

Continuous Curves

Justin Solomon

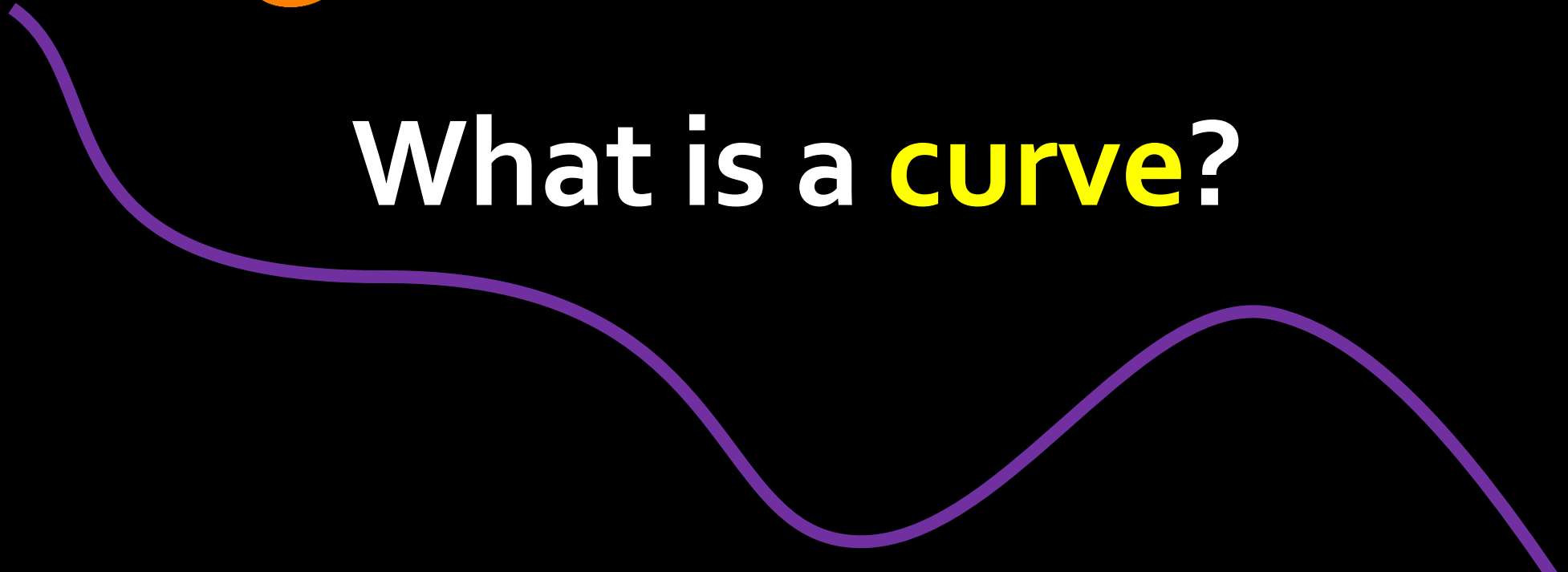
6.8410: Shape Analysis

Fall 2026

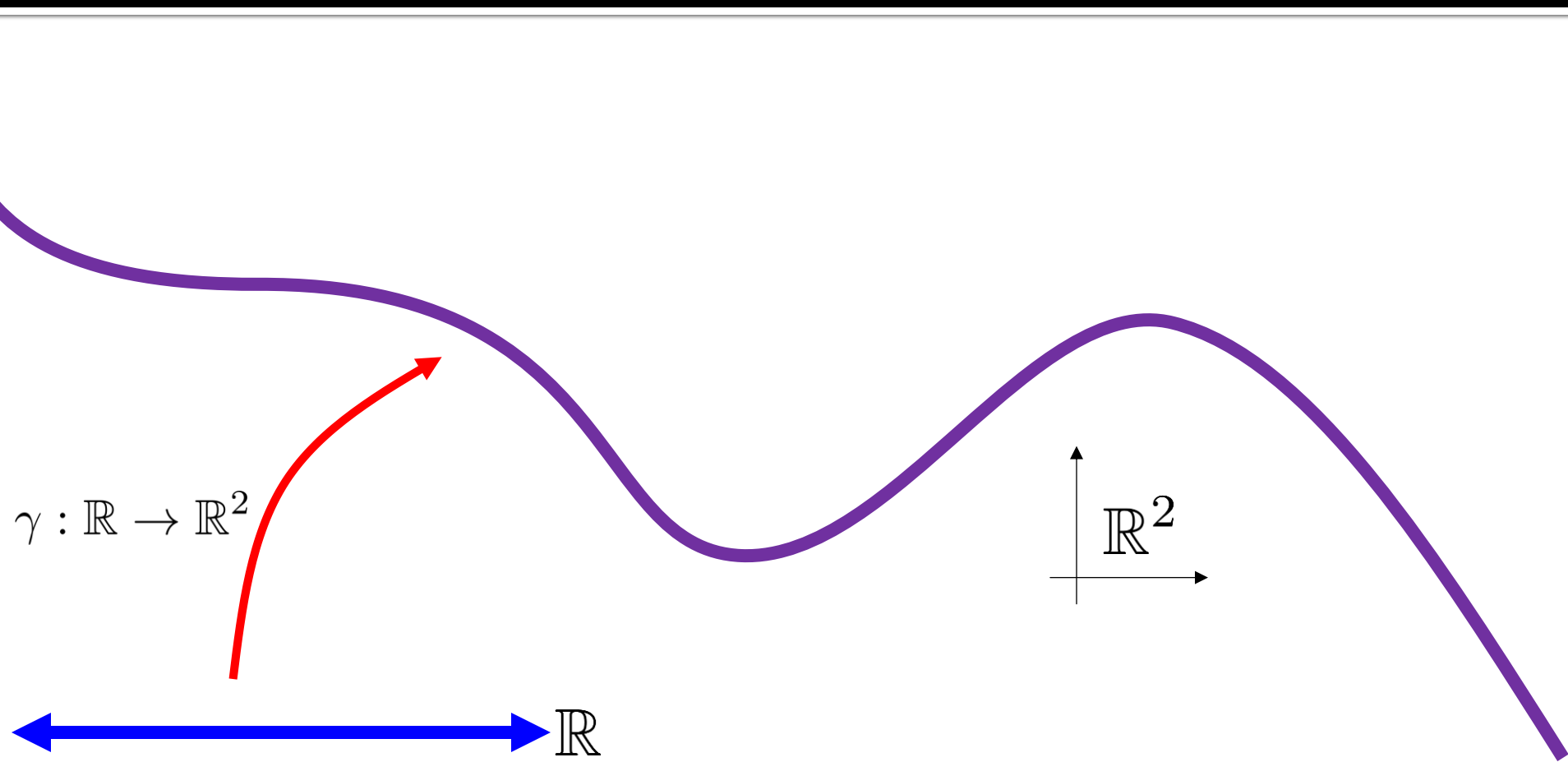




What is a **curve**?



Defining "Curve"



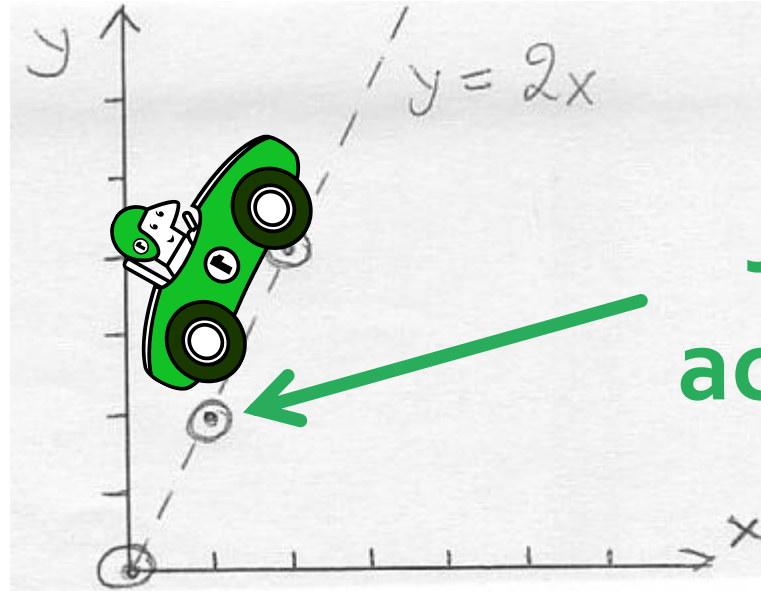
A function?

Subtlety

$$\gamma(t) \equiv (0, 0)$$

Not a curve

Different from Calculus

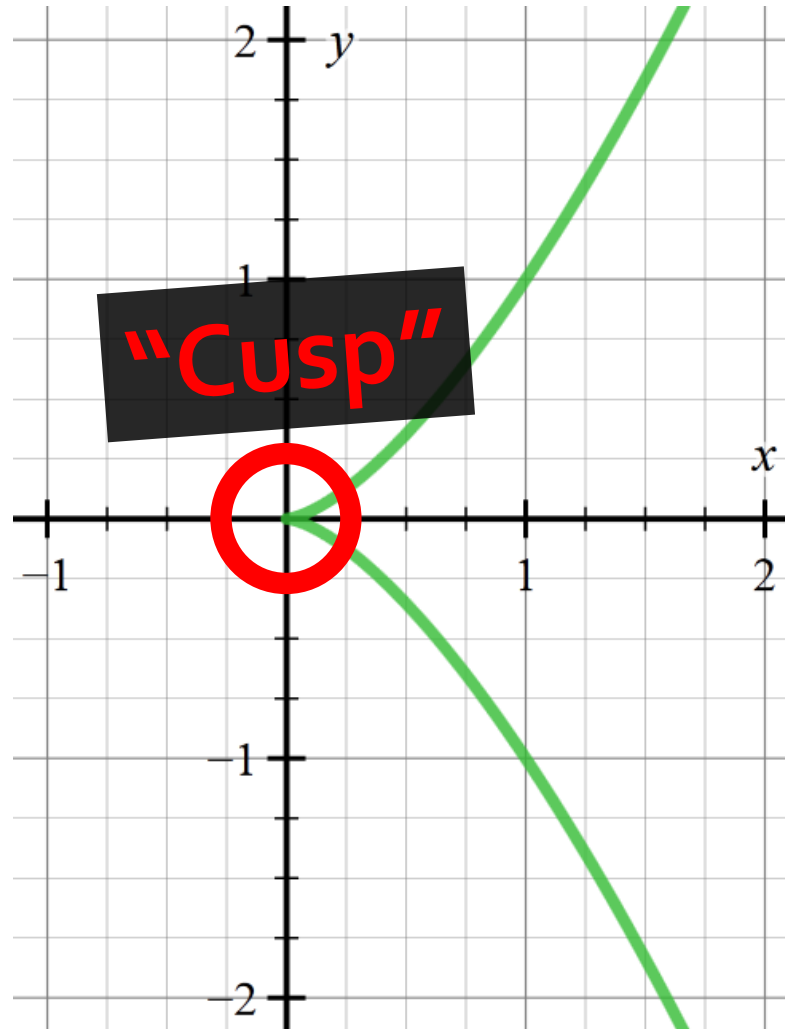


Jams on
accelerator

$$\gamma_1(t) = (t, 2t)$$

$$\gamma_2(t) = \begin{cases} (t, 2t) & t \leq 1 \\ (2(t - \frac{1}{2}), 4(t - \frac{1}{2})) & t > 1 \end{cases}$$

Graphs of Smooth Functions



$$\gamma(t) = (t^2, t^3)$$

Geometry of a Curve

A curve is a
set of points
with certain properties.

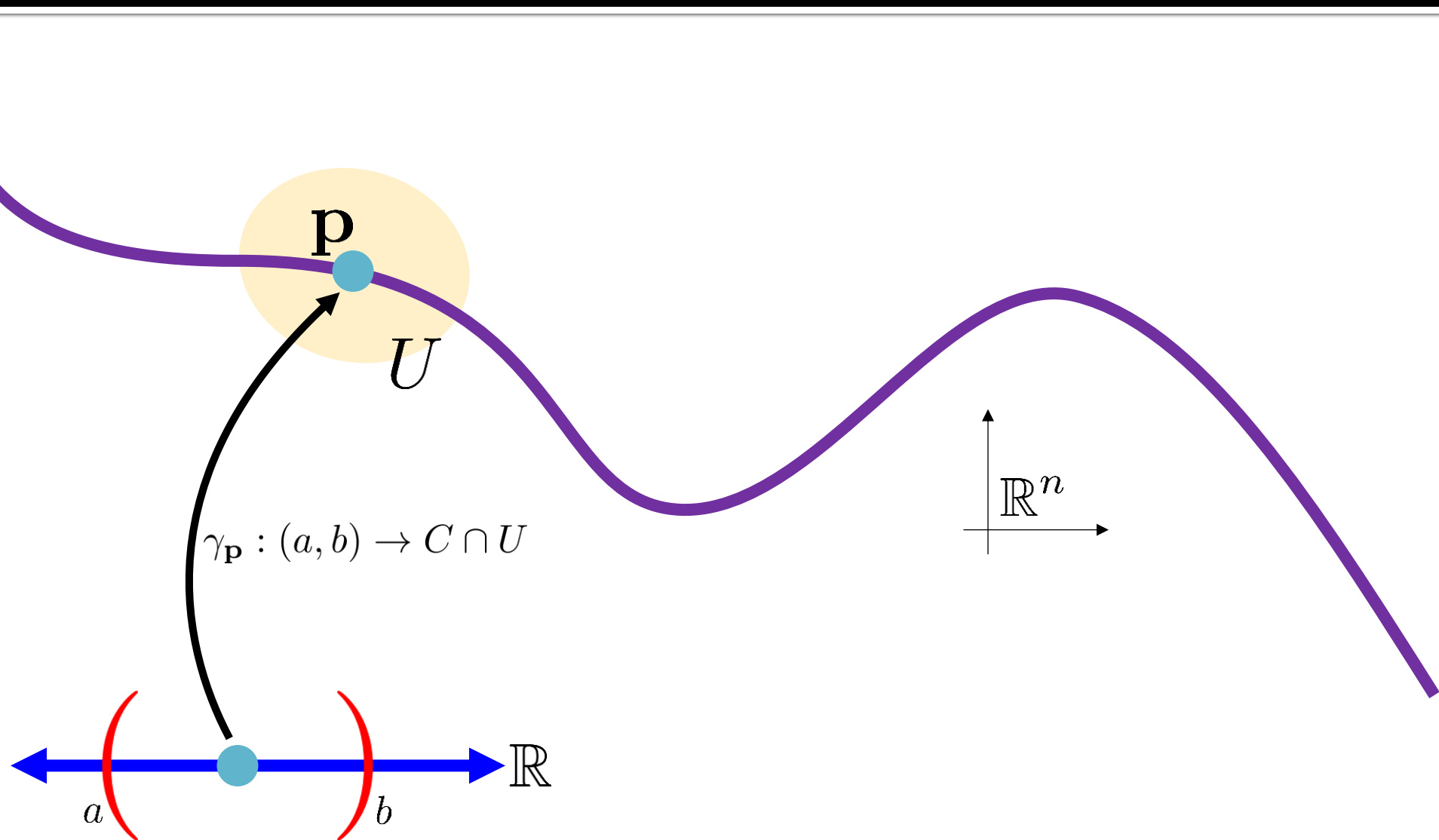
It is not a function.

Geometric Definition



Set of points that locally looks like a line.

Differential Geometry Definition

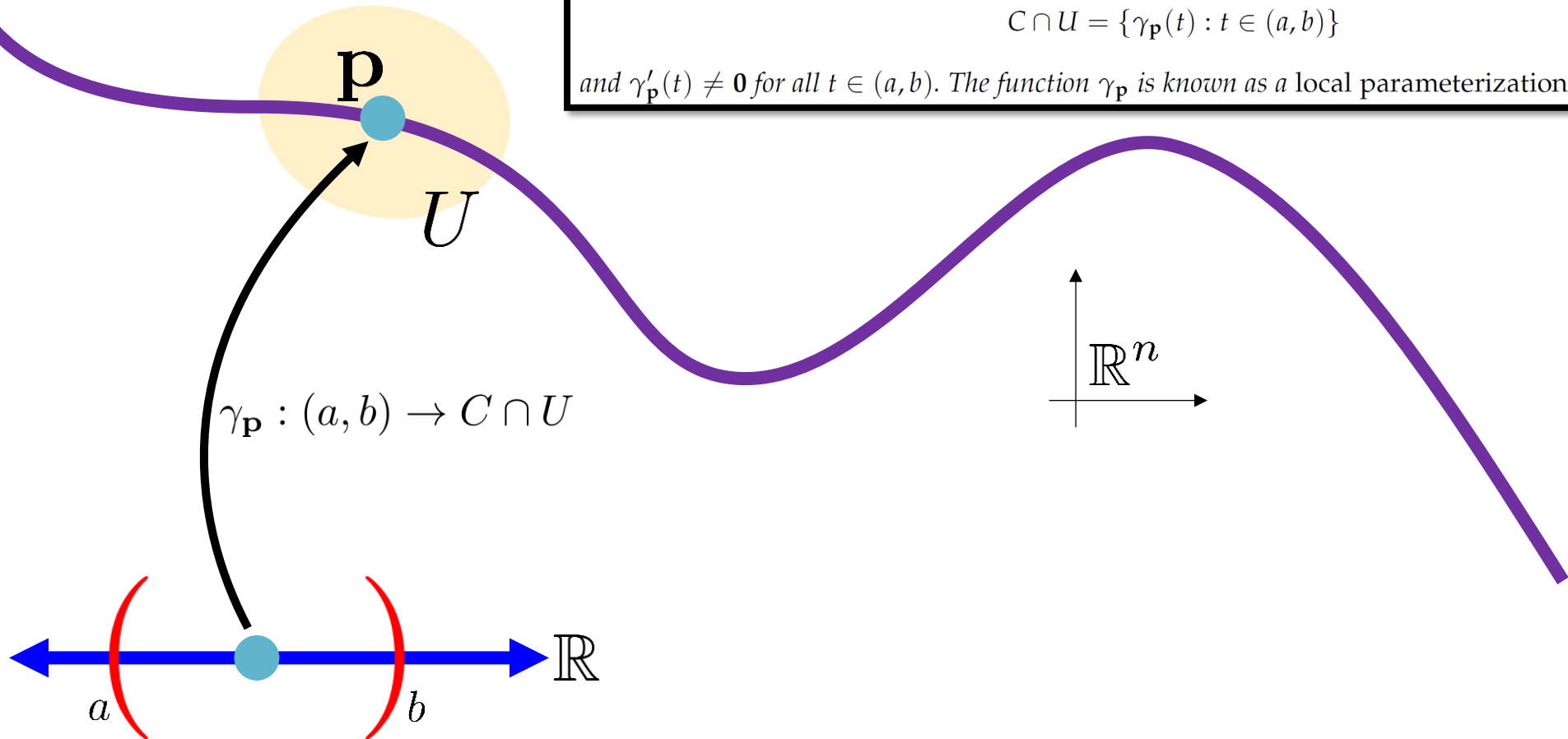


Formal Statement

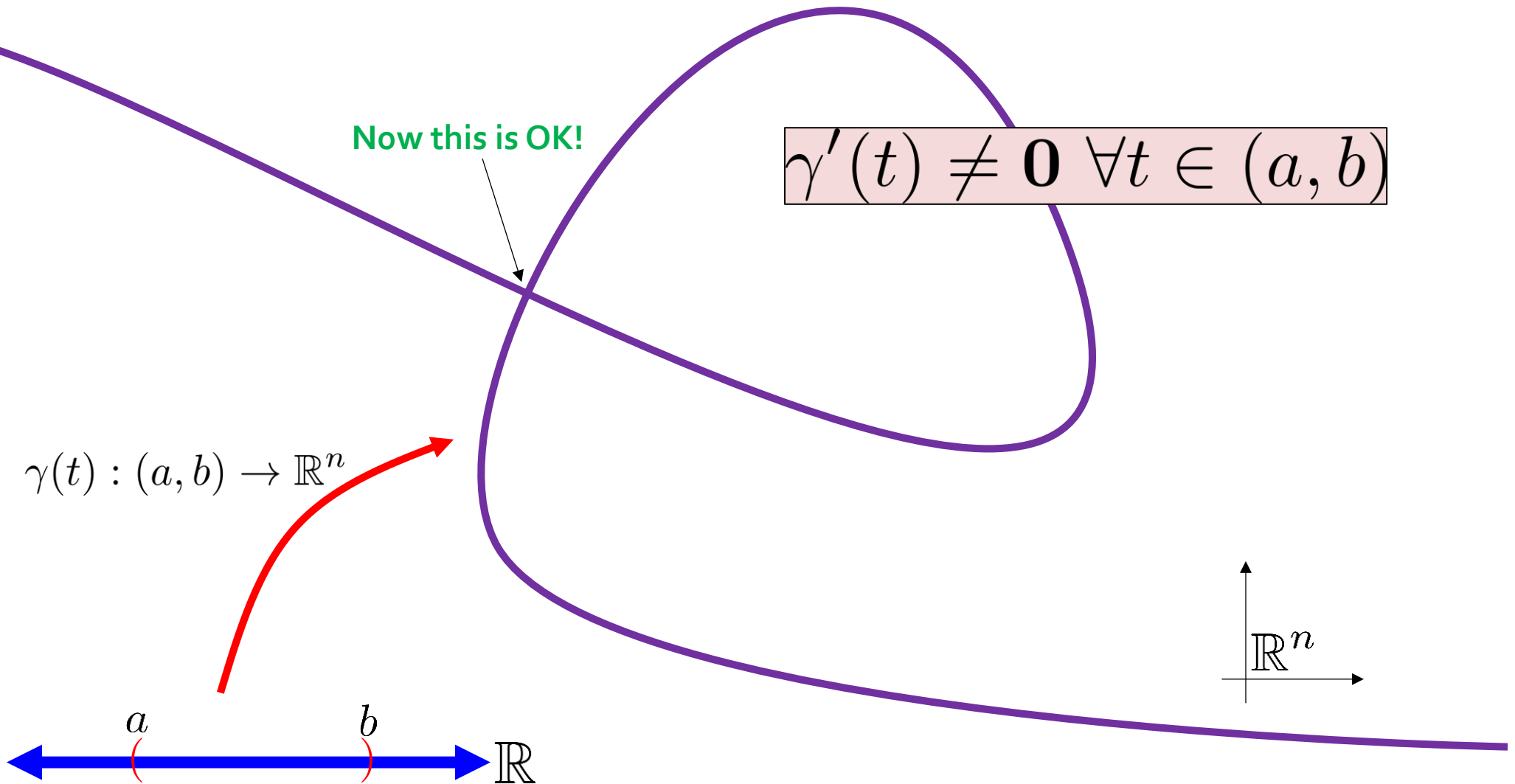
Definition 3.1 (Differentiable curve in $\mathbb{R}^n$). A differentiable curve in $\mathbb{R}^n$ is a set of points $C \subset \mathbb{R}^n$ with the property that for every $\mathbf{p} \in C$ there exists an open neighborhood $U \subseteq \mathbb{R}^n$ containing $\mathbf{p}$ and a one-to-one, smooth function $\gamma_{\mathbf{p}} : (a, b) \rightarrow C \cap U$, so that

$$C \cap U = \{\gamma_{\mathbf{p}}(t) : t \in (a, b)\}$$

and $\gamma'_{\mathbf{p}}(t) \neq \mathbf{0}$ for all $t \in (a, b)$. The function $\gamma_{\mathbf{p}}$ is known as a local parameterization of C at $\mathbf{p}$.



