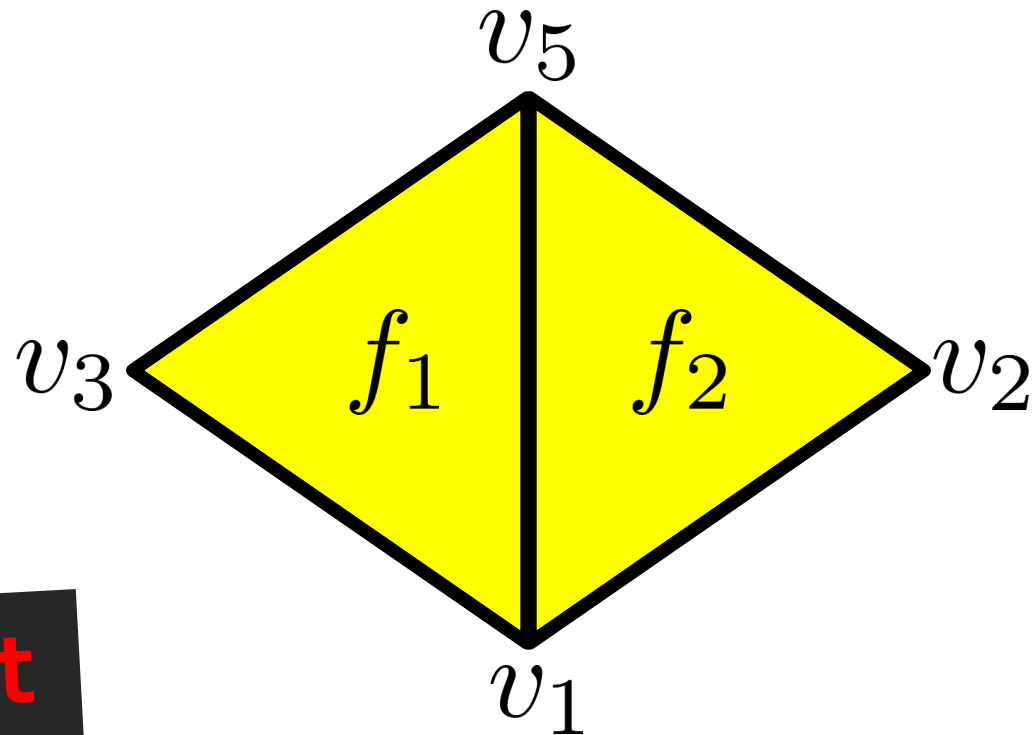
CS 468 2011 (M. Ben-Chen), other slides

Triangle soup

Factor Out Vertices

```
f 1 5 3  
f 5 1 2  
...  
v 0.2 1.5 3.2  
v 5.2 4.1 8.9  
...
```

.obj format



Shared vertex structure

Simple Mesh Smoothing

```
for i=1 to n
  for each vertex v
    v = .5*v +
      .5*(average of neighbors) ;
```

Typical Queries

- Neighboring vertices to a vertex
- Neighboring faces to an edge
- Edges adjacent to a face
- Edges adjacent to a vertex
- ...

Mostly localized

Typical Queries

- **Neighboring** vertices to a vertex
- **Neighboring** faces to an edge
- Edges **adjacent** to a face
- Edges **adjacent** to a vertex
- ...

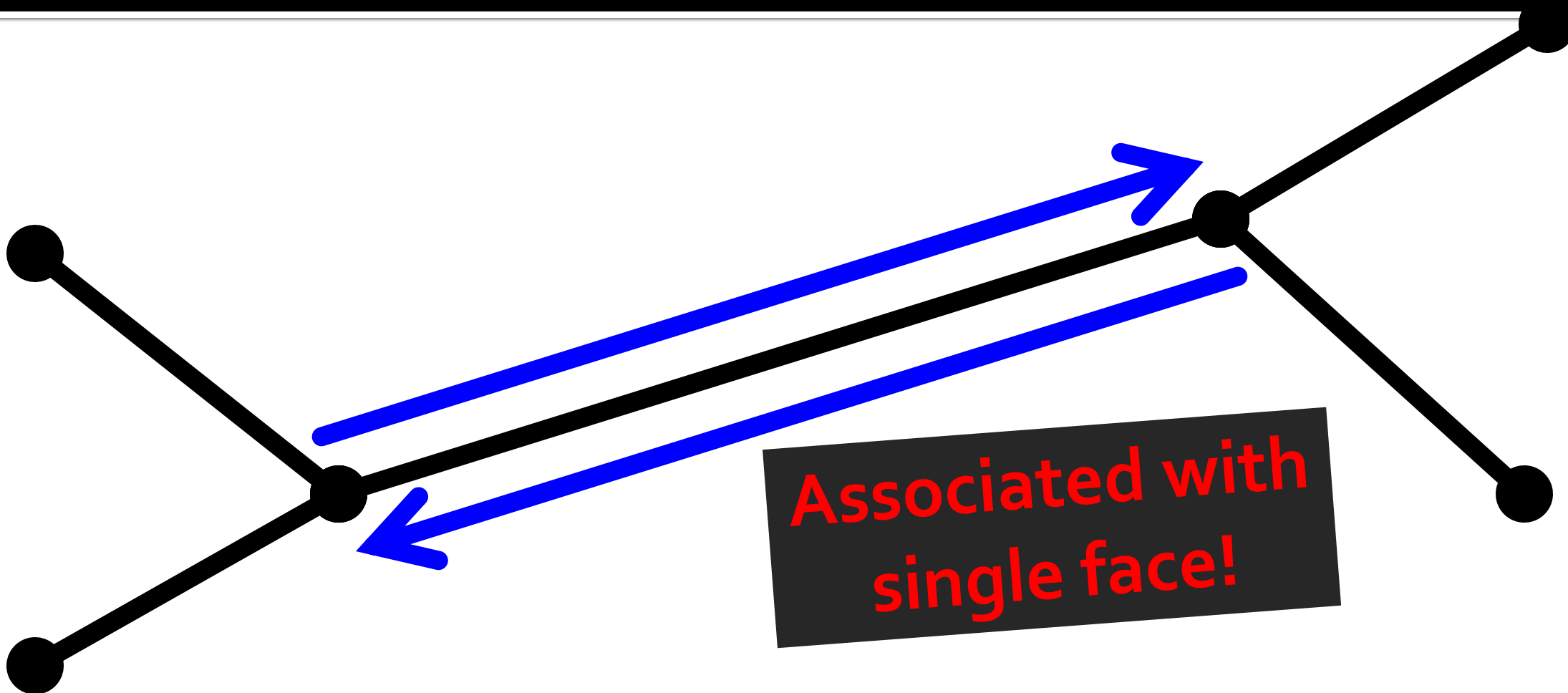
Mostly localized

Pieces of Halfedge Data Structure

- Vertices
- Faces
- Half-edges

Structure tuned for meshes

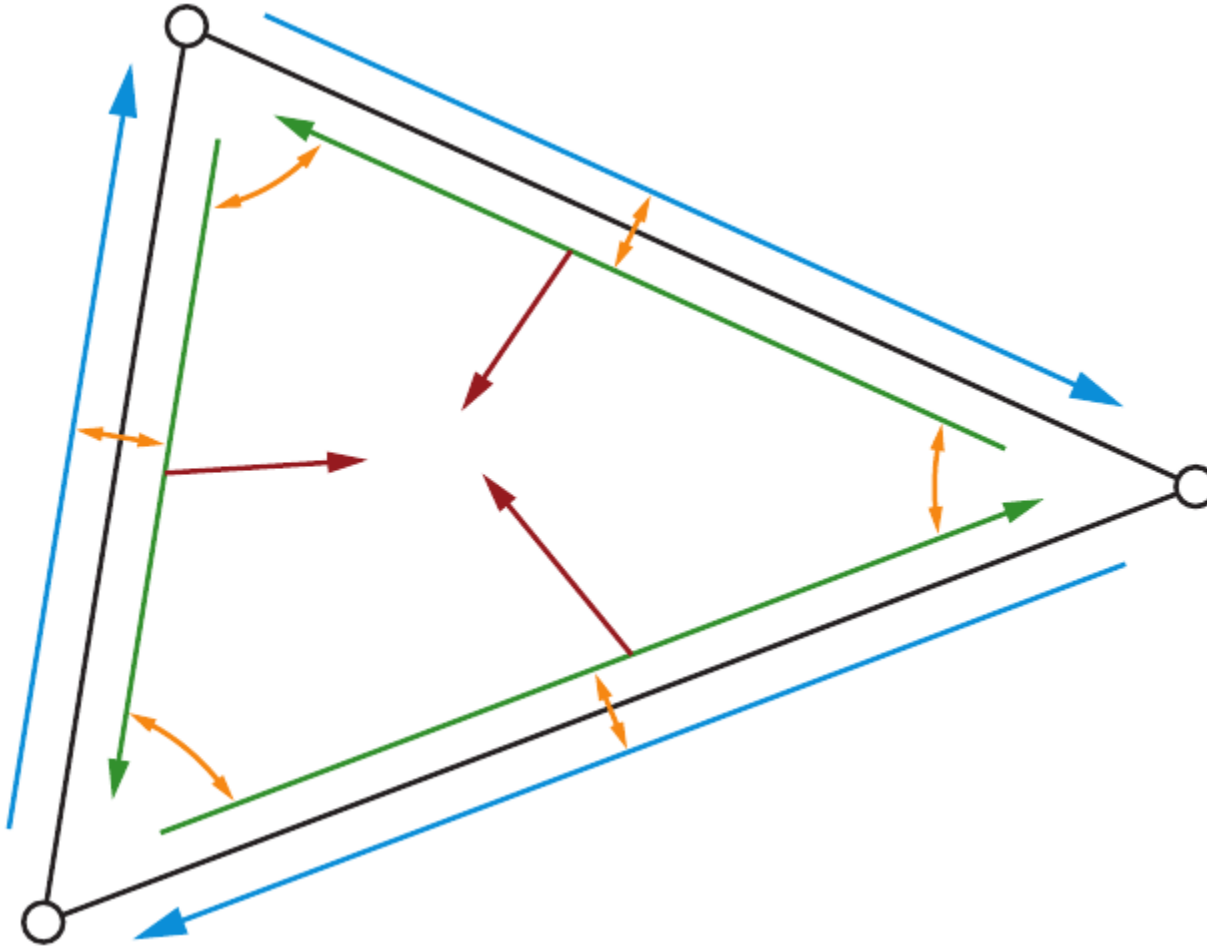
Halfedge?



Associated with
single face!

