The Laplacian Operator

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6.838: Shape Analysis

Spring 2021

MIT EECS


WARNING
SIGN
MISTAKES
LIKELY

Lots of (sloppy) math!

http://www.wgnflag.com/xcart/images/P/G-50_StopSeatBeltsStockSign.jpg

Famous Motivation

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

> "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

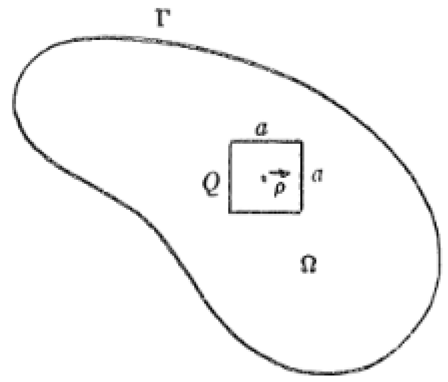
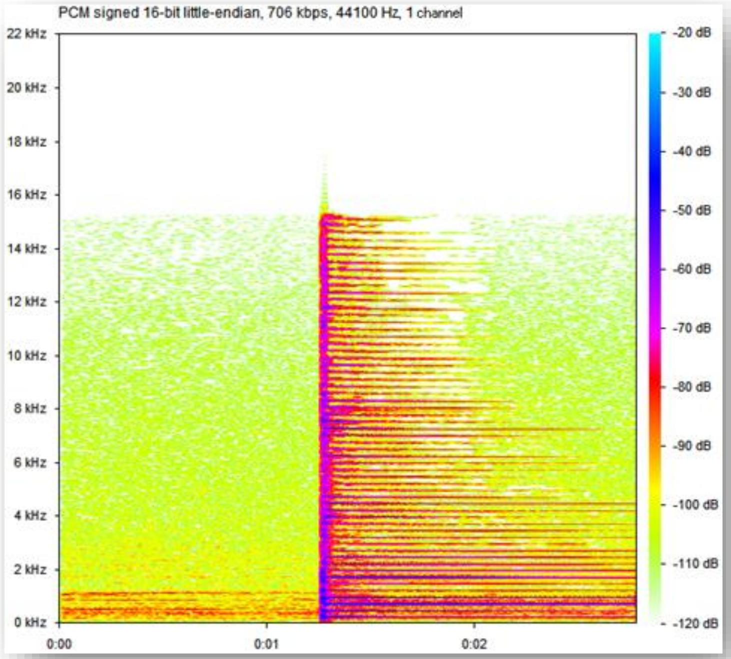


FIG. 1

An Experiment

Is this possible?

Unreasonable to Ask?

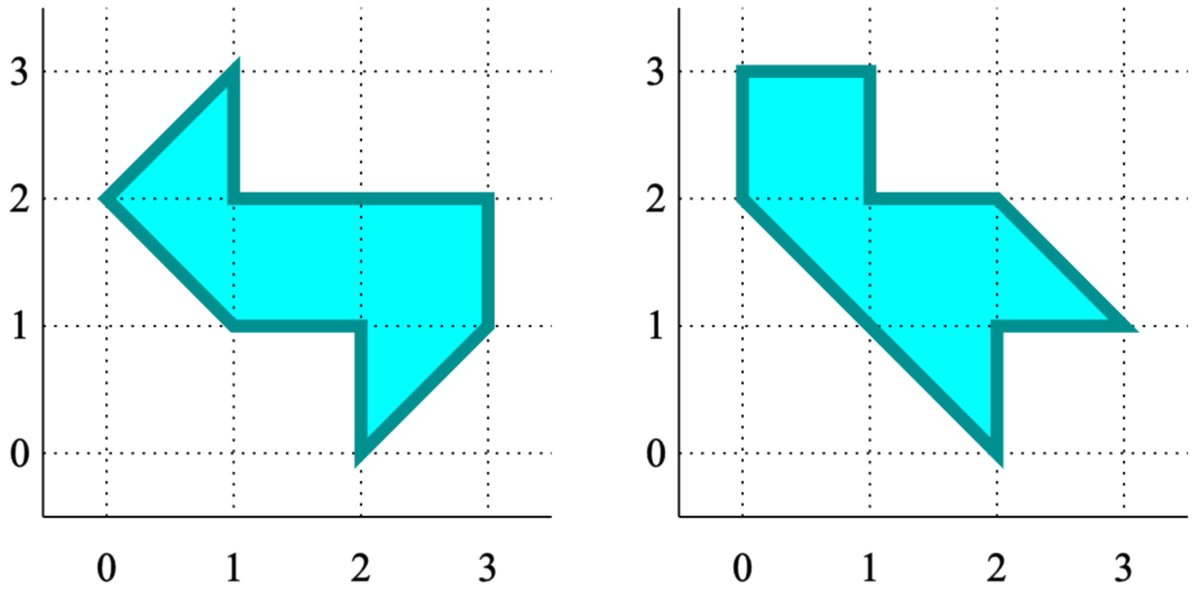


Length of string

http://www.takamine.com/templates/default/images/gclassical.png
http://firsttimeprogrammer.blogspot.com/2015/05/spectrogram-representation-of-b-note.html

Spoiler Alert

Extra credit: Make these!



"No, but…"

- **Has to be a weird drum**
- **Spectrum tells you a lot!**

https://en.wikipedia.org/wiki/Hearing_the_shape_of_a_drum

Rough Intuition

http://pngimg.com/upload/hammer_PNG3886.png

You can learn a lot about a shape by **hitting it** (lightly) **with a hammer**!

Spectral Geometry

What can you learn about its shape from **vibration frequencies and oscillation patterns?**

$$\Delta f = \lambda f$$

Objectives

- Make "vibration modes" more precise
- Progressively more complicated domains
  - Line segments
  - Regions in $\mathbb{R}^n$
  - Graphs
  - Surfaces/manifolds
- Coming up: Discretization, applications

Review:

Vector Spaces and Linear Operators

$$L[\mathbf{x}+\mathbf{y}] = L[\mathbf{x}] + L[\mathbf{y}]$$

$$L[c\mathbf{x}] = cL[\mathbf{x}]$$

$$L[\mathbf{x}] = A\mathbf{x}$$

Review:

In Finite Dimensions

$$\underbrace{A}_{\text{matrix}} \underbrace{\mathbf{x}}_{\text{vector}}$$

$$\underbrace{\mathbf{x} \mapsto A\mathbf{x}}_{\text{linear operator}}$$

Recall: Spectral Theorems in $\mathbb{C}^n$

Theorem. Suppose $A \in \mathbb{C}^{n\times n}$ is Hermitian. Then, A has an orthogonal basis of n eigenvectors. If A is positive definite, the corresponding eigenvalues are nonnegative.

Our Progression

- Line segments
- Regions in $\mathbb{R}^n$
- Graphs
- Surfaces/manifolds

Wave Equation

$$\frac{\partial^2 u}{\partial t^2} \overbrace{- \frac{\partial^2 u}{\partial x^2}}^{\text{Minus second derivative operator!}} = 0$$

Minus Second Derivative Operator

"Dirichlet boundary conditions"

$$\{f(\cdot) \in C^\infty([a,b]) : f(0) = f(\ell) = 0\}$$

$$\mathcal{L}[\cdot] : u \mapsto -\frac{\partial^2 u}{\partial x^2}$$

Interpretation as positive (semi-)definite operator.

$$\mathbf{u}^\top L \mathbf{u} \geq 0$$

$$\langle u, \mathcal{L} u \rangle \geq 0$$

Eigenfunctions of Second Derivative Operator

"Dirichlet boundary conditions"

$$\{f(\cdot) \in C^\infty([a,b]) : f(0) = f(\ell) = 0\}$$

$$\mathcal{L}[\cdot] : u \mapsto -\frac{\partial^2 u}{\partial x^2}$$

Eigenfunctions:

$$\phi_k(x) = \sqrt{\frac{2}{\ell}} \sin\left(\frac{\pi k x}{\ell}\right), \quad \lambda_k = \left(\frac{\pi k}{\ell}\right)^2$$

$$\mathbf{u}''(t) = -kL\mathbf{u}(t)$$

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \mathcal{L}[u]$$

Can you hear the length of an interval?

$$\phi_k(x) = \sqrt{\frac{2}{\ell}} \sin\left(\frac{\pi k x}{\ell}\right), \quad \lambda_k = \left(\frac{\pi k}{\ell}\right)^2$$

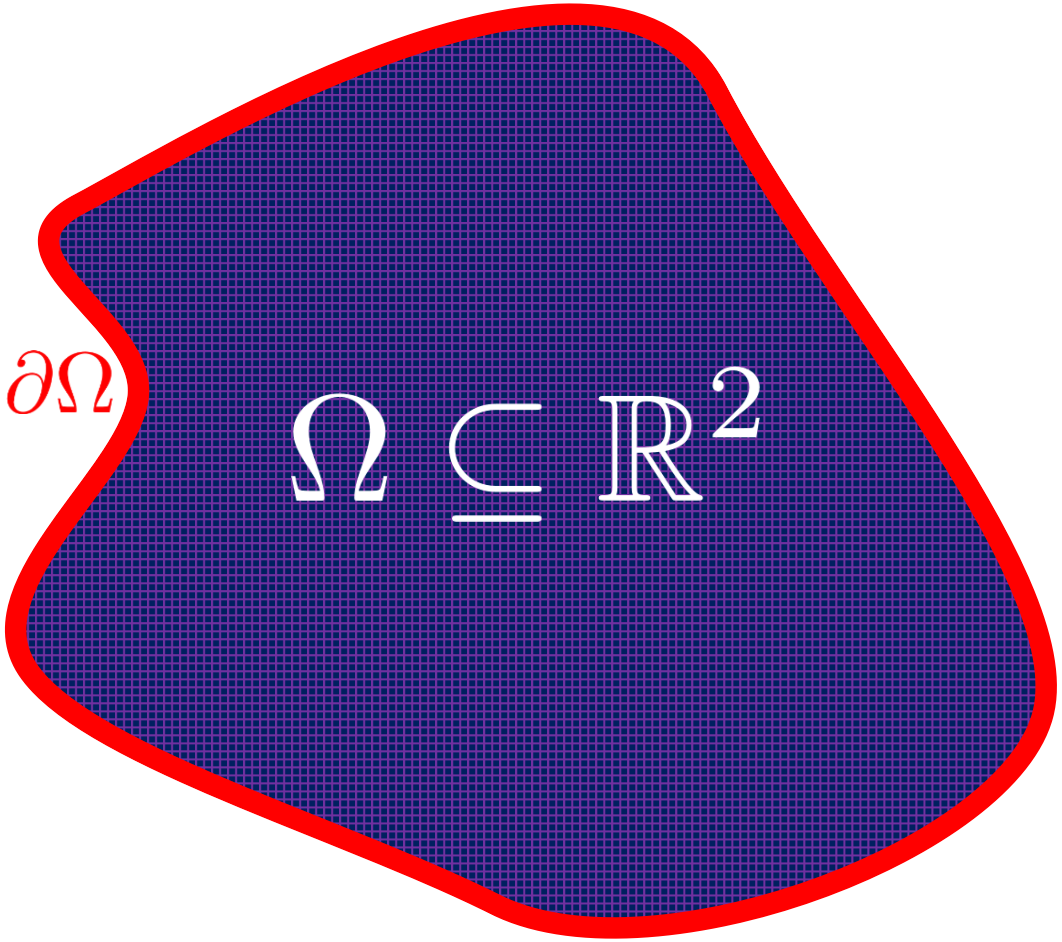
$$\boxed{\lambda_k = \left(\frac{\pi k}{\ell}\right)^2}$$

Yes!

