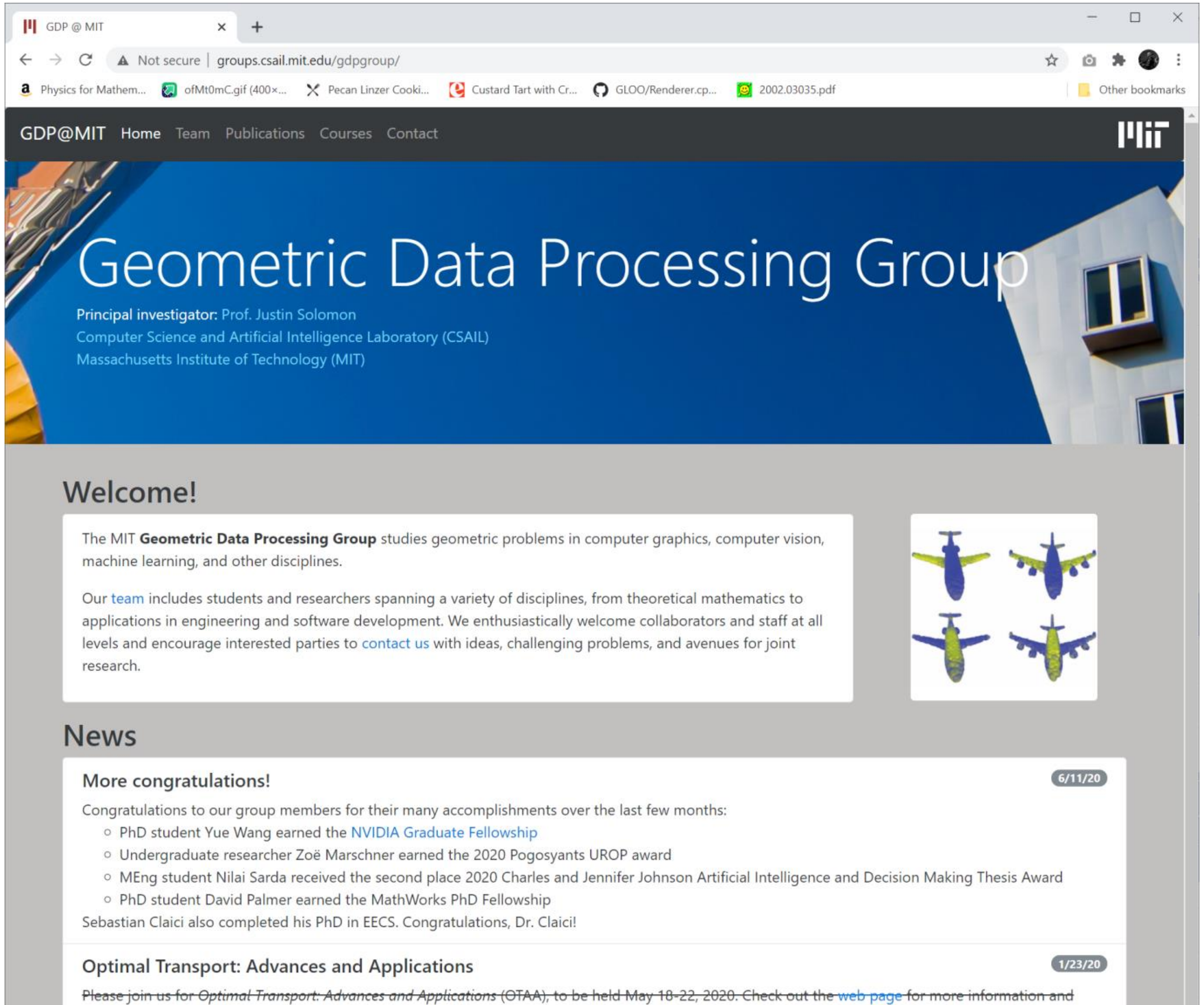
6.838:

Shape Analysis

Justin Solomon

Spring 2021

Course Instructor



Instructor: Justin Solomon
Email: jsolomon@mit.edu

Geometric Data Processing Group:
http://gdp.csail.mit.edu

Will cover administrative details over Zoom.

Prerequisites

- **Coding**
  Julia, Python, or Matlab preferred
- **Math**
  Fluency in linear algebra and multivariable calculus
- **Not required (won’t hurt):**
  Graphics, differential geometry, numerics, ML

Philosophy

We want you to take this course!

Assignments intended to be interesting
(may be unintentionally easy/hard!).

Image from postermywall.com

Theme

1. $\underbrace{\textit{Geometric data}}_{\text{Modifier}}\ \underbrace{\textit{analysis:}}_{\text{Noun}}$ The analysis of geometric data

2. $\underbrace{\textit{Geometric}}_{\text{Modifier}}\ \underbrace{\textit{data analysis:}}_{\text{Noun}}$ Data analysis using geometric techniques

Applied Geometry

I. Theoretical toolbox

II. Computational toolbox

III. Application areas

Mostly a picture book!