Oriented edge

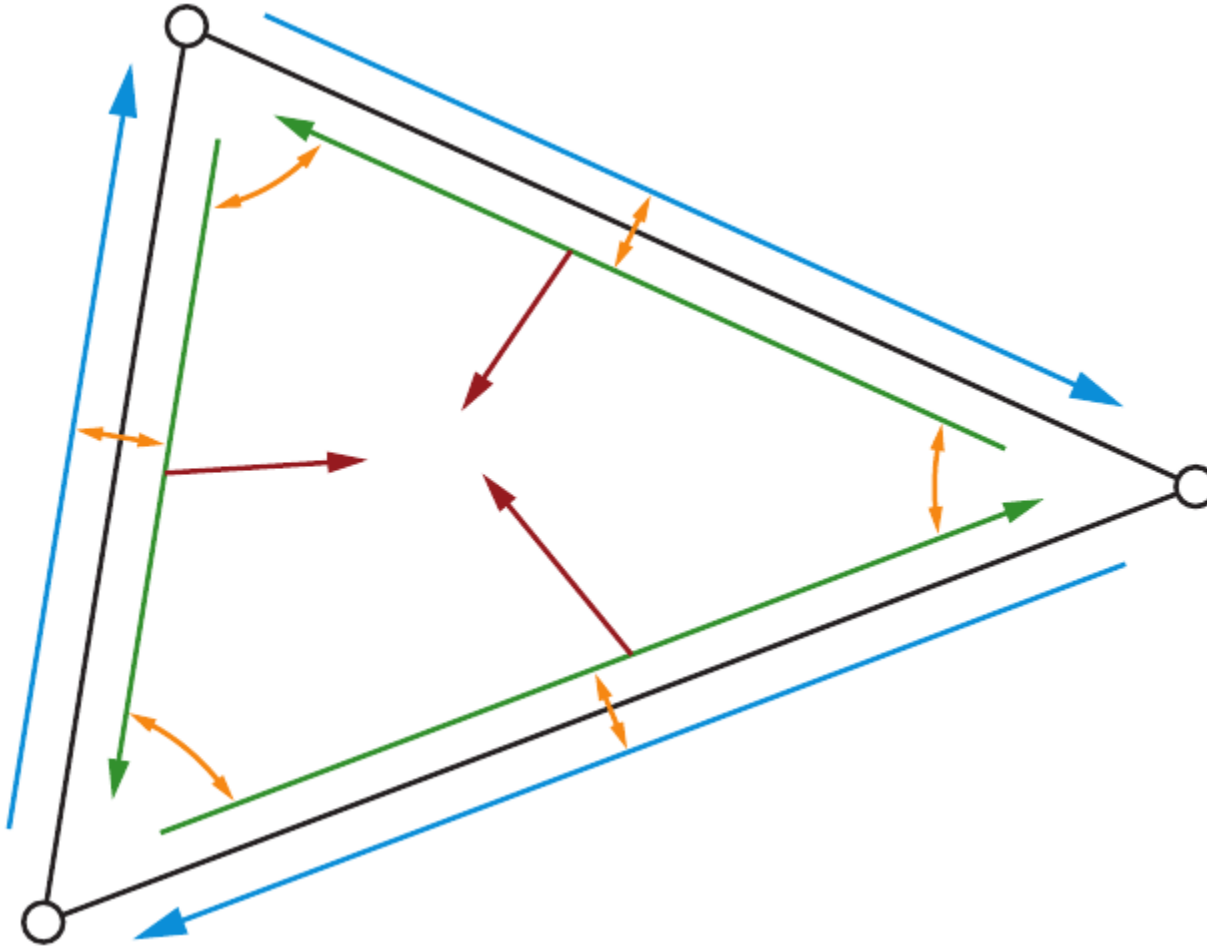
Halfedge Data Types



Vertex stores:

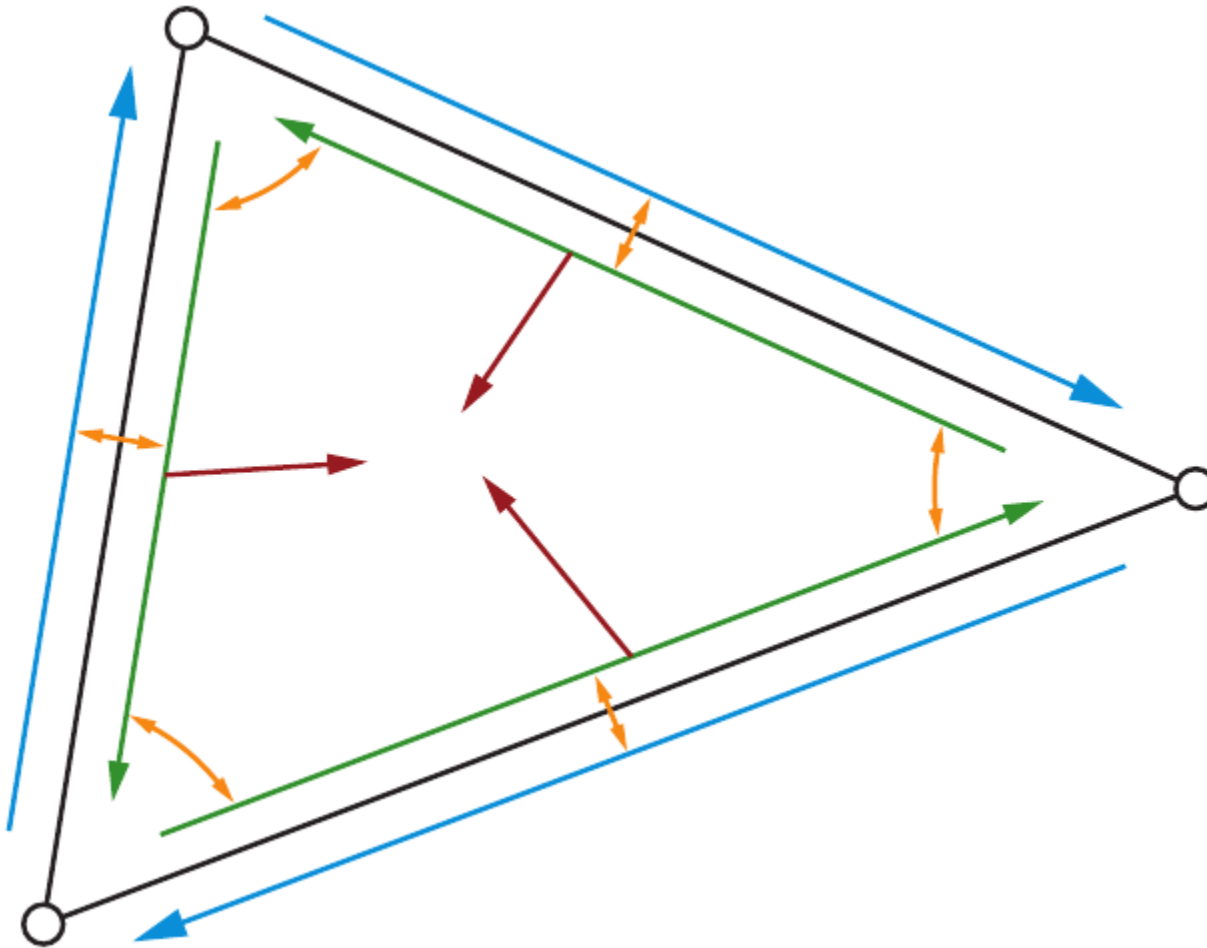
- Arbitrary outgoing halfedge

Halfedge Data Types



- Face stores:**
- Arbitrary adjacent halfedge

Halfedge Data Types

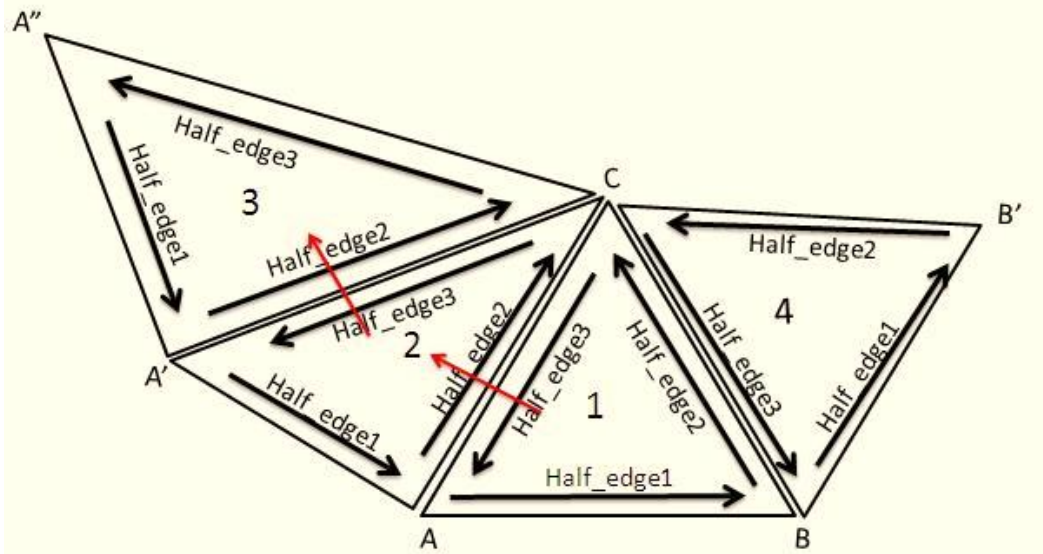


Halfedge

stores:

- Flip
- Next
- Face
- Vertex

Iterating Over Vertex Neighbors



```
Iterate(v) :  
startEdge = v.out;  
e = startEdge;  
do  
    process(e.flip.from)  
    e = e.flip.next  
while e != startEdge
```

Only Scratching the Surface

Eurographics Symposium on Geometry Processing (2005)

M. Desbrun, H. Pottmann (Editors)

Streaming Compression of Triangle Meshes

Martin Isenburg^{1†} Peter Lindstrom² Jack Snoeyink¹

¹ University of North Carolina at Chapel Hill ² Lawrence Livermore National Labs

EUROGRAPHICS 2011 / M. Chen and O. Deussen
(Guest Editors)