Parameterized Curve



Some Vocabulary

- **Trace** of parameterized curve

$$\{\gamma(t) : t \in (a, b)\} \subseteq \mathbb{R}^n$$

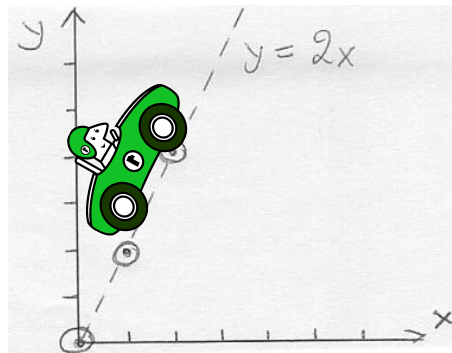
- **Component** functions

$$\gamma(t) = (x(t), y(t), z(t))$$

Change of Parameter

$$t \mapsto \gamma \circ \phi(t)$$

Geometric measurements should be
invariant
to changes of parameter.



Dependence of Velocity

$$\tilde{\gamma}(t) := \gamma(\phi(t))$$

Effect on velocity and acceleration?

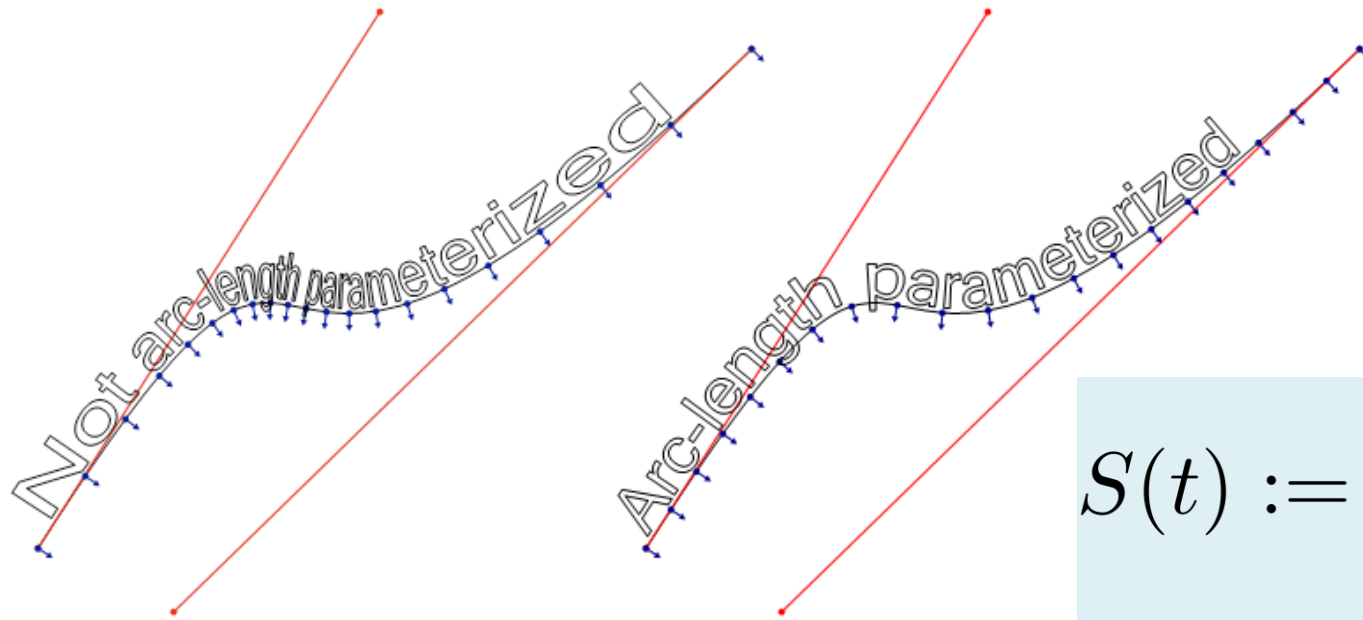
Arc Length

$$\int_a^b \|\gamma'(t)\|_2 dt$$

Independent of parameter!

Parameterization by Arc Length

<http://www.planetclegg.com/projects/WarpingTextToSplines.html>



$$S(t) := \int_{t_0}^t \|\gamma'(t)\|_2 dt$$

$$t = \phi \circ S(t)$$

$$\tilde{\gamma}(s) := \gamma \circ \phi(s)$$

← Inverse
functions

Constant-speed parameterization

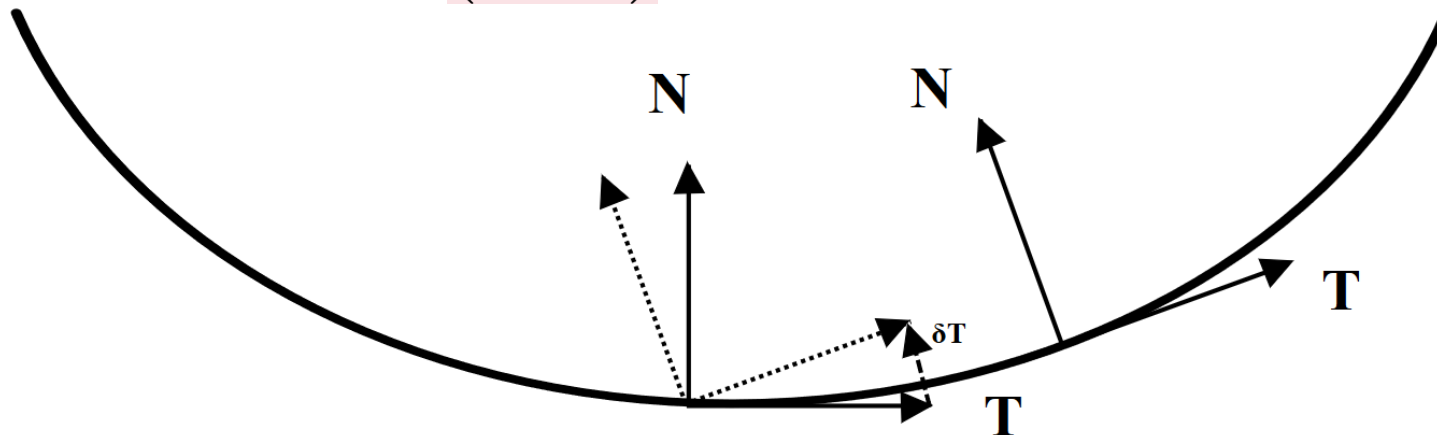
Moving Frame in 2D

$$\mathbf{T}(s) := \gamma'(s)$$

$$\implies \|\mathbf{T}(s)\|_2 \equiv 1$$

$$\mathbf{N}(s) := J\mathbf{T}(s) = \kappa(s)^{-1}\mathbf{T}'(s)$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



Philosophical Point

Differential geometry “should” be
coordinate-invariant.

Referring to x and y is a hack!

(but sometimes convenient...)



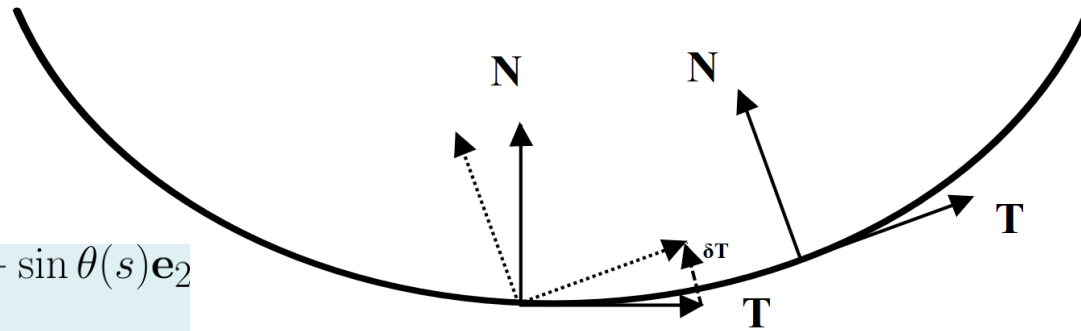
How do you
describe a curve
without coordinates?

Turtles All The Way Down

$$\frac{d}{ds} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \end{pmatrix} := \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \end{pmatrix}$$

Signed curvature κ is rate of change of turning angle θ .

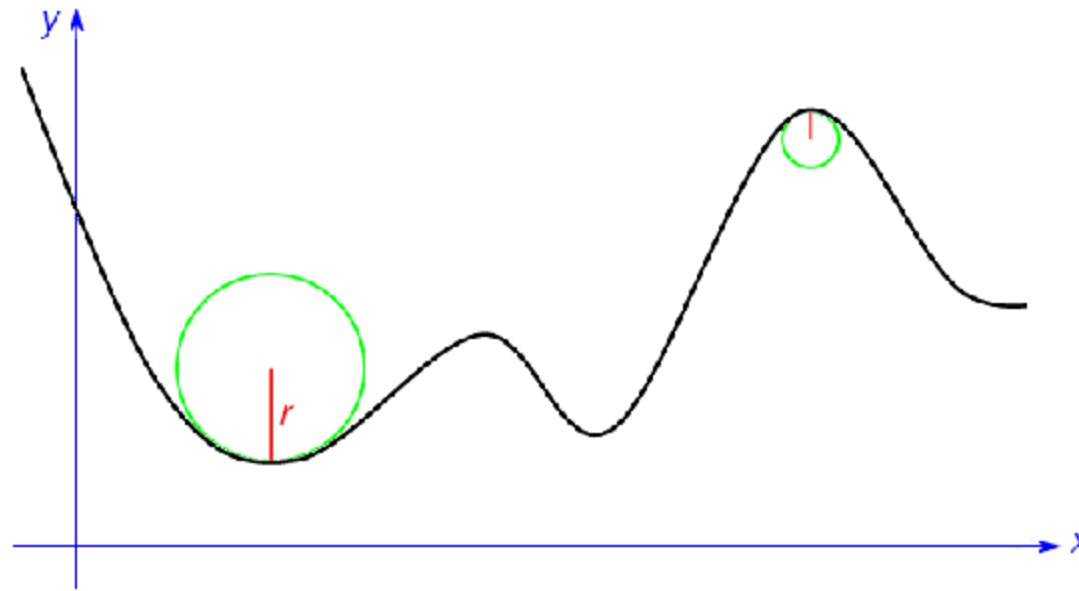
$$\begin{aligned} \mathbf{T}(s) &= \cos \theta(s) \mathbf{e}_1 + \sin \theta(s) \mathbf{e}_2 \\ \kappa(s) &:= \theta'(s) \end{aligned}$$



https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

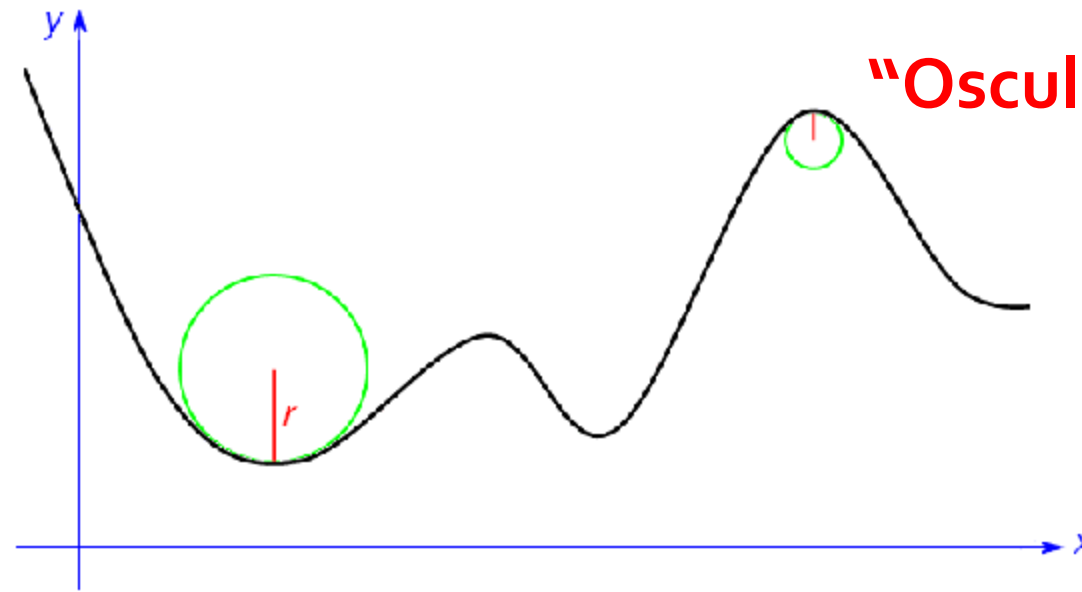
Use coordinates *from* the curve to
express its shape!

Radius of Curvature



$$r(s) := \frac{1}{\kappa(s)}$$

Radius of Curvature



$$r(s) := \frac{1}{\kappa(s)}$$

Fundamental theorem of the local theory of plane curves:

$\kappa(s)$ distinguishes a planar
curve up to rigid motion.

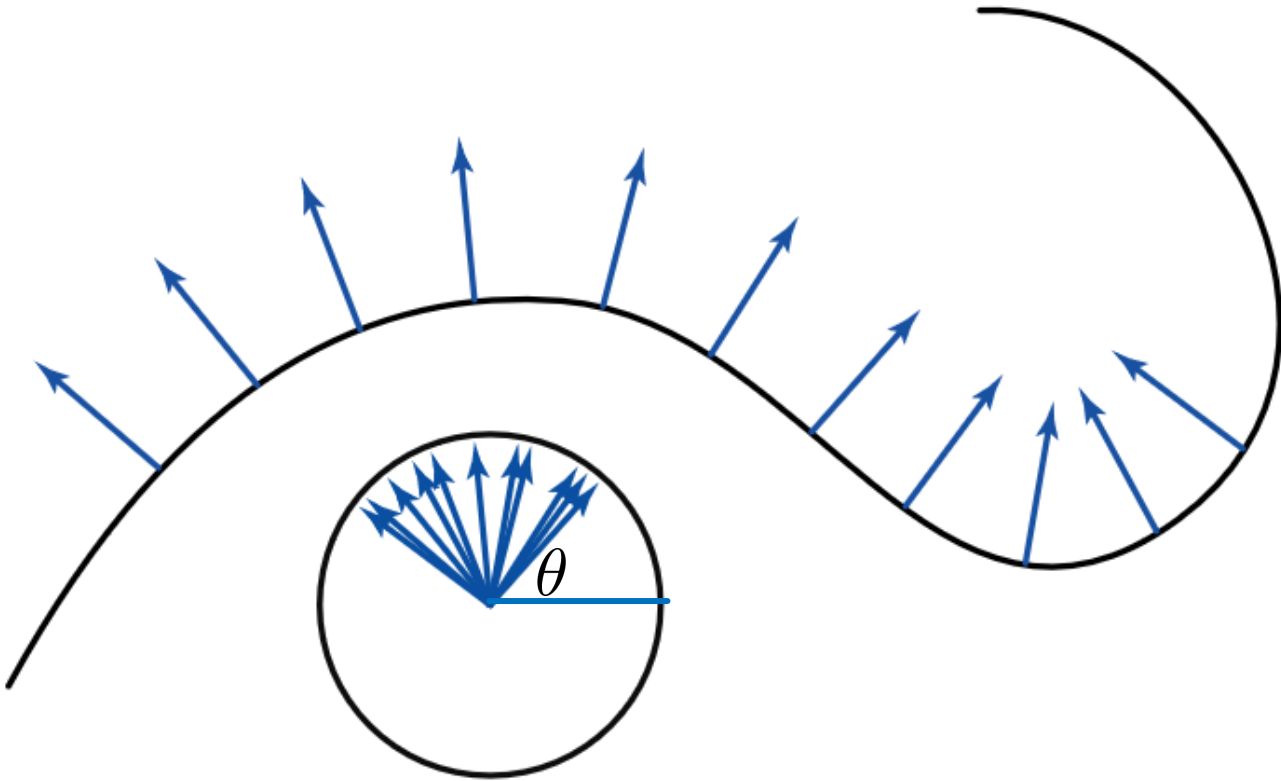
Fundamental theorem of the local theory of plane curves:

$\kappa(s)$ distinguishes a planar
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Statement shorter than the name!

Idea of Proof



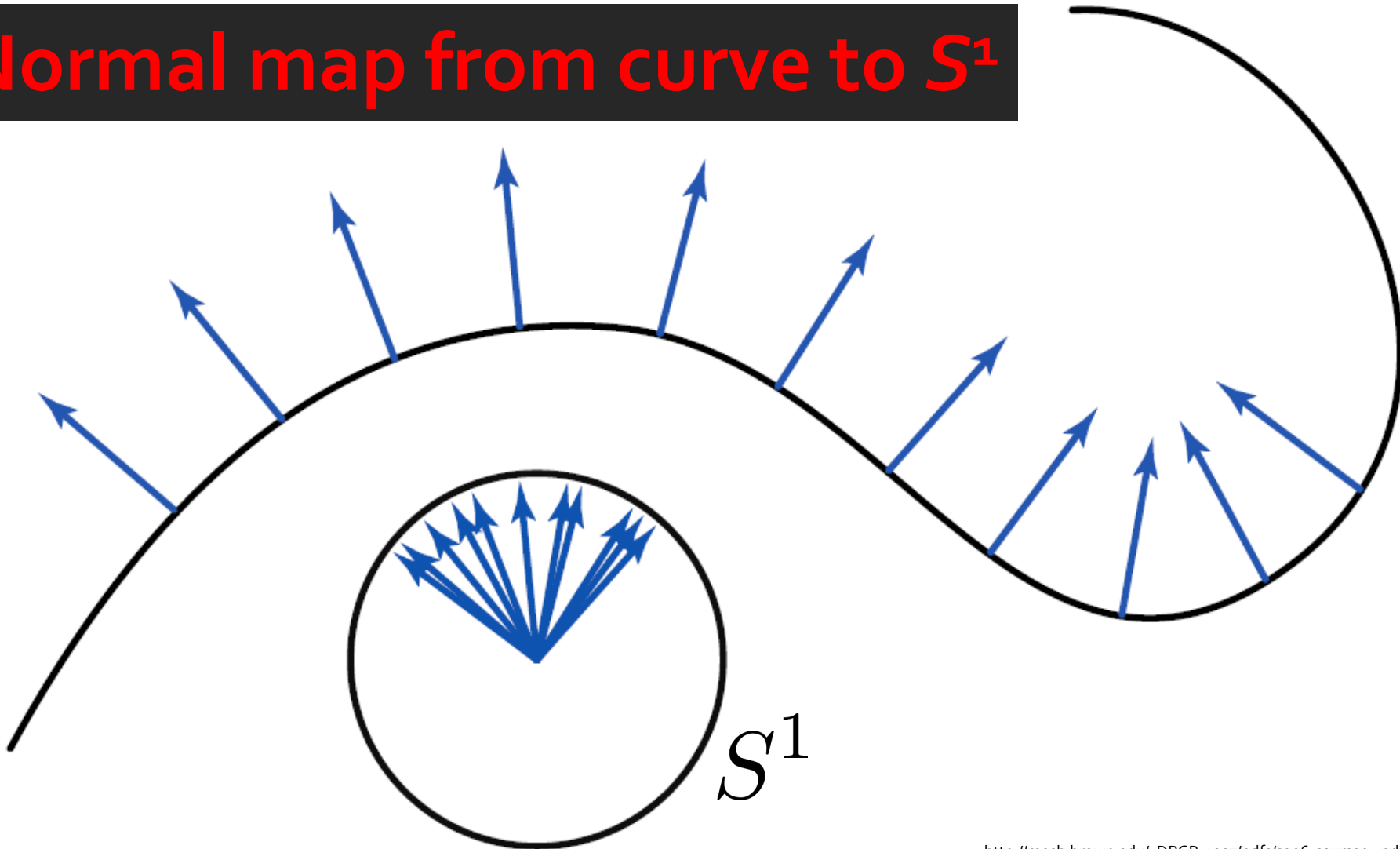
$$\begin{aligned}\mathbf{T}(s) &:= (\cos \theta(s), \sin \theta(s)) \\ \implies \kappa(s) &:= \theta'(s)\end{aligned}$$

