Useful Formulas for 6.838

Basic Geometry and Trigonometry

$A = \frac{1}{2}bh$
$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\cot\theta = (\tan\theta)^{-1}$
$\cos^2\theta + \sin^2\theta = 1$
$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$

$\sin(\theta \pm \frac{\pi}{2}) = \pm\cos\theta$
$\cos(\theta \pm \frac{\pi}{2}) = \mp\sin\theta$
$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$e^{i\theta} = \cos\theta + i\sin\theta$
$\frac{d}{dt}\sin t = -\cos t$
$\frac{d}{dt}\cos t = \sin t$

Linear Algebra

$(AB)^{-1} = B^{-1}A^{-1}$
$(AB)^\top = B^\top A^\top$
$\mathrm{tr}(A) = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i$
$\mathrm{tr}(A) = \mathrm{tr}(A^\top)$
$\mathrm{tr}(AB) = \mathrm{tr}(BA)$
$\langle A, B\rangle = \sum_{ij} a_{ij}b_{ij} = \mathrm{tr}(A^\top B)$

$\|A\|_{\text{Fro}} = \sqrt{\langle A, A\rangle}$
$v \cdot w = v^\top w = \mathrm{tr}(v^\top w) = \mathrm{tr}(wv^\top)$
$\|v\|_2^2 = v \cdot v = v^\top v$
$\|v\|_p = (\sum_i |v_i|^p)^{1/p}$
$\det(A) = \prod_{i=1}^n \lambda_i$
$\det(A^{-1}) = 1/\det(A)$

Matrix Calculus

Check out matrixcalculus.org for a handy matrix derivative calculation tool. The Matrix Cookbook also contains a comprehensive list of identities.

$\frac{dY^{-1}}{dt} = -Y^{-1}\frac{dY}{dt}Y^{-1}$

$\nabla_x(x^\top b) = b$

$\nabla_X(a^\top X b) = ab^\top$

$\nabla_x(x^\top Ax + b^\top x) = (A + A^\top)x + b$

$\nabla_X \mathrm{tr}(X) = I$

$\nabla_X \mathrm{tr}(XB) = B^\top$

$\nabla_X \mathrm{tr}(X^\top BXC) = BXC + B^\top XC^\top$

$\nabla_X \det(X) = \det(X) \cdot X^{-\top}$

$e^A = \sum_n \frac{1}{n!}A^n$

$e^{ABA^{-1}} = Ae^B A^{-1}$

$Ae^A = e^A A$

$e^A e^B = e^{A+B+1/2[A,B]+\cdots}$

$\frac{d}{dt}e^A(t) = A'(t)e^A(t)$

Differential Vector Calculus

See this Wikipedia page for many vector calculus identities.

$df_x(v) = \lim_{h\to 0}\frac{f(x+hv)-f(x)}{h} = \nabla f \cdot v$
$\nabla f = (\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n})$
$\mathrm{div}\, F = \nabla \cdot F = \sum_i \frac{\partial F^i}{\partial x^i}$
$\mathrm{curl}\, F = \nabla \times F = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (F^x, F^y, F^z)$ for $F : \mathbb{R}^3 \to \mathbb{R}^3$
$\Delta f = -\nabla^2 f = -\nabla \cdot \nabla f = -\sum_{i=1}^n \frac{\partial^2 f}{\partial (x^i)^2}$
(in 6.838 we use a positive semidefinite Laplacian)

$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$
$\nabla \cdot (\psi A) = \psi\nabla \cdot A + (\nabla\psi) \cdot A$
$\nabla \times (\psi A) = \psi\nabla \times A + (\nabla\psi) \times A$
$\nabla \cdot (\nabla \times A) = 0$
$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) + \Delta A$
$\nabla \times (\nabla\psi) = 0$
$(f \circ g)'(t) = f'(g(t))g'(t)$
$f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)^\top Hf(x_0)(x - x_0) + O(\|x - x_0\|_2^3)$

Derivatives and Integrals, Integration by Parts, Stokes, etc.

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)\,dx\right) = f(b(t),t)\frac{db(t)}{dt} - f(a(t),t)\frac{da(t)}{dt} + \int_{a(t)}^{b(t)}\frac{\partial f}{\partial t}(x,t)\,dx$$

$$\frac{d}{dt}\int_{D(t)} F(x,t)\,dV = \int_{D(t)}\frac{\partial F}{\partial t}(x,t)\,dV + \oint F(x,t)v_b \cdot \hat{n}\,dA$$

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,dx$$

$$\int_\Omega u\nabla \cdot V\,dA = \oint_{\partial\Omega} uV \cdot \hat{n}\,d\ell - \int_\Omega \nabla u \cdot V\,dA$$

$$\int_\Omega (\psi\nabla \cdot \Gamma + \Gamma \cdot \nabla\psi)\,dA = \oint_{\partial\Omega}\psi(\Gamma \cdot \hat{n})\,d\ell$$

$$\int_\Omega (\psi\nabla \cdot (\varepsilon\nabla\phi) - \phi\nabla \cdot (\varepsilon\nabla\psi))\,dV = \oint_{\partial\Omega}\varepsilon(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n})\,dA$$

$$\int_\Omega [G \cdot (\nabla \times F) - F \cdot (\nabla \times G)]\,dV = \oint_{\partial\Omega}(F \times G) \cdot \hat{n}\,dA$$

$$\int_\Omega G \cdot \nabla f\,dV = \oint_{\partial\Omega}(fG) \cdot \hat{n}\,dA - \int_\Omega f(\nabla \cdot G)\,dV$$

$$\oint_{\partial\Omega} F \cdot \hat{n}\,dA = \int_\Omega \nabla \cdot F\,dV$$

$$\int_\Omega [F \cdot \nabla g + g(\nabla \cdot F)]\,dV = \oint_{\partial\Omega} gF \cdot \hat{n}\,dA$$