Volume 30 (2011), Number 2

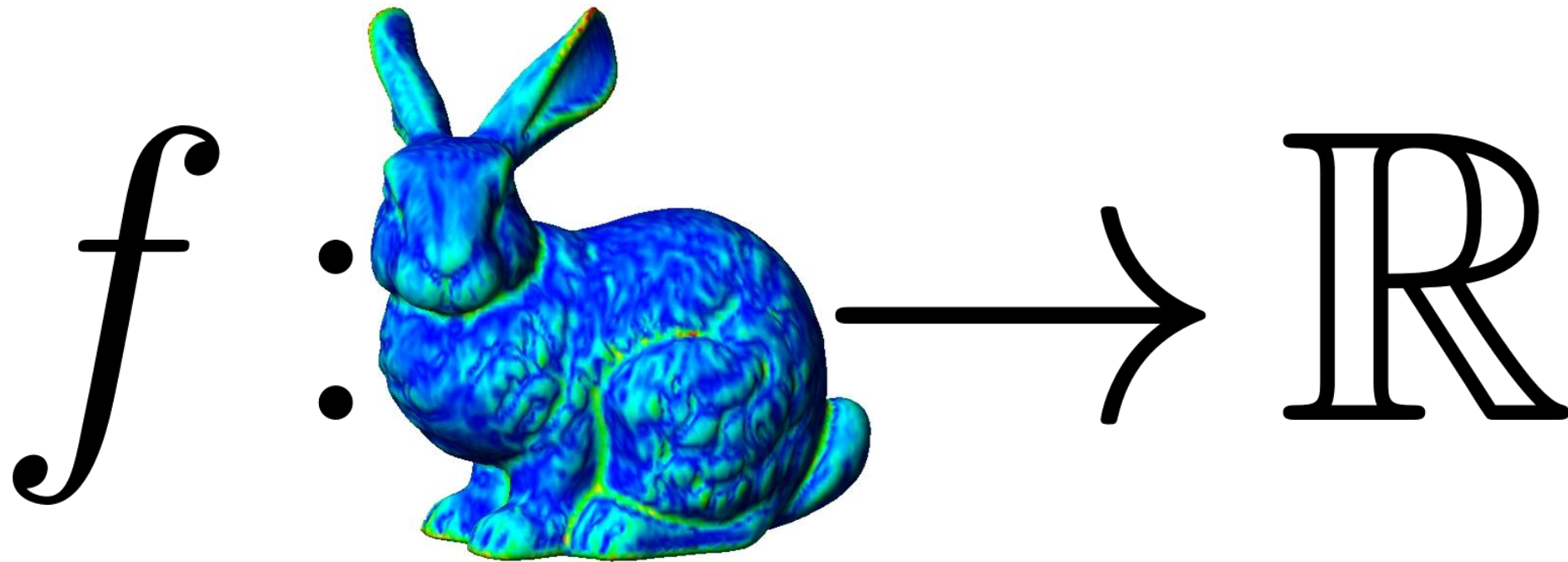
SQuad: Compact Representation for Triangle Meshes

Topraj Gurung¹, Daniel Laney², Peter Lindstrom², Jarek Rossignac¹

¹Georgia Institute of Technology

²Lawrence Livermore National Laboratory

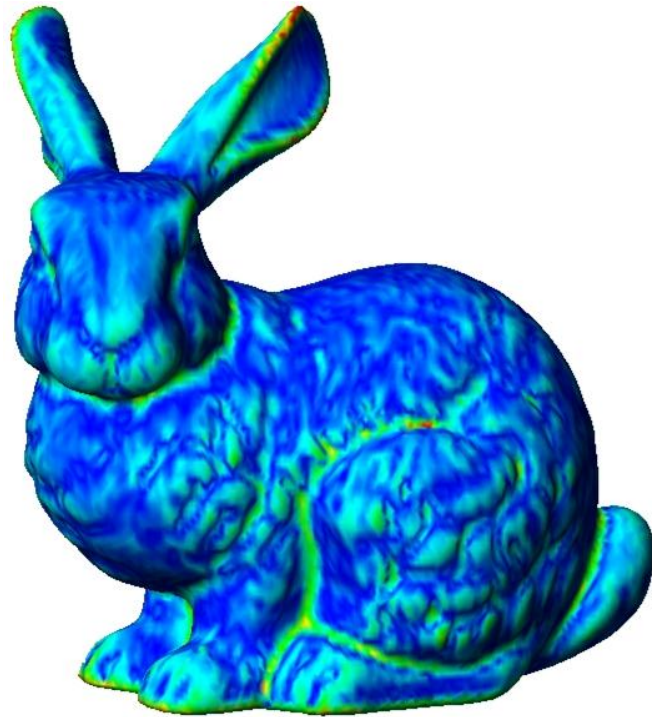
Scalar Functions



http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

Map points to real numbers

Discrete Scalar Functions



$$f \in \mathbb{R}^{|V|}$$

http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

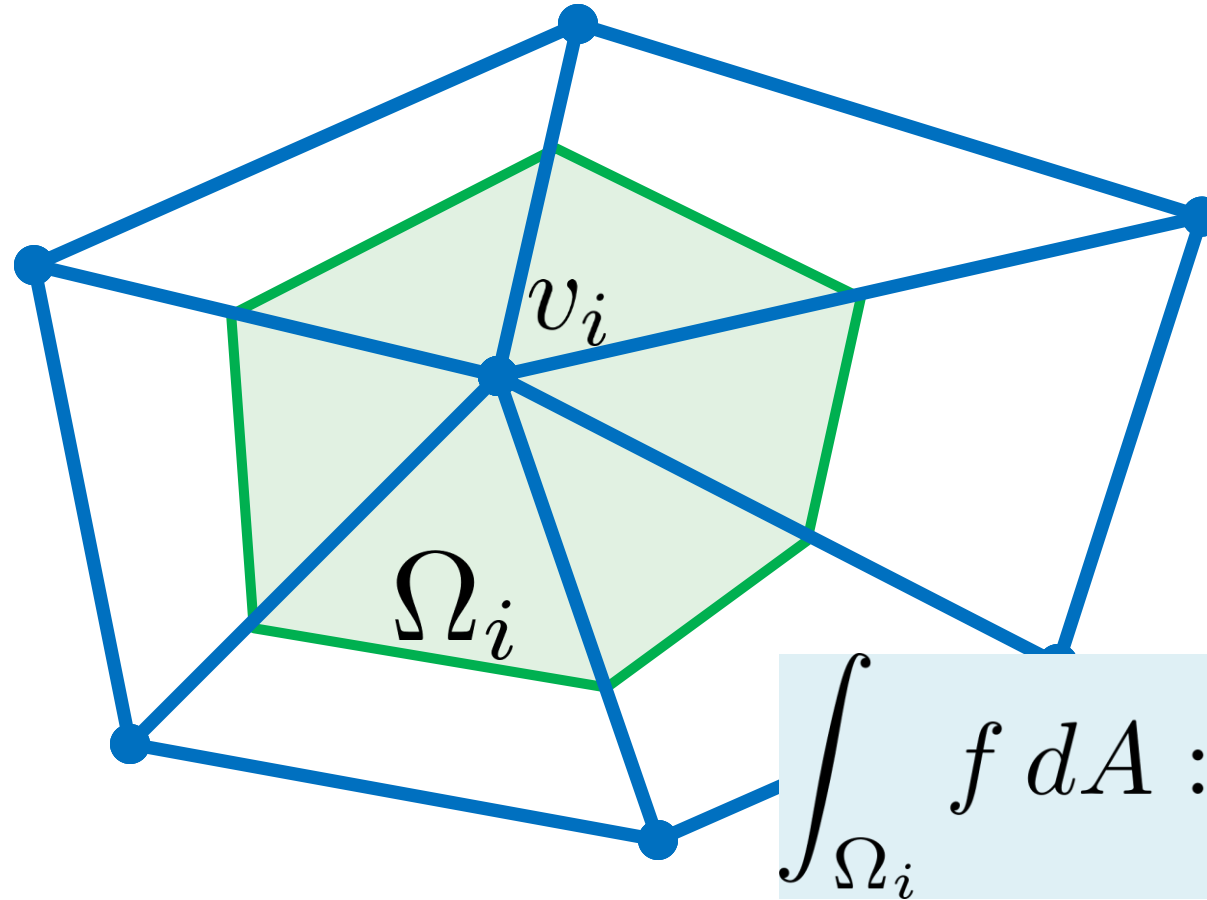
Map vertices to real numbers

Question

What is the integral of f ?