Our Progression

- Line segments
- Regions in $\mathbb{R}^n$
- Graphs
- Surfaces/manifolds

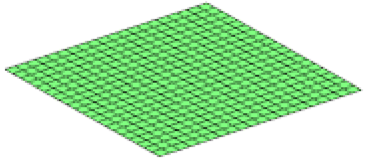
Planar Region



Wave equation:

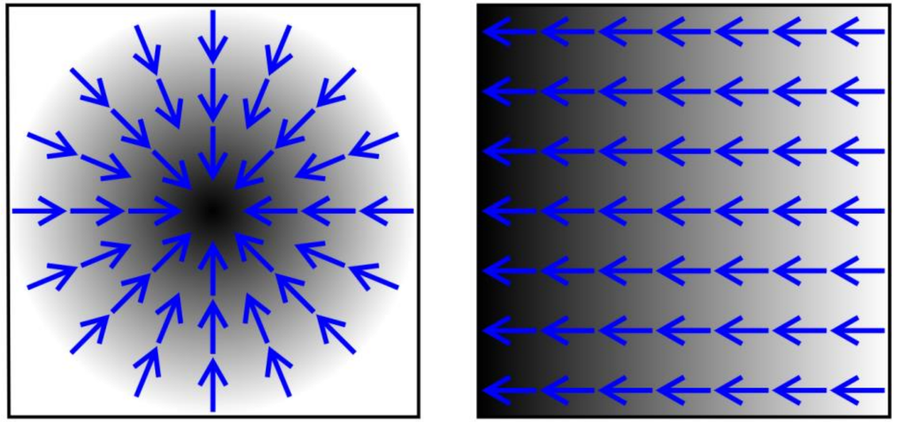
$$\frac{\partial^2 u}{\partial t^2} = -\Delta u$$

$$\Delta := -\sum_i \frac{\partial^2}{\partial (x^i)^2}$$



Typical Notation

$$\text{“}\Delta = -\underbrace{\nabla \cdot}_{\text{divergence}} \underbrace{\nabla}_{\text{gradient}}\text{”}$$

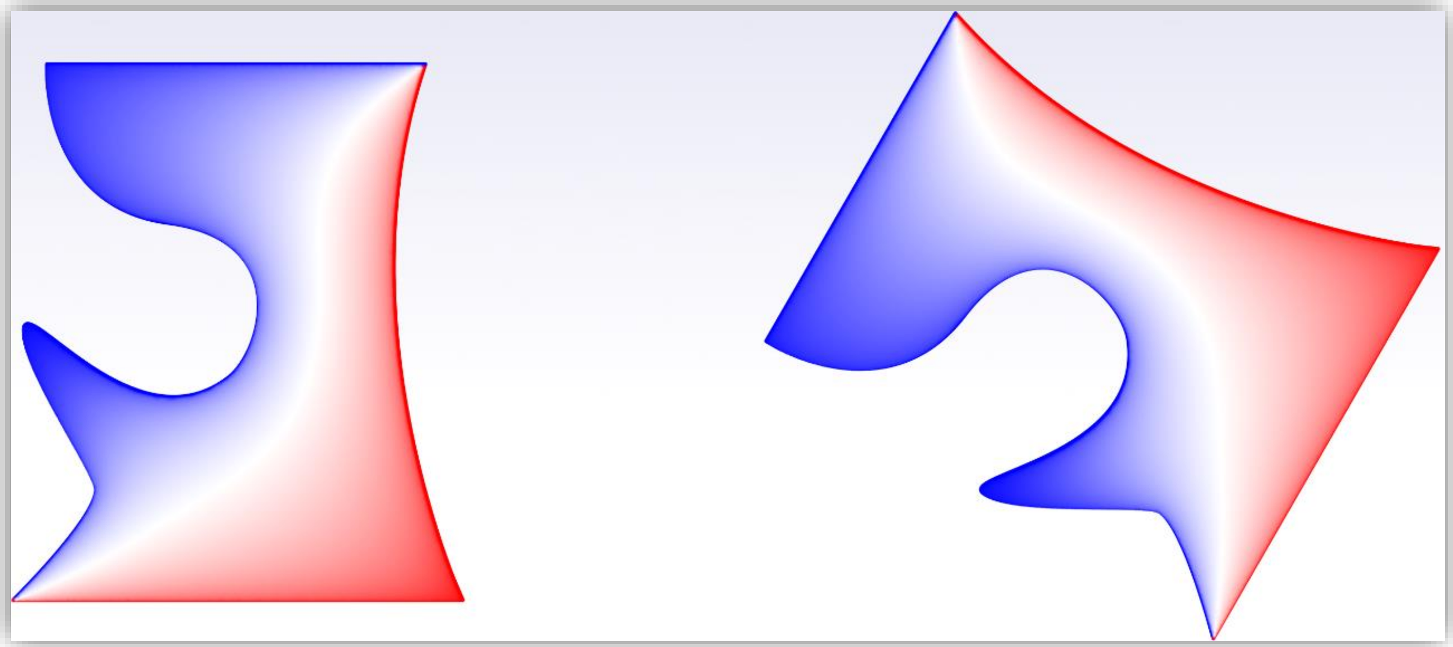


Gradient operator:

$$\nabla := \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n} \right)$$

http://www.gamasutra.com/db_area/images/feature/4164/figy.png, https://en.wikipedia.org/wiki/Gradient

Intrinsic Operator



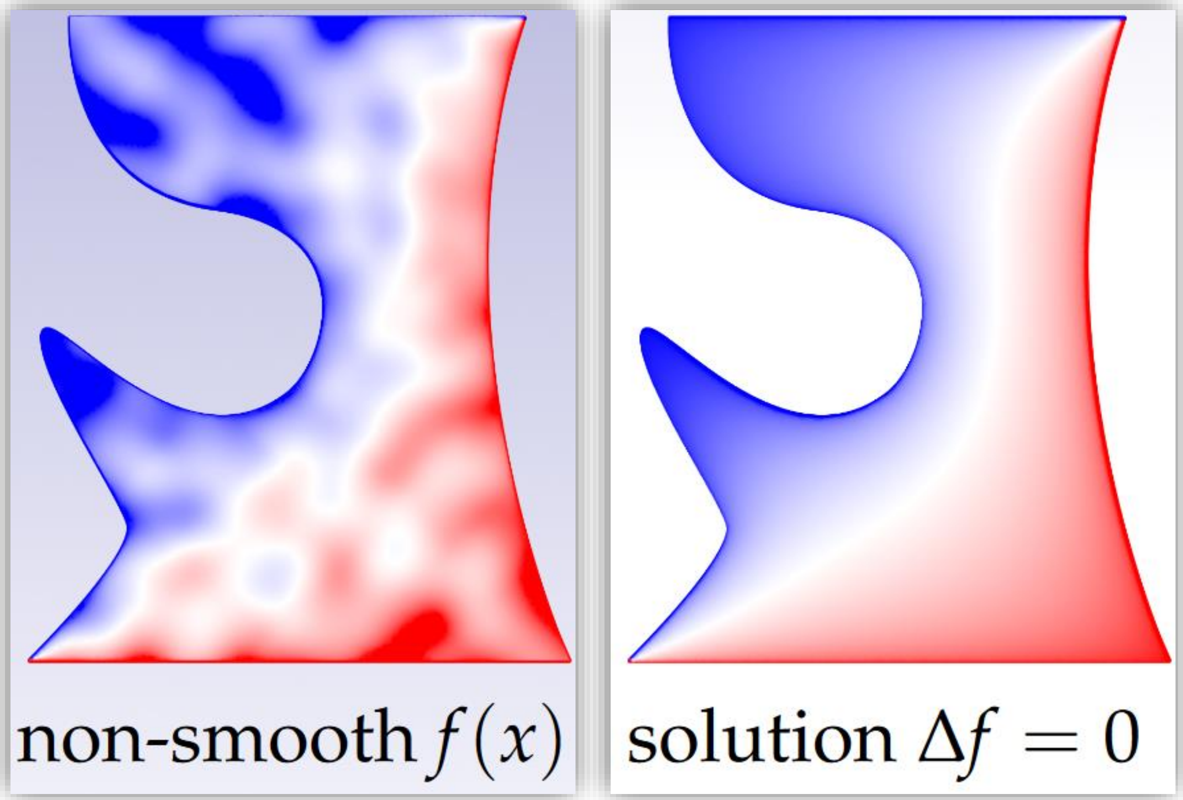
Images made by E. Vouga

Coordinate-independent

$$g(\mathbf{x}) = f(R\mathbf{x} + \mathbf{t})$$

Dirichlet Energy

$$E[u] := \frac{1}{2}\int_\Omega \|\nabla u(\mathbf{x})\|_2^2 \, dA(\mathbf{x})$$