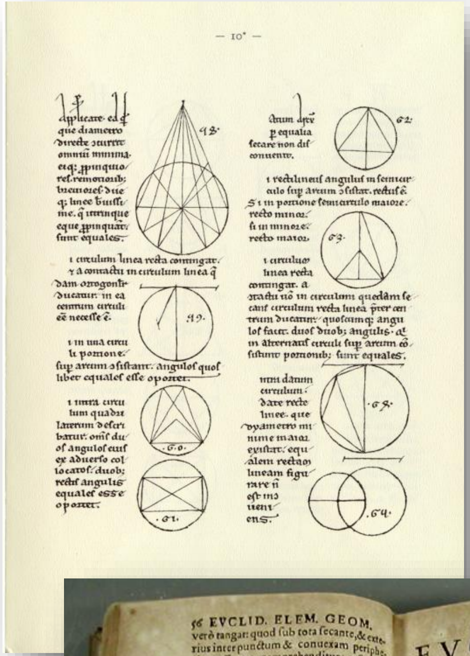
Applied Geometry

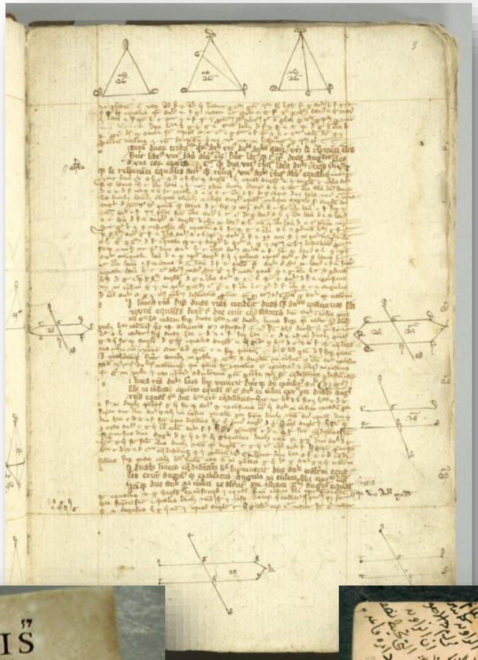
I. Theoretical toolbox

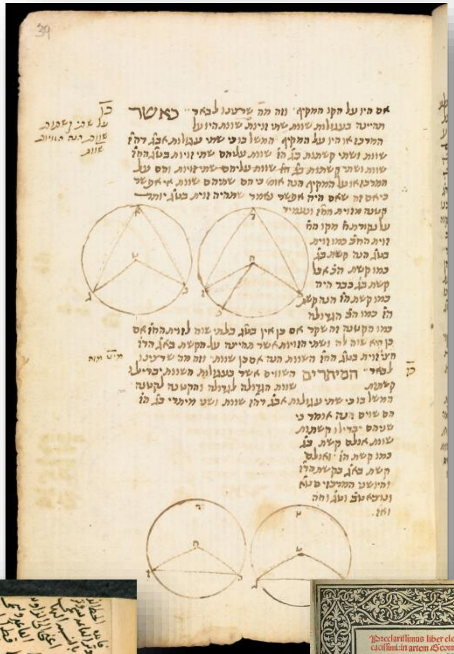
II. Computational toolbox

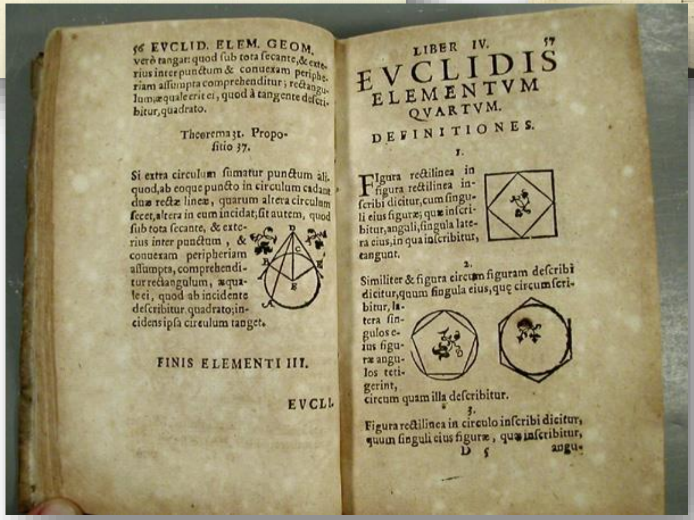
III. Application areas

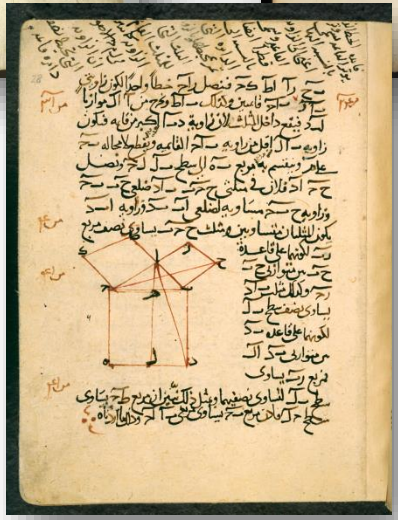
Euclidean Geometry

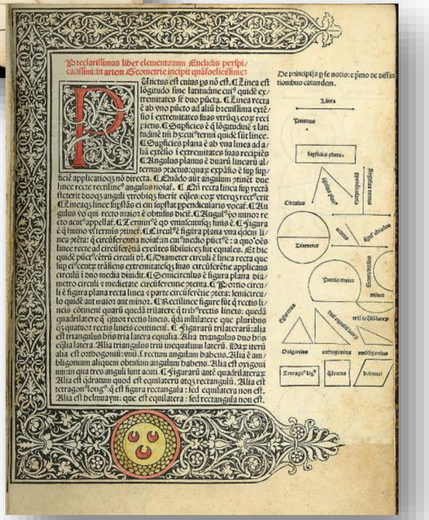






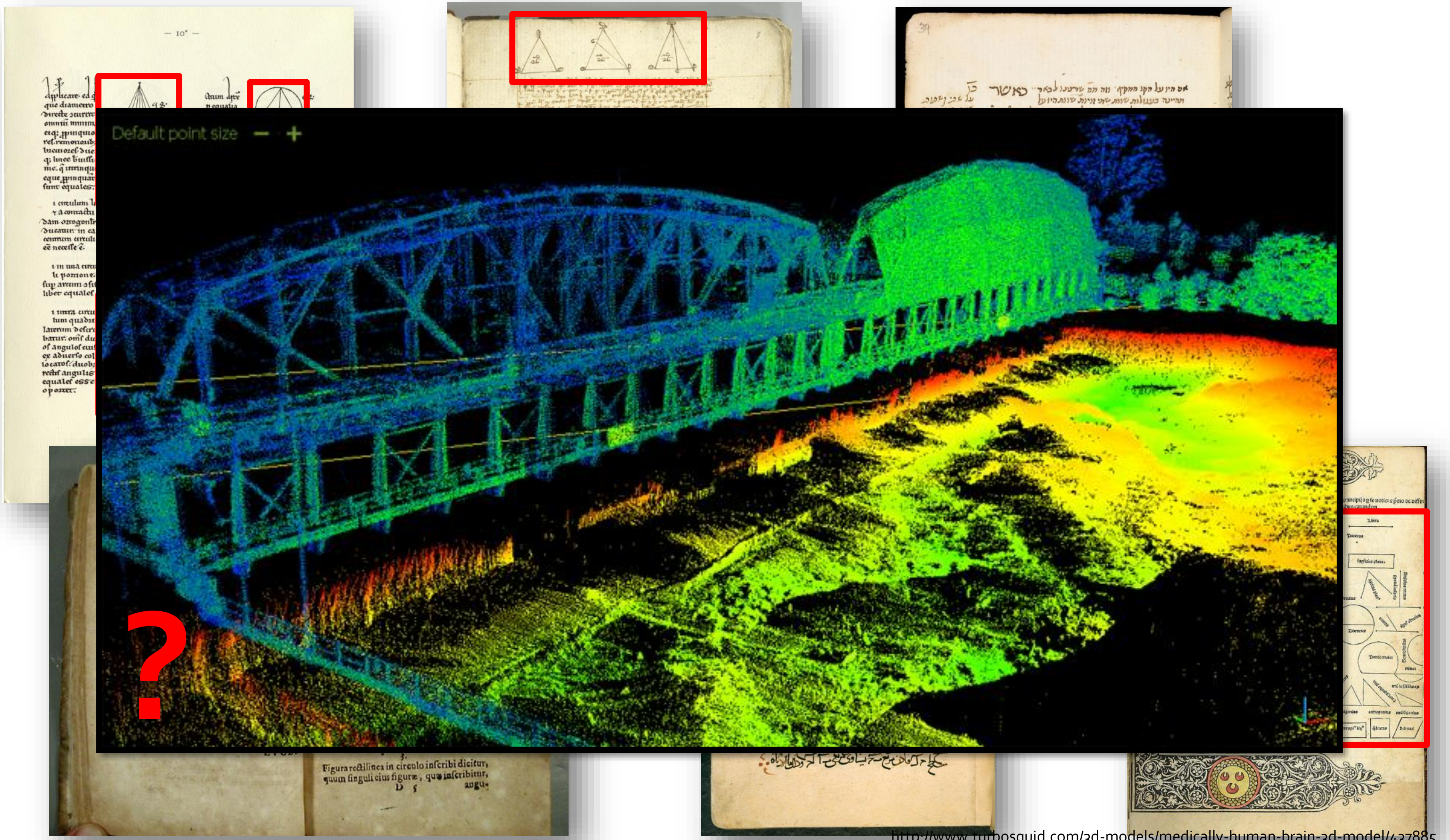






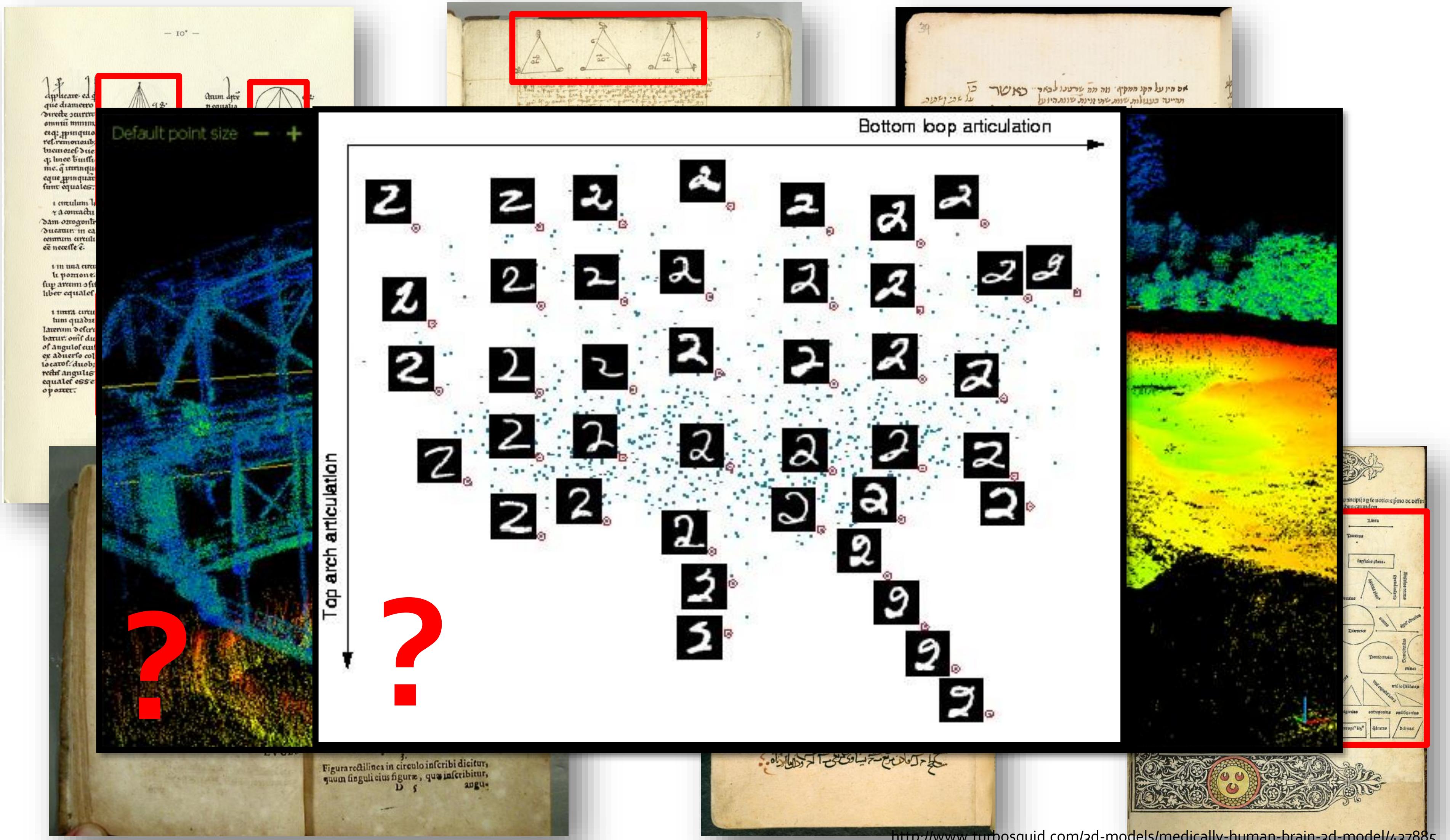
Euclidean Geometry

Euclidean Geometry



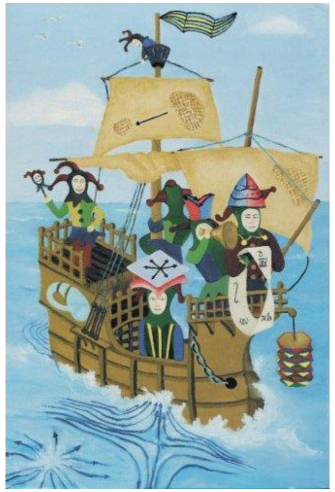
http://www.turbosquid.com/3d-models/medically-human-brain-3d-model/437885

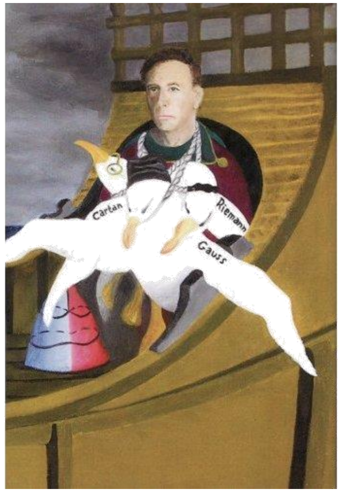
Euclidean Geometry

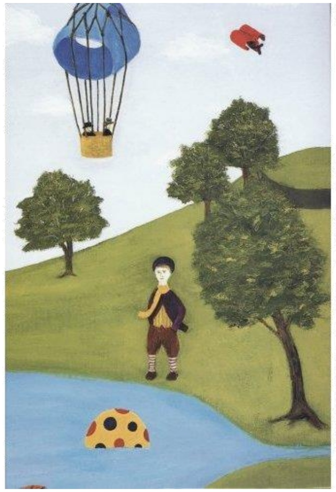


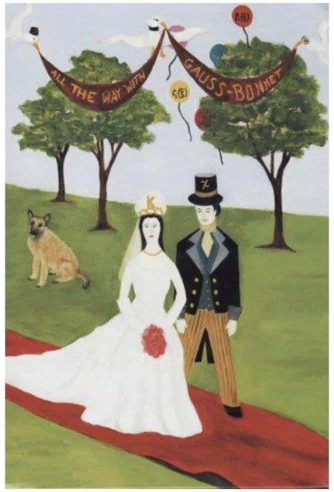
http://www.turbosquid.com/3d-models/medically-human-brain-3d-model/437885