$$\int_M f \, dA$$

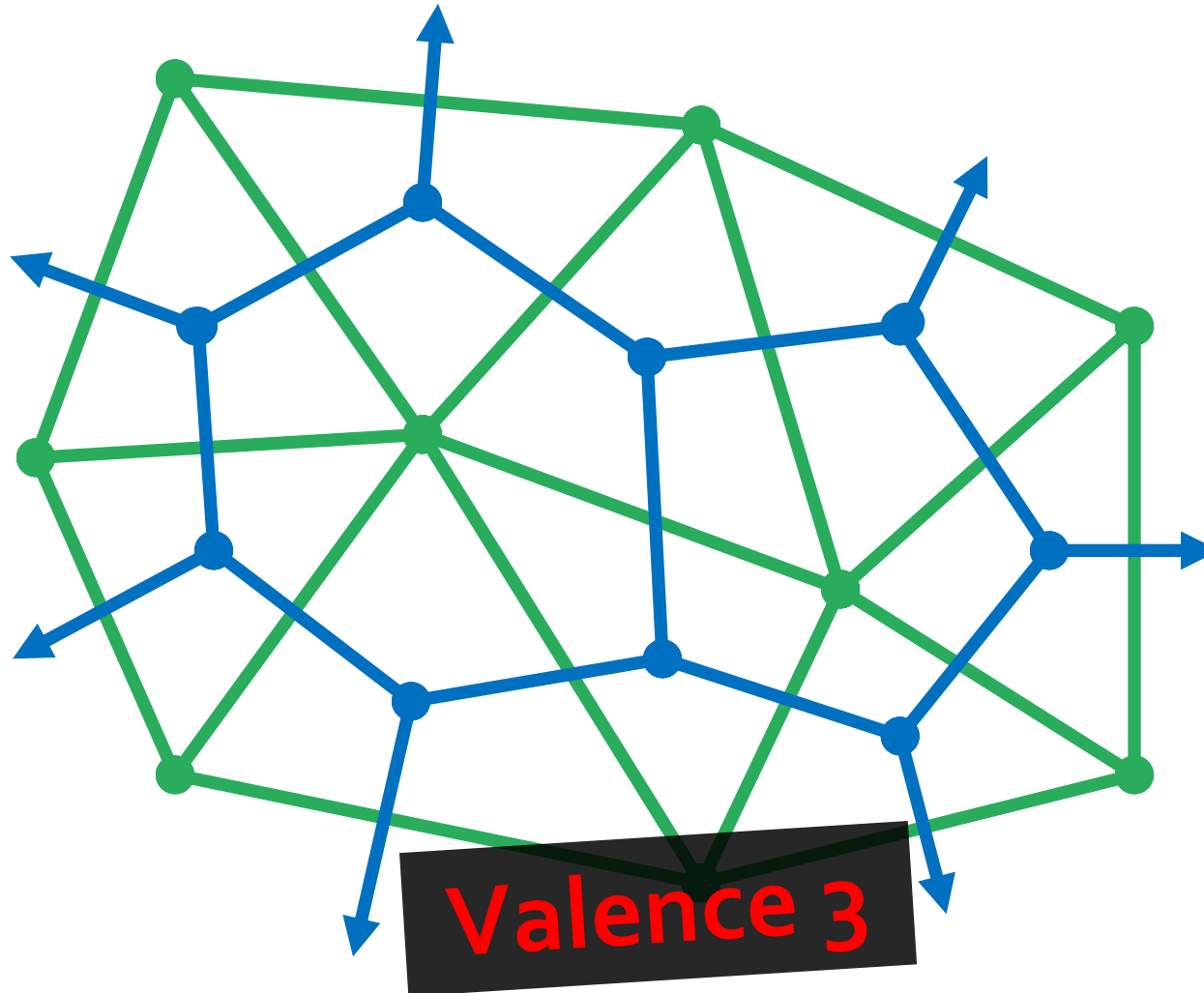
Dual Cell



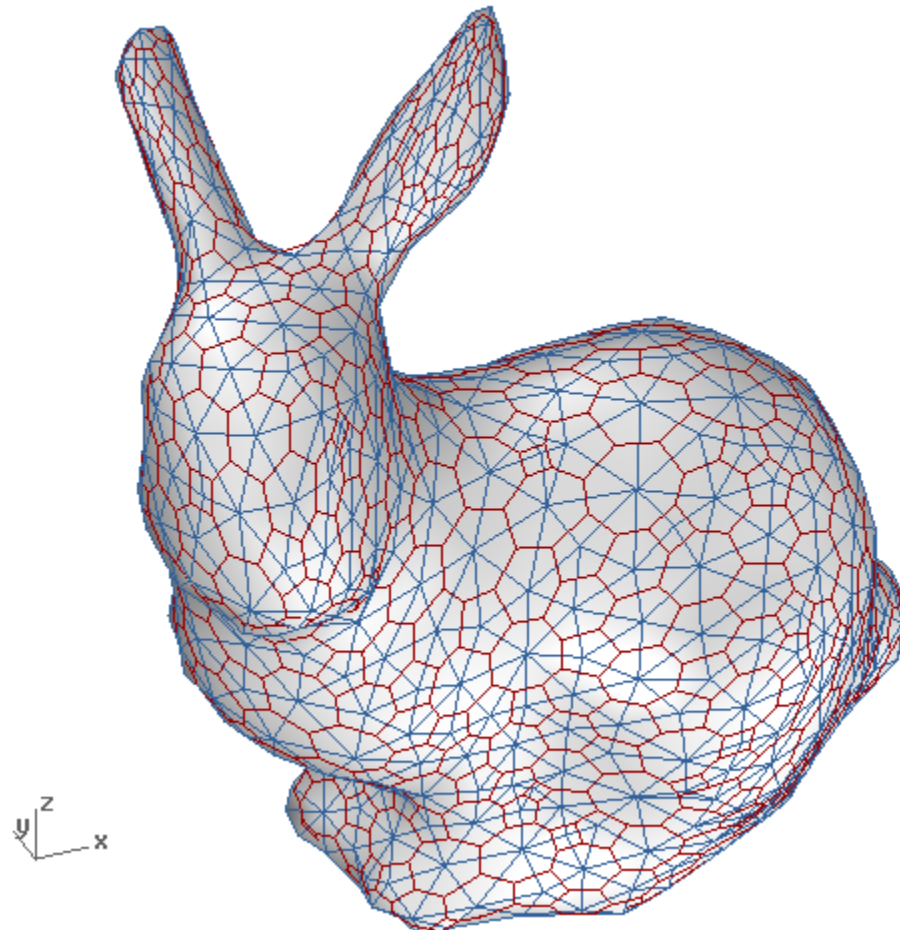
$$\int_{\Omega_i} f \, dA := f_i |\Omega_i|$$

Discrete version of dA

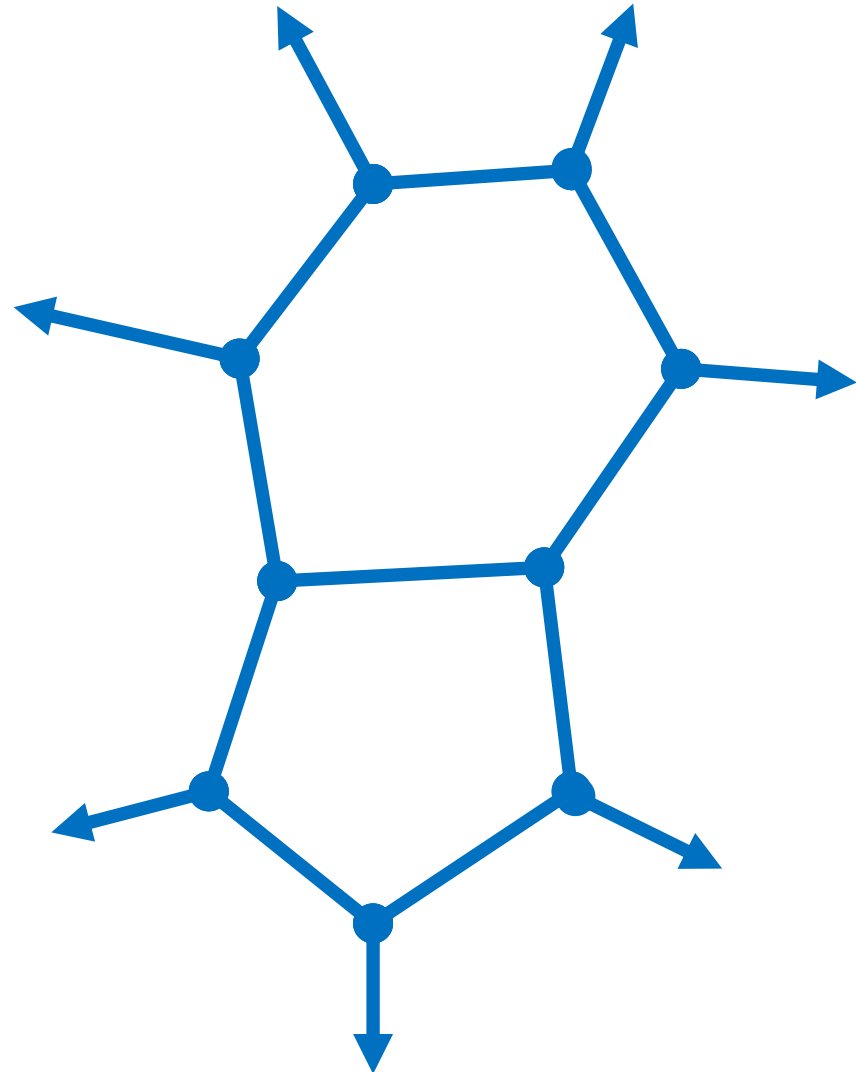
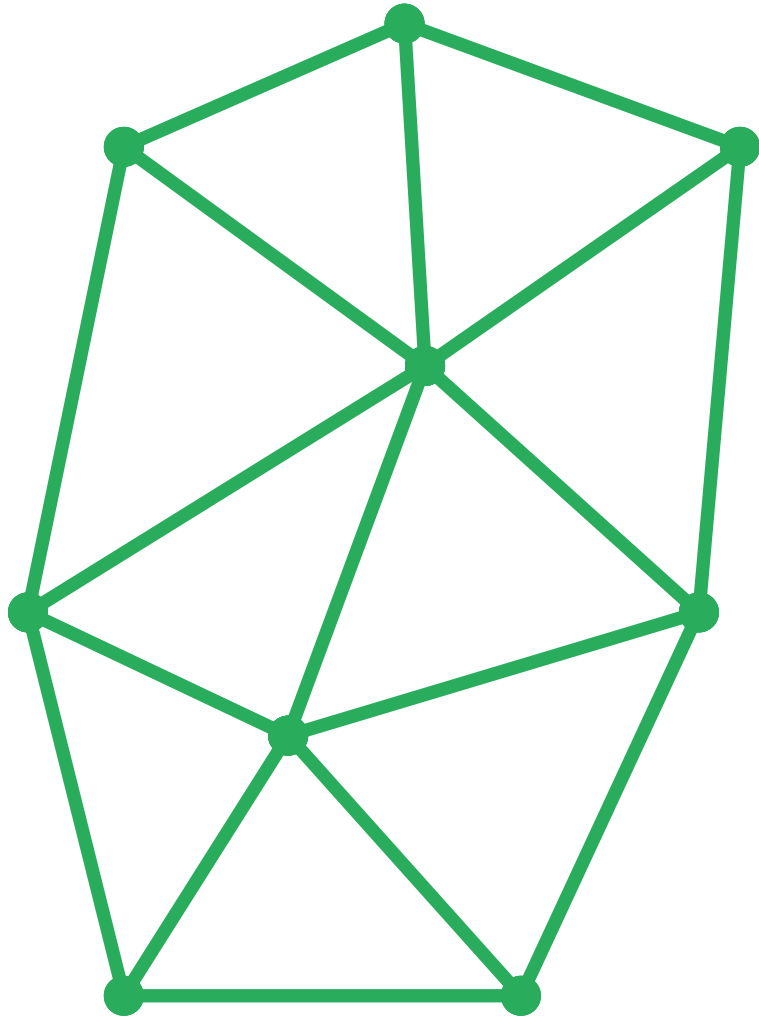
Dual Complex