Image from DDG course notes by E. Grinspun

Provides intuition for curvature

Gauss Map

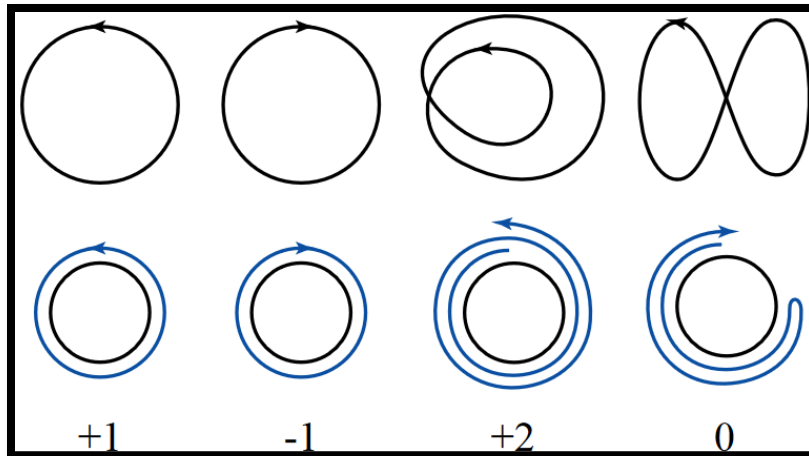
Normal map from curve to S^1



Winding Number

$$W[\gamma] := \frac{1}{2\pi} \int_a^b \kappa(s) ds \in \mathbb{Z}$$

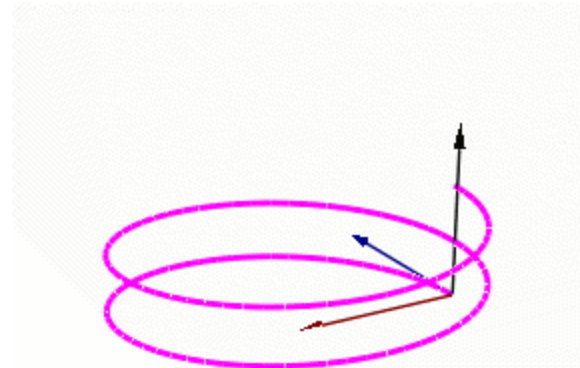
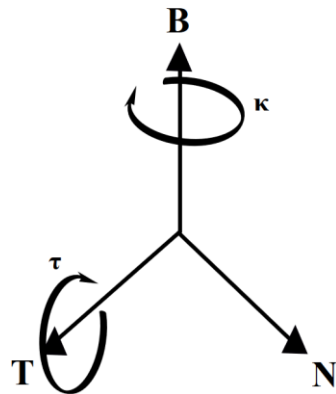
$W[\gamma]$ is an integer, and smoothly deforming γ does not affect $W[\gamma]$.



Frenet Frame: Curves in $\mathbb{R}^3$

$$\frac{d}{ds} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}$$

- **Binormal:** $\mathbf{T} \times \mathbf{N}$
- **Curvature:** In-plane motion
- **Torsion:** Out-of-plane motion



Fundamental theorem of the local theory of space curves:

Curvature and torsion
distinguish a 3D curve up to
rigid motion.

Aside: Generalized Frenet Frame

$$\gamma(s) : \mathbb{R} \rightarrow \mathbb{R}^n$$

$$\frac{d}{ds} \begin{pmatrix} \mathbf{e}_1(s) \\ \mathbf{e}_2(s) \\ \vdots \\ \mathbf{e}_n(s) \end{pmatrix} = \begin{pmatrix} 0 & \chi_1(s) & & \\ -\chi_1(s) & \ddots & \ddots & \\ & \ddots & 0 & \chi_{n-1}(s) \\ & & -\chi_{n-1}(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1(s) \\ \mathbf{e}_2(s) \\ \vdots \\ \mathbf{e}_n(s) \end{pmatrix}$$

Suspicion: Application to time series analysis? ML?

C. Jordan, 1874

Gram-Schmidt on first n derivatives

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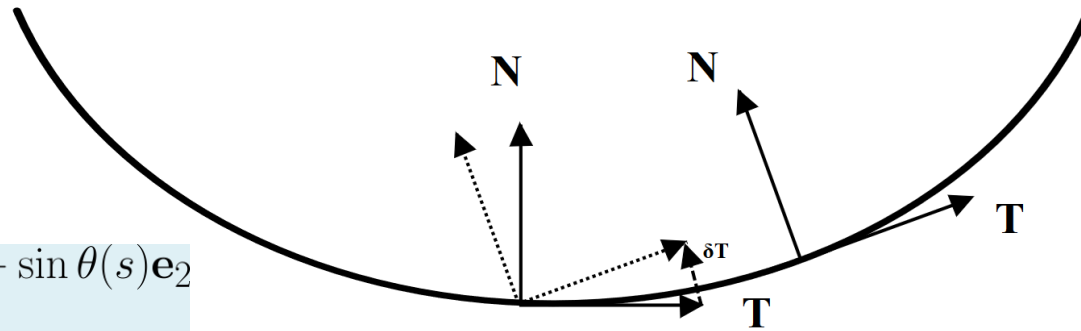


Frenet Frame: Curves in $\mathbb{R}^2$

$$\frac{d}{ds} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \end{pmatrix} := \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \end{pmatrix}$$

Signed curvature κ is rate of change of turning angle θ .

$$\begin{aligned} \mathbf{T}(s) &= \cos \theta(s) \mathbf{e}_1 + \sin \theta(s) \mathbf{e}_2 \\ \kappa(s) &:= \theta'(s) \end{aligned}$$



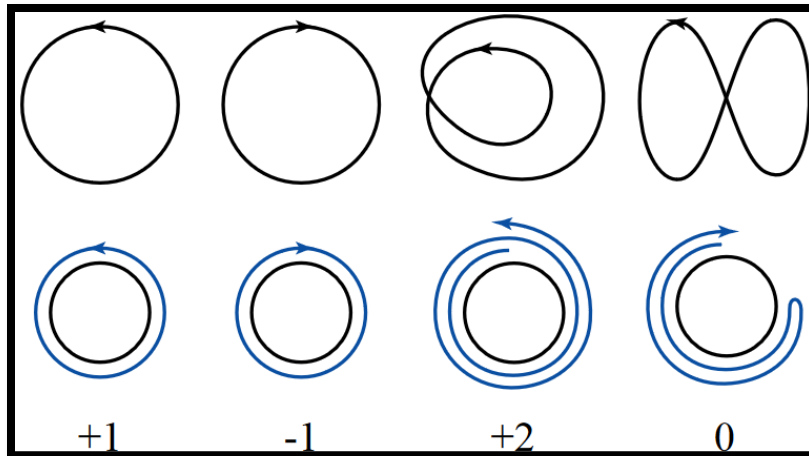
https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

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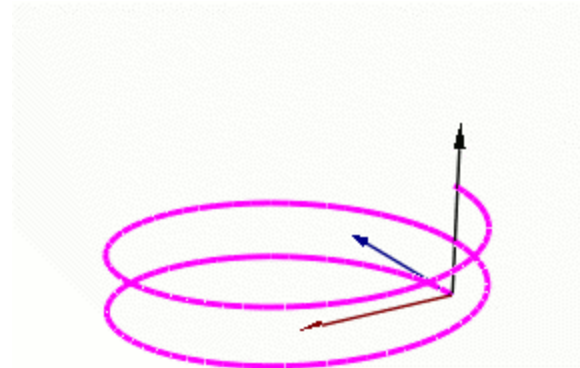
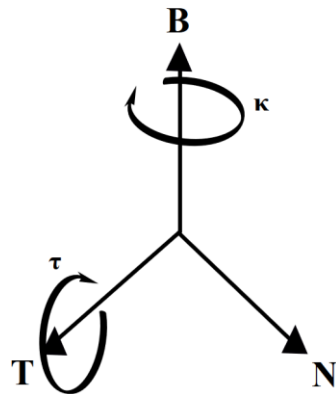
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Frenet Frame: Curves in $\mathbb{R}^3$

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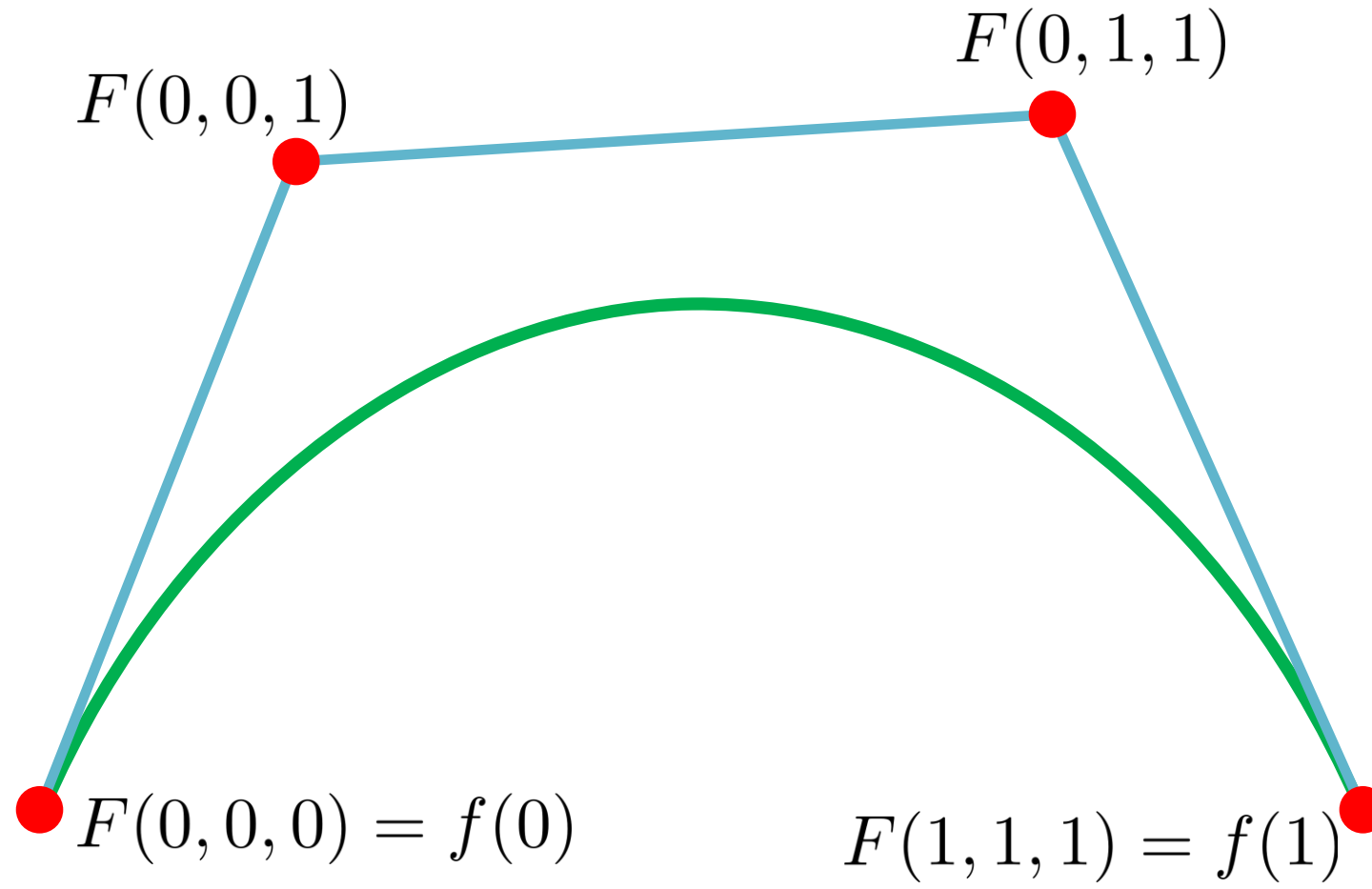
- **Binormal:** $\mathbf{T} \times \mathbf{N}$
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What do these
calculations look like
in software?

Old-School Approach



Piecewise smooth approximations

Question

What is the arc length of a cubic Bézier curve?

$$\int_a^b \|\gamma'(t)\|_2 dt$$

Question

What is the arc length of a cubic Bézier curve?

$$\int_a^b \|\gamma'(t)\|_2 dt$$

Not known in closed form.

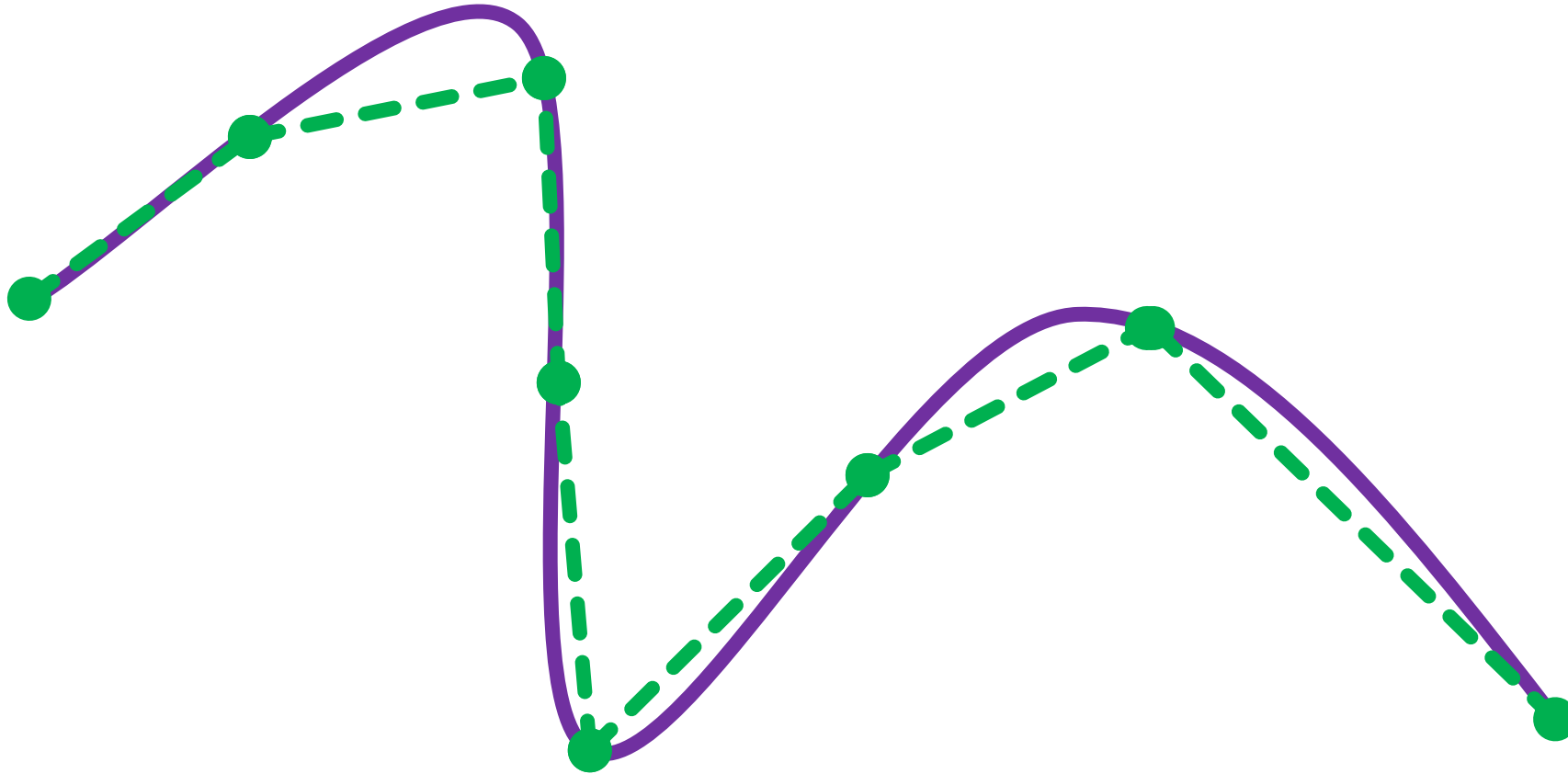
Sad fact:

Closed-form
expressions **rarely exist**.
When they do exist, they
usually are **messy**.

Only Approximations Anyway

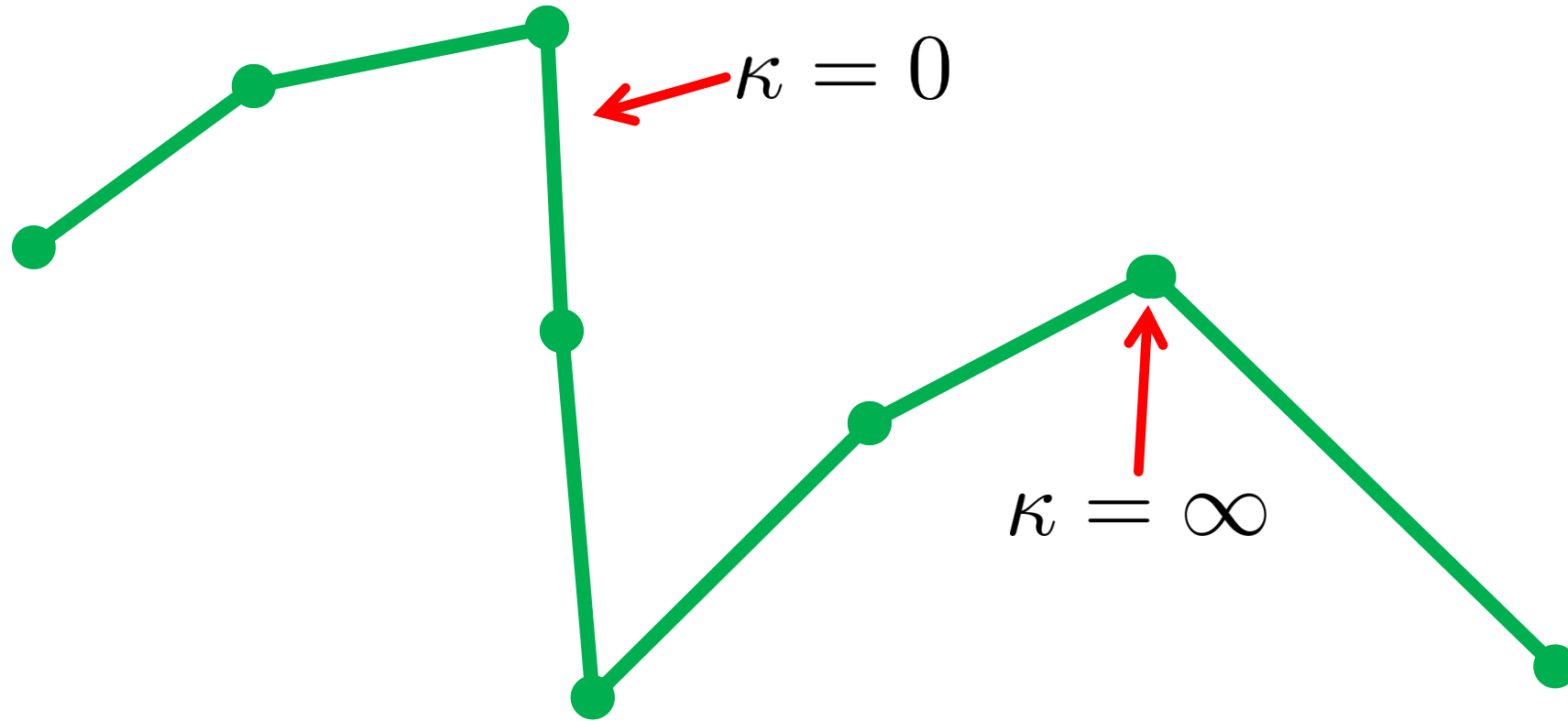
$$\{\text{Bézier curves}\} \subsetneq \{\gamma : \mathbb{R} \rightarrow \mathbb{R}^3\}$$

Simpler Approximation



Piecewise linear: Poly-line

Big Problem



Boring differential structure

Finite Difference Approach

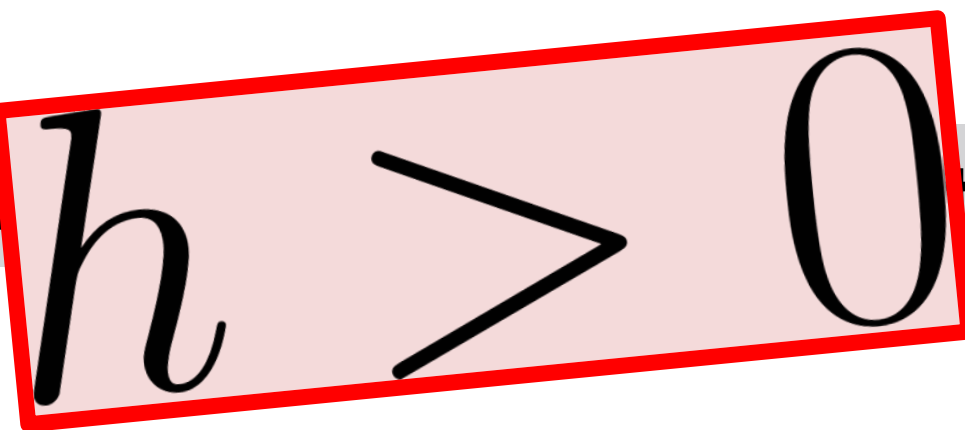
$$f'(x) \approx \frac{1}{h} [f(x + h) - f(x)]$$

THEOREM: As $\Delta h \rightarrow 0$, [insert statement].