Differential Geometry



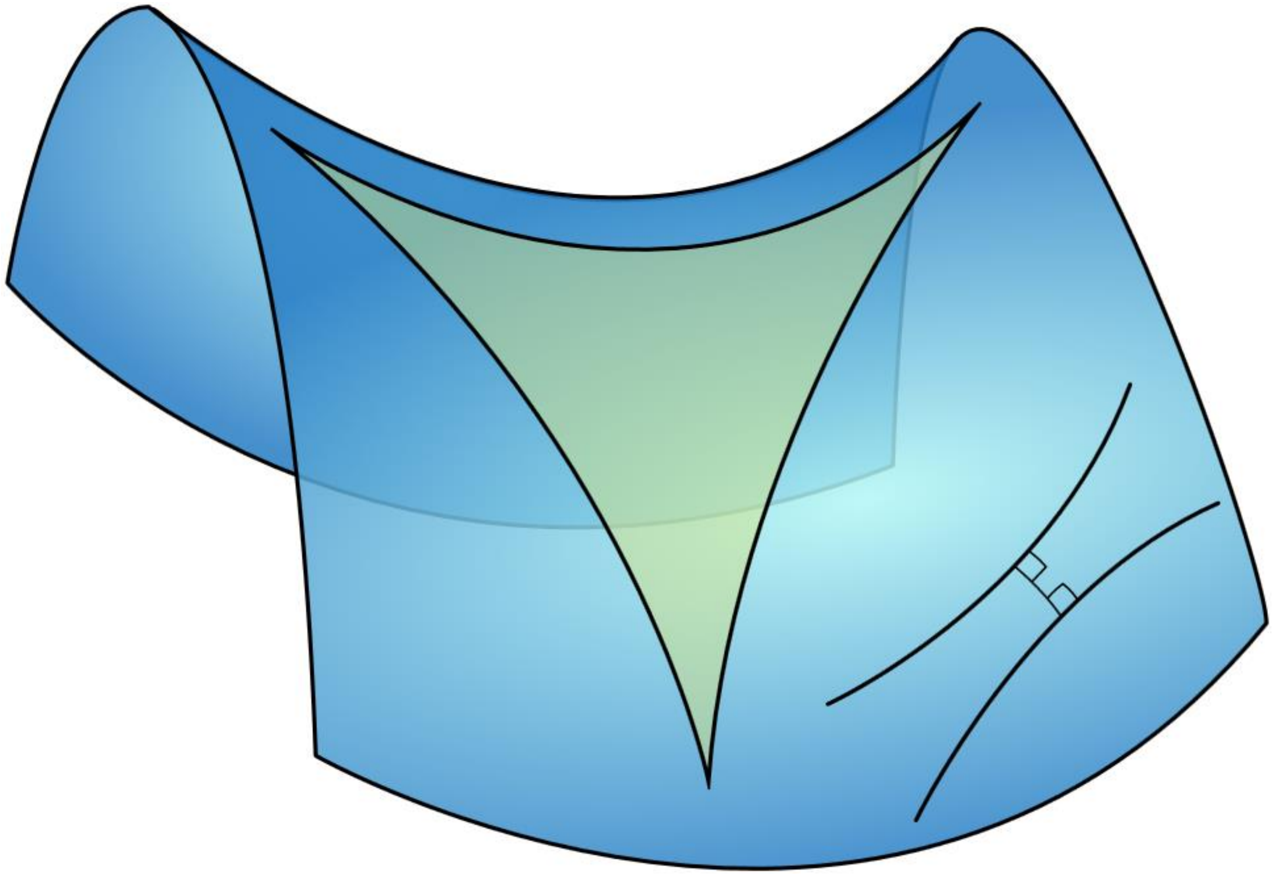






Spivak: *A Comprehensive Introduction to Differential Geometry*

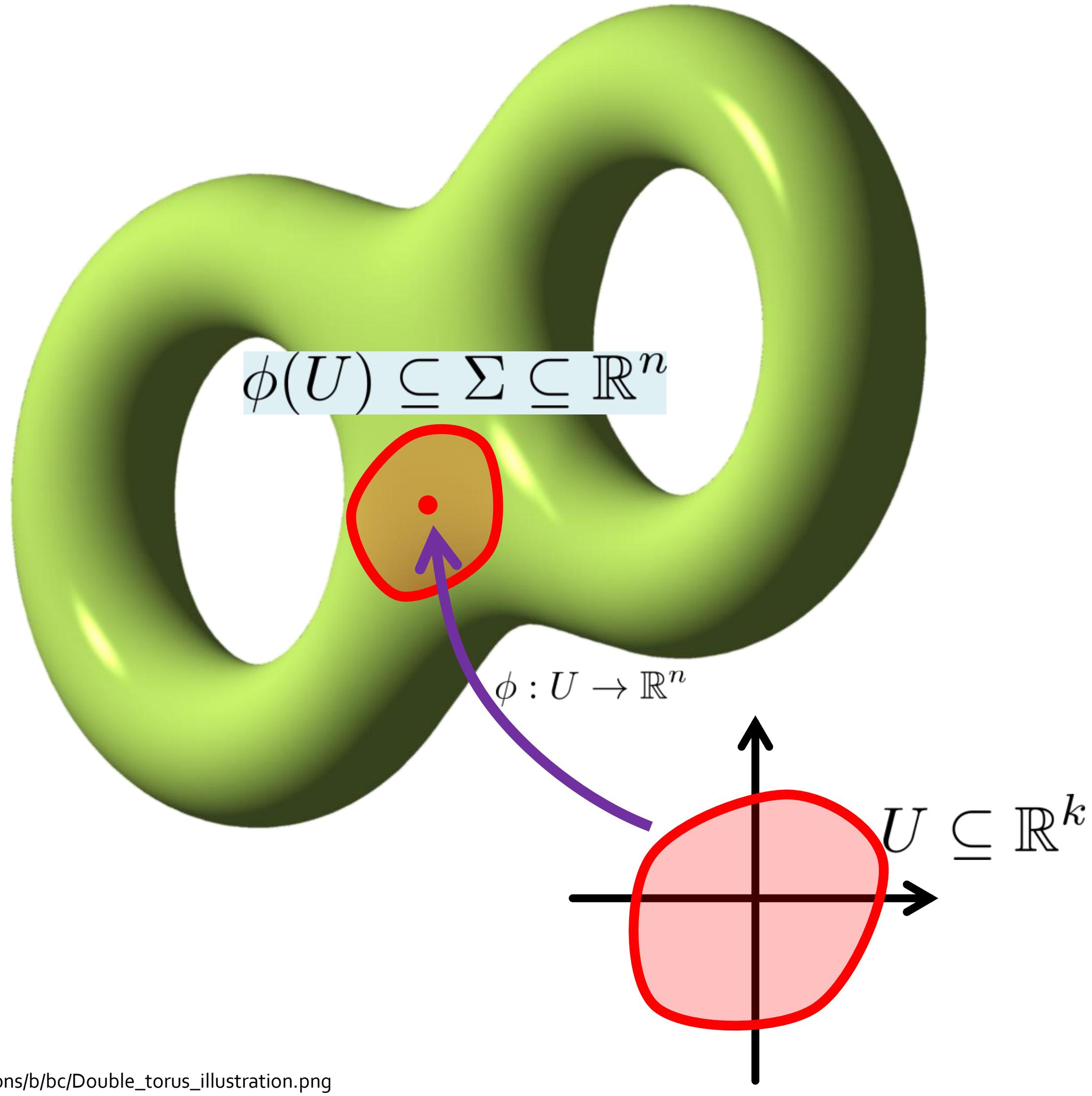
Differential Geometry



http://en.wikipedia.org/wiki/Differential_geometry

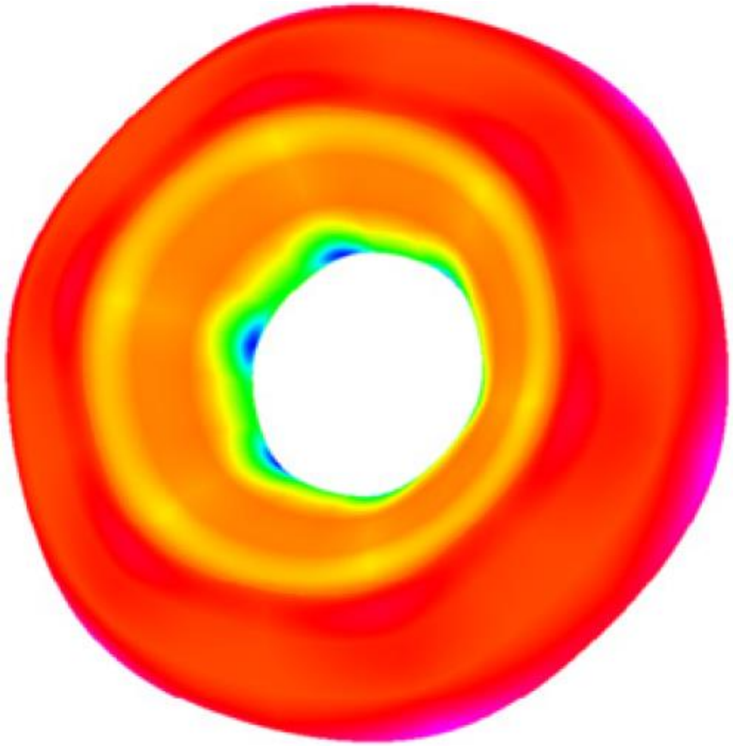
Study of smooth manifolds

Manifold

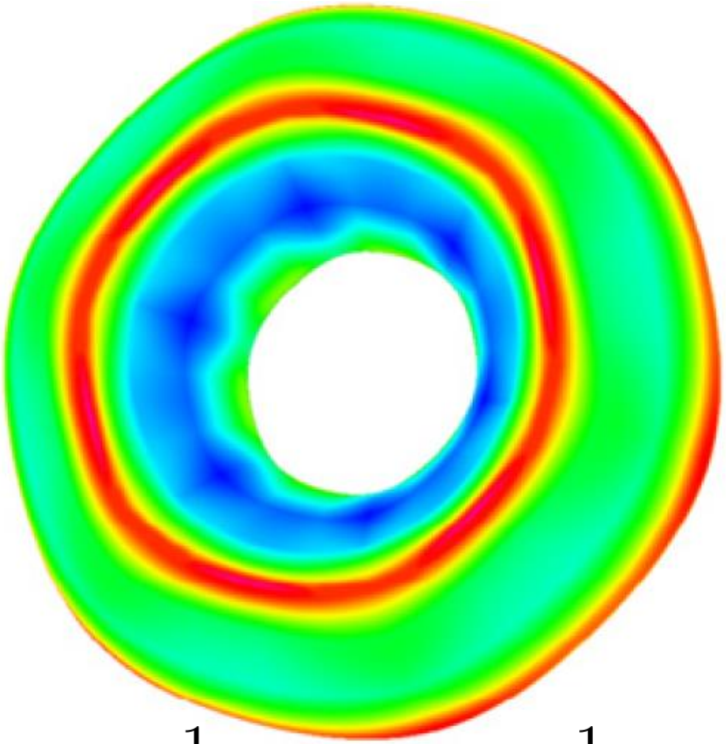


http://upload.wikimedia.org/wikipedia/commons/b/bc/Double_torus_illustration.png

Differential Geometry Toolbox