Images made by E. Vouga

$$\min_{u(\mathbf{x}):\Omega\to\mathbb{R}} E[u]$$

$$\text{s.t. } u|_{\partial\Omega} \text{ prescribed}$$

$$\Downarrow$$

$$\Delta u \equiv 0$$

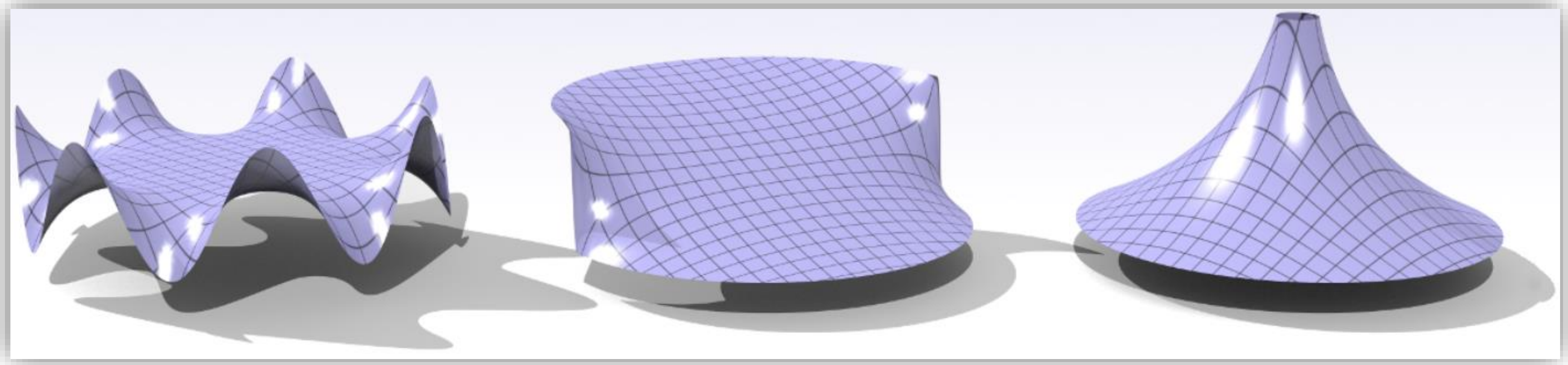
"Laplace equation"
"Harmonic function"

$$\min_{u(\mathbf{x}):\Omega\to\mathbb{R}} E[u]$$

$$\text{s.t. } u|_{\partial\Omega} \text{ prescribed}$$

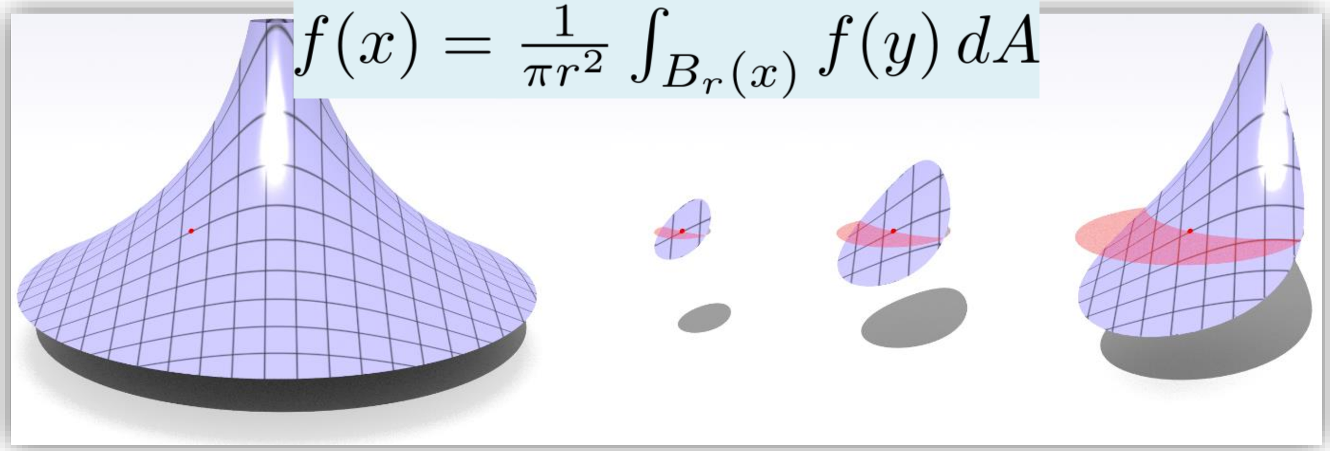
Harmonic Functions

$$\Delta f \equiv 0$$



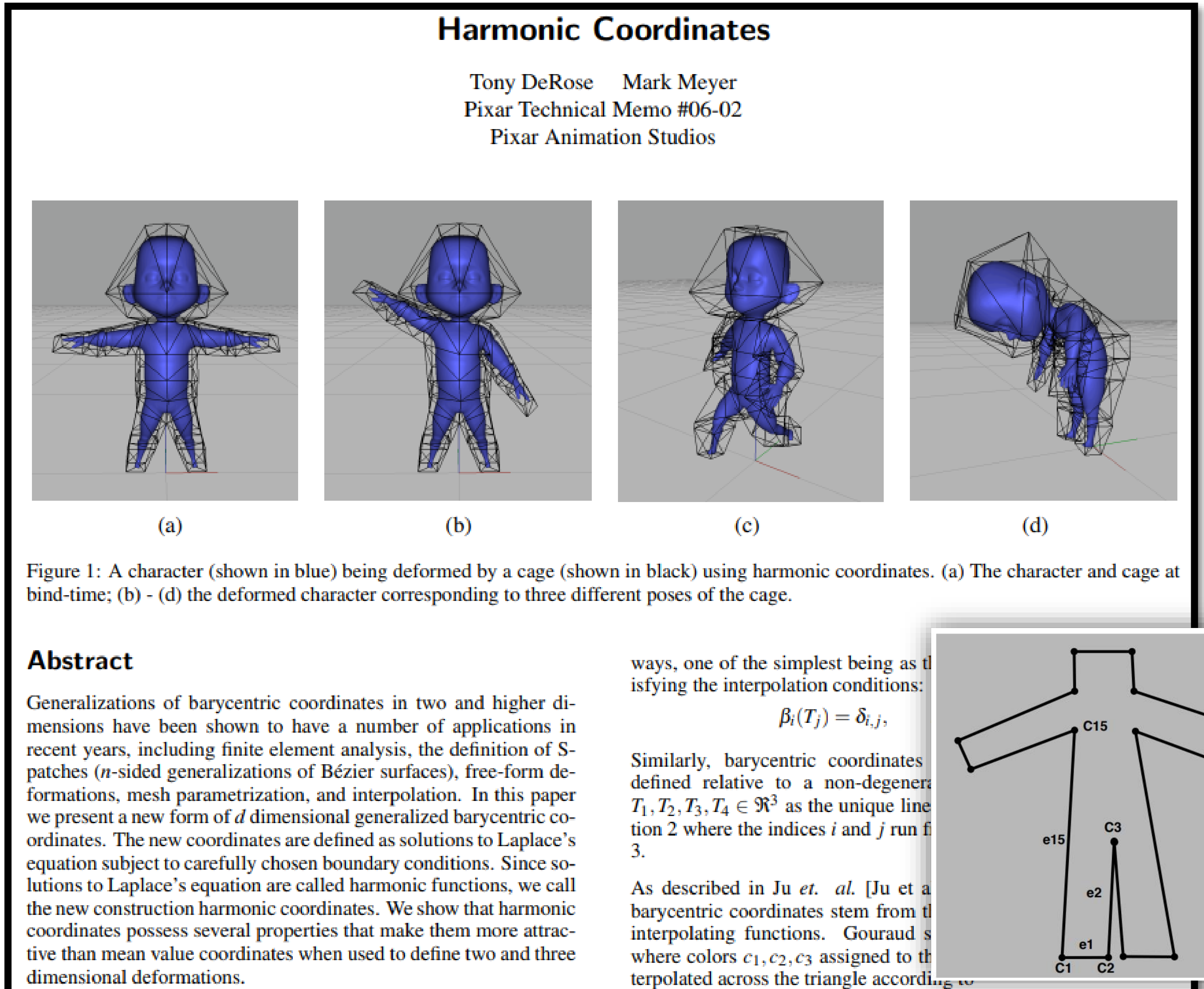
Mean value property:

$$f(x) = \frac{1}{\pi r^2} \int_{B_r(x)} f(y)\, dA$$



Images made by E. Vouga

Application



Harmonic Coordinates

Tony DeRose Mark Meyer
Pixar Technical Memo #06-02
Pixar Animation Studios

(a) (b) (c) (d)

Figure 1: A character (shown in blue) being deformed by a cage (shown in black) using harmonic coordinates. (a) The character and cage at bind-time; (b) - (d) the deformed character corresponding to three different poses of the cage.

Abstract

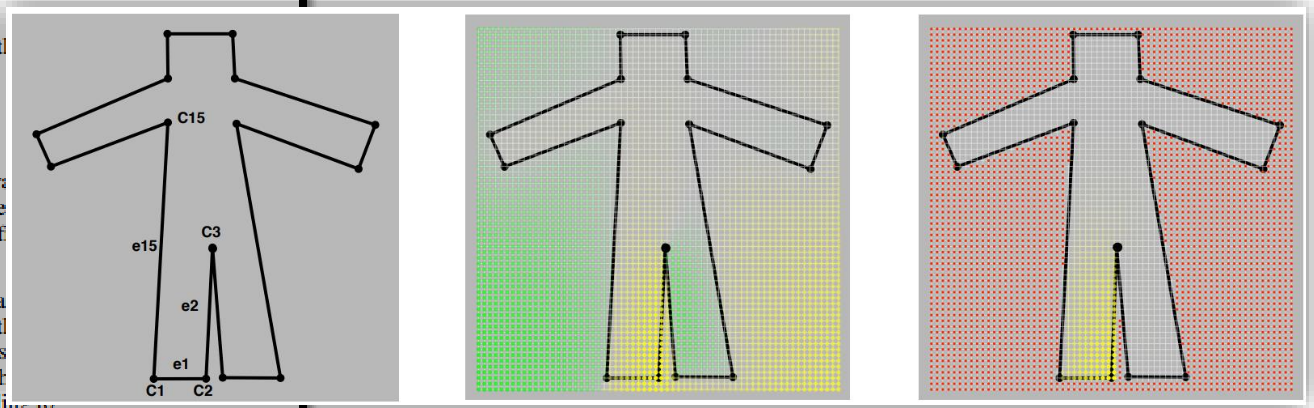
Generalizations of barycentric coordinates in two and higher dimensions have been shown to have a number of applications in recent years, including finite element analysis, the definition of S-patches (n-sided generalizations of Bézier surfaces), free-form deformations, mesh parametrization, and interpolation. In this paper we present a new form of d dimensional generalized barycentric coordinates. The new coordinates are defined as solutions to Laplace's equation subject to carefully chosen boundary conditions. Since solutions to Laplace's equation are called harmonic functions, we call the new construction harmonic coordinates. We show that harmonic coordinates possess several properties that make them more attractive than mean value coordinates when used to define two and three dimensional deformations.

ways, one of the simplest being as t
isfying the interpolation conditions:

$$\beta_i(T_j) = \delta_{i,j},$$

Similarly, barycentric coordinates
defined relative to a non-degenera
$T_1, T_2, T_3, T_4 \in \Re^3$ as the unique line
tion 2 where the indices i and j run f
3.