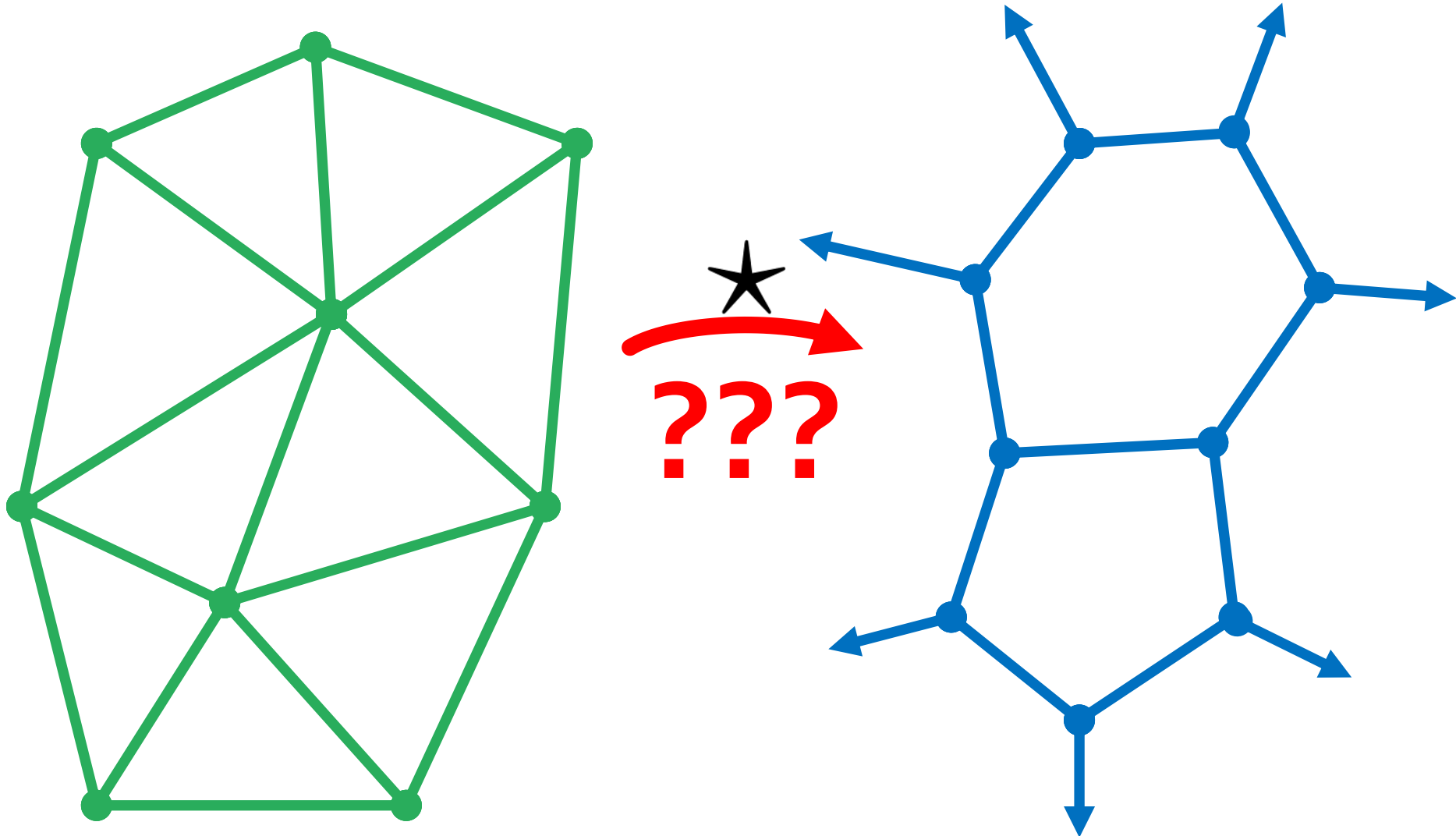
One Surface, Two Meshes



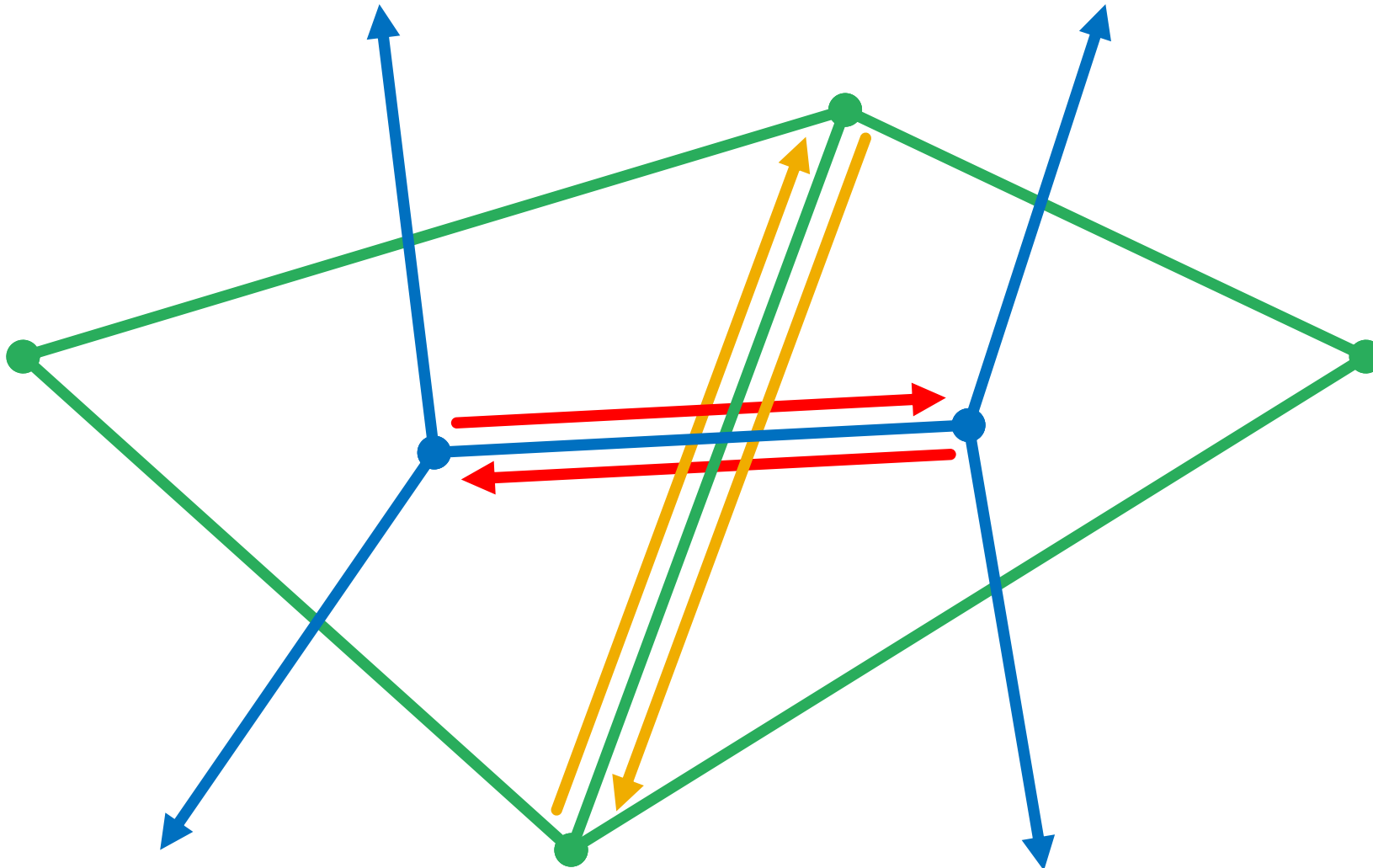
One Surface, Two Halfedges



Missing Operation

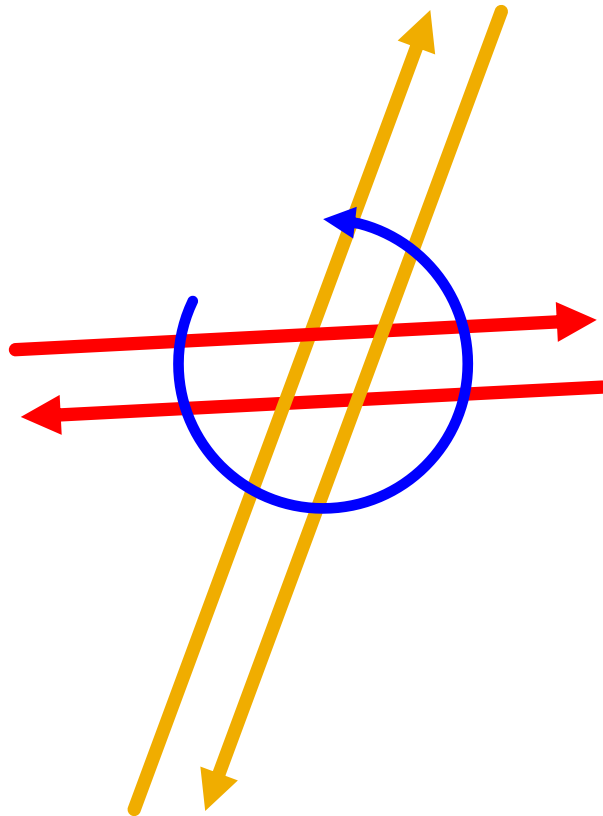


Quad Edge

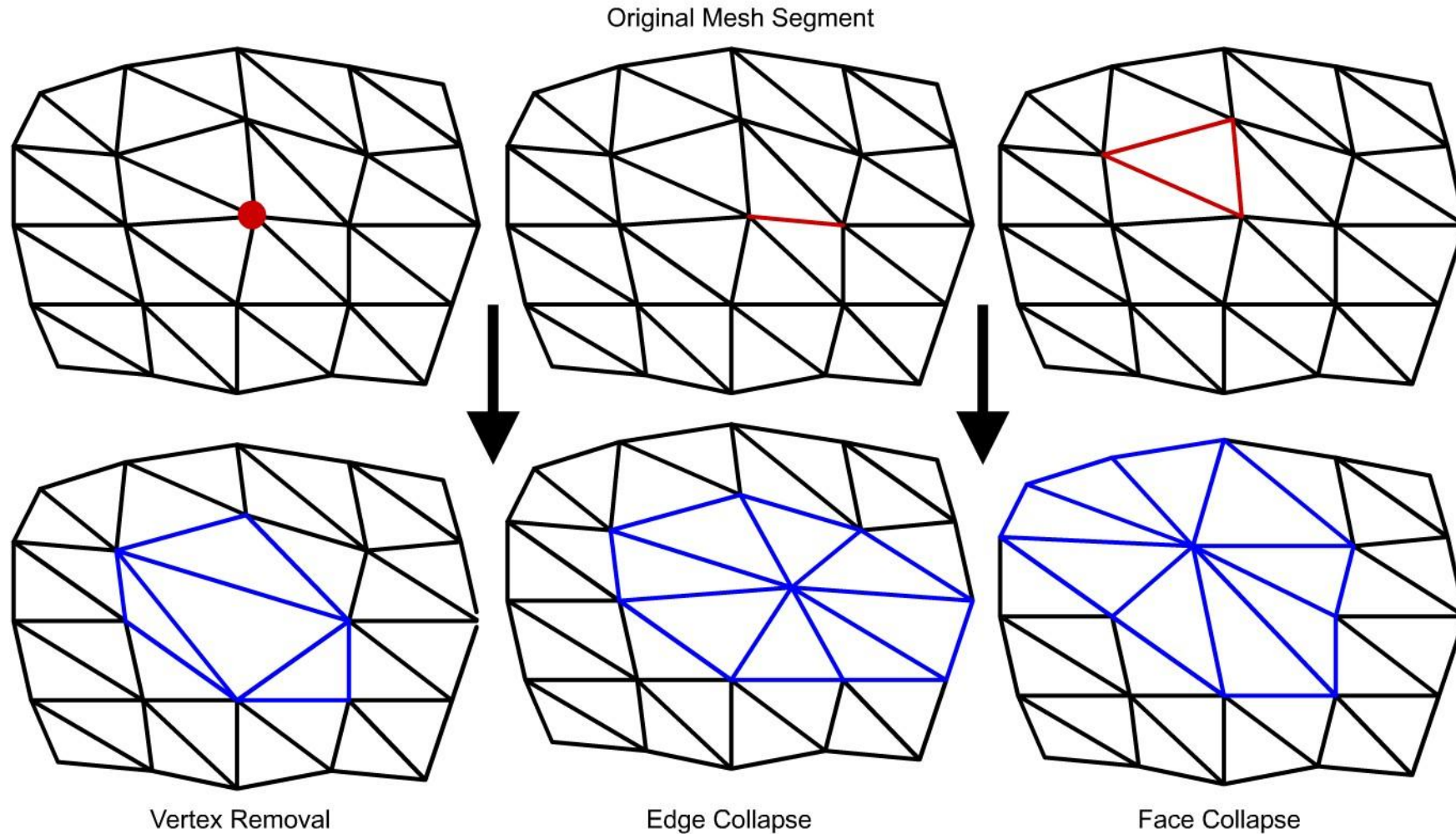


Rotation Operation

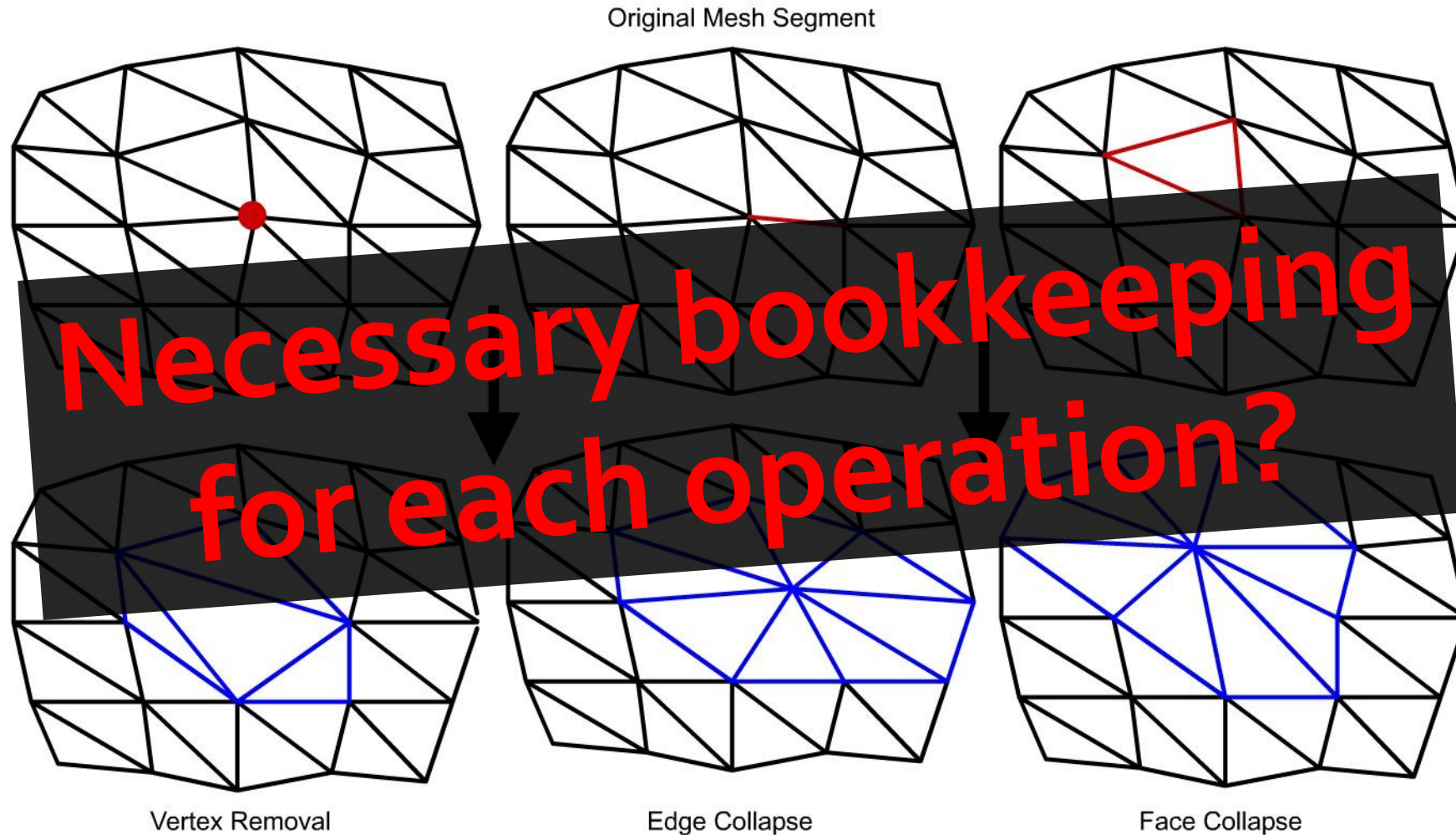
$e \rightarrow \text{Rot} \rightarrow \text{Rot} = e \rightarrow \text{Flip}$