$$K := \kappa_1 \kappa_2 = \det \mathrm{I\!I}$$

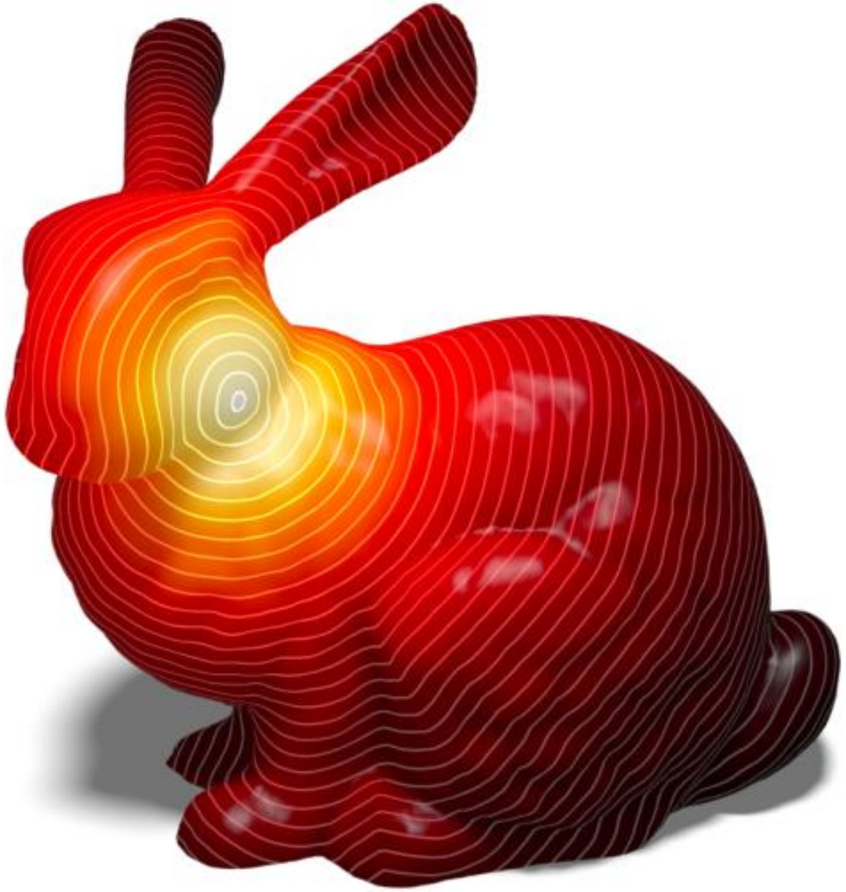


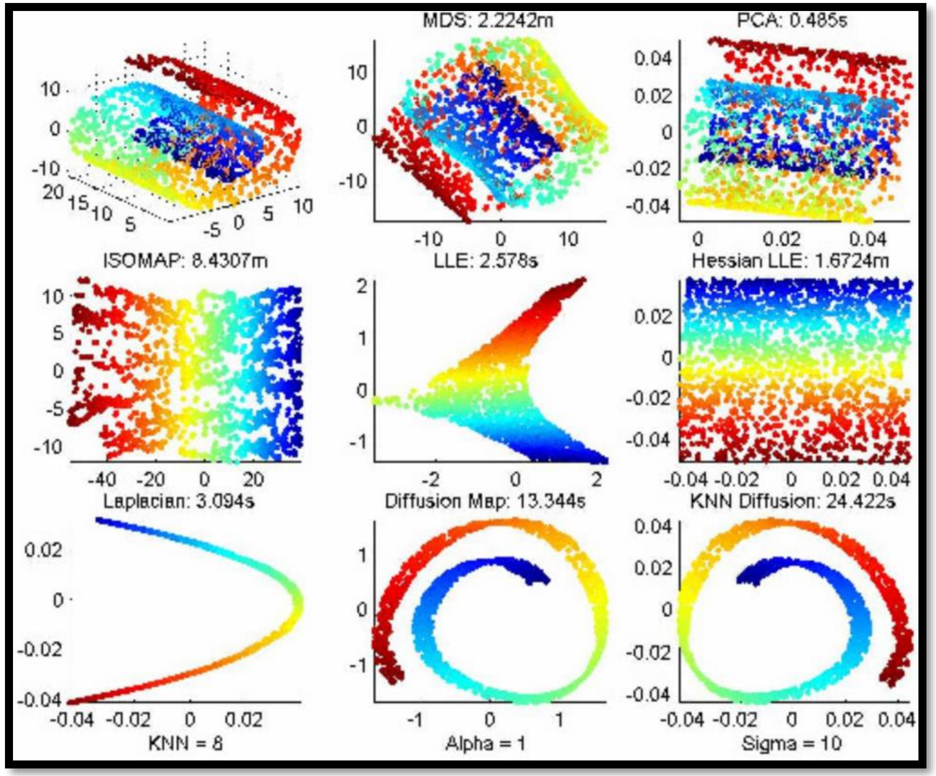
$$H := \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2} \mathrm{tr}\, \mathrm{I\!I}$$

http://www.sciencedirect.com/science/article/pii/S0010448510001983

Curvature and shape properties

Differential Geometry Toolbox

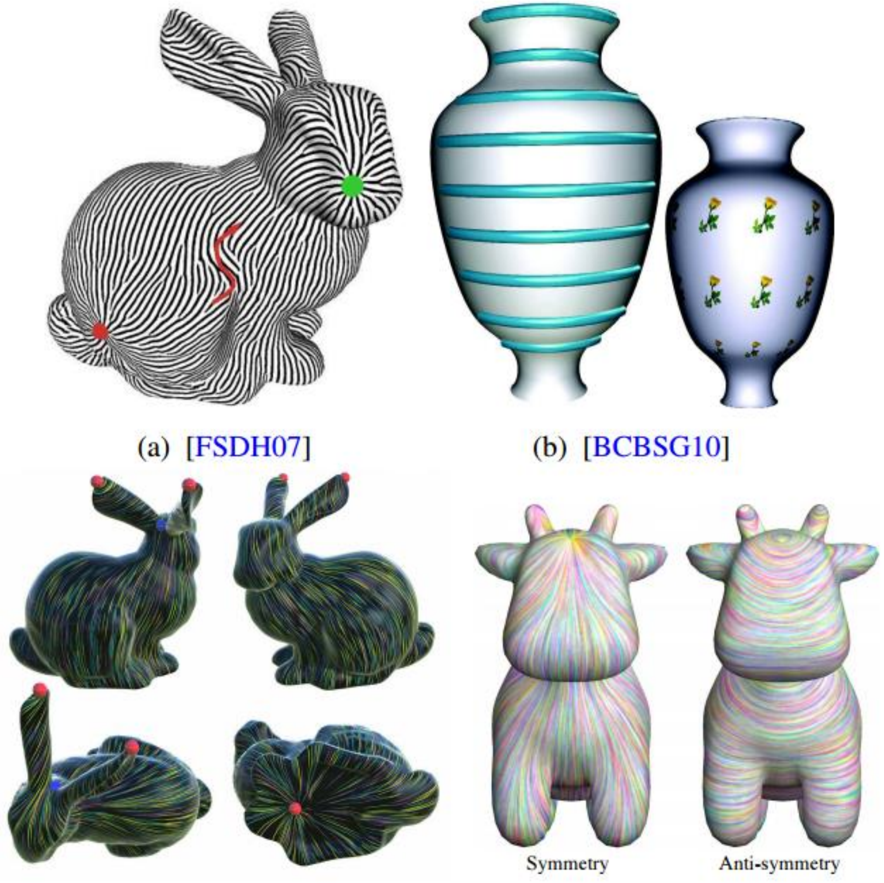




Crane, Weischedel, Wardetzky. *Geodesics in heat*. TOG 2013.
Wittman. Manifold learning techniques.