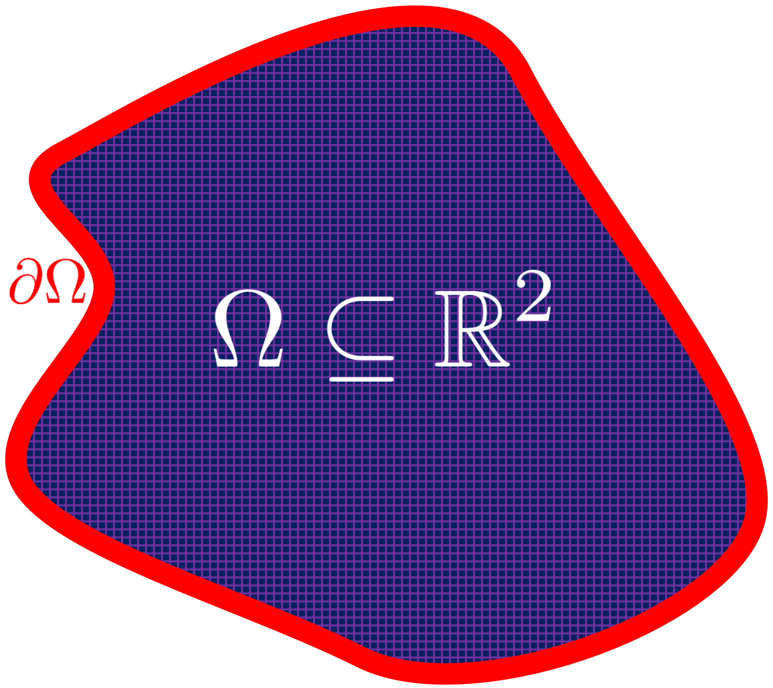
As described in Ju *et. al.* [Ju et a
barycentric coordinates stem from t
interpolating functions. Gouraud s
where colors c_1, c_2, c_3 assigned to th
terpolated across the triangle accordi



Positivity, Self-Adjointness

$$\{f(\cdot) \in C^\infty(\Omega) : f|_{\partial\Omega} \equiv 0\}$$

"Dirichlet boundary conditions"



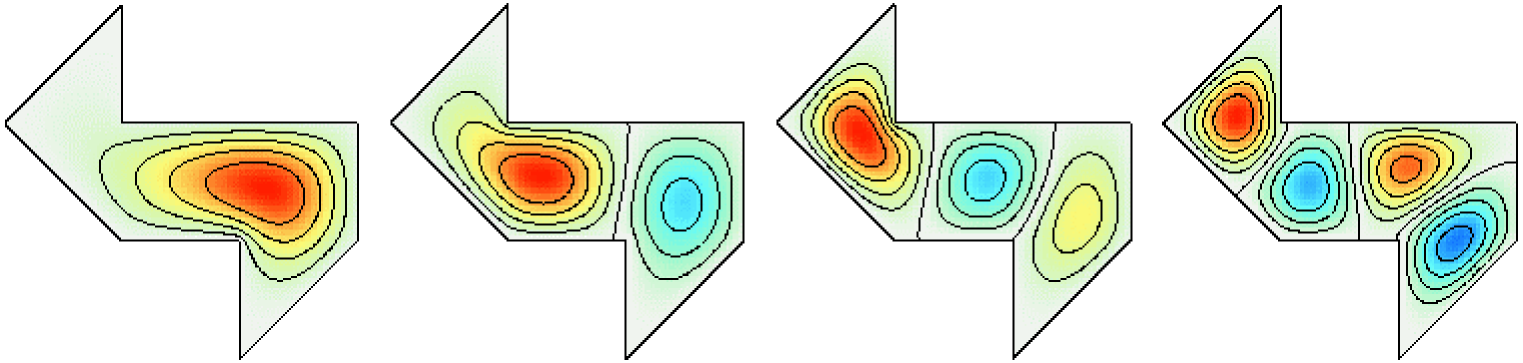
$$\mathcal{L}[f] := \Delta f$$

$$\langle f, g \rangle := \int_\Omega f(\mathbf{x}) g(\mathbf{x})\, dA(\mathbf{x})$$

1. **Positive:** $\langle f, \mathcal{L}[f] \rangle \geq 0$
2. **Self-adjoint:** $\langle f, \mathcal{L}[g] \rangle = \langle \mathcal{L}[f], g \rangle$

$$\langle f, \mathcal{L}[f] \rangle \geq 0$$
$$\langle f, \mathcal{L}[g] \rangle = \langle \mathcal{L}[f], g \rangle$$

Laplacian Eigenfunctions



$$\min_u \frac{1}{2} \int_\Omega \|\nabla u(\mathbf{x})\|_2^2 \, d\mathbf{x}$$
$$\text{s.t.} \int_\Omega u(\mathbf{x})^2 \, d\mathbf{x} = 1$$

Critical points on the "unit sphere"

Theorem (Weyl's Law). *Let $N(\lambda)$ be the number of Dirichlet eigenvalues of the Laplacian Δ for a domain $\Omega \subseteq \mathbb{R}^d$ less than or equal to λ. Then,*

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = (2\pi)^{-d} \omega_d \mathrm{vol}(\Omega),$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

http://www.math.udel.edu/~driscoll/research/gww1-4.gif

Small eigenvalue: Small Dirichlet Energy

Aside:

Common Misconception

$$\min_f \; E[f] \text{ s.t. } f(p) = \text{const.}$$

Point constraints are ill-advised

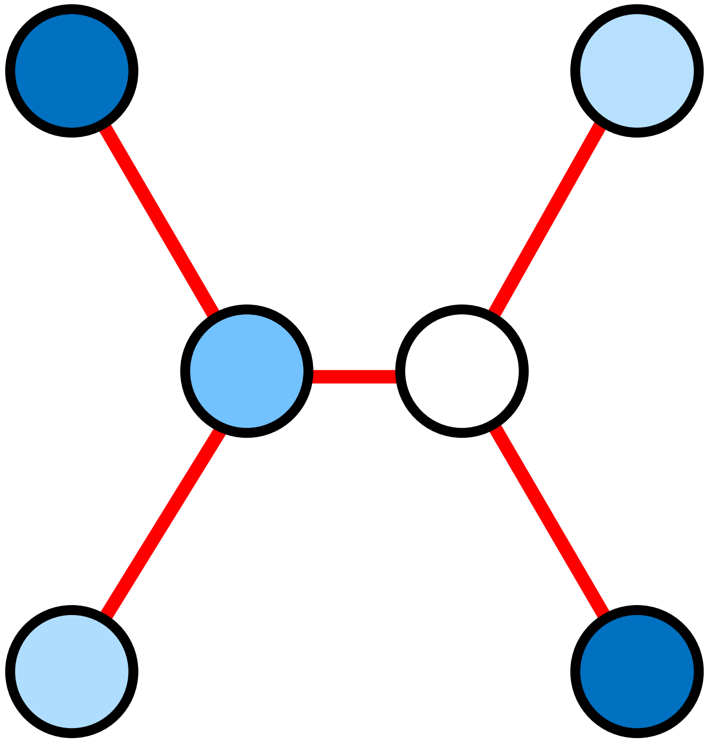
Our Progression

- Line segments
- Regions in $\mathbb{R}^n$
- Graphs
- Surfaces/manifolds

Basic Setup

- **Function:**

  One value per vertex



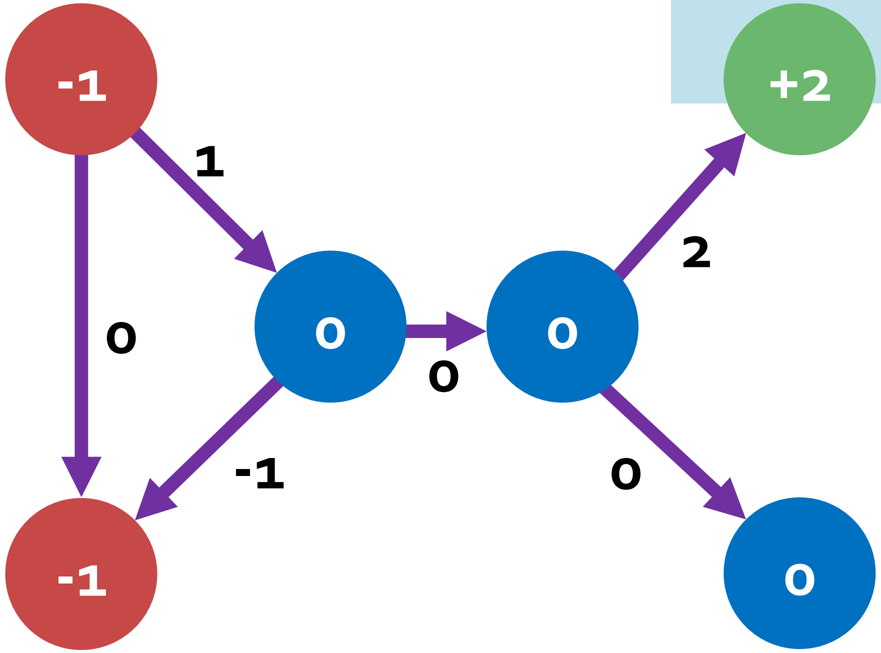
Q:

What is the Dirichlet energy of a function on a graph?