Reality Check

$$f'(x) \approx \frac{1}{h} [f(x+h) - f(x)]$$

THEOREM


$$h > 0$$

statement].

Two Key Considerations

- **Convergence** to continuous theory
- Discrete **behavior**

Goal

**Examine discrete theories
of differentiable curves.**

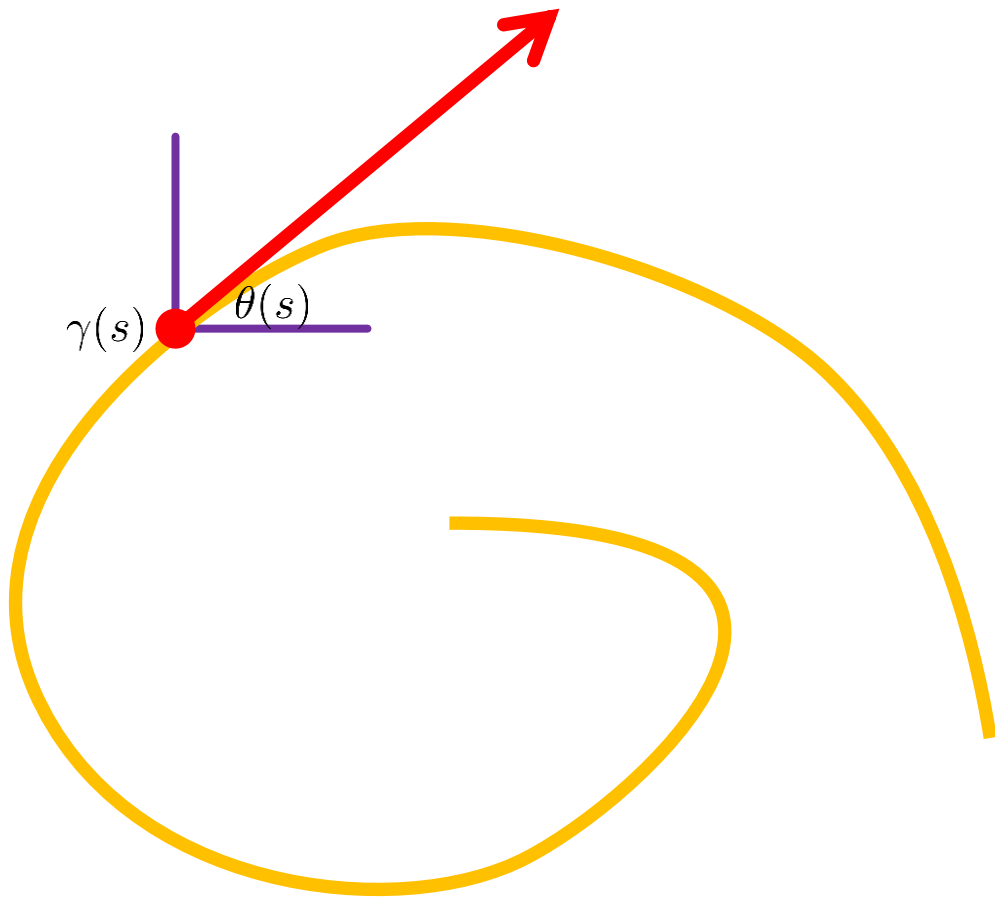
Goal

**Examine discrete theories
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Recall:

Signed Curvature on Plane Curves

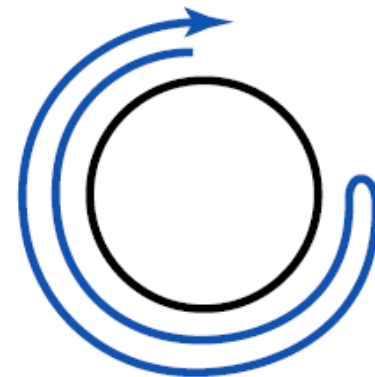
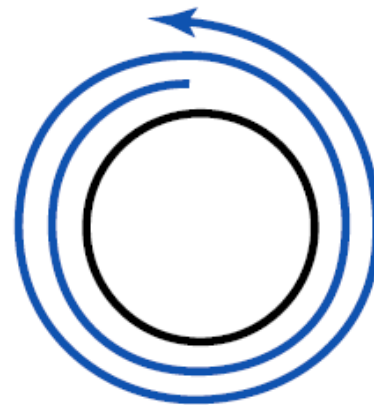
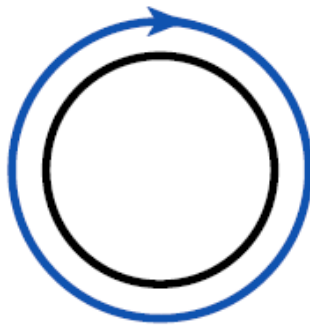
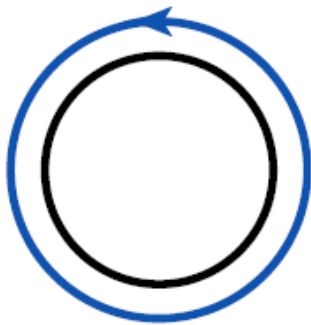
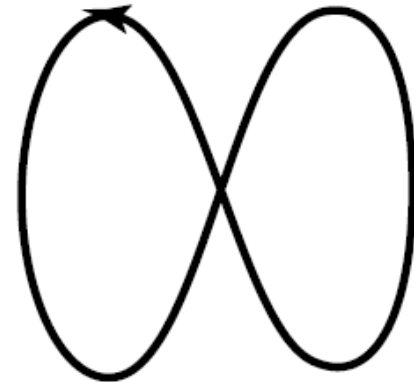
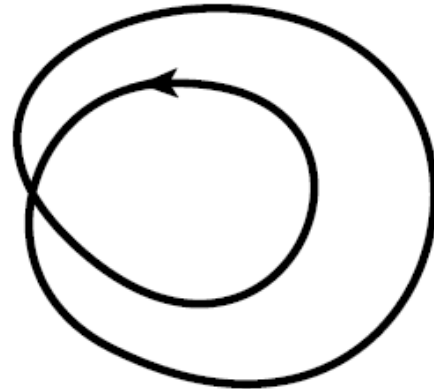
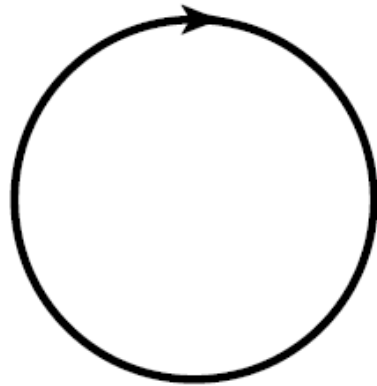
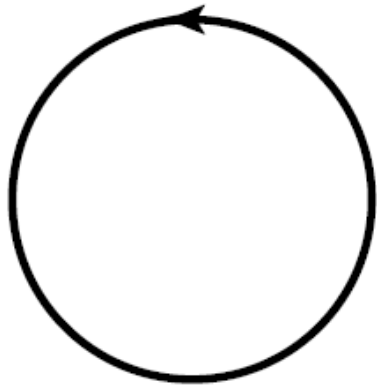
$$\mathbf{T}(s) = (\cos \theta(s), \sin \theta(s))$$



Gauss map:
Map from curve to its normals.

$$\begin{aligned}\mathbf{T}'(s) &= \theta'(s)(-\sin \theta(s), \cos \theta(s)) \\ &:= \kappa(s)\mathbf{N}(s)\end{aligned}$$

Turning Numbers



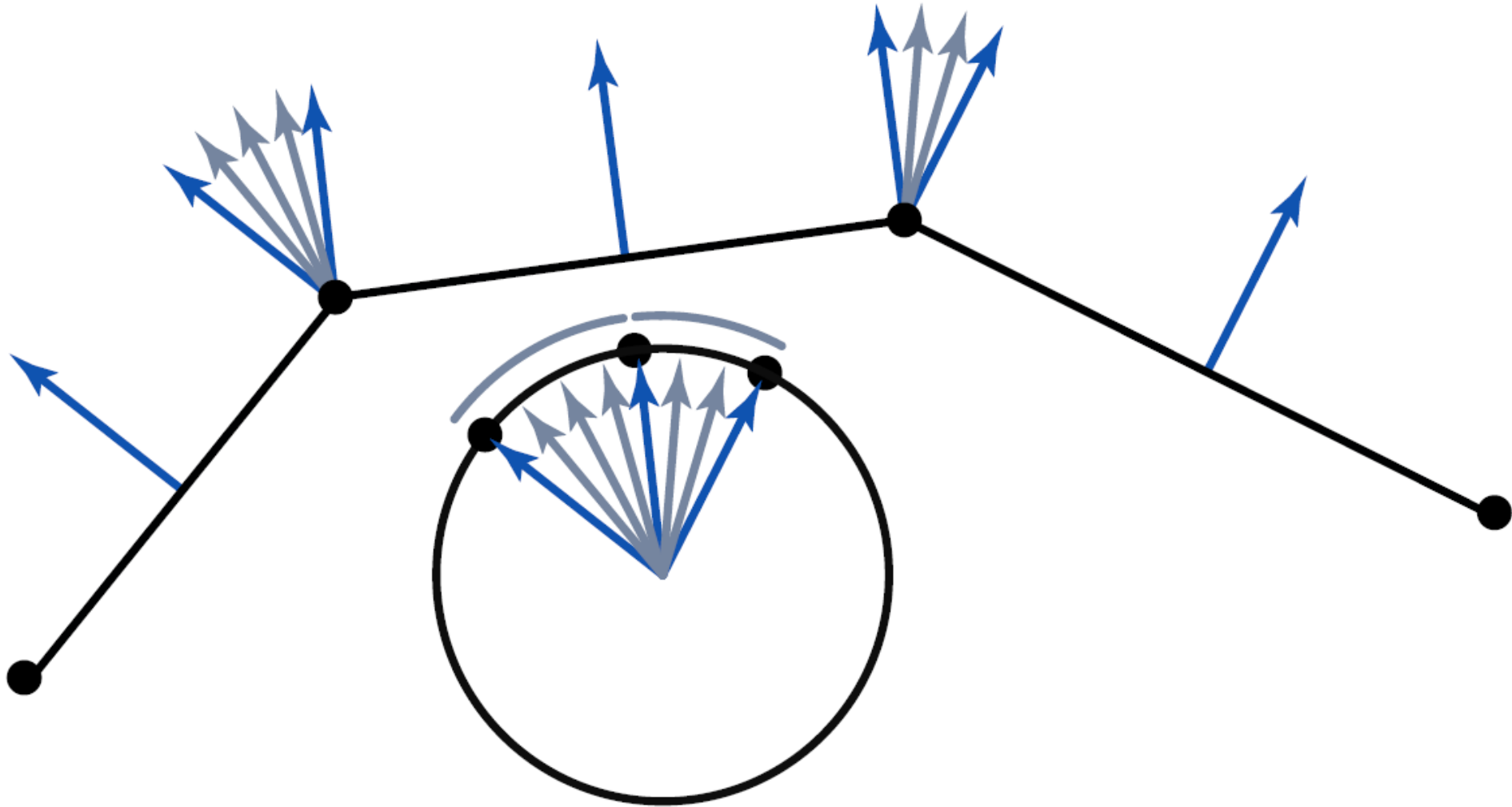
+1

-1

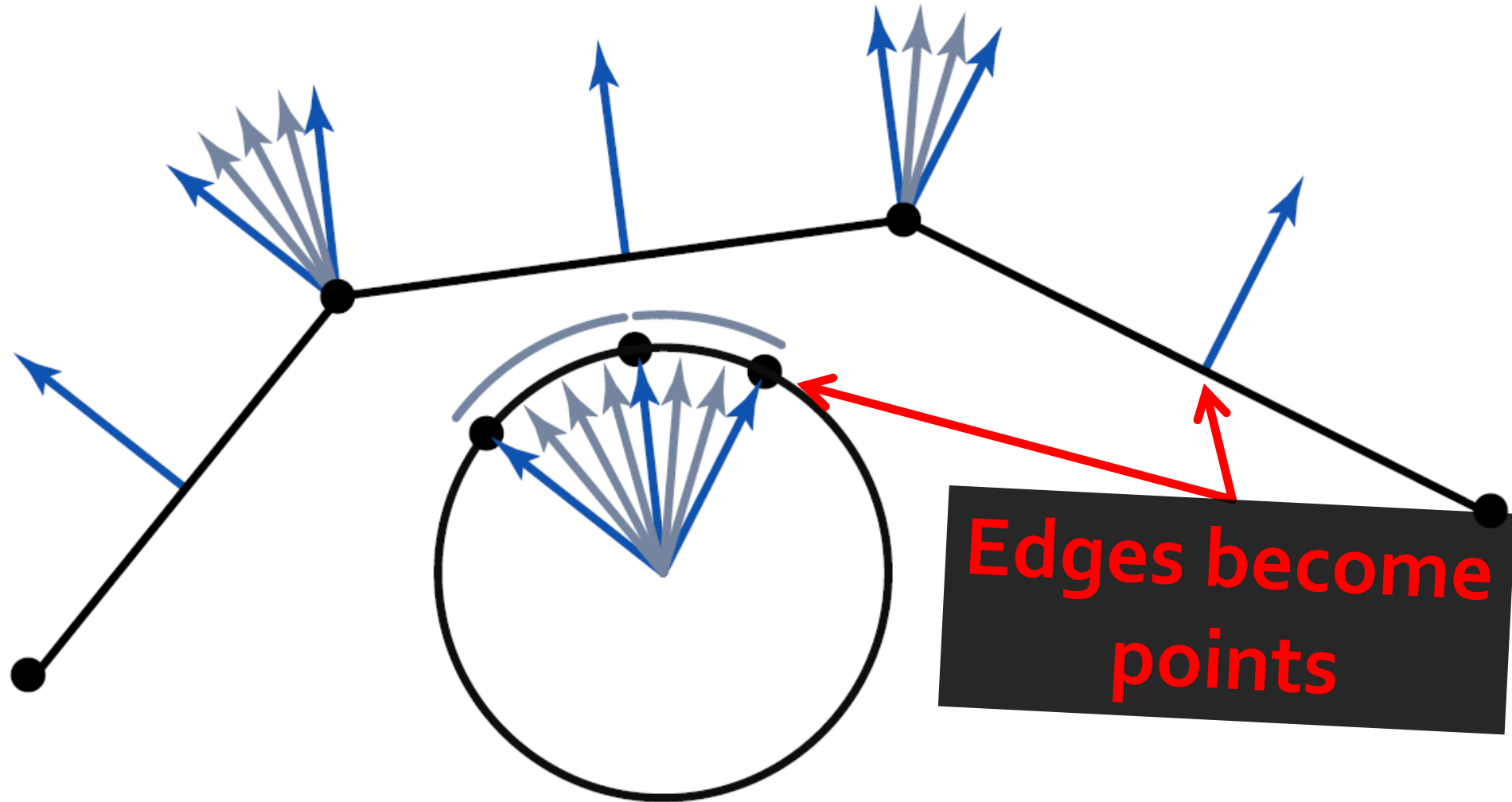
+2

0

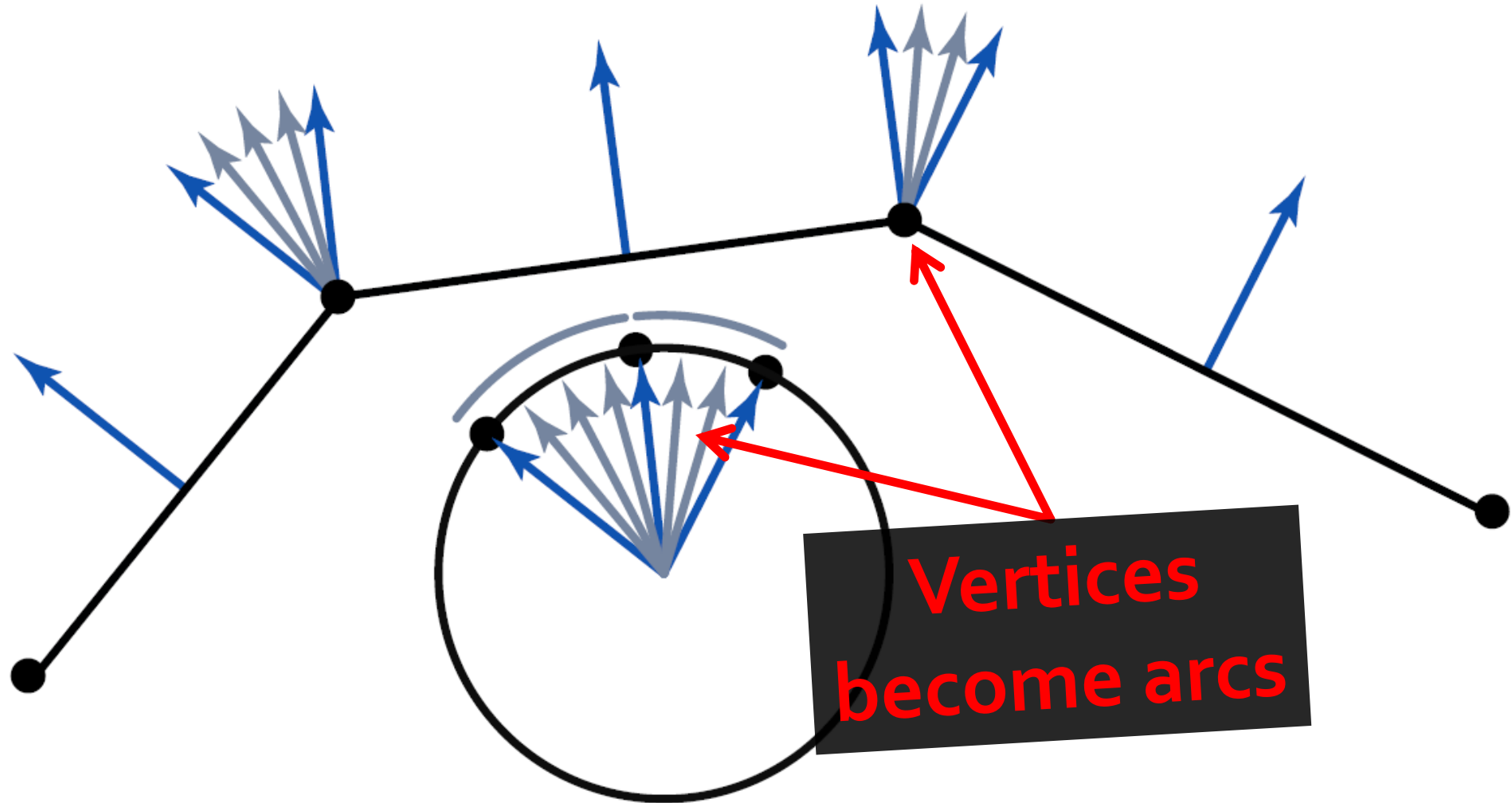
Discrete Gauss Map



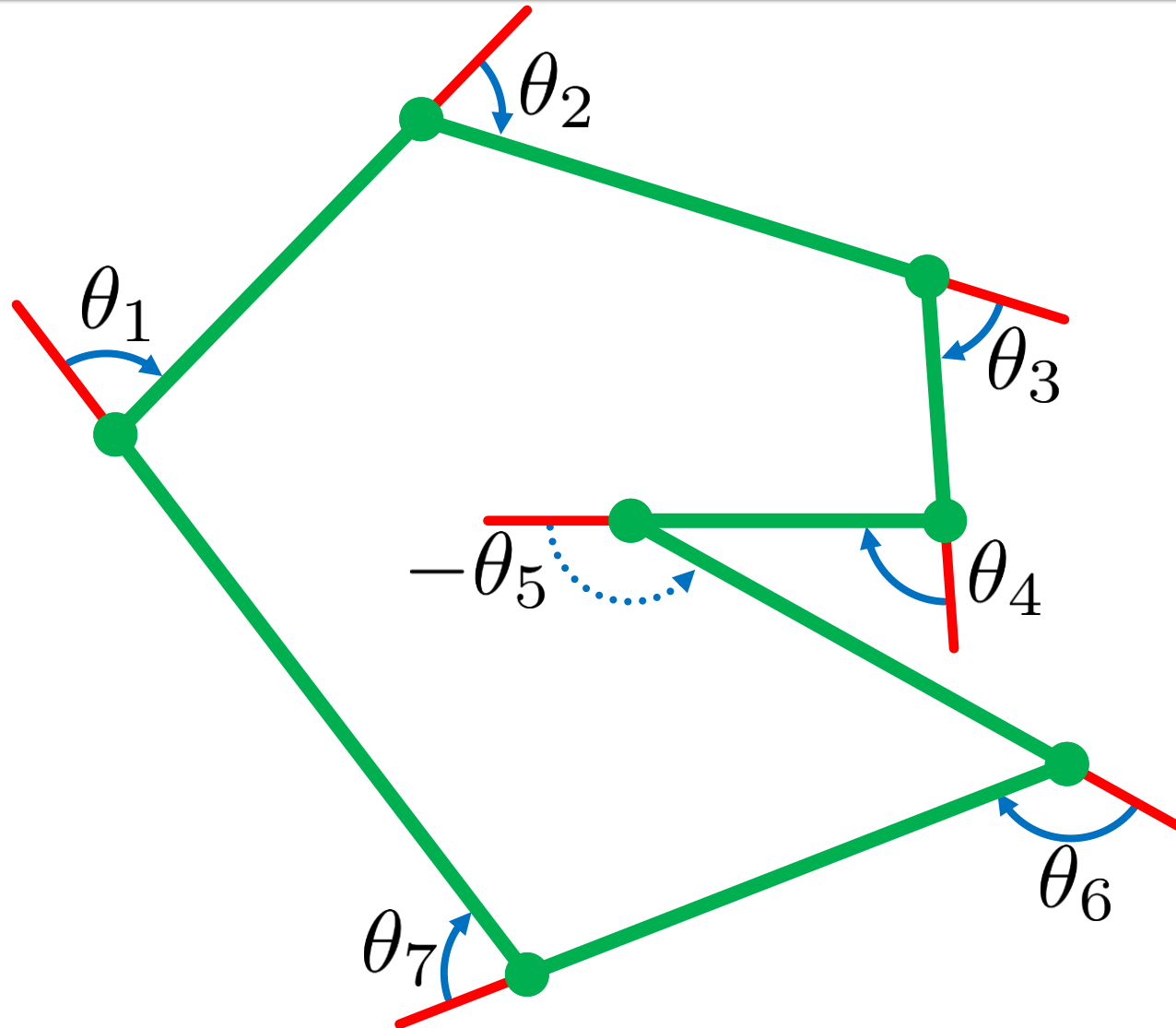
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Discrete Gauss Map