Topological Operations



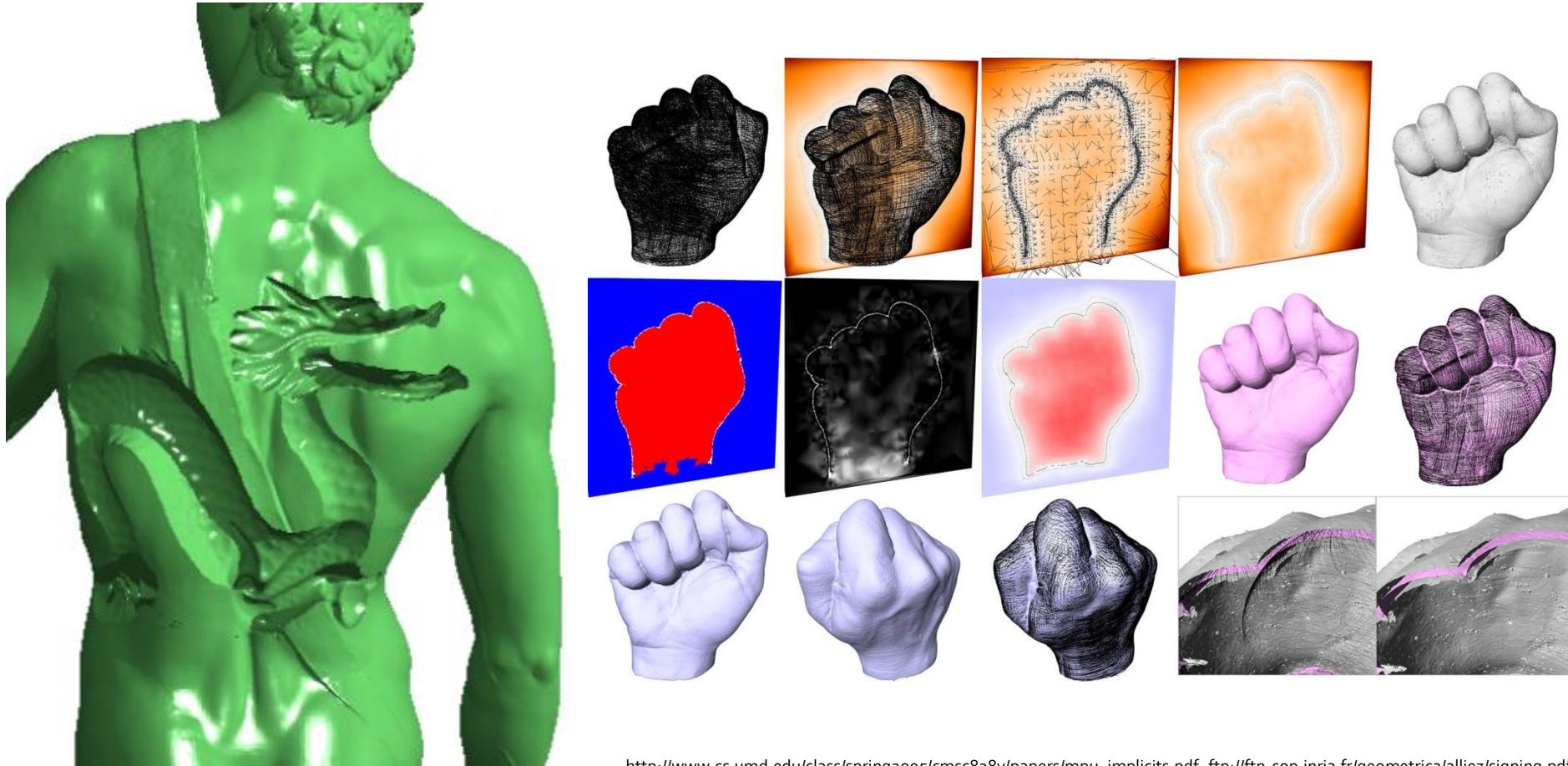
Topological Operations



Take-Away

Complex data structures
enable simpler traversal at
cost of more bookkeeping.

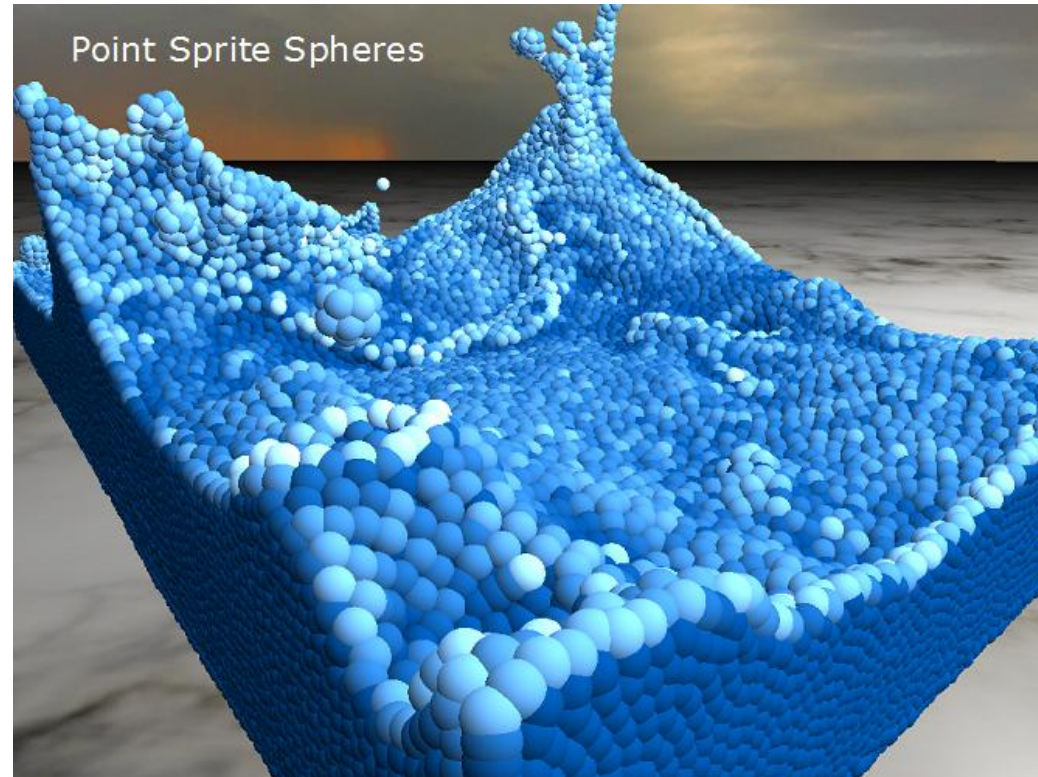
Not the Only Geometric Representation



http://www.cs.umd.edu/class/spring2005/cmsc828v/papers/mpu_implicit.pdf <ftp://ftp-sop.inria.fr/geometrica/alliez/signing.pdf>

