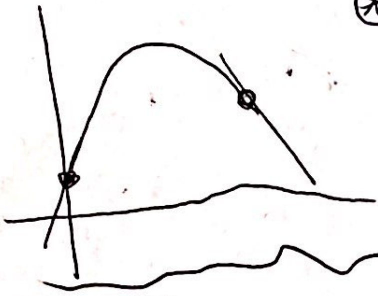
①

(*) Given: $f(0)=a,\ f(1)=b,\ f'(0)=c,\ f'(1)=d$



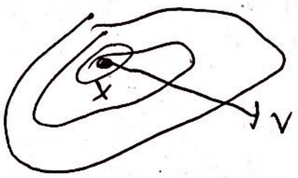
$$\boxed{\min_{f:[0,1]\to\mathbb{R}} \int_0^1 (f''(t))^2\,dt}$$

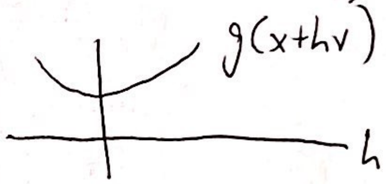
FINITE DIMENSIONAL INTUITION

Want: $\min_{x\in\mathbb{R}^n} g(x) \longrightarrow \underbrace{\nabla g(x)}_{??} = 0$

Instead: $\forall v,\ \nabla g(x)\cdot v = 0 \iff dg_x(v) = 0 \quad \forall v$

$$\iff 0 = \frac{d}{dh} g(x+hv)\Big|_{h=0}$$

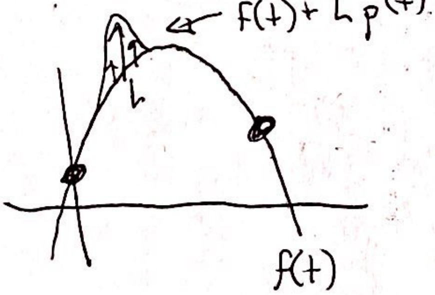




$\boxed{v \text{ like a perturbation}}$

Back to our problem:

$E[f] := \int_0^1 f''(t)^2\,dt$



Gâteaux derivative:

$$dE_f[p] = \frac{d}{dh} E[\underset{\uparrow\ \text{fn}}{f} + \underset{\text{scalar}}{h}p]\Big|_{h=0}$$

key!

$$dE_f[p] = \frac{d}{dh}\int_0^1 (f''(t)+hp''(t))^2\,dt\Big|_{h=0}$$

$$= \int_0^1 2(f''(t)+hp''(t))\,p''(t)\Big|_{h=0}\,dt \quad \text{by differentiation under integral}$$

$$= \int_0^1 2f''(t)\,p''(t)\,dt$$

If in doubt, integrate by parts!

$$\Rightarrow dE_f[p] = \left[2 f''(t)\, p'(t)\right]_0^1 - 2\int_0^1 f'''(t)\, p'(t)\, dt$$

$$(**) = \left[2 f''(t)\, p'(t)\right]_0^1 - \left[2 f'''(t)\, p(t)\right]_0^1 + 2\int_0^1 f^{(4)}(t)\, p(t)\, dt$$

Suppose $f(t)$ optimal for our <u>constrained</u> problem (constraints $(*)$).
Then, $dE_f[p] = 0$ <u>so long as</u> $f(t) + h\,p(t)$ is <u>feasible</u> for small h $\Rightarrow$ $p(0) = p(1) = p'(0) = p'(1) = 0$ since f has prescribed value/derivative at $t \in \{0,1\}$.

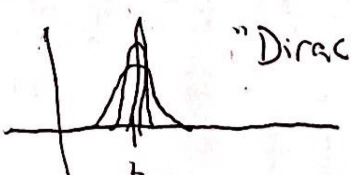
In this case, by $(**)$

$$(*) \left[\begin{array}{l} 0 = dE_f[p] = 2\int_0^1 f^{(4)}(t)\, p(t)\, dt \\ \qquad \forall p \in C^\infty([0,1]) \text{ with } p(0) = p(1) = p'(0) = p'(1) = 0. \end{array}\right.$$

VAGUE/INFORMAL ARGUMENT AHEAD:

$(*)$ includes a <u>lot</u> of functions $p(t)$, and the expression equals zero for <u>all</u> of them!

Example: $p(t) \to \delta_{t_0}$



"Dirac δ-function"

t_0

Conclusion: Only way this can happen is if thing integrated against $p(t)$ is zero $\Rightarrow f^{(4)}(t) \equiv 0 \Rightarrow f(t) = a^{0} + a^{(1)} t + a^{(2)} t^2 + a^{(3)} t^3$

"Cubic spline" !