Differencing Operator

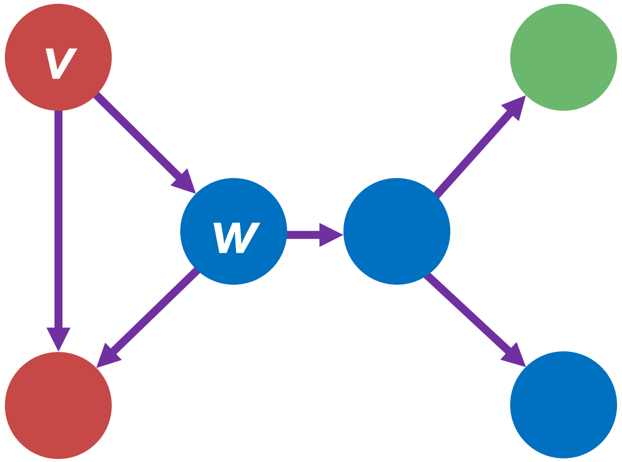
$$D_{ev} := \begin{cases} -1 & \text{if } e = (v, w) \\ 1 & \text{if } e = (w, v) \\ 0 & \text{otherwise} \end{cases}$$

$D \in \{-1, 0, 1\}^{|E| \times |V|}$



Orient edges arbitrarily

Dirichlet Energy on a Graph



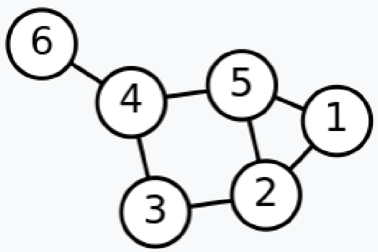
$$D_{ev} := \begin{cases} -1 & \text{if } e = (v,w) \\ 1 & \text{if } e = (w,v) \\ 0 & \text{otherwise} \end{cases}$$

$$E[\mathbf{f}] := \|D\mathbf{f}\|_2^2 = \sum_{(v,w)\in E} (f^v - f^w)^2$$

(Unweighted) Graph Laplacian

$$E[\mathbf{f}] = \|D\mathbf{f}\|_2^2 = \mathbf{f}^\top (D^\top D)\mathbf{f} := \mathbf{f}^\top L \mathbf{f}$$

$$L_{vw} = A - \overline{D} = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

Labeled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2&0&0&0&0&0 \\ 0&3&0&0&0&0 \\ 0&0&2&0&0&0 \\ 0&0&0&3&0&0 \\ 0&0&0&0&3&0 \\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&0&1&0 \\ 1&0&1&0&1&0 \\ 0&1&0&1&0&0 \\ 0&0&1&0&1&1 \\ 1&1&0&1&0&0 \\ 0&0&0&1&0&0 \end{pmatrix}$	$\begin{pmatrix} 2&-1&0&0&-1&0 \\ -1&3&-1&0&-1&0 \\ 0&-1&2&-1&0&0 \\ 0&0&-1&3&-1&-1 \\ -1&-1&0&-1&3&0 \\ 0&0&0&-1&0&1 \end{pmatrix}$

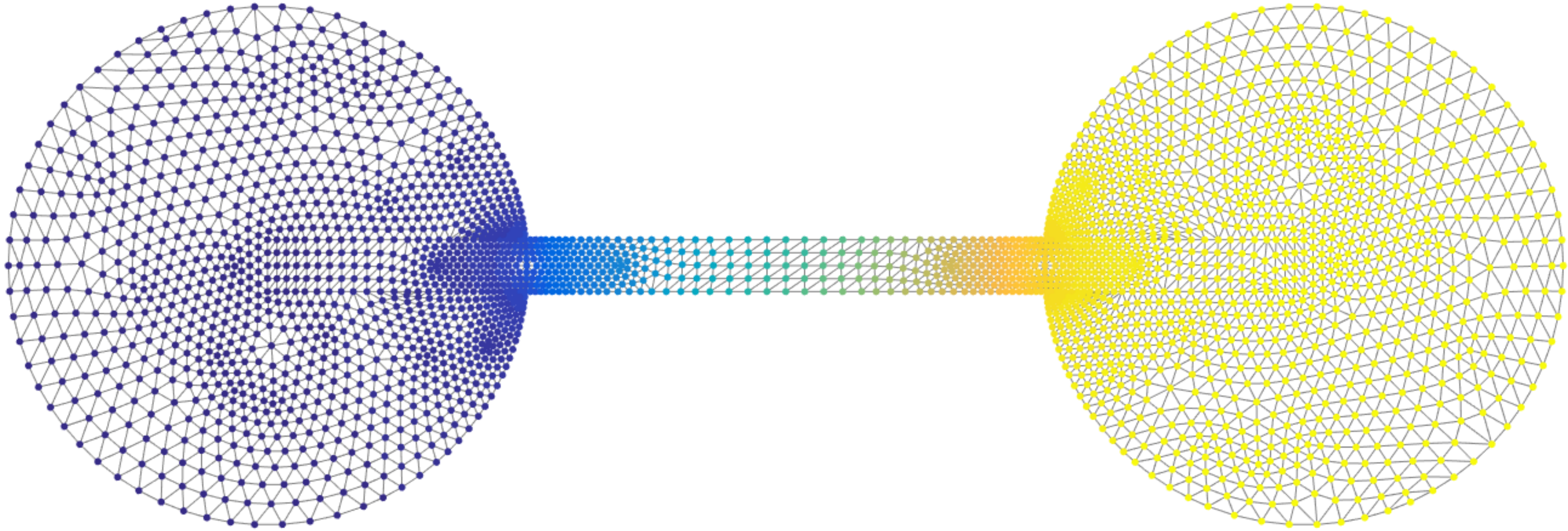
- **Symmetric**
- **Positive semidefinite**

https://en.wikipedia.org/wiki/Laplacian_matrix

Q:

What is the
smallest eigenvalue
of the graph Laplacian?

Second-Smallest Eigenvector



$$L\mathbf{x} = \lambda\mathbf{x}$$

Used for graph partitioning

Fiedler vector ("algebraic connectivity")

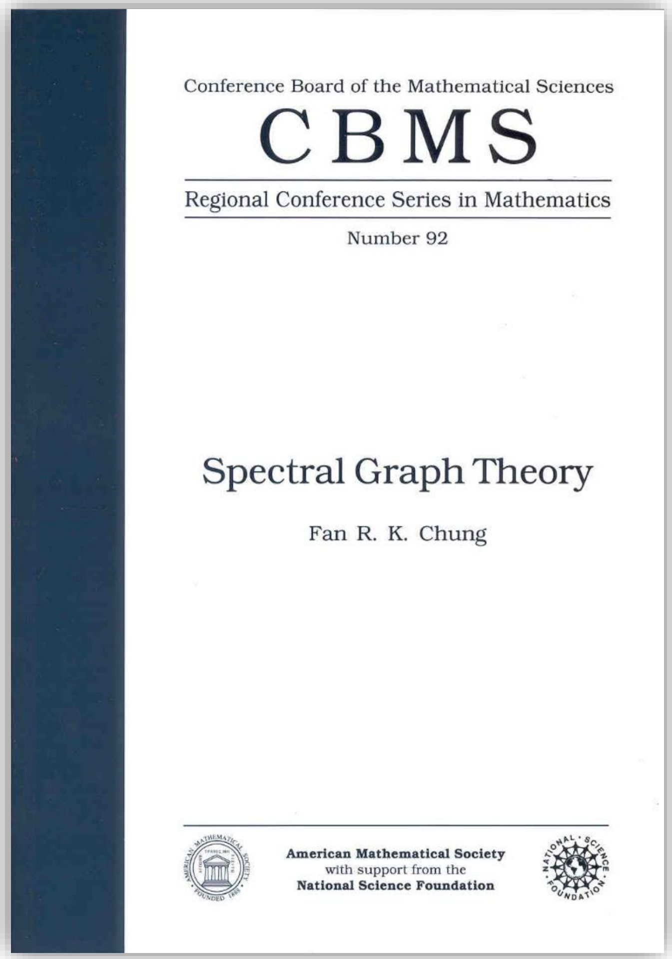
Mean Value Property

$$L_{vw} = A - D = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$$(L\mathbf{x})^v = 0$$

Value at *v* is average of neighboring values

For More Information...



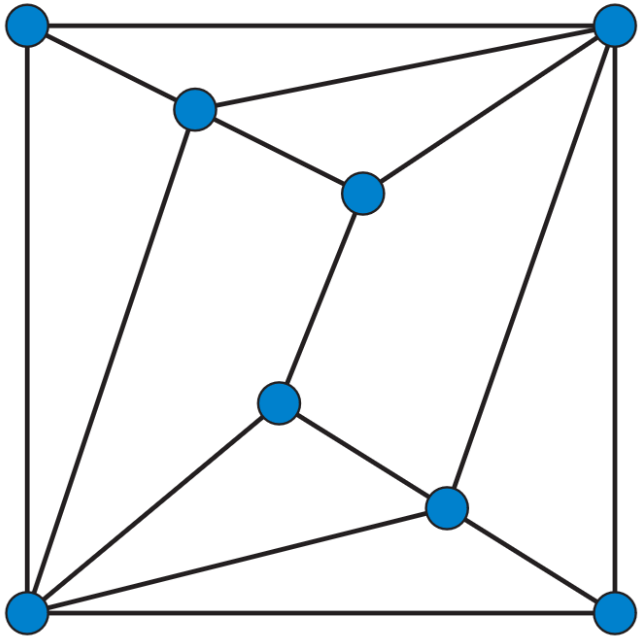
Graph Laplacian encodes lots of information!

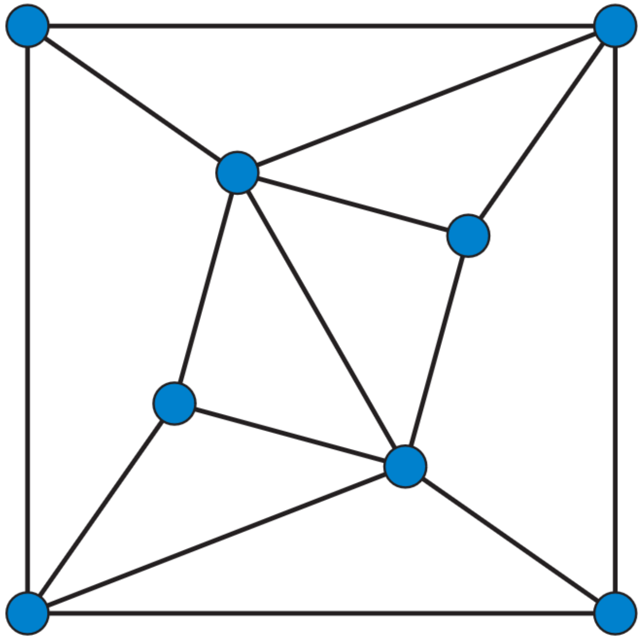
Example: Kirchoff's Theorem

Number of spanning trees equals

$$t(G) = \frac{1}{|V|} \prod_{k=2}^{|V|} \lambda_k$$

Hear the Shape of a Graph?