Implicit surfaces

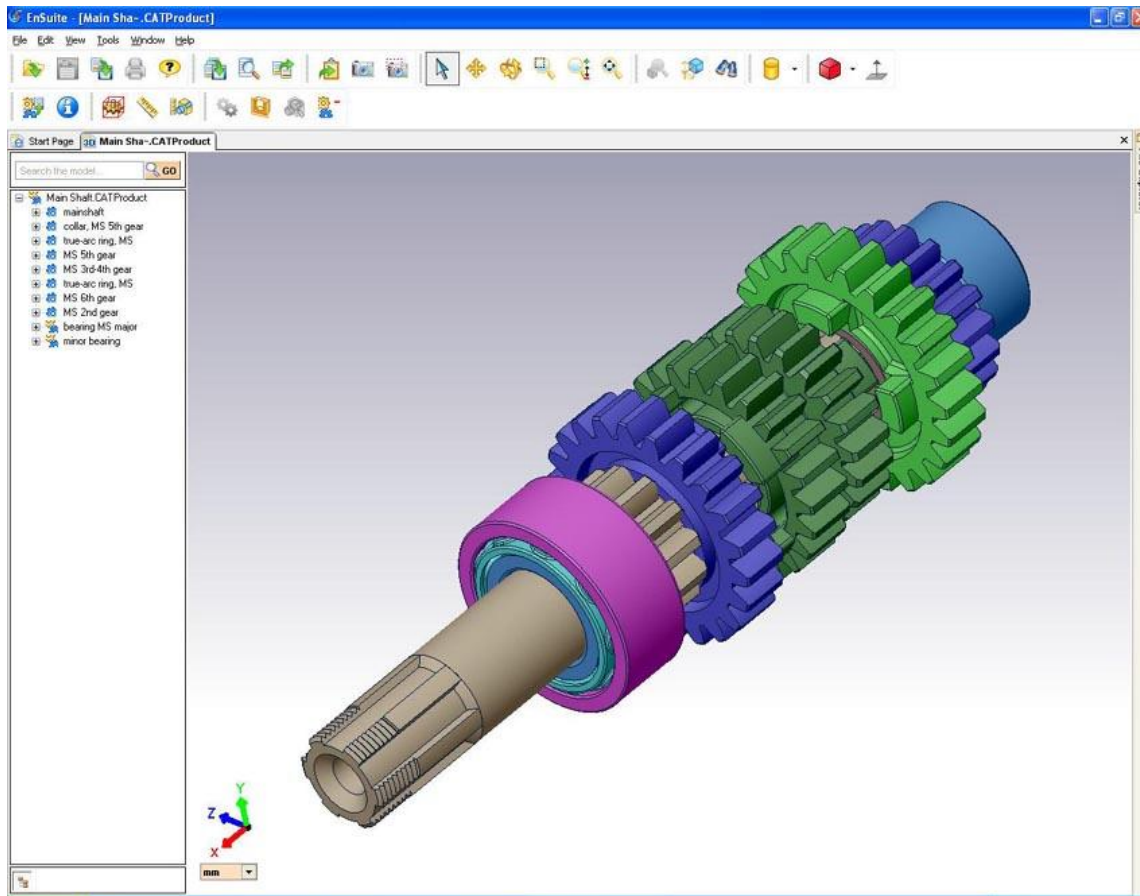
Application of Implicit Surfaces



<http://www.itsartmag.com/features/cgfluids/>
<https://developer.nvidia.com/content/fluid-simulation-alice-madness-returns>

Smoothed-particle hydrodynamics (SPH)

Not the Only Geometric Representation

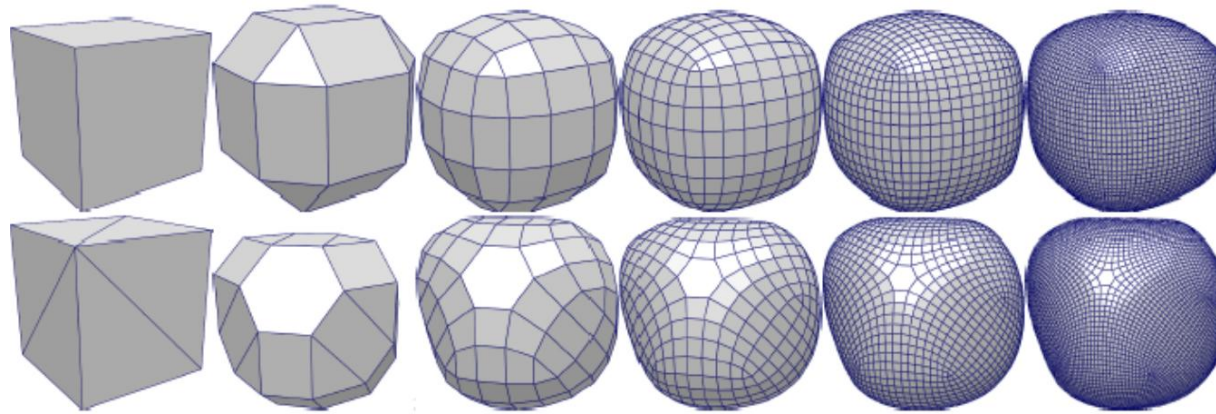


Computer-Aided
Design (CAD)