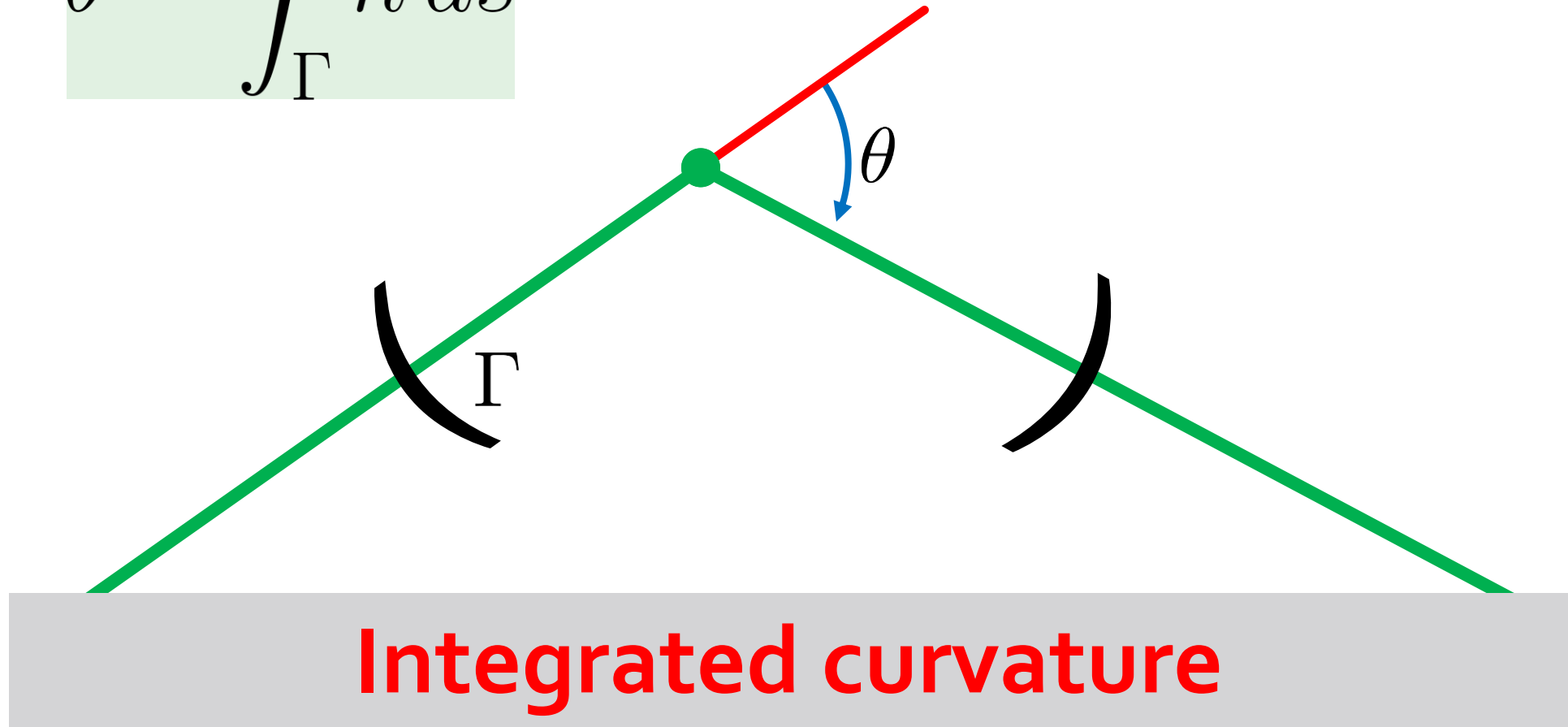
Key Observation



$$\sum_i \theta_i = 2\pi k$$

What's Going On?

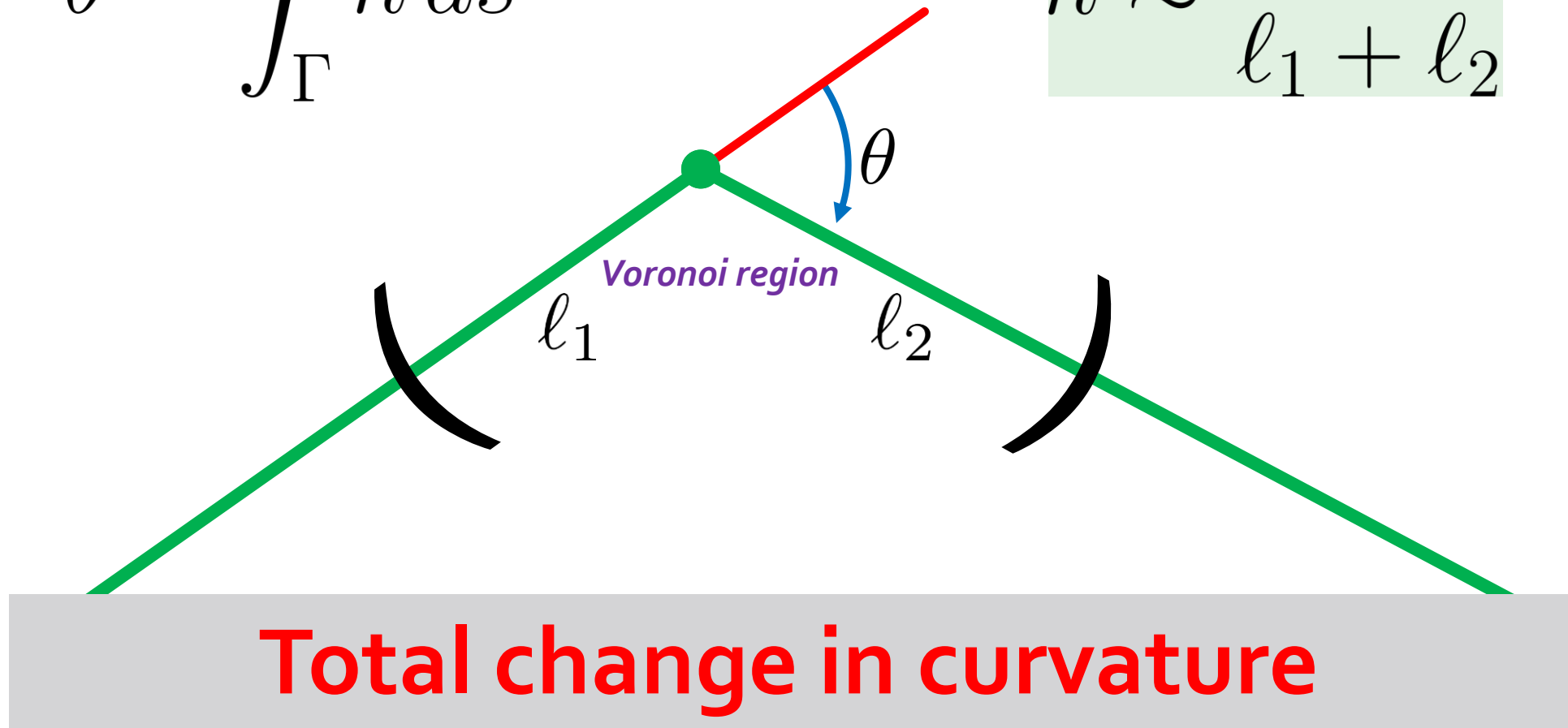
$$\theta = \int_{\Gamma} \kappa ds$$



What's Going On?

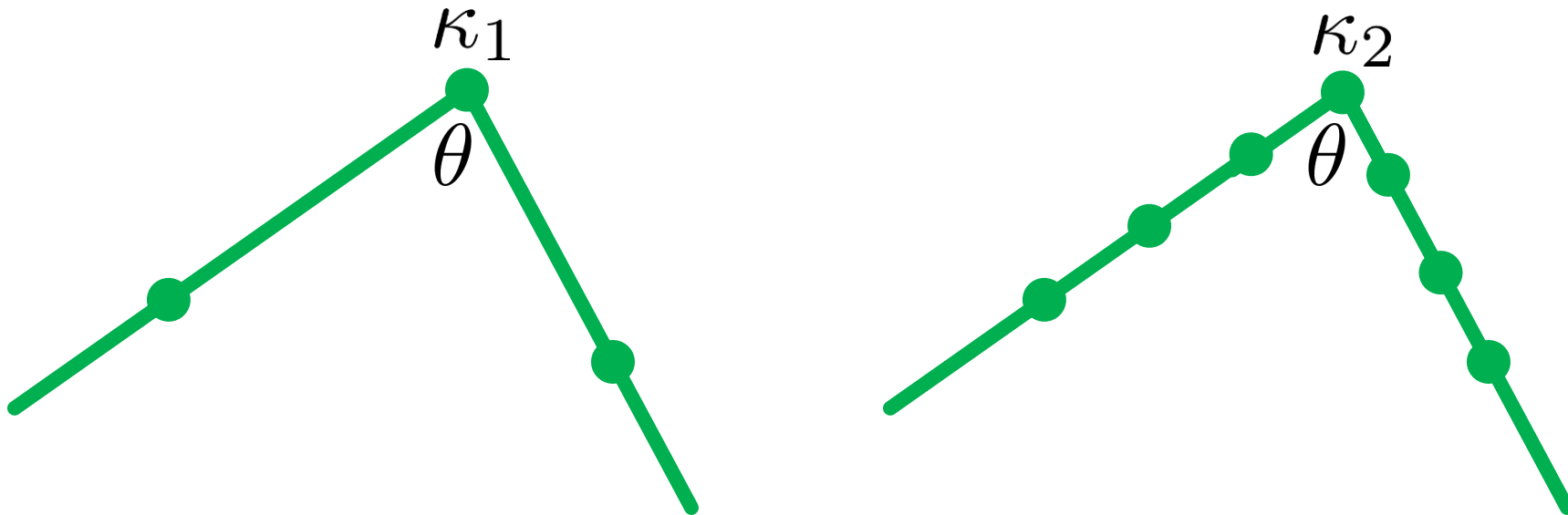
$$\theta = \int_{\Gamma} \kappa \, ds$$

$$\kappa \approx \frac{\theta}{\ell_1 + \ell_2}$$



Interesting Distinction

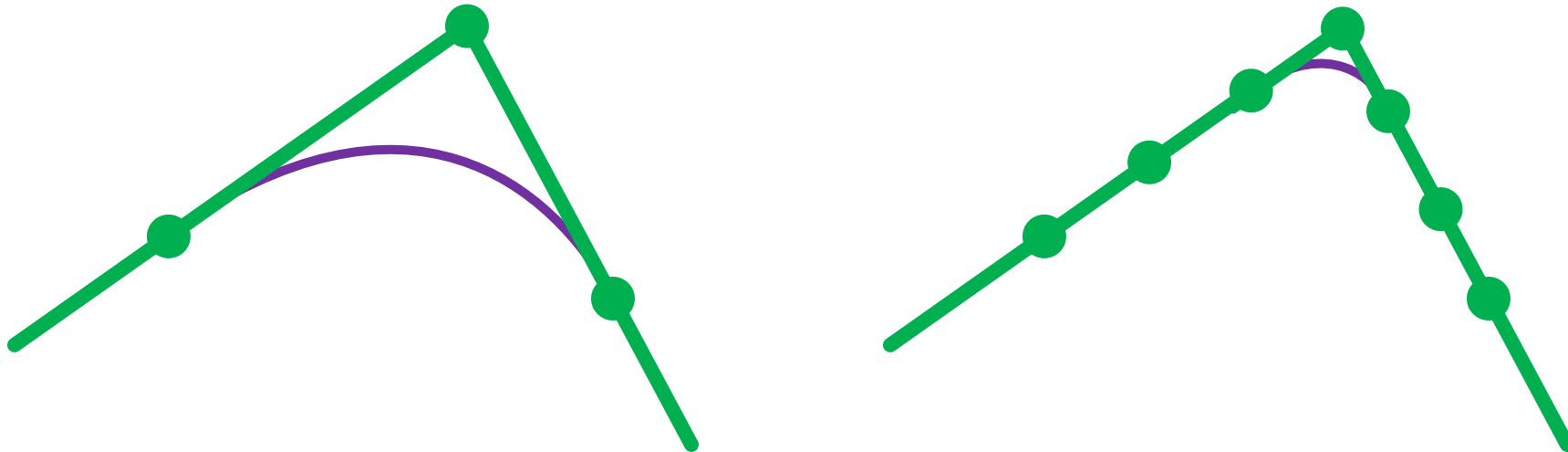
$$\kappa_1 \neq \kappa_2$$



Same integrated curvature

Interesting Distinction

$$\kappa_1 \neq \kappa_2$$

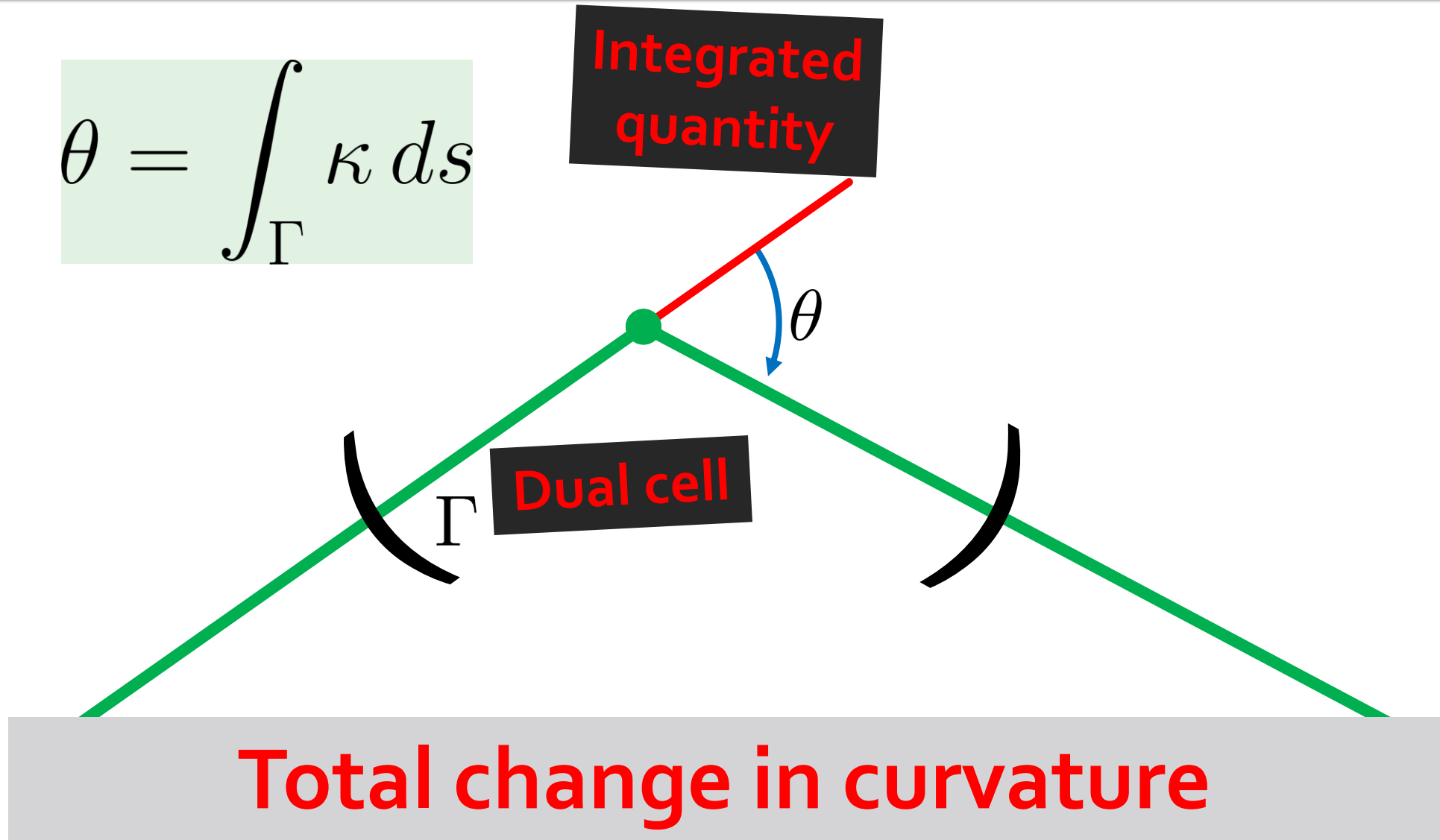


Same integrated curvature

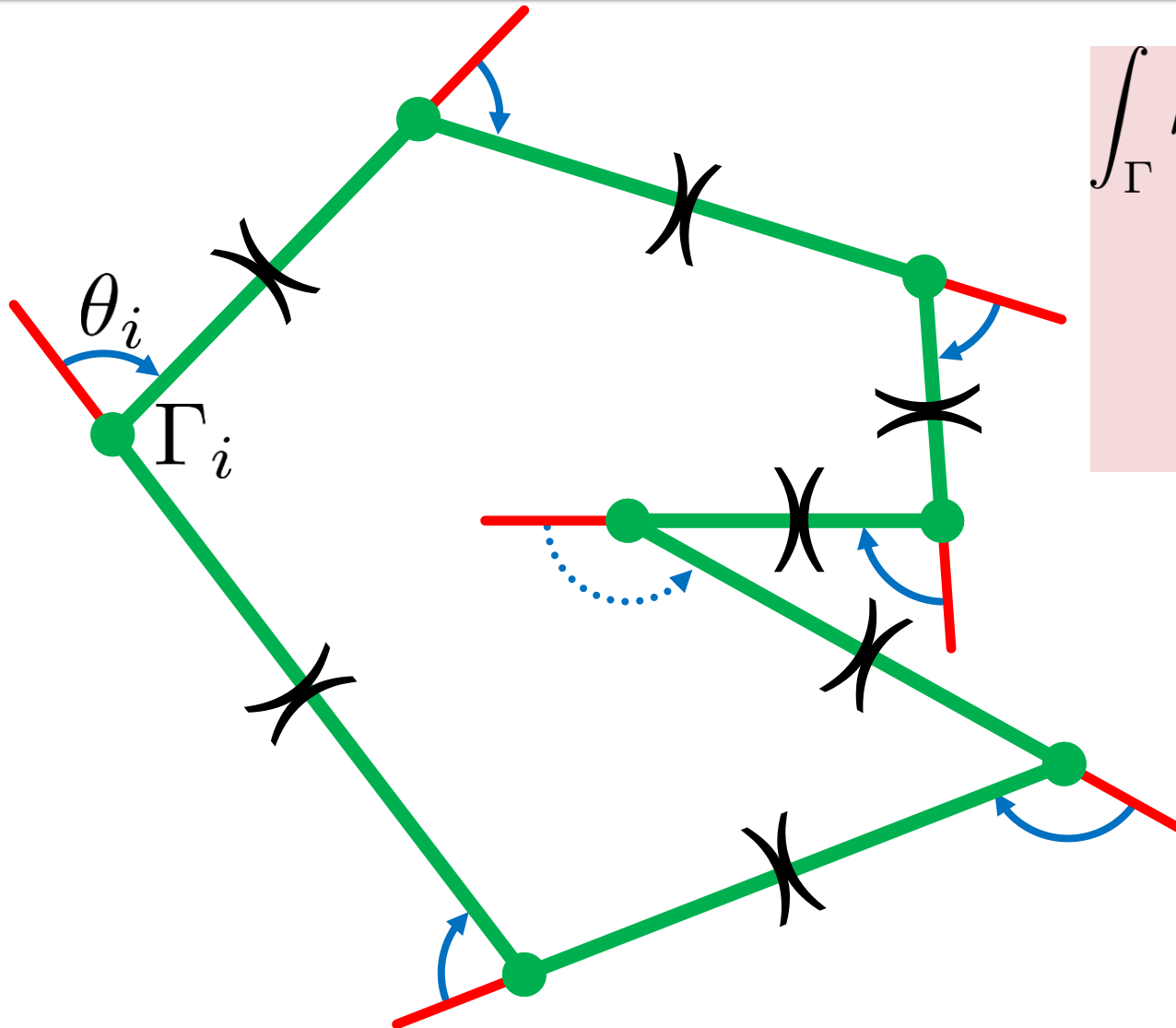
What's Going On?