"Enneahedra"

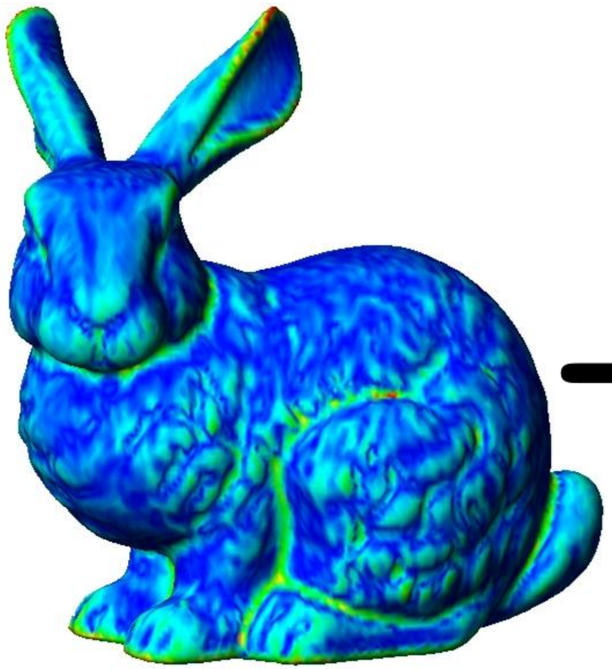
No!

Our Progression

- Line segments
- Regions in $\mathbb{R}^n$
- Graphs
- Surfaces/manifolds

Recall:

Scalar Functions

$f :$  $\longrightarrow \mathbb{R}$

http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

Map points to real numbers

Recall:

Differential of a Map

Definition (Differential). *Suppose* $\varphi : \mathcal{M} \to \mathcal{N}$ *is a map from a submanifold* $\mathcal{M} \subseteq \mathbb{R}^k$ *into a submanifold* $\mathcal{N} \subseteq \mathbb{R}^\ell$. *Then, the* differential $d\varphi_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \to T_{\varphi(\mathbf{p})}\mathcal{N}$ *of* φ *at a point* $\mathbf{p} \in \mathcal{M}$ *is given by*

$$d\varphi_{\mathbf{p}}(\mathbf{v}) := (\varphi \circ \gamma)'(0),$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ *is any curve with* $\gamma(0) = \mathbf{p}$ *and* $\gamma'(0) = \mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$.

Linear map of tangent spaces

$$d\varphi_{\mathbf{p}}(\gamma'(0)) := (\varphi \circ \gamma)'(0)$$

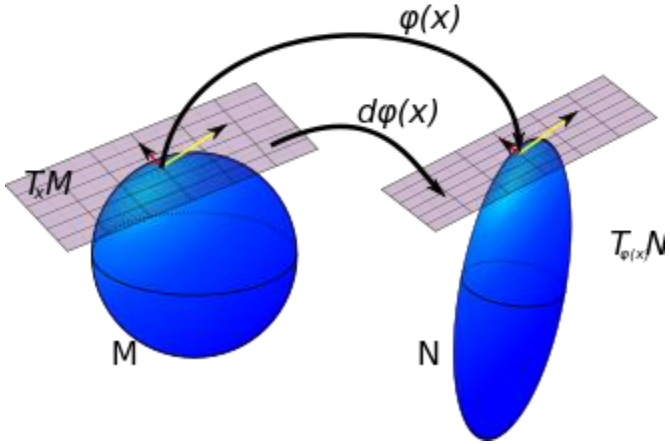
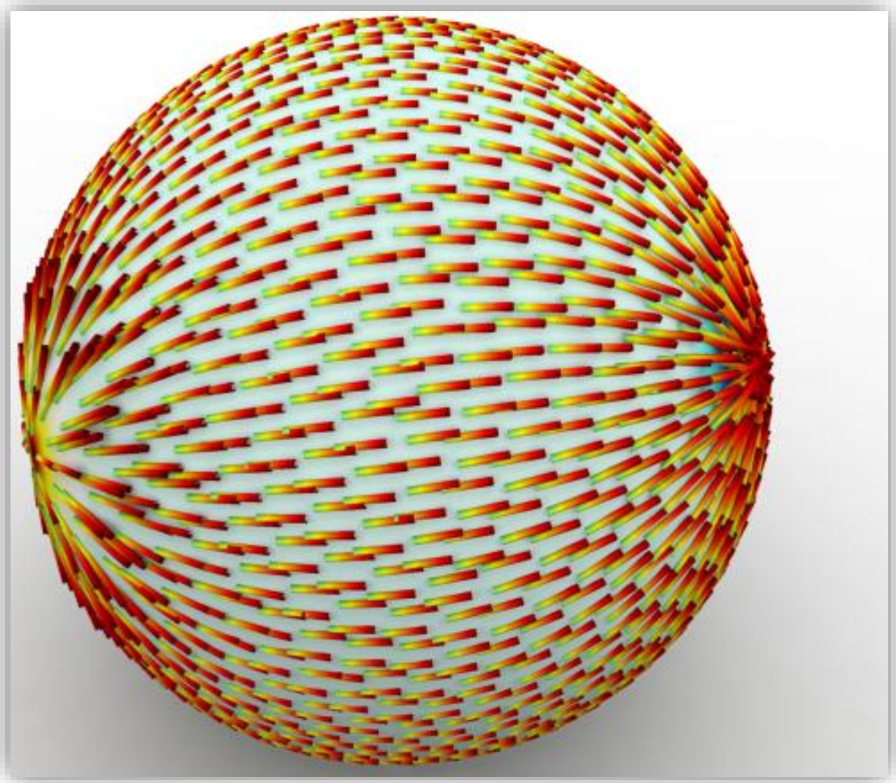


Image from Wikipedia

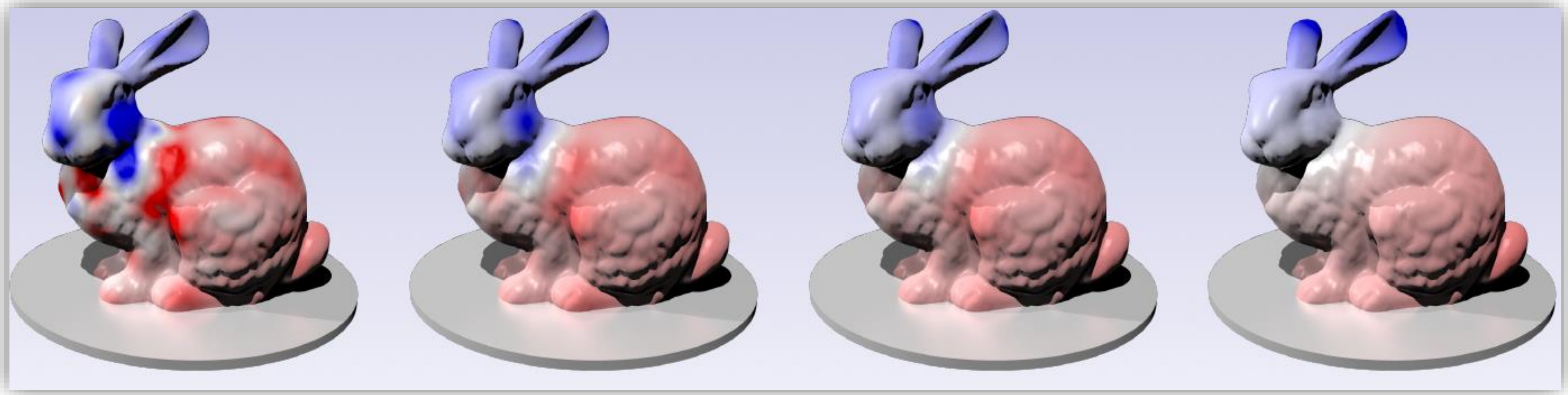
Gradient Vector Field

Proposition *For each* $\mathbf{p} \in \mathcal{M}$, *there exists a unique vector* $\nabla f(\mathbf{p}) \in T_{\mathbf{p}}\mathcal{M}$ *so that* $df_{\mathbf{p}}(\mathbf{v}) = \mathbf{v} \cdot \nabla f(\mathbf{p})$ *for all* $\mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$.



$$df_{\mathbf{p}}(\mathbf{v}) = \mathbf{v} \cdot \nabla f(\mathbf{p})$$

Dirichlet Energy



Decreasing ***E***

$$E[f] := \int_S \|\nabla f\|_2^2 \, dA$$

Images made by E. Vouga

From Inner Product to Operator

$$\langle f, g \rangle_\Delta := \int_S \nabla f(x) \cdot \nabla g(x) \, dA$$
$$:= \langle f, \Delta g \rangle$$

Implies
$\langle f, f \rangle \geq 0$

"Motivated" by finite-dimensional linear algebra.

Laplace-Beltrami operator

What is Divergence?

$$\mathbf{v} : \mathcal{M} \to \mathbb{R}^3 \text{ where } \mathbf{v}(\mathbf{p}) \in T_{\mathbf{p}}\mathcal{M}$$
$$d\mathbf{v}_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \to \mathbb{R}^3$$
$$\{\mathbf{e}_1, \mathbf{e}_2\} \subset T_{\mathbf{p}}\mathcal{M} \text{ orthonormal basis}$$

$$(\nabla \cdot \mathbf{v})_{\mathbf{p}} := \sum_{i=1}^{2} \langle \mathbf{e}_i, d\mathbf{v}(\mathbf{e}_i) \rangle_{\mathbf{p}}$$

Things we should check (but probably won't):

- Independent of choice of basis
- $\boldsymbol{\Delta} = -\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$

Extra lecture segment: Motivation for this formula

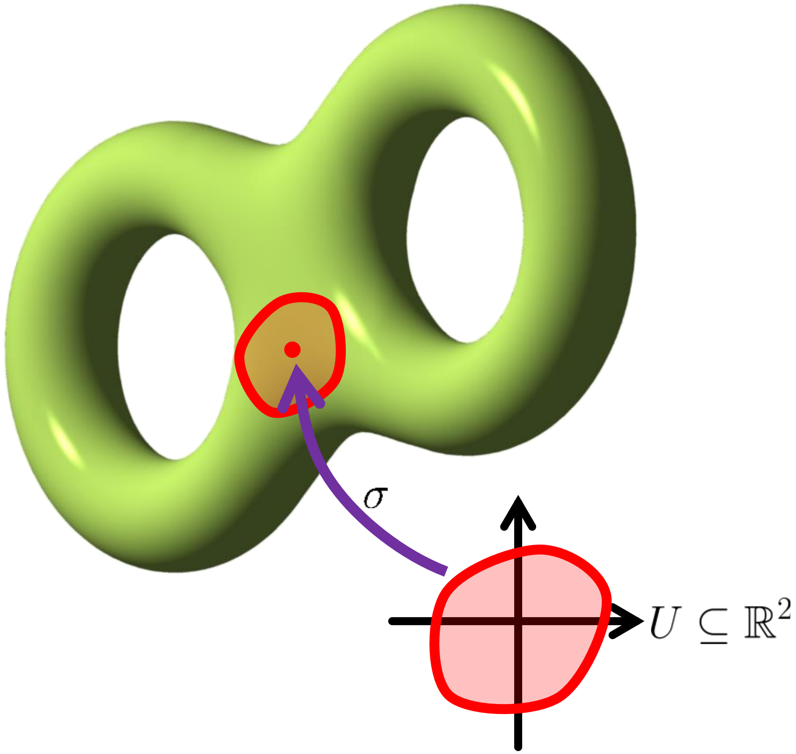
Flux Density: Backward Definition

$$\nabla \cdot \mathbf{v}(\mathbf{p}) := \lim_{r \to 0} \frac{\oint_{\partial B_r(\mathbf{p})} \mathbf{v} \cdot \mathbf{n}_{\text{tangent}} \, d\ell}{\mathrm{vol}(B_r(\mathbf{p}))}$$