<http://www.cad-sourcing.com/wp-content/uploads/2011/12/free-cad-software.jpg>

Polynomial/rational patches

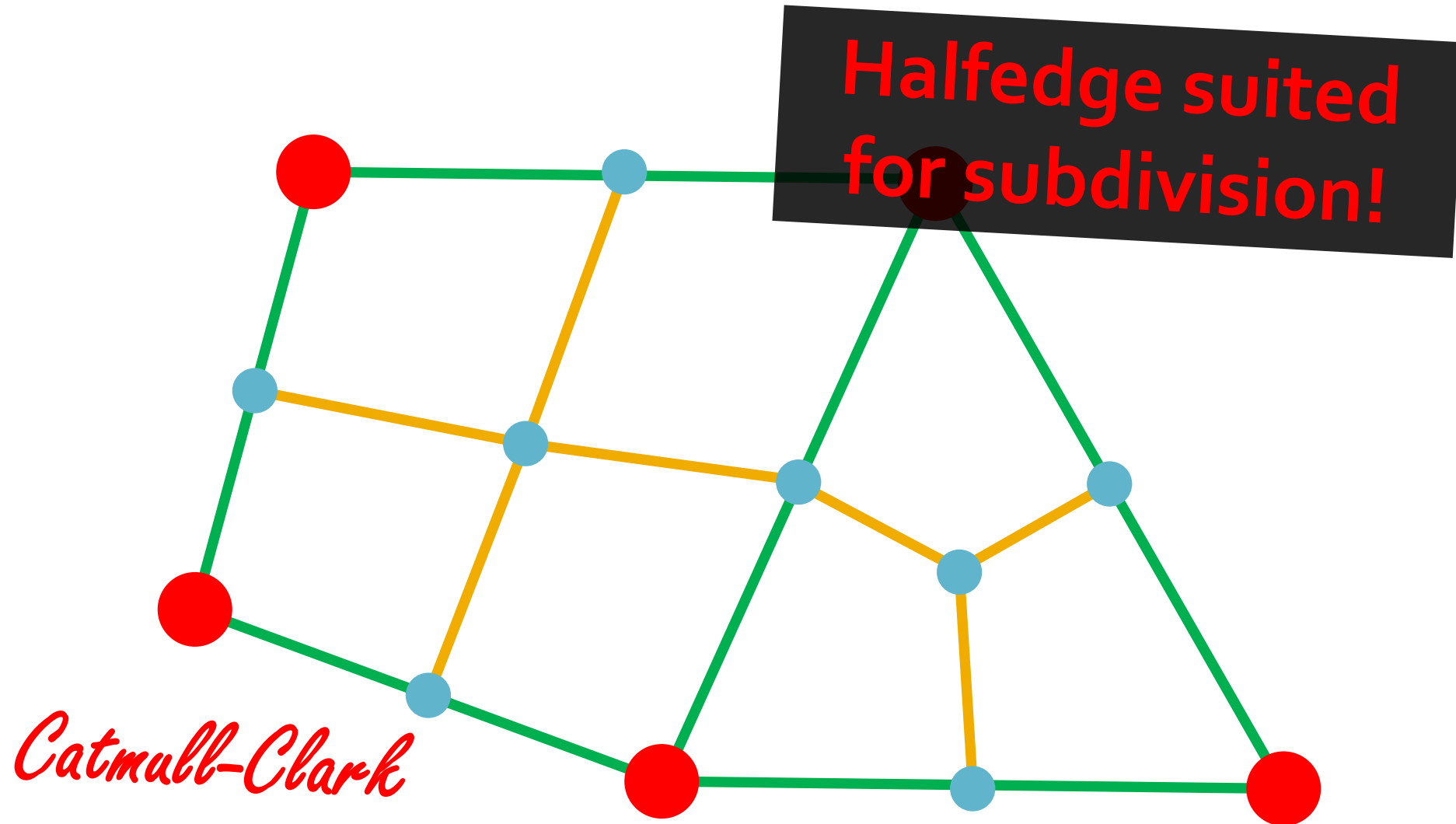
Not the Only Geometric Representation



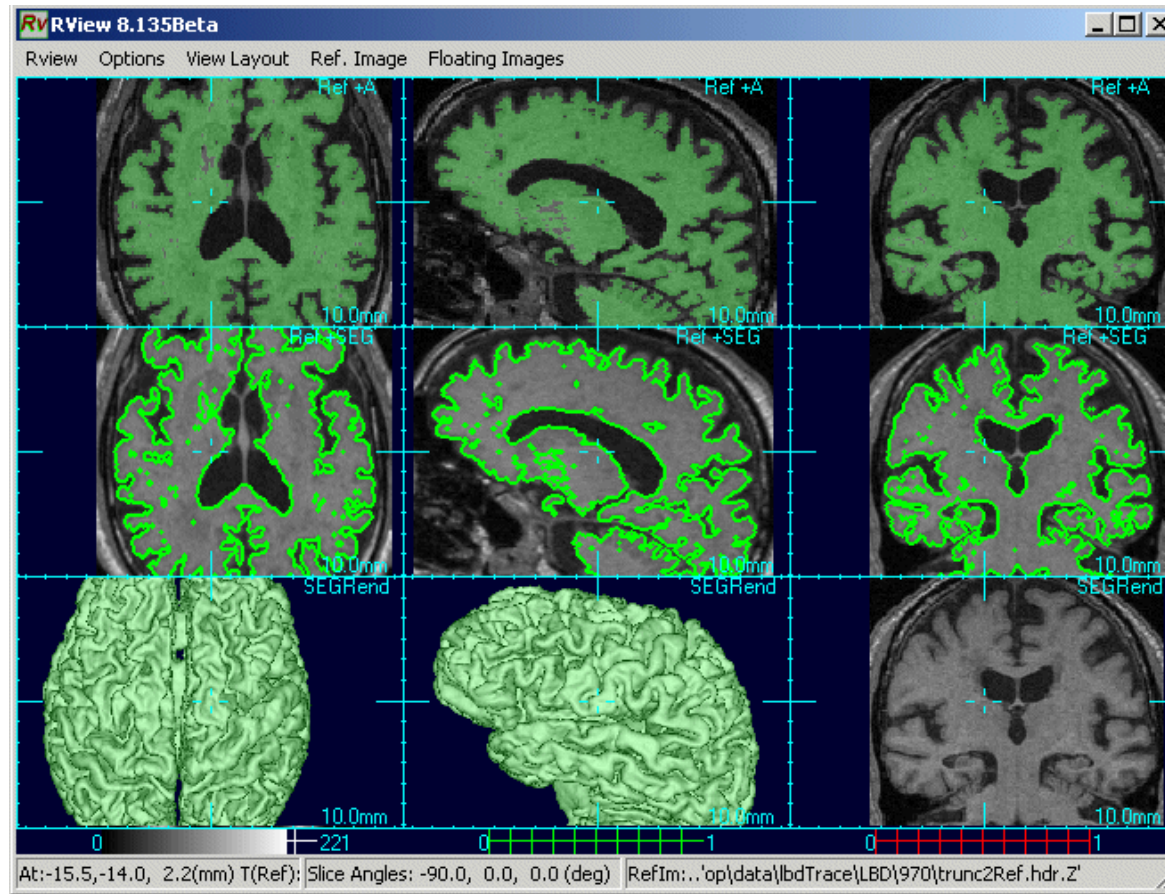
https://imagecomputing.net/damien.rohmer/teaching/2018_2019/semester_1/m2_mpri_cg_viz/class/01_surface_representation/content/035_subdivision_surfaces/index.html

Subdivision Surfaces

Aside



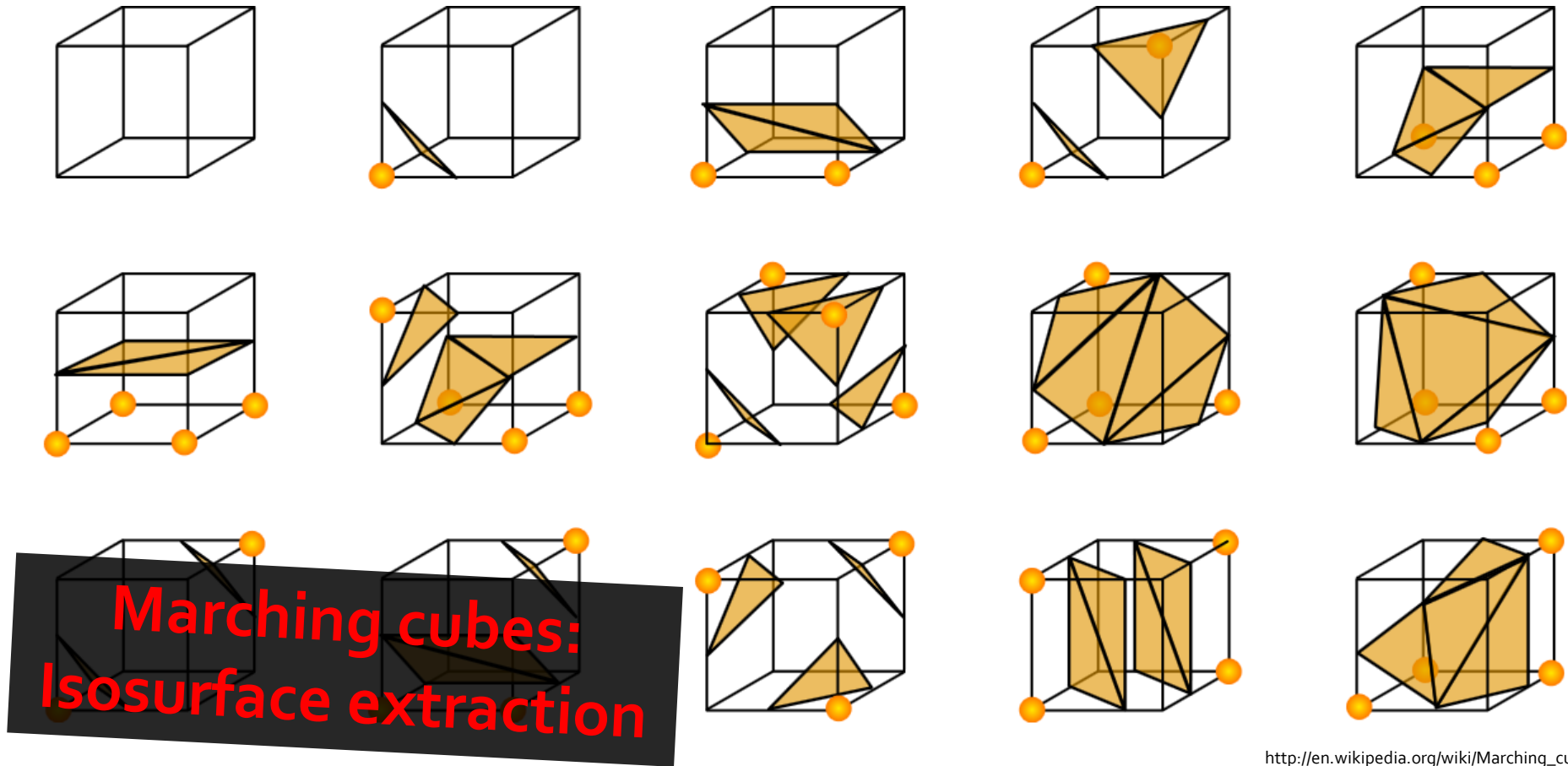
Not the Only Geometric Representation



<http://www.colin-studholme.net/software/rview/rvmanual/morphtool5.gif>

Volumetric imaging

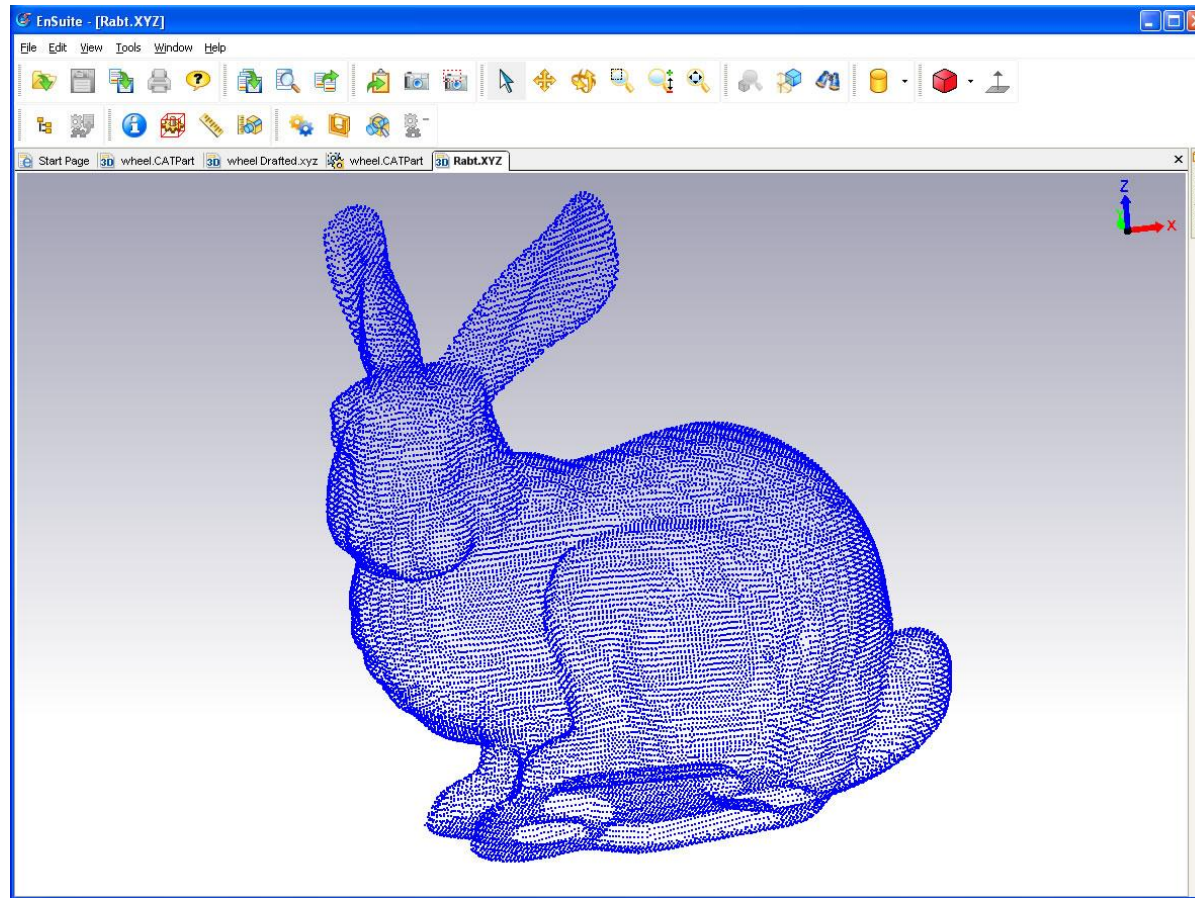
Surfaces from Volumes



http://en.wikipedia.org/wiki/Marching_cubes

Volumetric extraction

Not the Only Geometric Representation



<http://www.engineeringspecifier.com/public/primages/pr1200.jpg>

Point clouds

Surfaces: Smooth and Discrete

Justin Solomon

6.8410: Shape Analysis

Fall 2026