Distances

Differential Geometry Toolbox

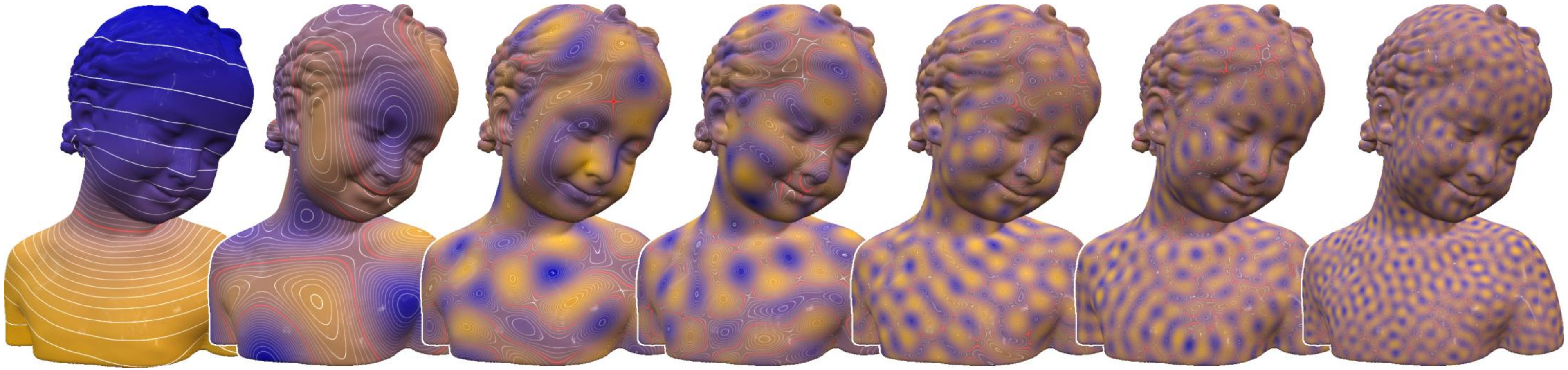


(a) [FSDH07] (b) [BCBSG10]
(c) [KCPS13] (d) [ABCCO13]

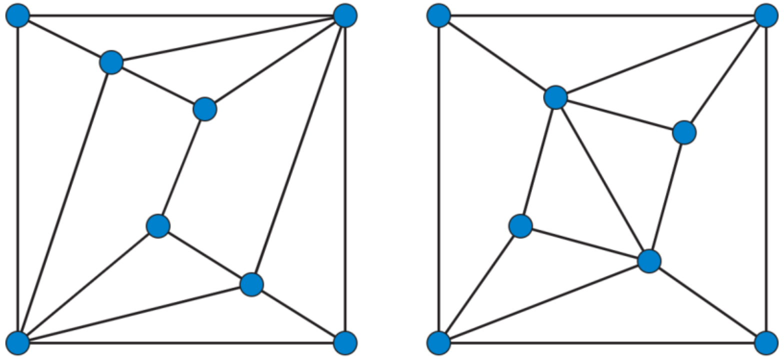
Vaxman et al.
Directional field synthesis, design, and processing.
EG STAR 2016.

Flows and vector fields

Differential Geometry Toolbox



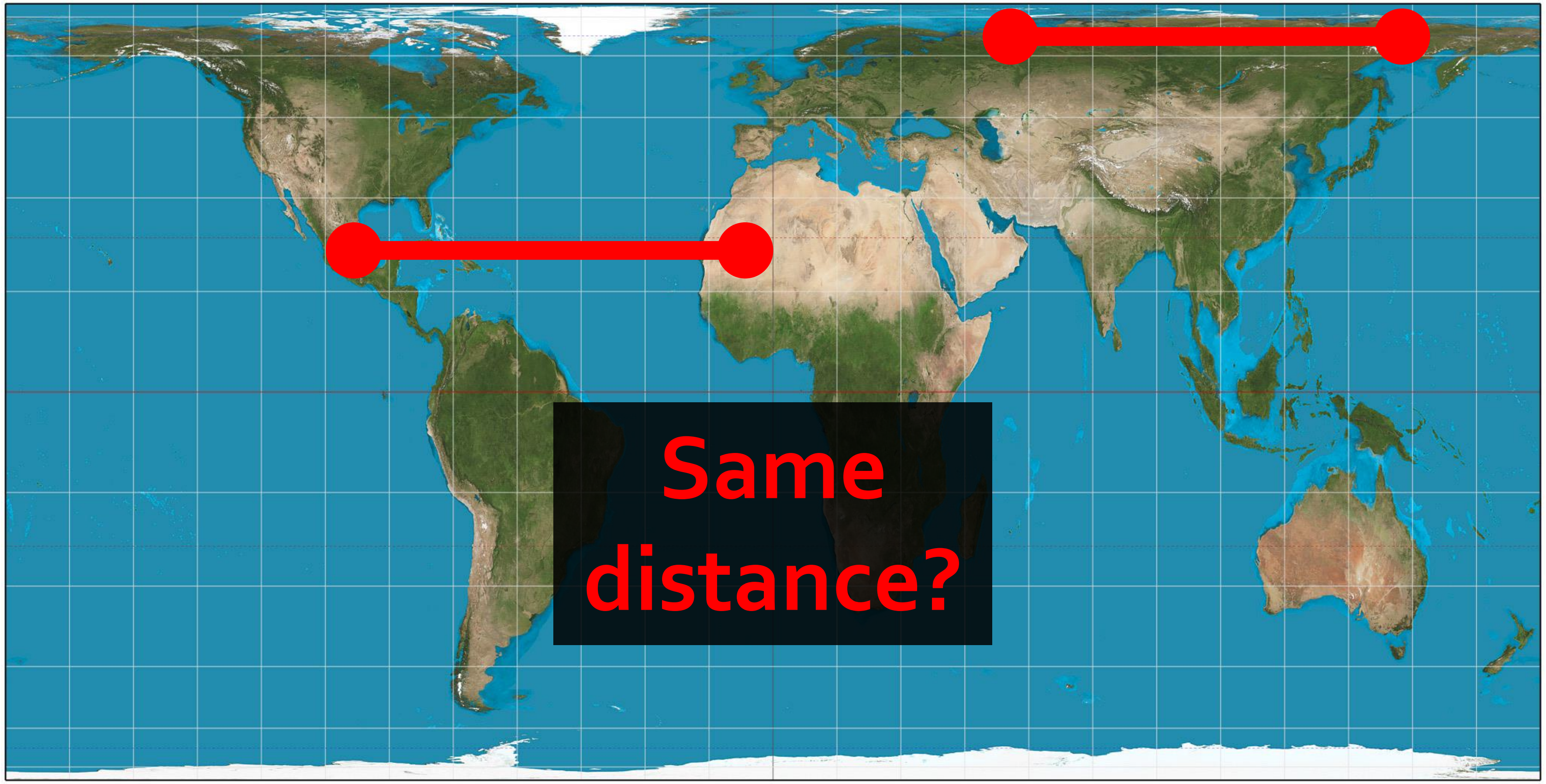
Vallet and Lévy. *Spectral Geometry Processing with Manifold Harmonics*. EG 2008



"Enneahedra"

Differential operators

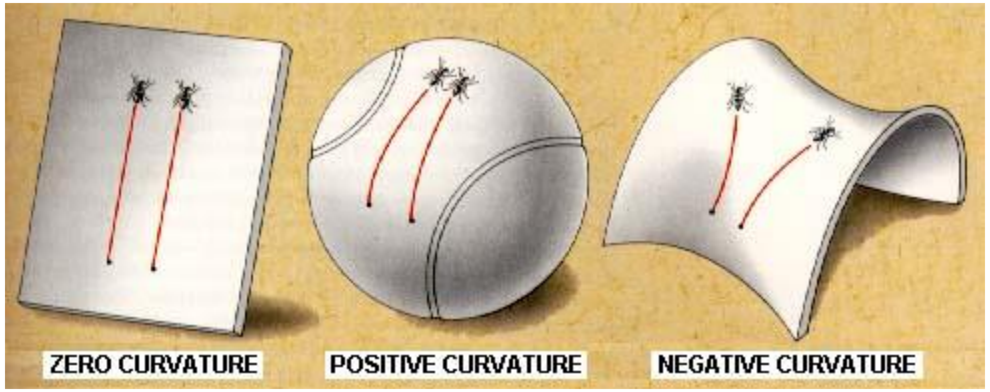
Riemannian Geometry

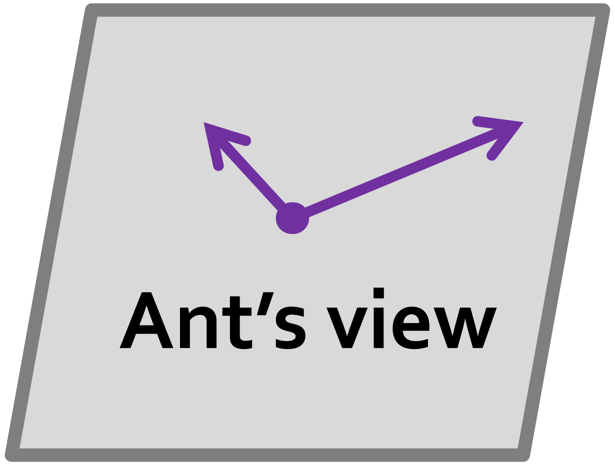


http://upload.wikimedia.org/wikipedia/commons/2/2c/Hobo%E2%80%93Dyer_projection_SW.jpg