$$\theta = \int_{\Gamma} \kappa ds$$

Integrated
quantity



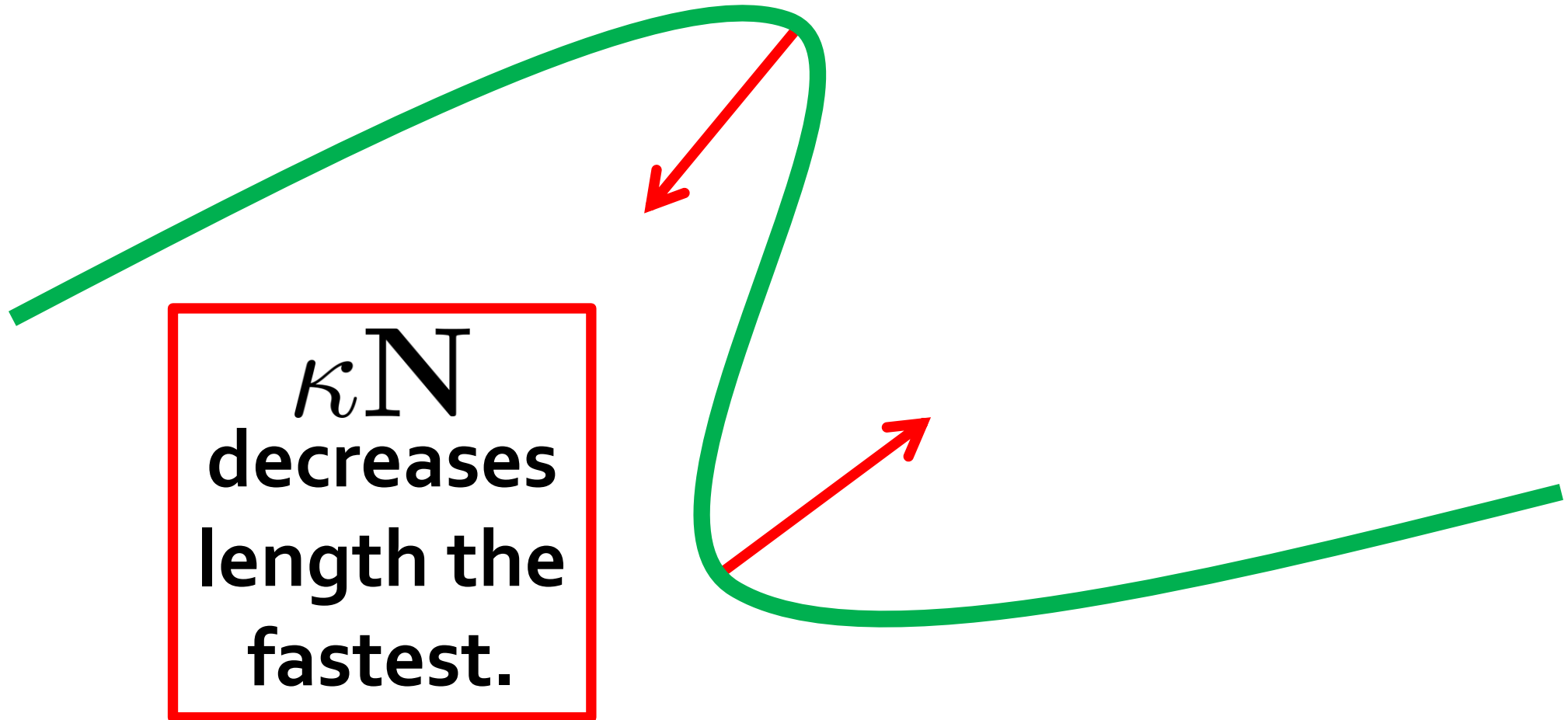
Discrete Turning Angle Theorem



$$\begin{aligned}\int_{\Gamma} \kappa ds &= \sum_i \int_{\Gamma_i} \kappa ds \\ &= \sum_i \theta_i \\ &= 2\pi k\end{aligned}$$

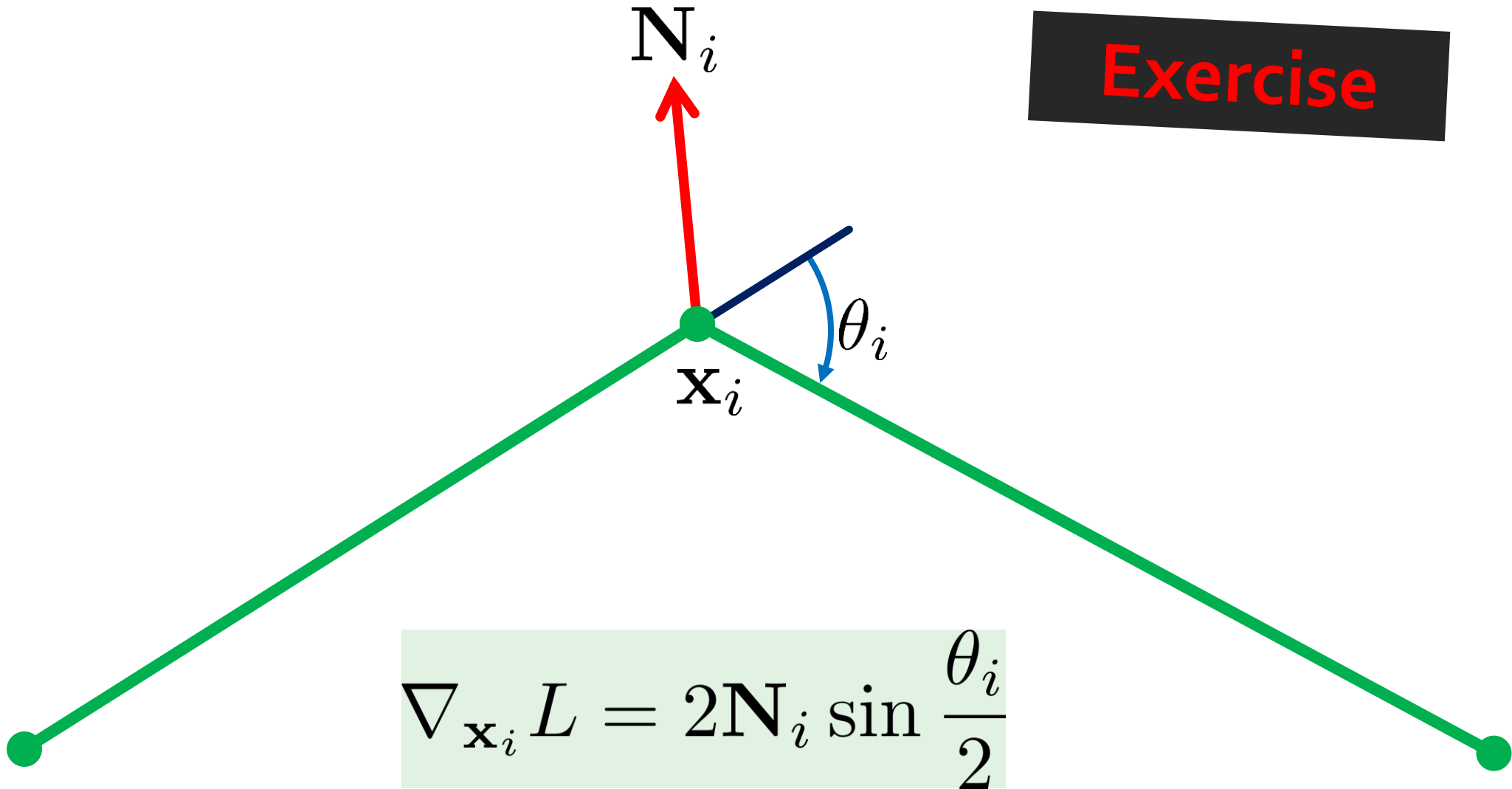
**Preserved
structure!**

First Variation Formula

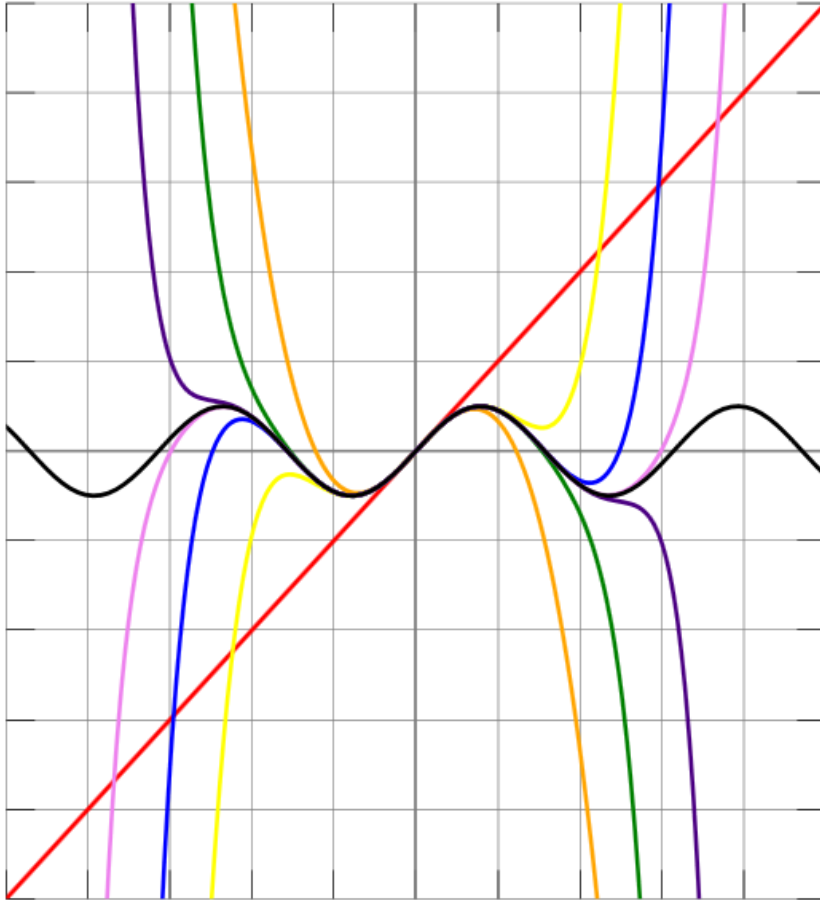


Discrete Case

Exercise



For Small θ



http://en.wikipedia.org/wiki/Taylor_series

$$2 \sin \frac{\theta}{2} \approx 2 \cdot \frac{\theta}{2} \\ = \theta$$

Same behavior in the limit

No Free Lunch

Choose one:

- Discrete curvature with
turning angle theorem
- Discrete curvature from
gradient of arc length



Remaining Question

**Does discrete curvature
converge in limit?**

Yes!
Under some assumptions!

Remaining Question

**Does discrete curvature
converge in limit?**

Questions:

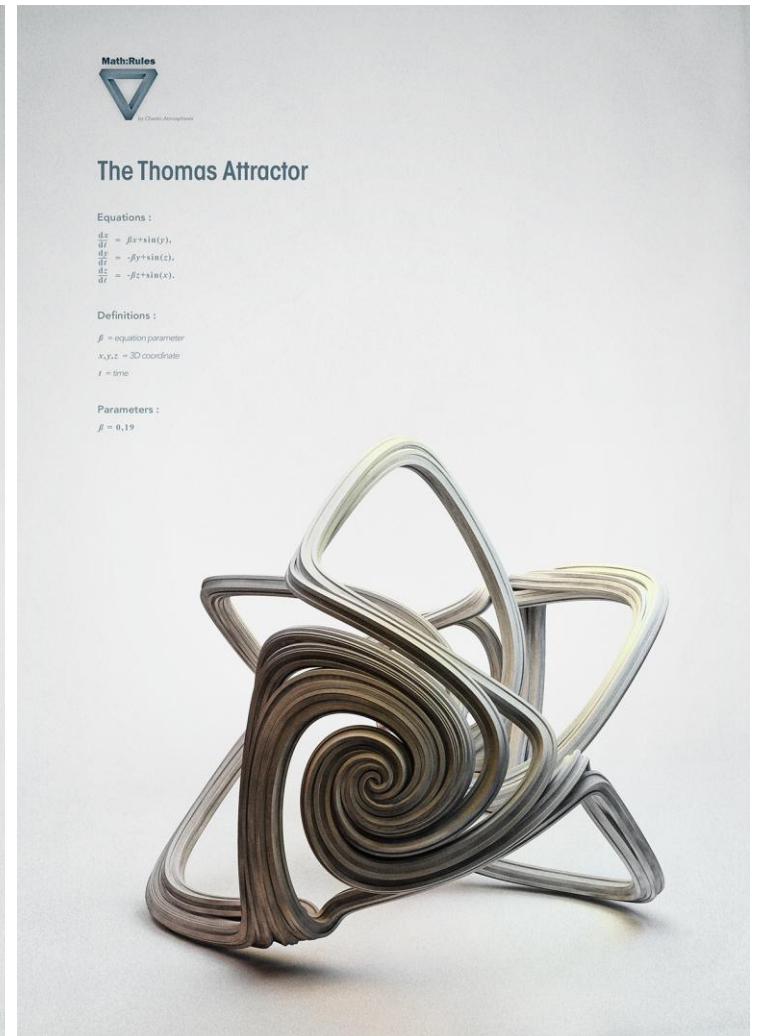
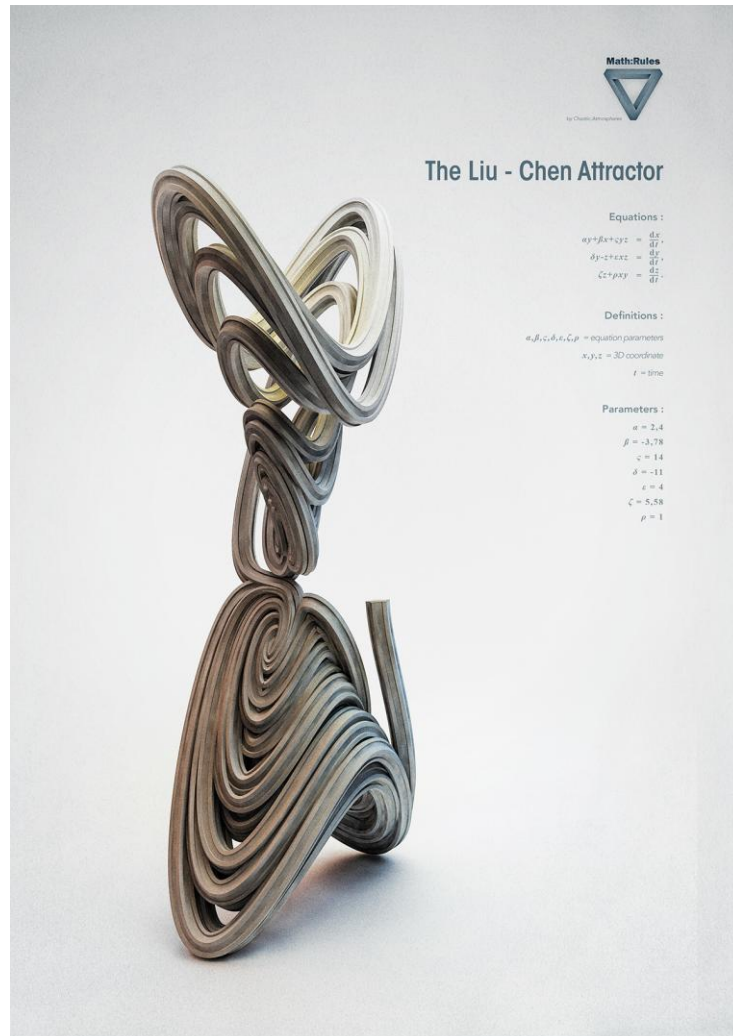
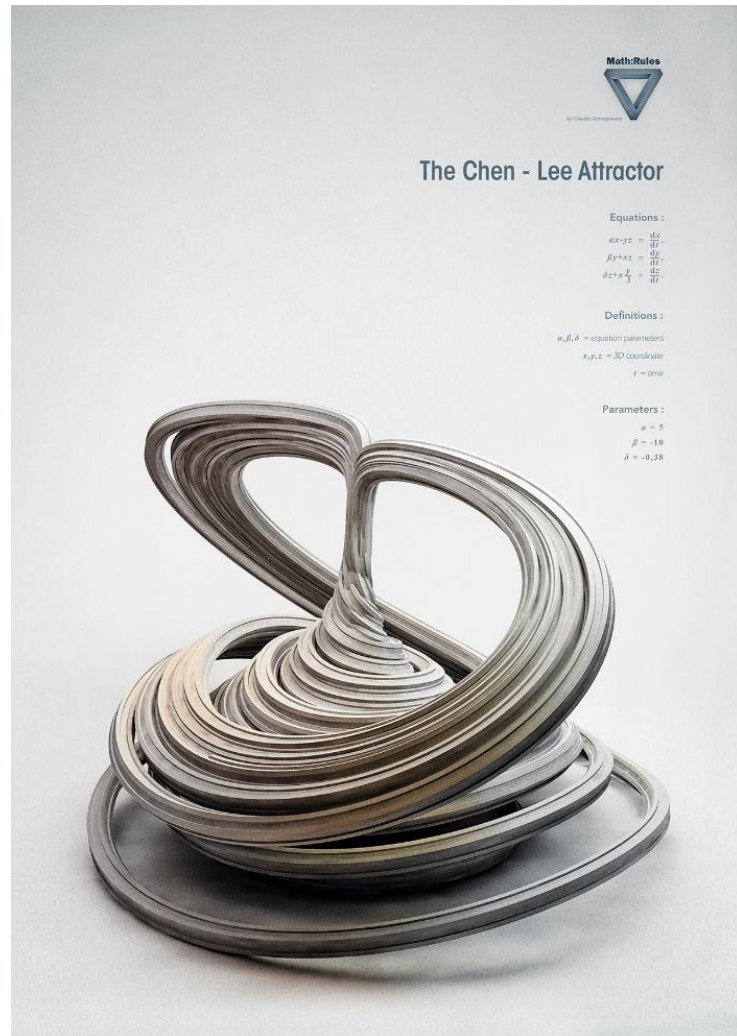
- Type of convergence?
- Sampling?
- Class of curves?

Yes!
Under some assumptions!

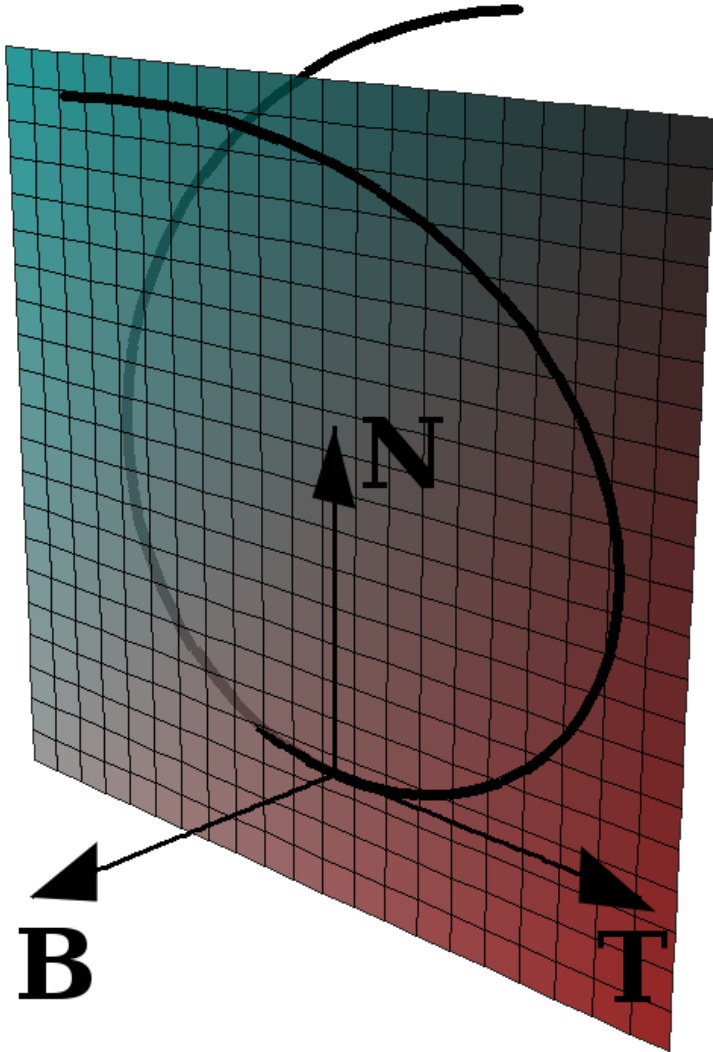
Discrete Differential Geometry

- **Different** discrete behavior
- **Same** convergence

Curves in 3D?