Sanity Check: Local Version

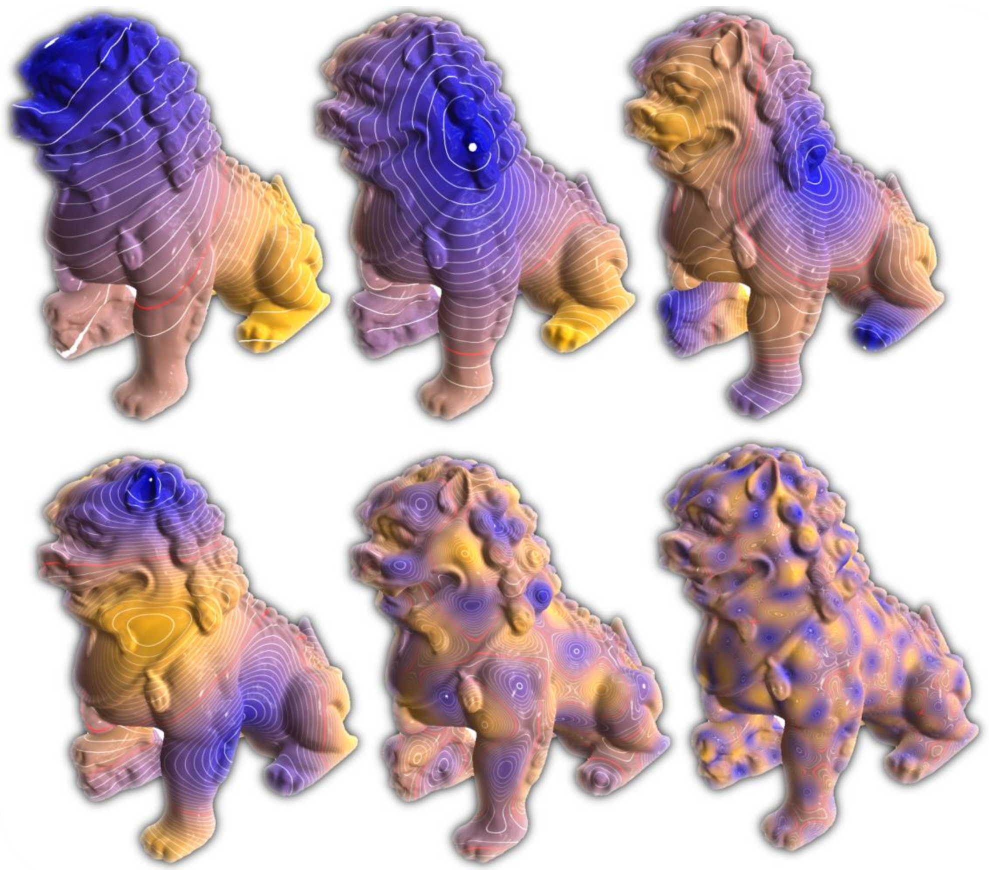
$$f : \mathcal{M} \to \mathbb{R}$$

$$\text{Pullback: } \sigma^* f := f \circ \sigma : U \to \mathbb{R}$$



Laplace-Beltrami **coincides with Laplacian** on $\mathbb{R}^2$ when σ takes x, y axes to orthonormal vectors.

Eigenfunctions



$$\Delta\psi_i = \lambda_i\psi_i$$

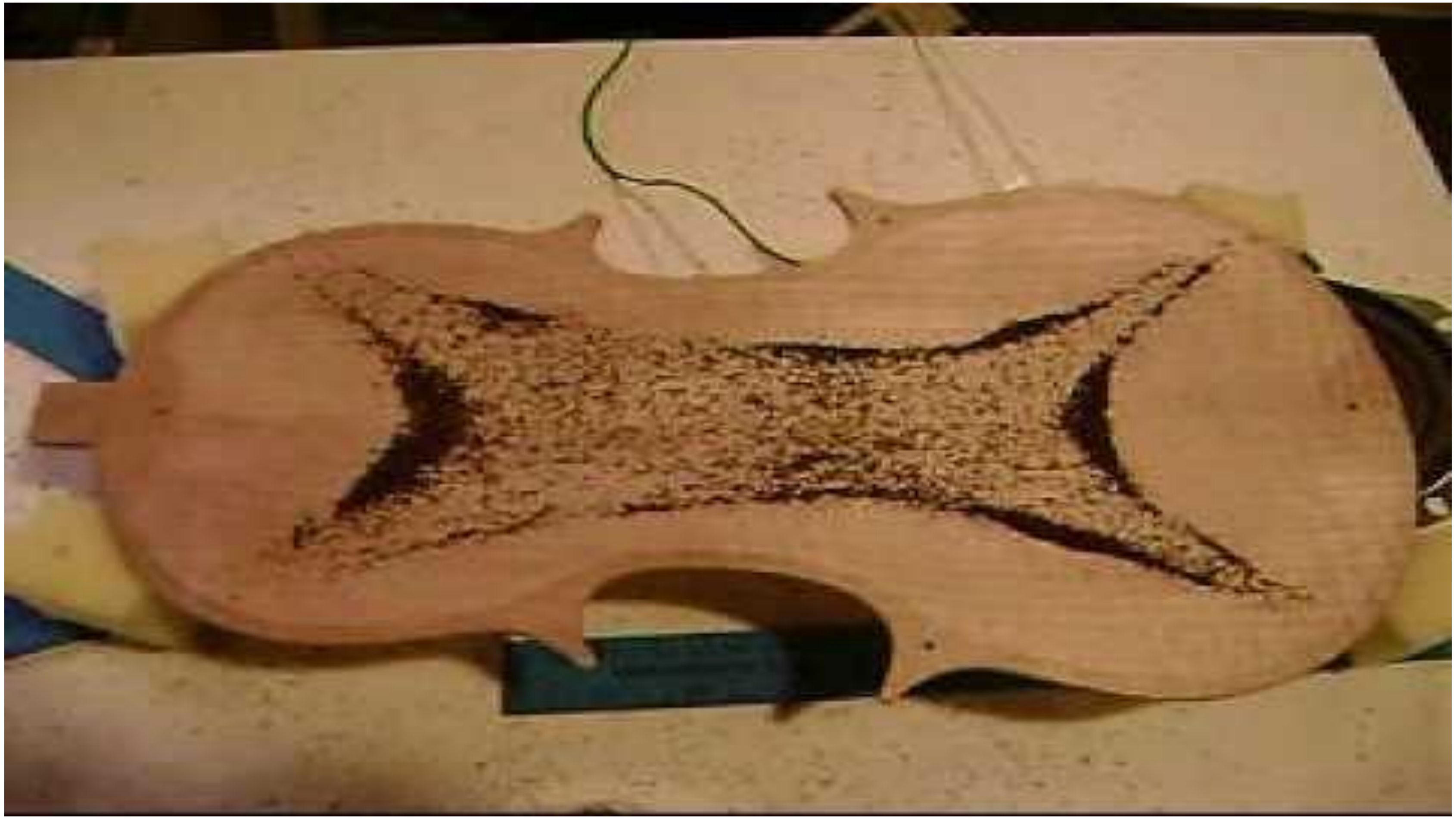
Vibration modes of surface (not volume!)

http://alice.loria.fr/publications/papers/2008/ManifoldHarmonics//photo/dragon_mhb.png