Only need angles and distances

Riemannian Viewpoint

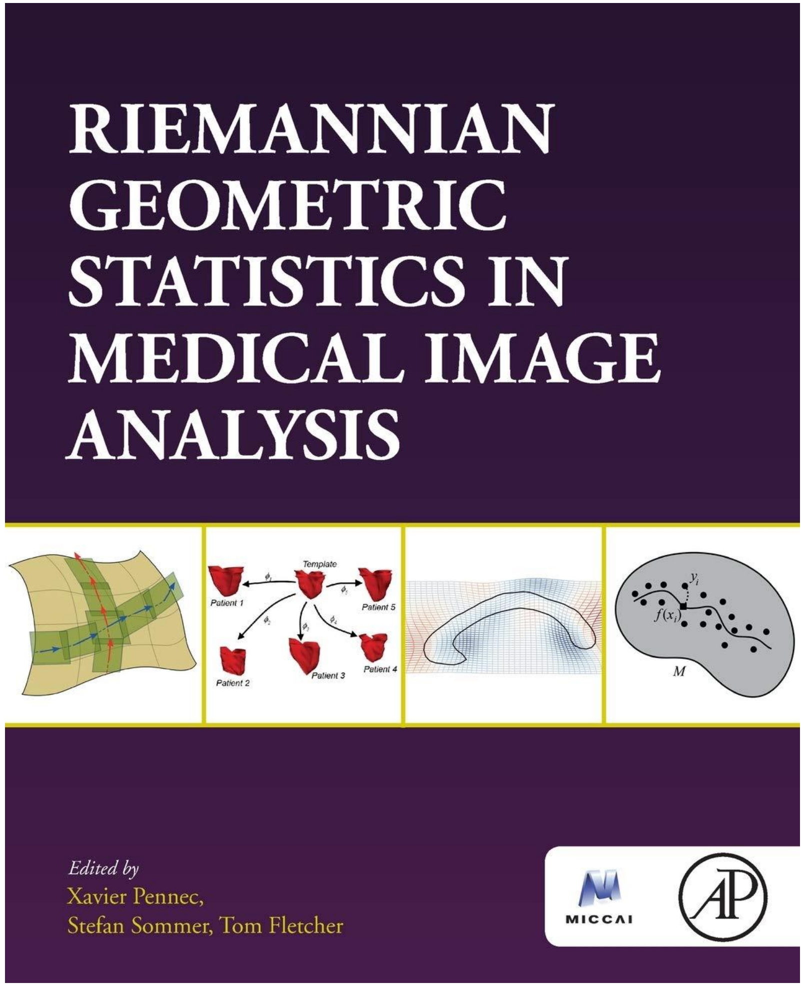


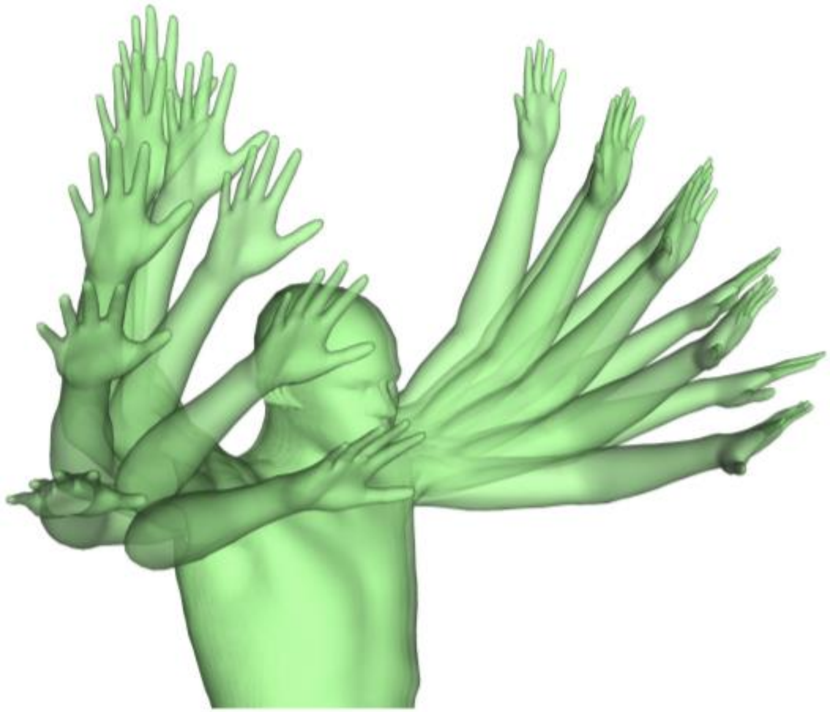


http://www.phy.syr.edu/courses/modules/LIGHTCONE/pics/curved.jpg

Only need angles and distances

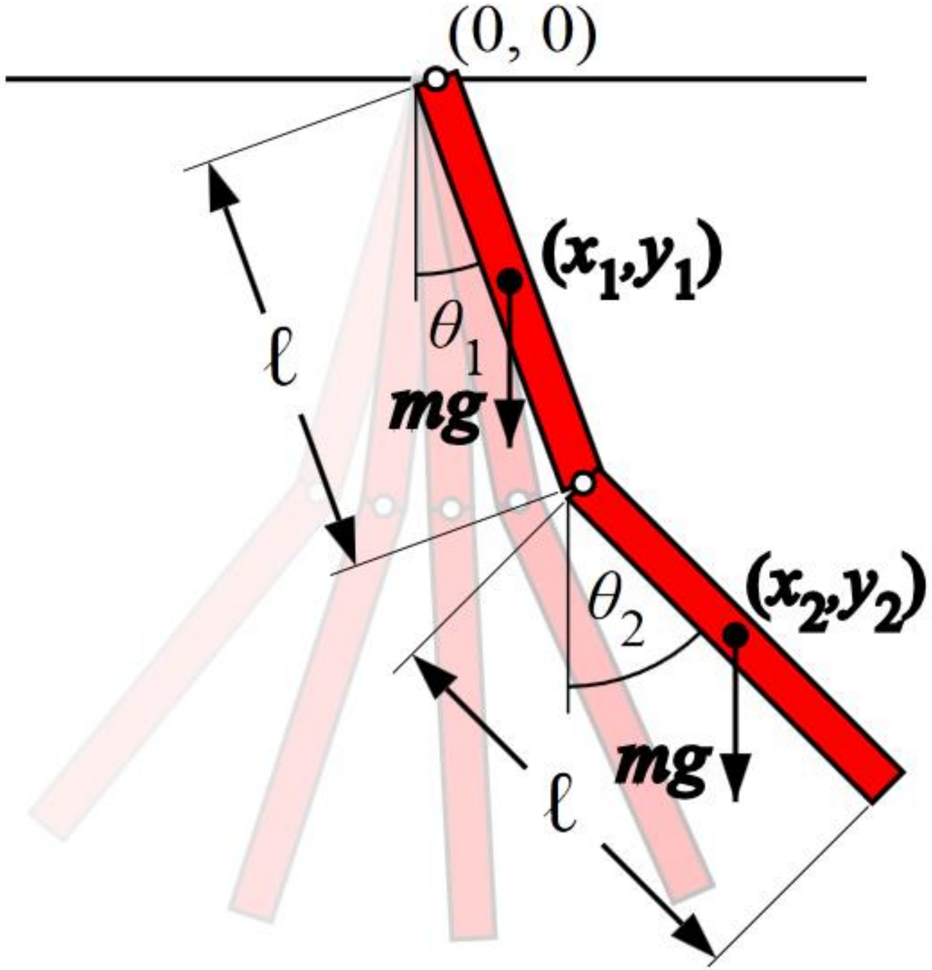
High-Dimensional Geometry

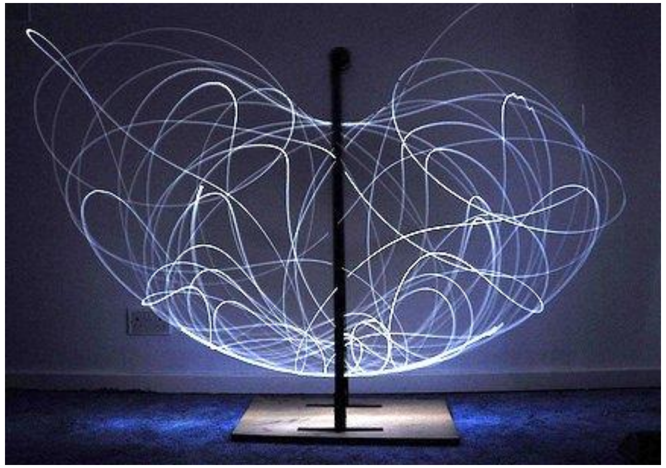


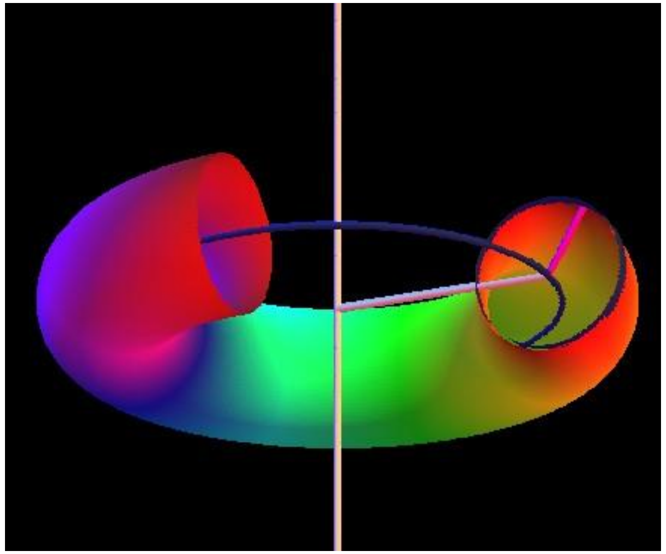


Heeren et al.
Splines in the Space of Shells.
SGP 2016.