Frenet Frame

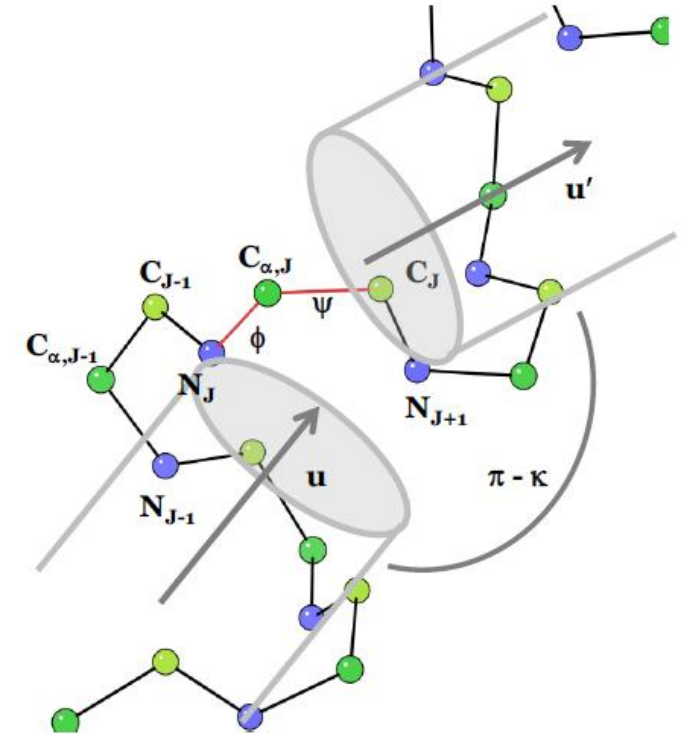
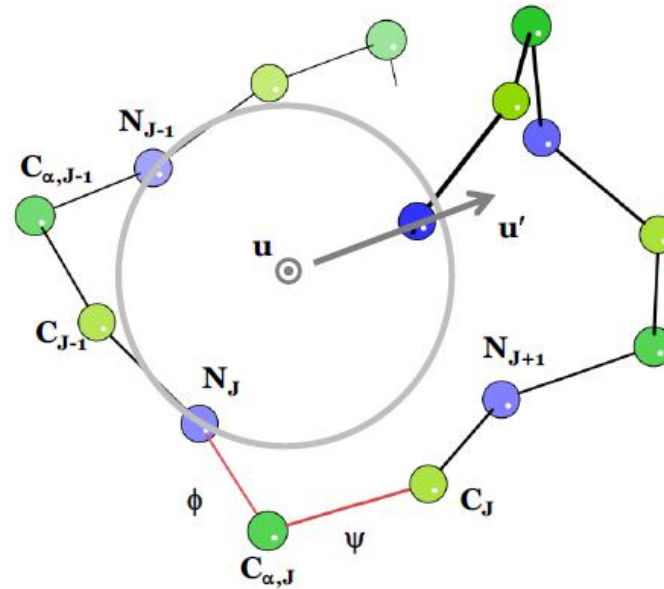


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Application



NMR scanner



Kinked alpha helix

Structure Determination of Membrane Proteins Using Discrete Frenet Frame and Solid State NMR Restraints

Achuthan and Quine

Discrete Mathematics and its Applications, ed. M. Sethumadhavan (2006)

Potential Discretization

$$\begin{aligned}\mathbf{T}_j &= \frac{\mathbf{p}_{j+1} - \mathbf{p}_j}{\|\mathbf{p}_{j+1} - \mathbf{p}_j\|_2} \\ \mathbf{B}_j &= \mathbf{T}_{j-1} \times \mathbf{T}_j \\ \mathbf{N}_j &= \mathbf{B}_j \times \mathbf{T}_j\end{aligned}$$

Discrete Frenet frame

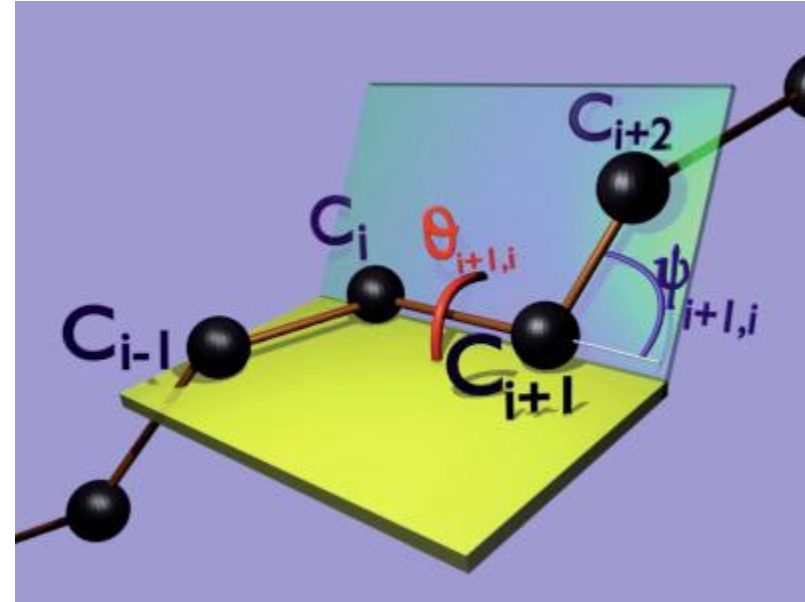
$$\begin{aligned}\mathbf{T}_k &= R(\mathbf{B}_k, \theta_k) \mathbf{T}_{k-1} \\ \mathbf{B}_{k+1} &= R(\mathbf{T}_k, \phi_k) \mathbf{B}_k\end{aligned}$$

“Bond and torsion angles”
(derivatives converge to κ
and τ , resp.)

Discrete frame introduced in:
The resultant electric moment of complex molecules
Eyring, Physical Review, 39(4):746—748, 1932.

Transfer Matrix

$$\begin{pmatrix} \mathbf{T}_{i+1} \\ \mathbf{N}_{i+1} \\ \mathbf{B}_{i+1} \end{pmatrix} = R_{i+1,i} \begin{pmatrix} \mathbf{T}_i \\ \mathbf{N}_i \\ \mathbf{B}_i \end{pmatrix}$$



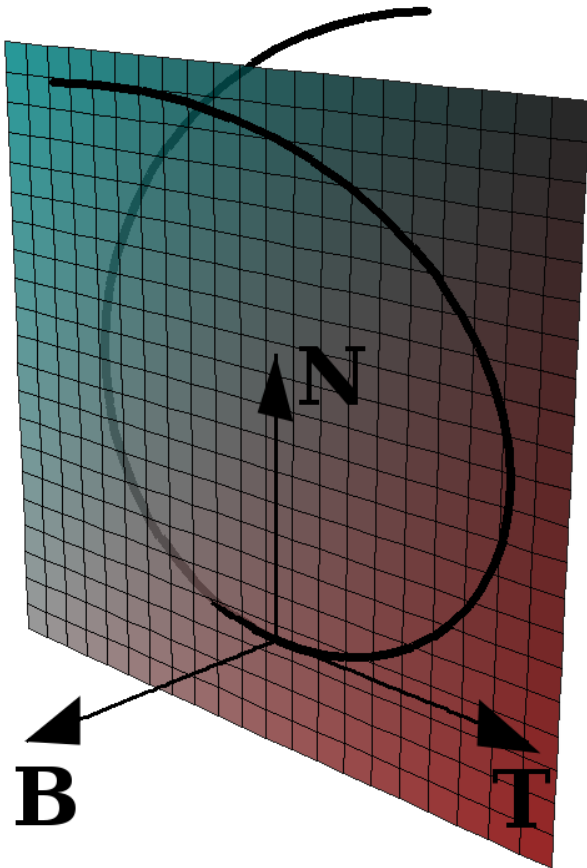
**Discrete construction that works for fractal curves
and converges in continuum limit.**

Discrete Frenet Frame, Inflection Point Solitons, and Curve Visualization
with Applications to Folded Proteins

Hu, Lundgren, and Niemi
Physical Review E 83 (2011)

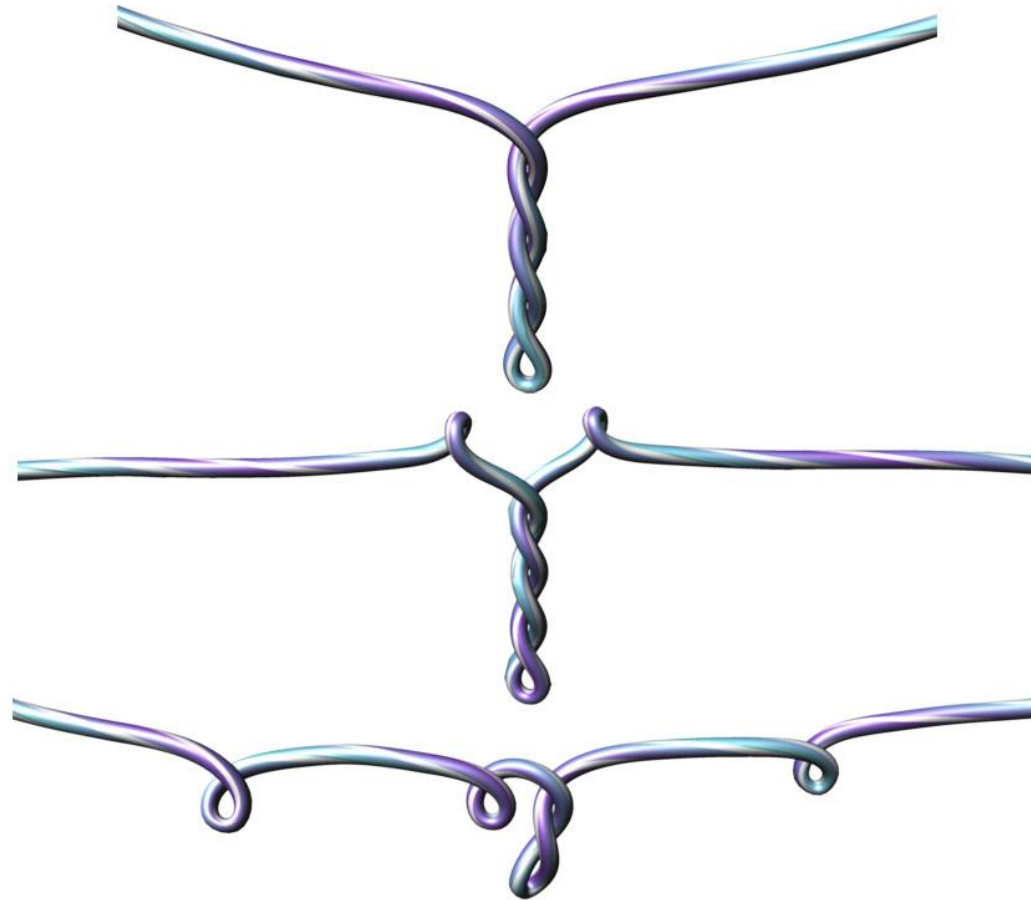
Frenet Frame: Issue

$\kappa = 0?$



$$\frac{d}{ds} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & \boxed{\kappa(s)} & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}$$

Segments Not Always Enough

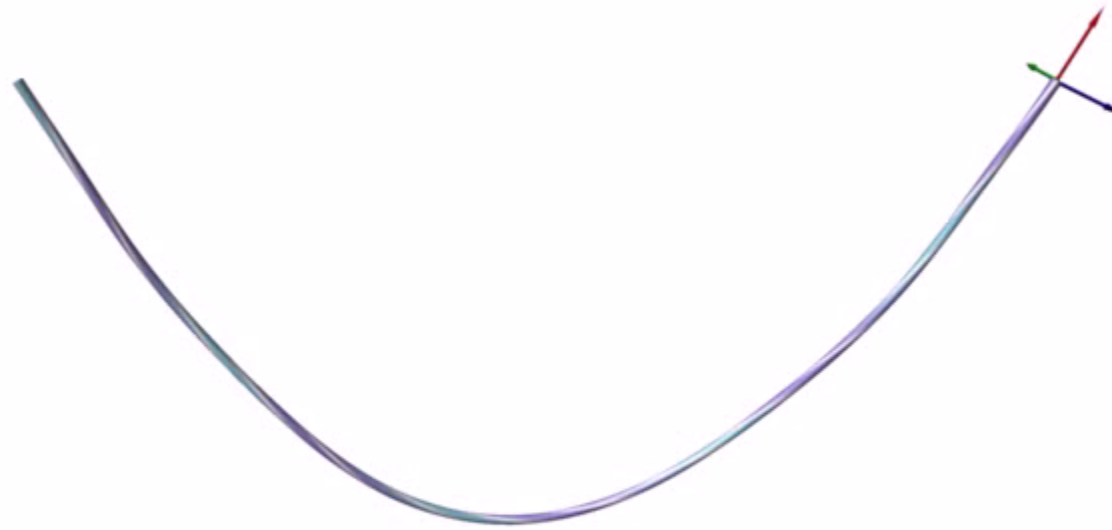


Discrete Elastic Rods

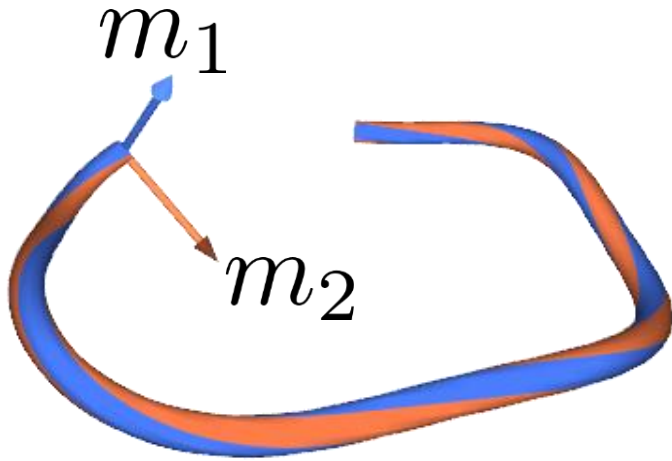
Bergou, Wardetzky, Robinson, Audoly, and Grinspun

SIGGRAPH 2008

Simulation Goal



Adapted Framed Curve



$$\Gamma = \{\gamma(s); \mathbf{T}, \mathbf{m}_1, \mathbf{m}_2\}$$

Material frame

Normal part encodes twist

Bending Energy

$$E_{\text{bend}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \alpha \kappa^2 ds$$

Penalize turning the steering wheel

$$\begin{aligned} \kappa \mathbf{N} &= \mathbf{T}' \\ &= (\mathbf{T}' \cdot \mathbf{T})\mathbf{T} + (\mathbf{T}' \cdot \mathbf{m}_1)\mathbf{m}_1 + (\mathbf{T}' \cdot \mathbf{m}_2)\mathbf{m}_2 \\ &= (\mathbf{T}' \cdot \mathbf{m}_1)\mathbf{m}_1 + (\mathbf{T}' \cdot \mathbf{m}_2)\mathbf{m}_2 \\ &:= \omega_1 \mathbf{m}_1 + \omega_2 \mathbf{m}_2 \end{aligned}$$

Bending Energy

$$E_{\text{bend}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \alpha(\omega_1^2 + \omega_2^2) ds$$

Penalize turning the steering wheel

$$\begin{aligned} \kappa \mathbf{N} &= \mathbf{T}' \\ &= (\mathbf{T}' \cdot \mathbf{T})\mathbf{T} + (\mathbf{T}' \cdot \mathbf{m}_1)\mathbf{m}_1 + (\mathbf{T}' \cdot \mathbf{m}_2)\mathbf{m}_2 \\ &= (\mathbf{T}' \cdot \mathbf{m}_1)\mathbf{m}_1 + (\mathbf{T}' \cdot \mathbf{m}_2)\mathbf{m}_2 \\ &:= \omega_1 \mathbf{m}_1 + \omega_2 \mathbf{m}_2 \end{aligned}$$

Twisting Energy

$$E_{\text{twist}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \beta m^2 ds$$

Penalize non-tangent change in material frame

$$m := \mathbf{m}'_1 \cdot \mathbf{m}_2$$

$$= \frac{d}{dt} (\mathbf{m}_1 \cdot \mathbf{m}_2) - \mathbf{m}_1 \cdot \mathbf{m}'_2$$

$$= -\mathbf{m}_1 \cdot \mathbf{m}'_2$$

← Swapping m_1 and m_2
does not affect E_{twist} !

Bishop Frame: The Hipster Framed Curve

THERE IS MORE THAN ONE WAY TO FRAME A CURVE

RICHARD L. BISHOP

The Frenet frame of a 3-times continuously differentiable (that is, C^3) non-degenerate curve in euclidean space has long been the standard vehicle for analysing properties of the curve invariant under euclidean motions. For arbitrary moving frames, that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the curve parameter in terms of the frame itself, and due to orthonormality the coefficient matrix is always skew-symmetric. Thus it generally has three nonzero entries. The Frenet frame gains part of its special significance from the fact that one of the three derivatives is always zero. Another feature of the Frenet frame is that it is *adapted* to the curve: the members are either tangent to or perpendicular to the curve. It is the purpose of this paper to show that there are other frames which have these same advantages and to compare them with the Frenet frame.



Relatively parallel fields. We say that a normal vector field M along a curve is *relatively parallel* if its derivative is tangential. Such a field turns only whatever is necessary for it to remain normal, so it is as close to being parallel as possible without losing normality. Since its derivative is perpendicular to it, a relatively parallel normal field (couldn't decide on a meme) fields occur classically in



Bishop Frame

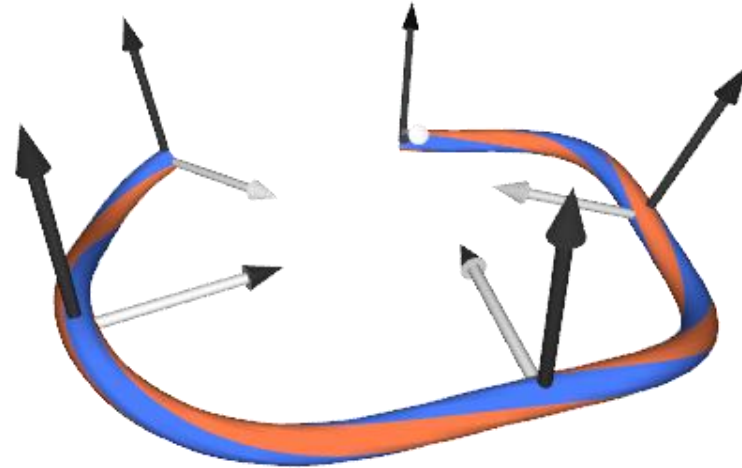
$$\mathbf{T}' = \boldsymbol{\Omega} \times \mathbf{T}$$

$$\mathbf{u}' = \boldsymbol{\Omega} \times \mathbf{u}$$

$$\mathbf{v}' = \boldsymbol{\Omega} \times \mathbf{v}$$

$$\boldsymbol{\Omega} := \kappa \mathbf{B} \text{ (“curvature binormal”)}$$

Darboux vector



Most relaxed frame

Bishop Frame

$$\mathbf{T}' = \Omega \times \mathbf{T}$$

$$\mathbf{u}' = \Omega \times \mathbf{u}$$

$$\mathbf{v}' = \Omega \times \mathbf{v}$$

$$\mathbf{u}' \cdot \mathbf{v} \equiv 0$$

No twist
("parallel transport")

$$\Omega := \kappa \mathbf{B} \text{ ("curvature binormal")}$$

Darboux vector

Most relaxed frame

Curve-Angle Representation

$$\mathbf{m}_1 = \mathbf{u} \cos \theta + \mathbf{v} \sin \theta$$

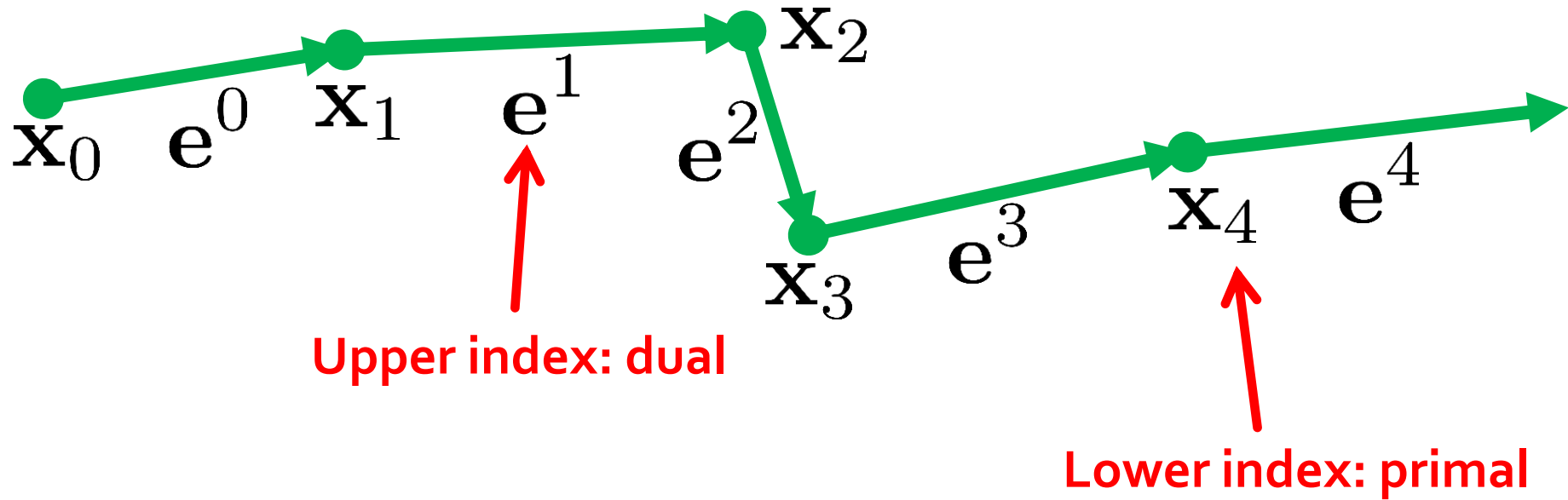
$$\mathbf{m}_2 = -\mathbf{u} \sin \theta + \mathbf{v} \cos \theta$$

$$E_{\text{twist}}(\Gamma) := \frac{1}{2} \int_{\Gamma} \beta(\theta')^2 ds$$

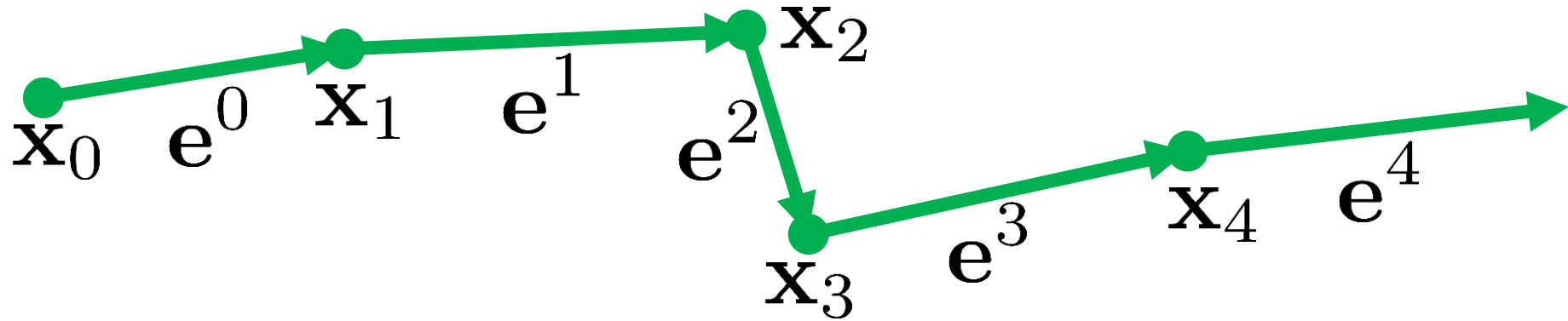
Degrees of freedom for elastic energy:

- Shape of curve
- Twist angle θ

Discrete Kirchhoff Rods



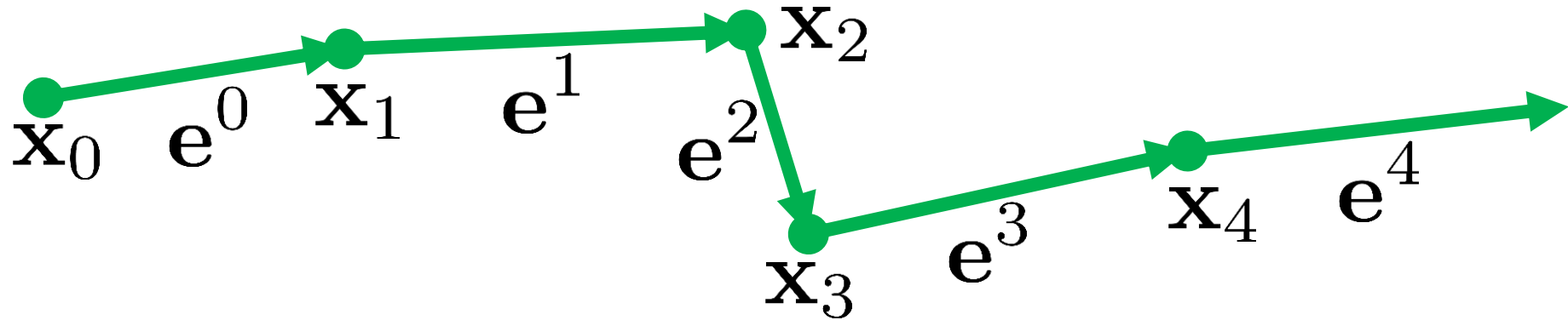
Discrete Kirchhoff Rods



$$\mathbf{T}^i := \frac{\mathbf{e}^i}{\|\mathbf{e}^i\|_2}$$

Tangent unambiguous on edge

Discrete Kirchhoff Rods



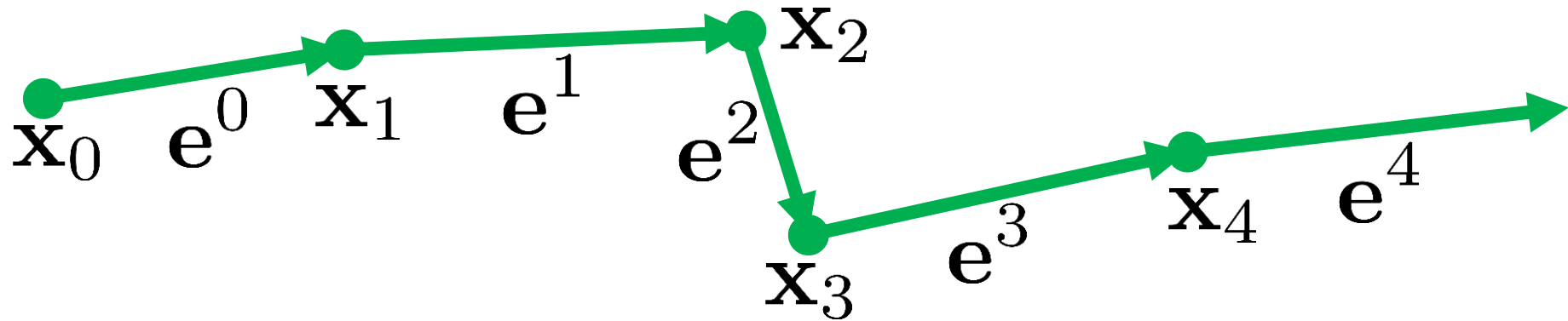
$$\kappa_i := 2 \tan \frac{\phi_i}{2}$$

Turning angle

Yet another curvature!

Integrated curvature

Discrete Kirchhoff Rods



$$\kappa_i := 2 \tan \frac{\phi_i}{2}$$

Yet another curvature!

$$(\kappa \mathbf{B})_i := \frac{2\mathbf{e}^{i-1} \times \mathbf{e}^i}{\|\mathbf{e}^{i-1}\|_2 \|\mathbf{e}^i\|_2 + \mathbf{e}^{i-1} \cdot \mathbf{e}^i}$$

Orthogonal to osculating plane,
norm κ_i

Darboux vector

Bending Energy

$$\begin{aligned} E_{\text{bend}}(\Gamma) &:= \frac{\alpha}{2} \sum_i \left(\frac{(\kappa \mathbf{B})_i}{\ell_i/2} \right)^2 \frac{\ell_i}{2} \\ &= \alpha \sum_i \frac{\|(\kappa \mathbf{B})_i\|_2^2}{\ell_i} \end{aligned}$$

Convert to pointwise and integrate

Discrete Parallel Transport

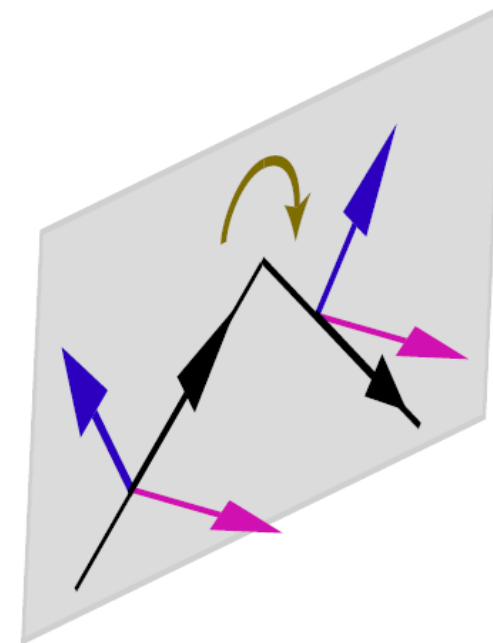
$$P_i(\mathbf{T}^{i-1}) = \mathbf{T}^i$$

$$P_i(\mathbf{T}^{i-1} \times \mathbf{T}^i) = \mathbf{T}^{i-1} \times \mathbf{T}^i$$

- Map tangent to tangent
- Preserve binormal
- Orthogonal

$$\mathbf{u}^i = P_i(\mathbf{u}^{i-1})$$

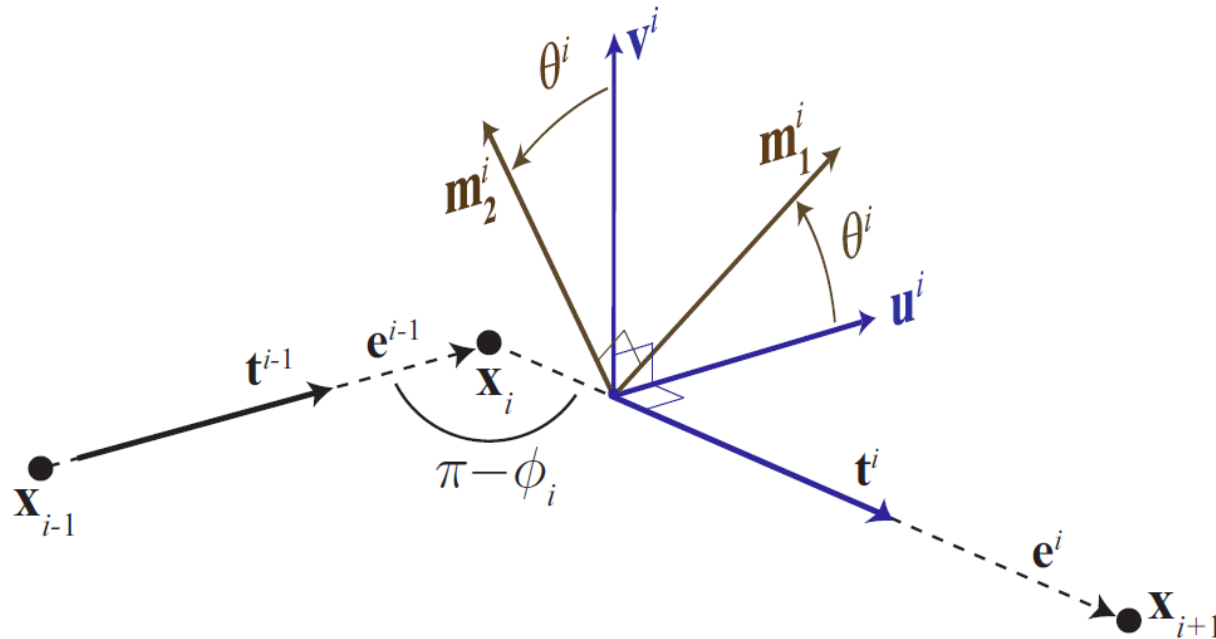
$$\mathbf{v}^i = \mathbf{T}^i \times \mathbf{u}^i$$



Discrete Material Frame

$$\mathbf{m}_1^i = \mathbf{u}^i \cos \theta^i + \mathbf{v}^i \sin \theta^i$$

$$\mathbf{m}_2^i = -\mathbf{u}^i \sin \theta^i + \mathbf{v}^i \cos \theta^i$$



Discrete Twisting Energy

$$E_{\text{twist}}(\Gamma) := \beta \sum_i \frac{(\theta^i - \theta^{i-1})^2}{\ell_i}$$


Note θ_0 can be arbitrary

Simulation

`\omit{physics}`

Worth reading!

Extension and Speedup

Discrete Viscous Threads

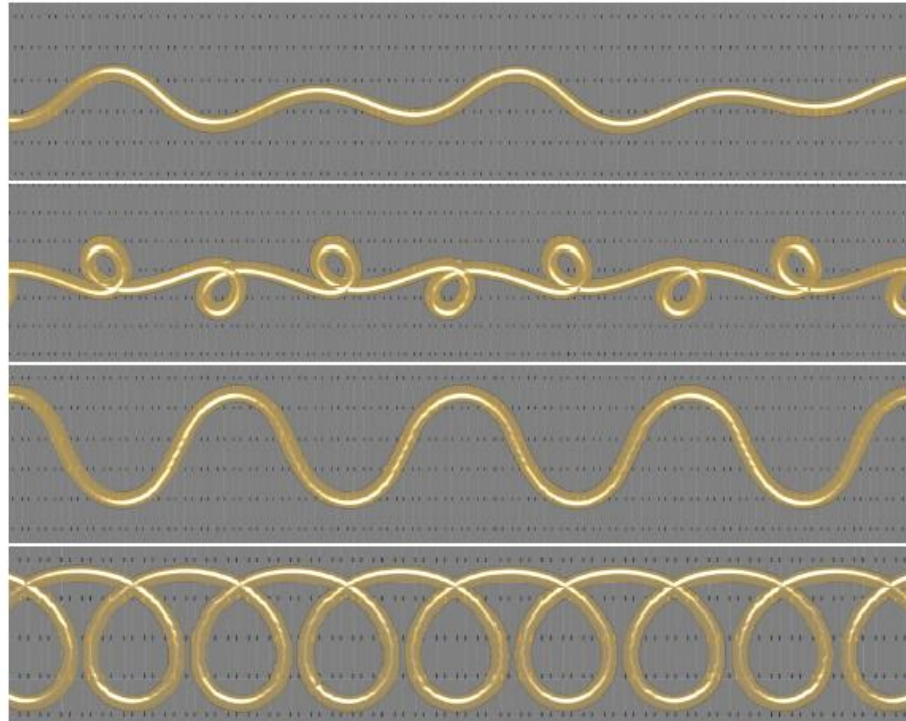
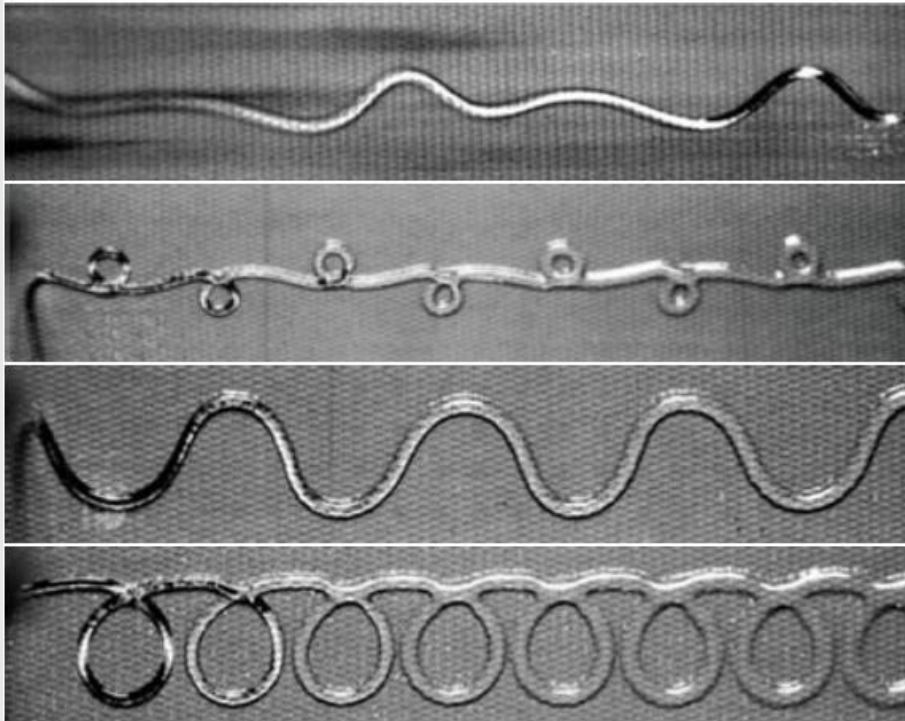
Miklós Bergou
Columbia University

Basile Audoly
UPMC Univ. Paris 06 & CNRS

Etienne Vouga
Columbia University

Max Wardetzky
Universität Göttingen

Eitan Grinspun
Columbia University



Extension and Speedup

Discrete Viscous Threads

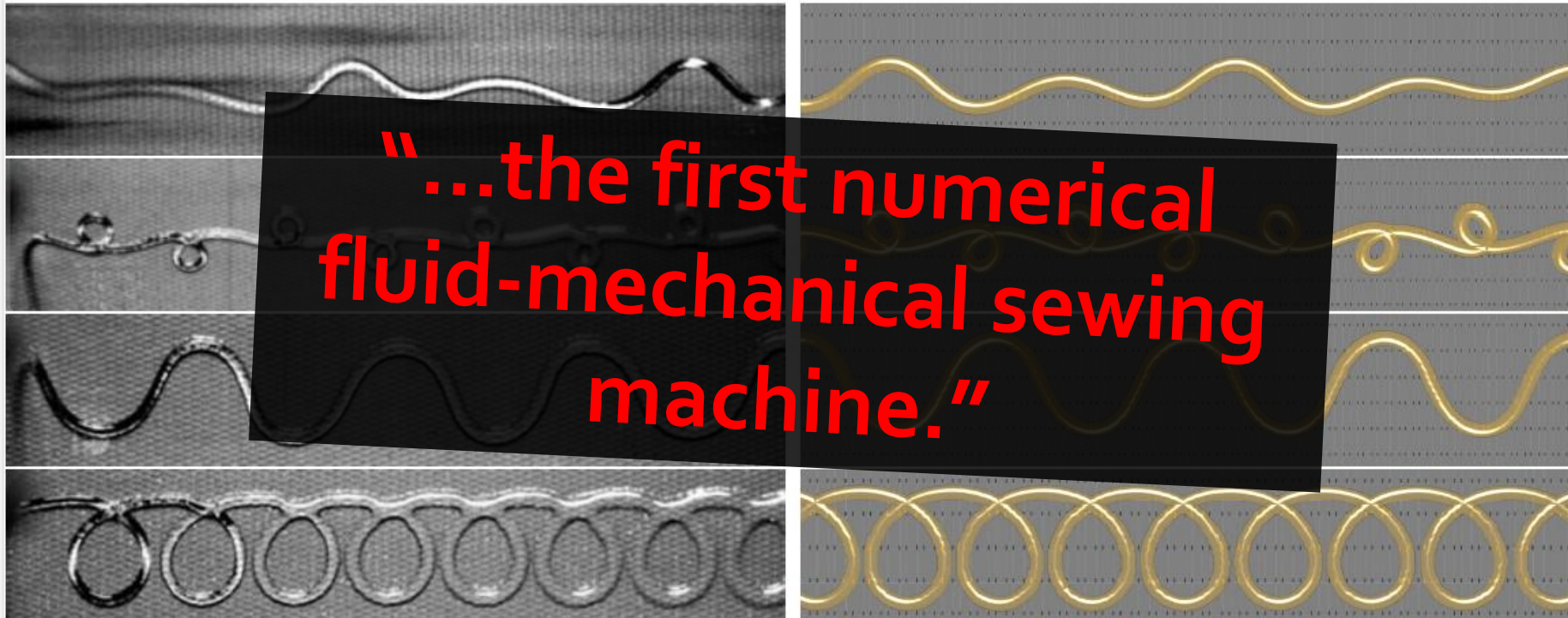
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Columbia University



Morals

One curve,
three curvatures.

$$\theta$$

$$2 \sin \frac{\theta}{2}$$

$$2 \tan \frac{\theta}{2}$$

Morals

**Easy theoretical object,
hard to use.**

$$\frac{d}{ds} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}$$

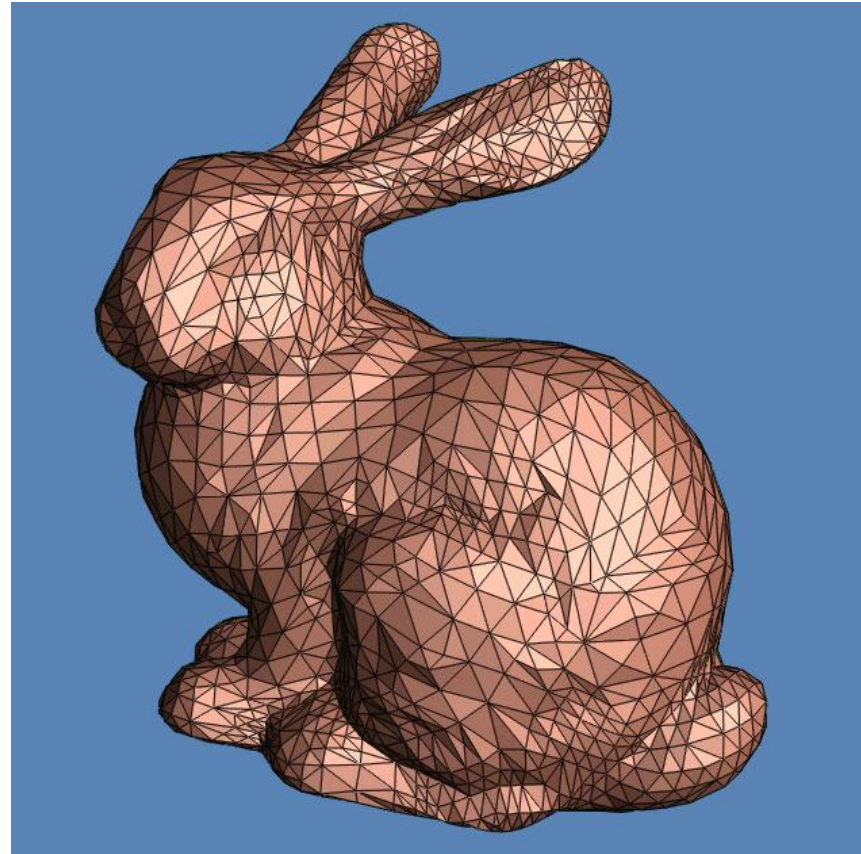
Morals

Proper frames and DOFs
go a long way.

$$\mathbf{m}_1^i = \mathbf{u}^i \cos \theta^i + \mathbf{v}^i \sin \theta^i$$

$$\mathbf{m}_2^i = -\mathbf{u}^i \sin \theta^i + \mathbf{v}^i \cos \theta^i$$

Next



<http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg>
<http://www.stat.washington.edu/wxs/images/BUNMID.gif>

Surfaces

Discrete Curves

Justin Solomon

6.8410: Shape Analysis

Fall 2026