Chladni Plates

https://www.youtube.com/watch?v=CGiiSlMFFlI

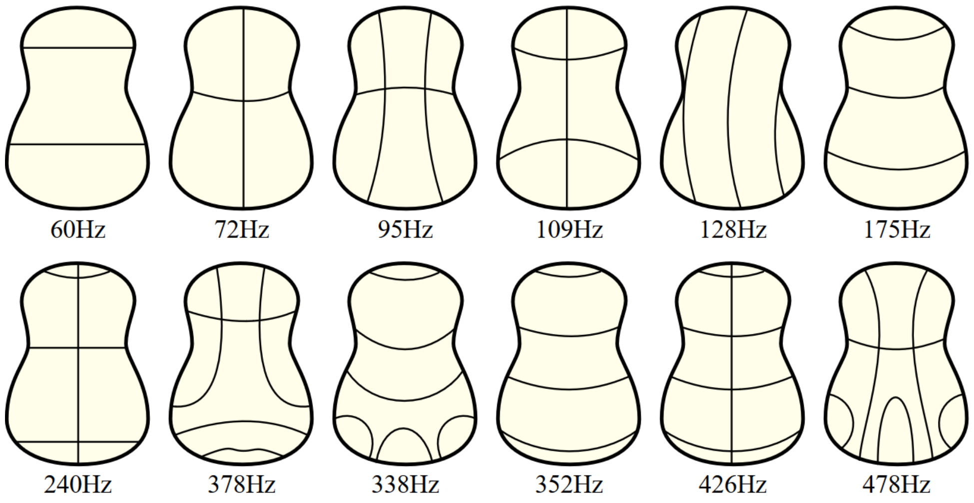
Practical Application



https://www.youtube.com/watch?v=3uMZzVvnSiU

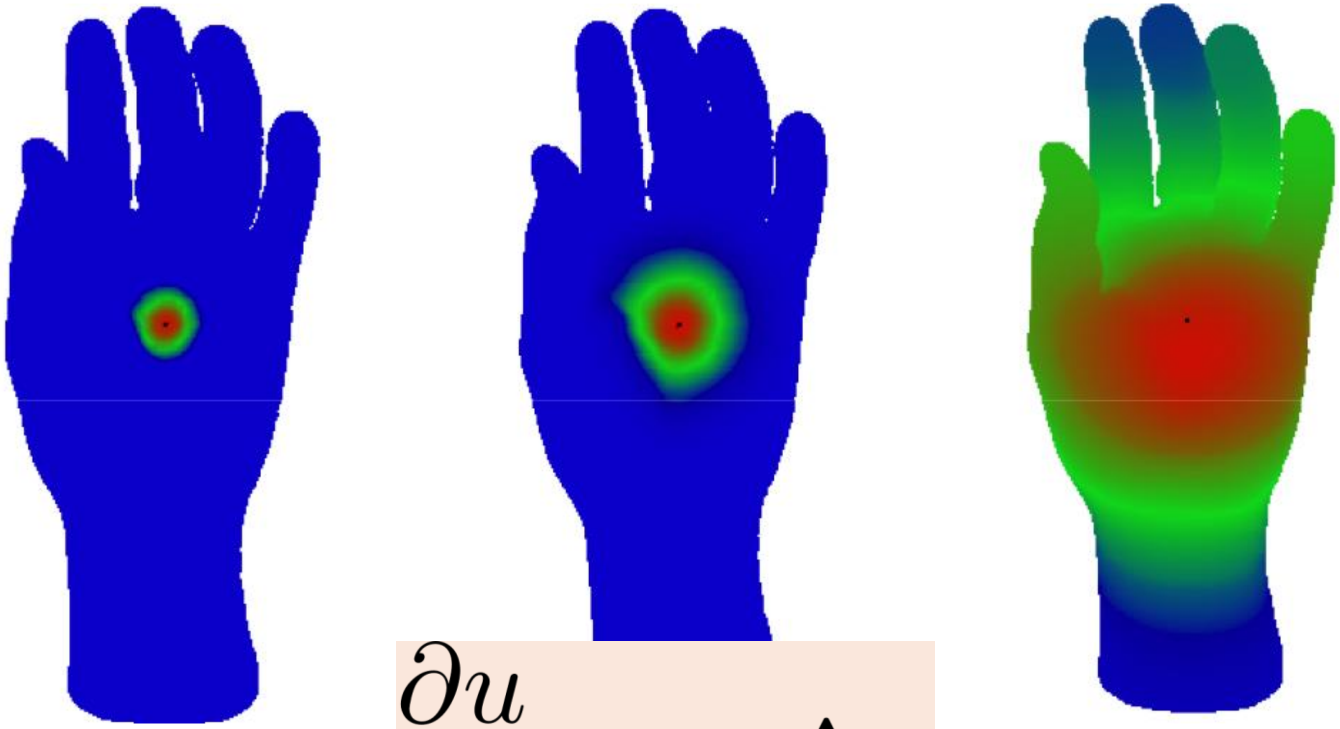
Nodal Domains

Theorem (Courant). The n-th eigenfunction of the Dirichlet boundary value problem has at most n nodal domains.



https://en.wikipedia.org/wiki/Ernst_Chladni

Additional Connection to Physics

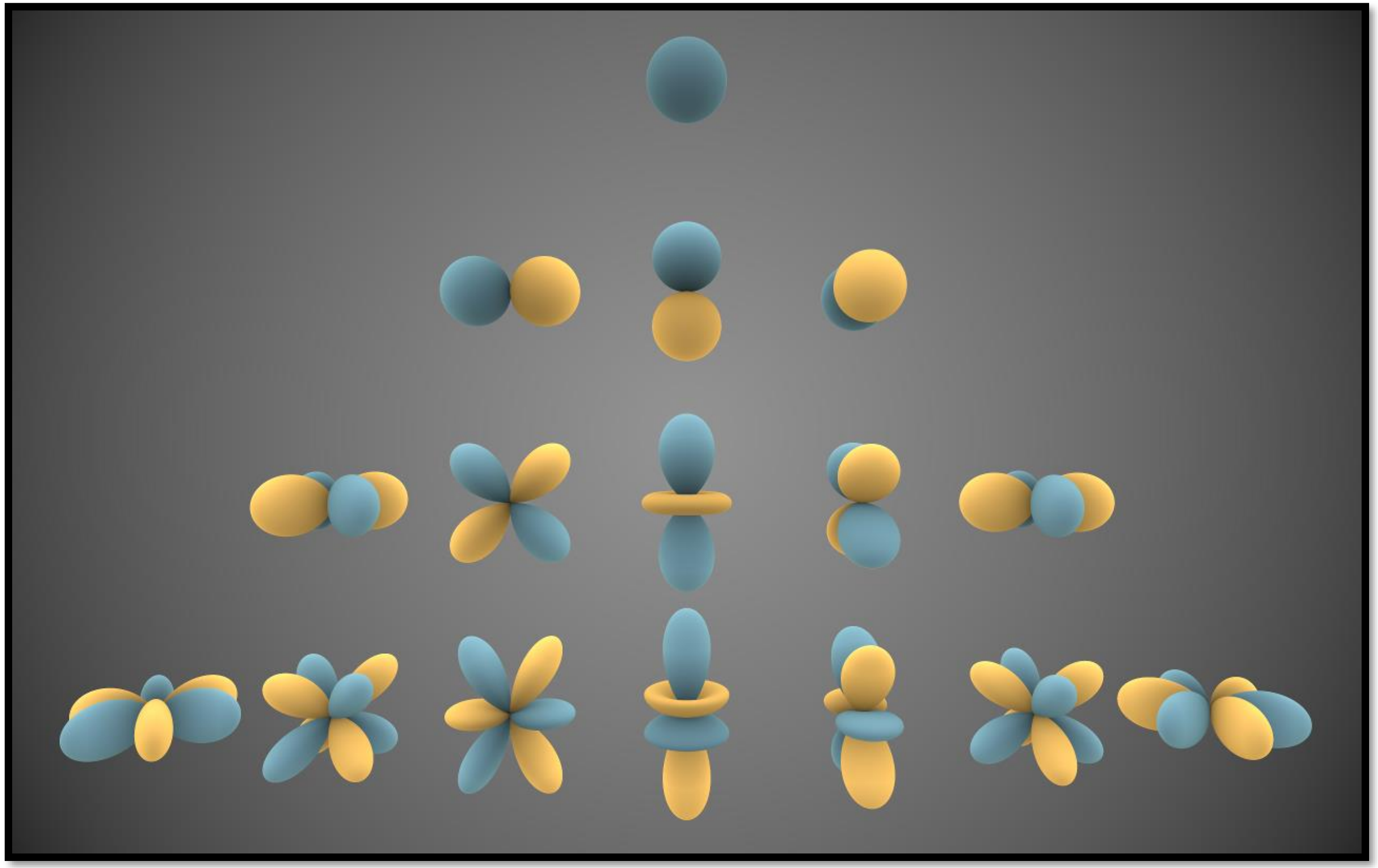


$$\frac{\partial u}{\partial t} = -\Delta u$$

http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

Spherical Harmonics



https://en.wikipedia.org/wiki/Spherical_harmonics

Weyl's Law

$$N(\lambda) := \#\ \text{eigenfunctions} \le \lambda$$

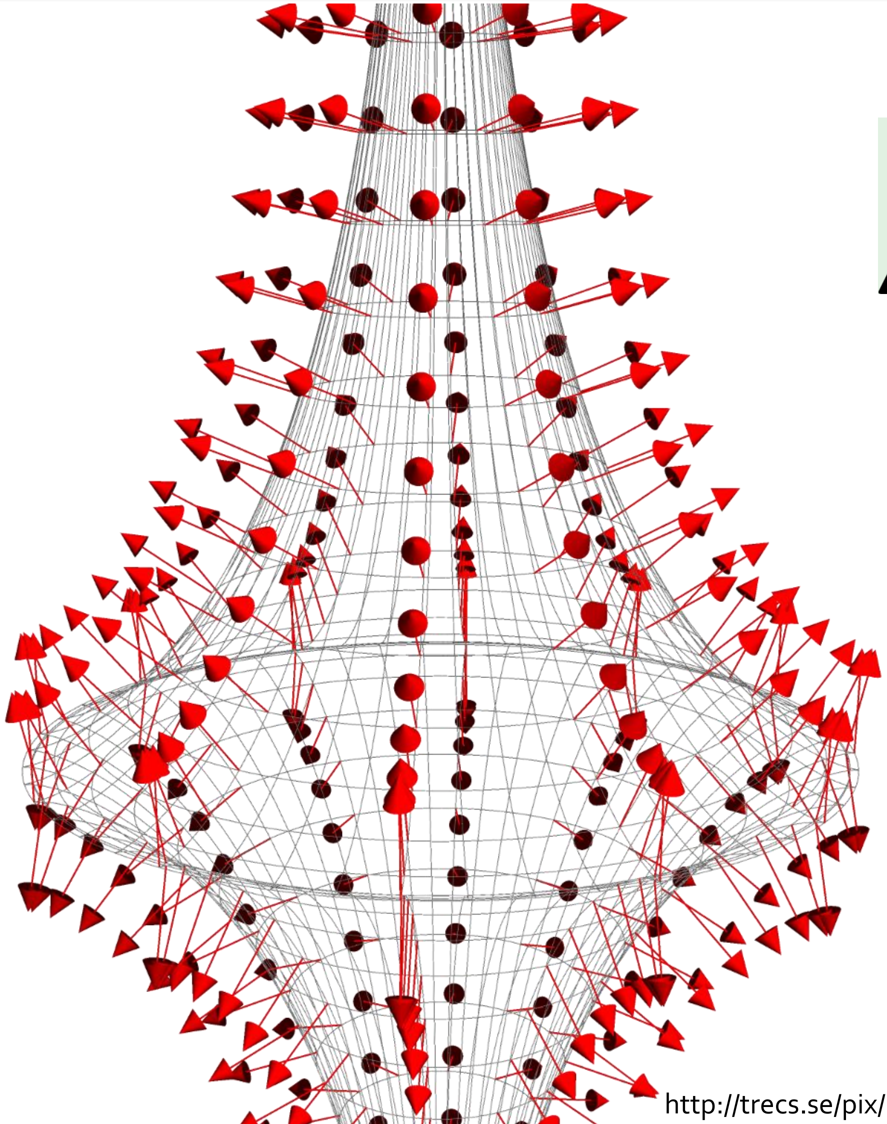
$$\omega_d := \text{volume of unit ball in } \mathbb{R}^d$$

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = (2\pi)^{-d} \omega_d \text{vol}(\Omega)$$

Corollary: $\text{vol}(\Omega) = (2\pi)^d \lim_{R \to \infty} \frac{N(R)}{R^{d/2}}$

$$\text{For surfaces: } \lambda_n \sim \frac{4\pi}{\text{vol}(\Omega)} n$$

Laplacian of *xyz* function



$$\Delta \mathbf{x} = H\mathbf{n}$$

Intuition:

Laplacian measures difference with neighbors.

http://trecs.se/pix/pseudosphereNormalVectorField4096.png

The Laplacian Operator

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6.838: Shape Analysis

Spring 2021

Extra: Divergence

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