Geometric Mechanics and Lie Groups

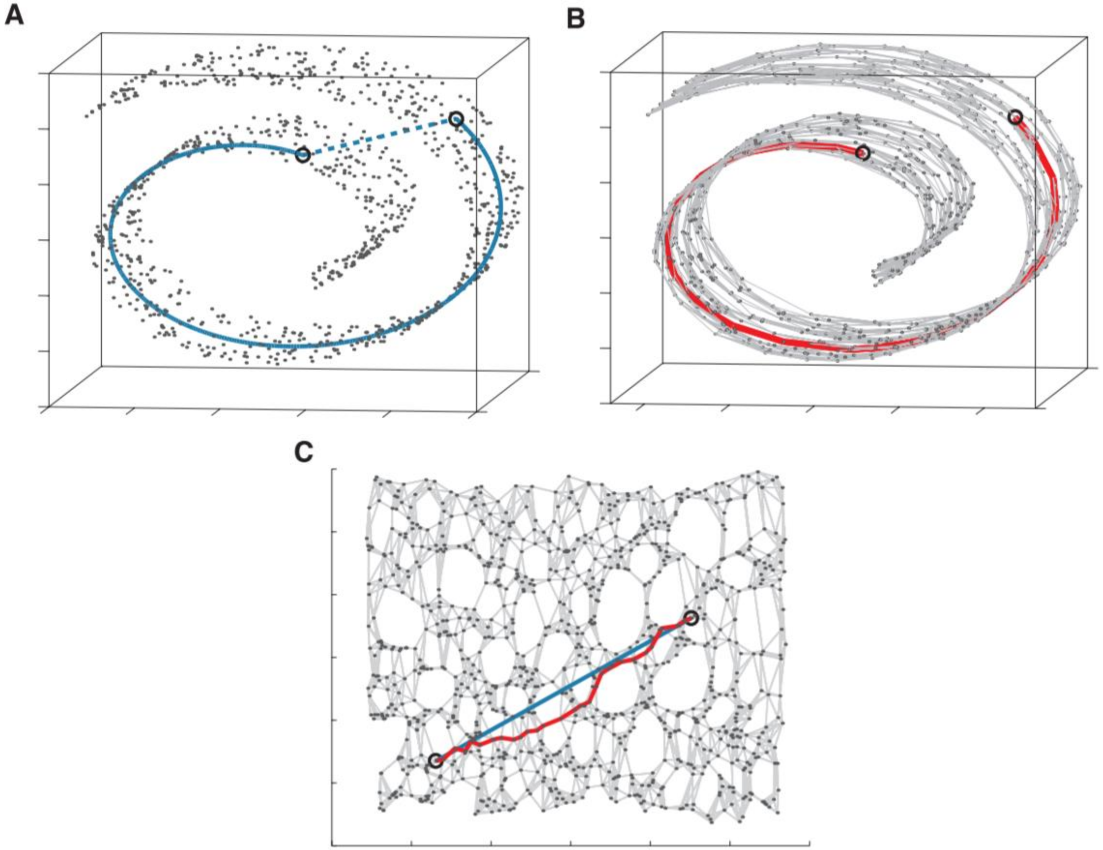


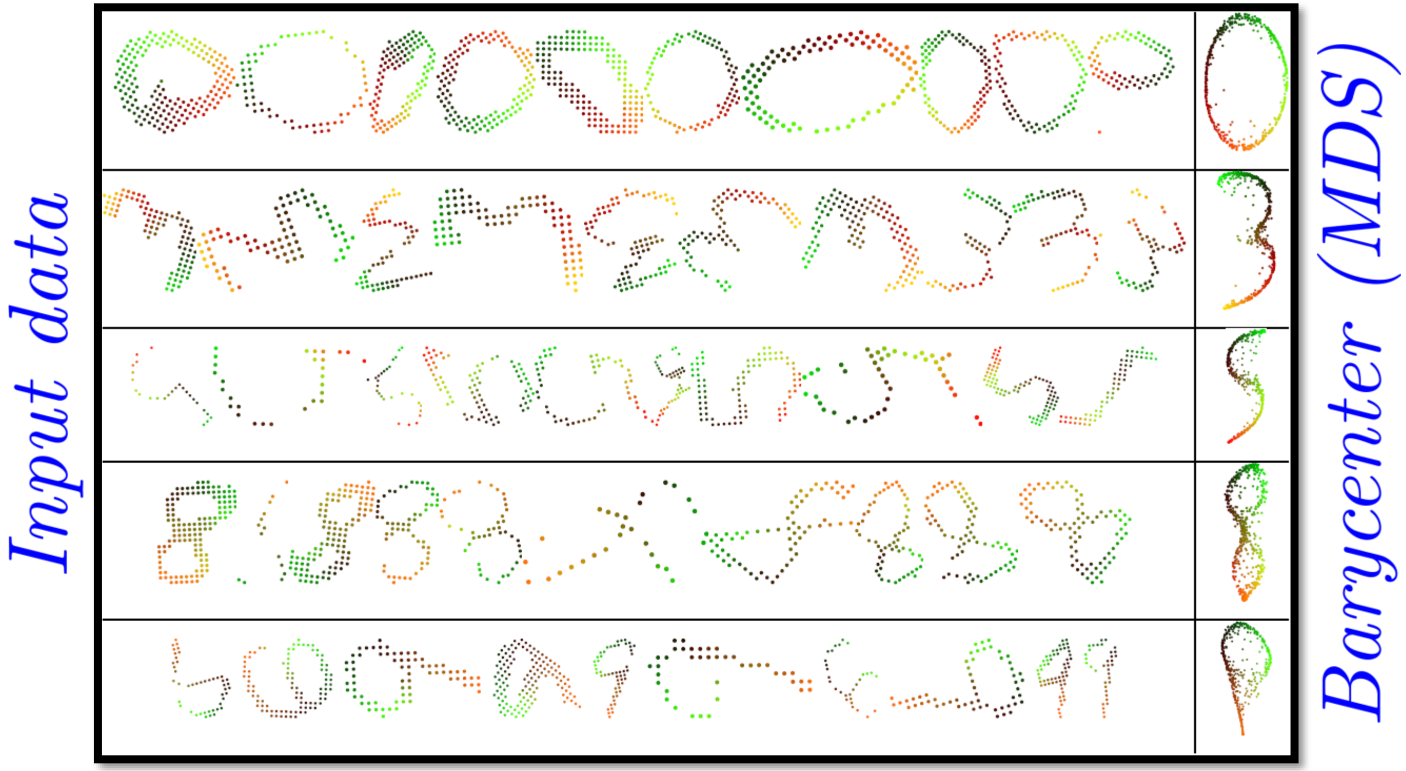




http://en.wikipedia.org/wiki/Double_pendulum
http://www.ualberta.ca/dept/math/gauss/fcm/BscIdeas/SpcDmnsn/pndlm2.htm

Metric Geometry and Metric Embedding

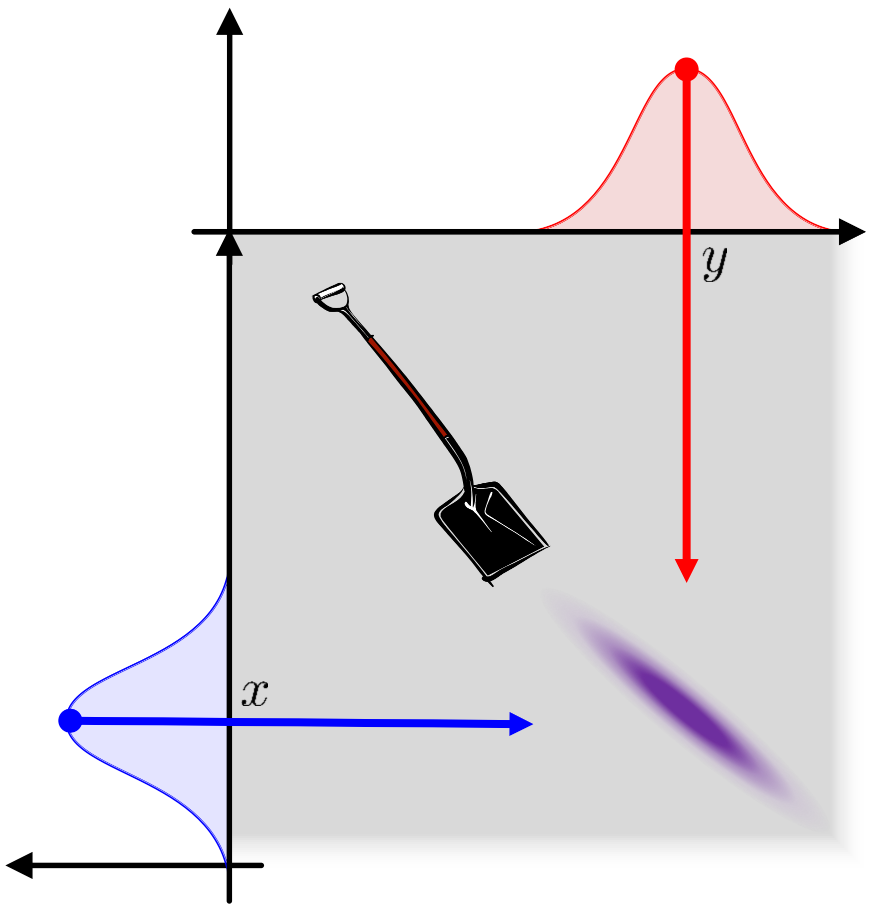


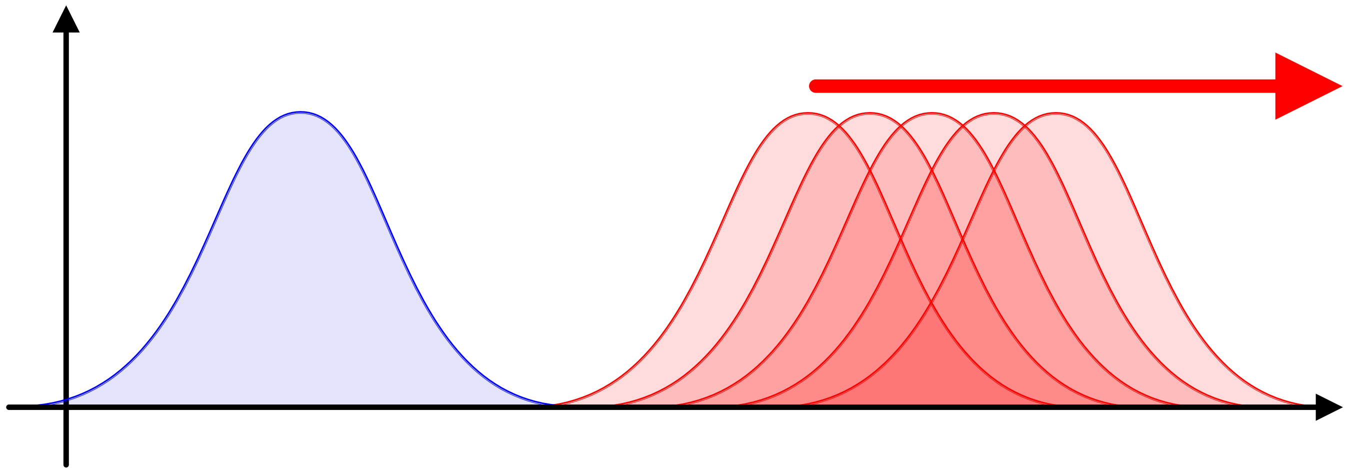


Tenenbaum et al.
A Global Geometric Framework for Nonlinear Dimensionality Reduction.
Science 2000.

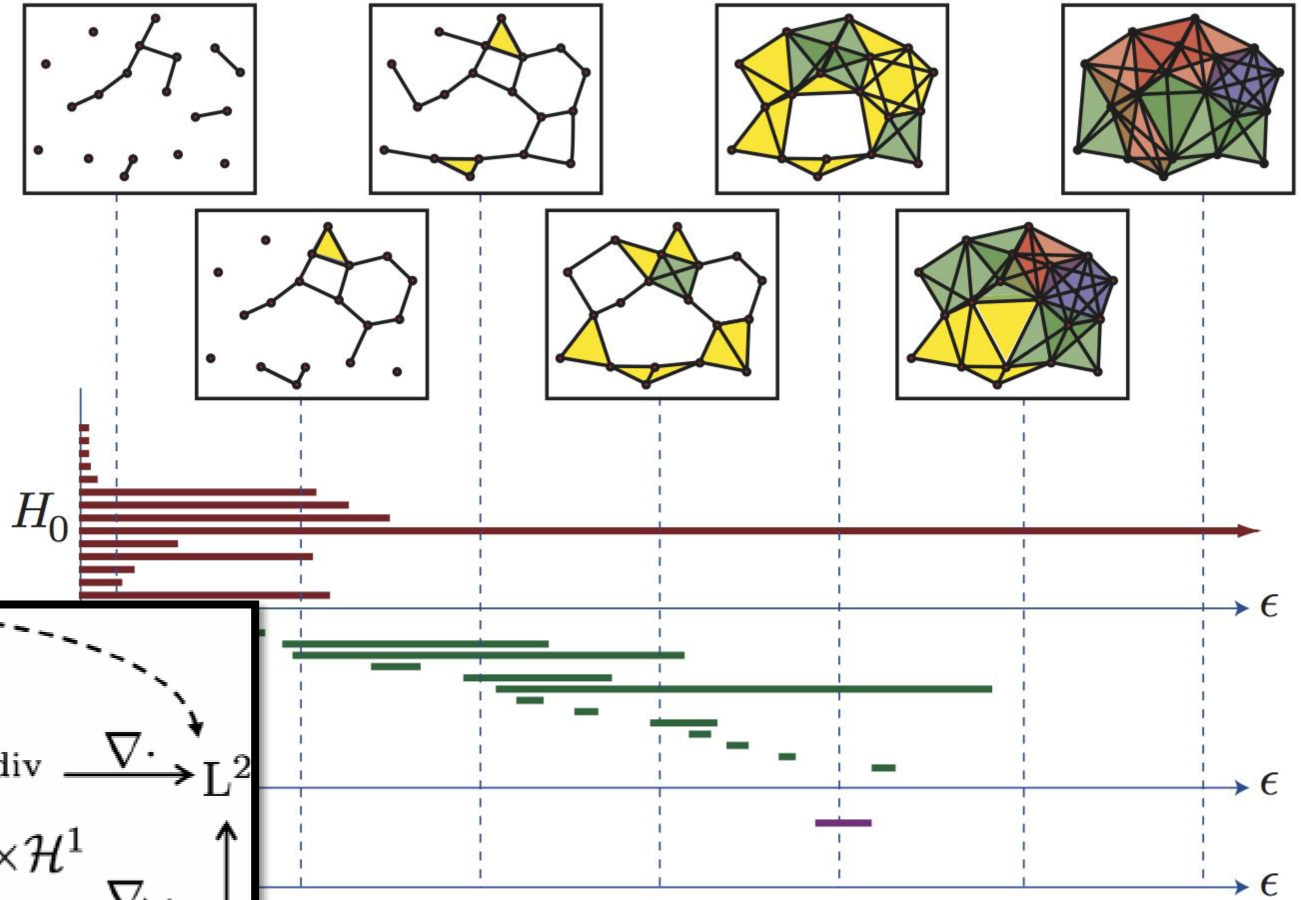
Peyré, Cuturi, and Solomon.
Gromov-Wasserstein Averaging of Kernel and Distance Matrices.
ICML 2016.

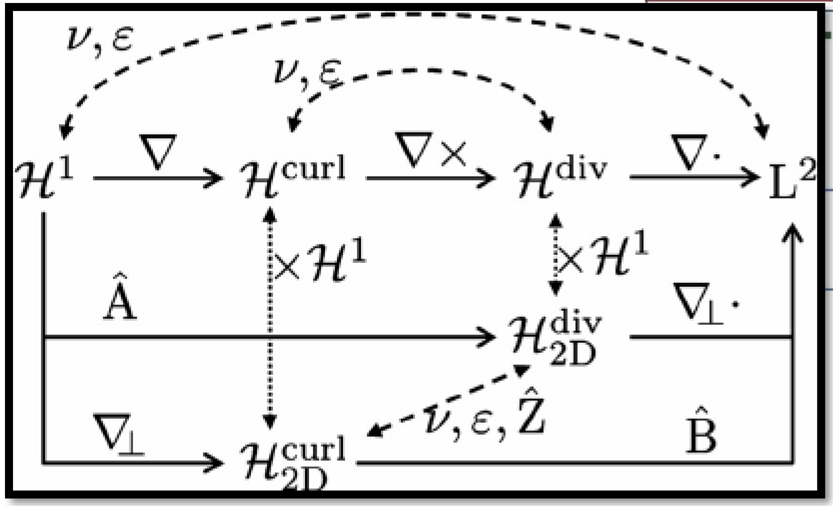
Optimal Transport





{Differential/Morse/Persistent/…} Topology





https://www.researchgate.net/publication/274071948_Space_charge_and_resistive_wall_impedance_computation_in_the_frequency_domain_using_the_finite_element_method
http://blog.forty.to/wp-content/uploads/2013/05/Bars1.jpg

Plan for Today

I. Theoretical toolbox

II. Computational toolbox

III. Application areas

Many Notions of Shape

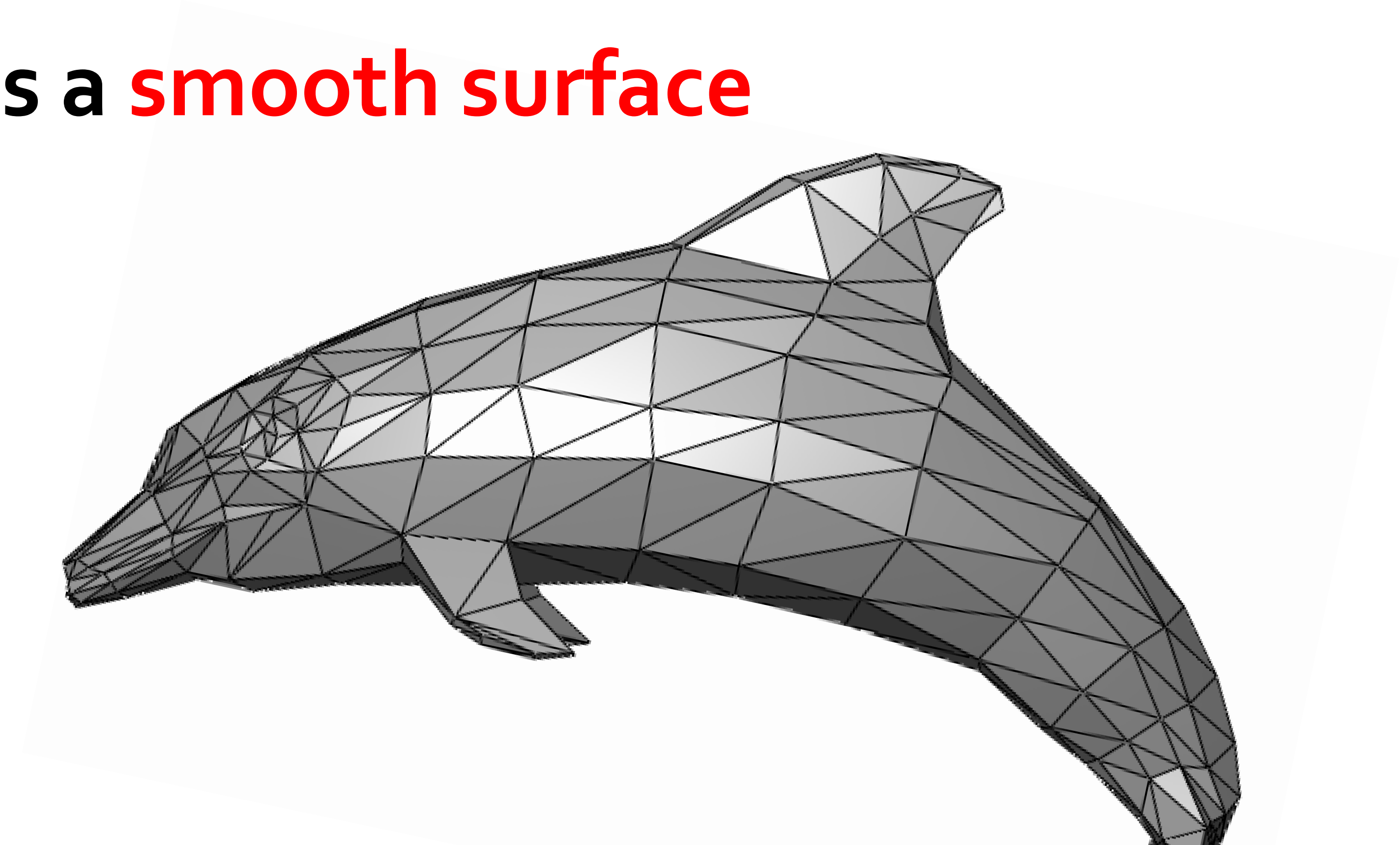
- Triangle mesh
- Triangle soup
- Graph
- Point cloud
- Pairwise distance matrix
- Dataset
- Network

Nearly anything with a notion of proximity/distance/curvature/…

Typical issue:

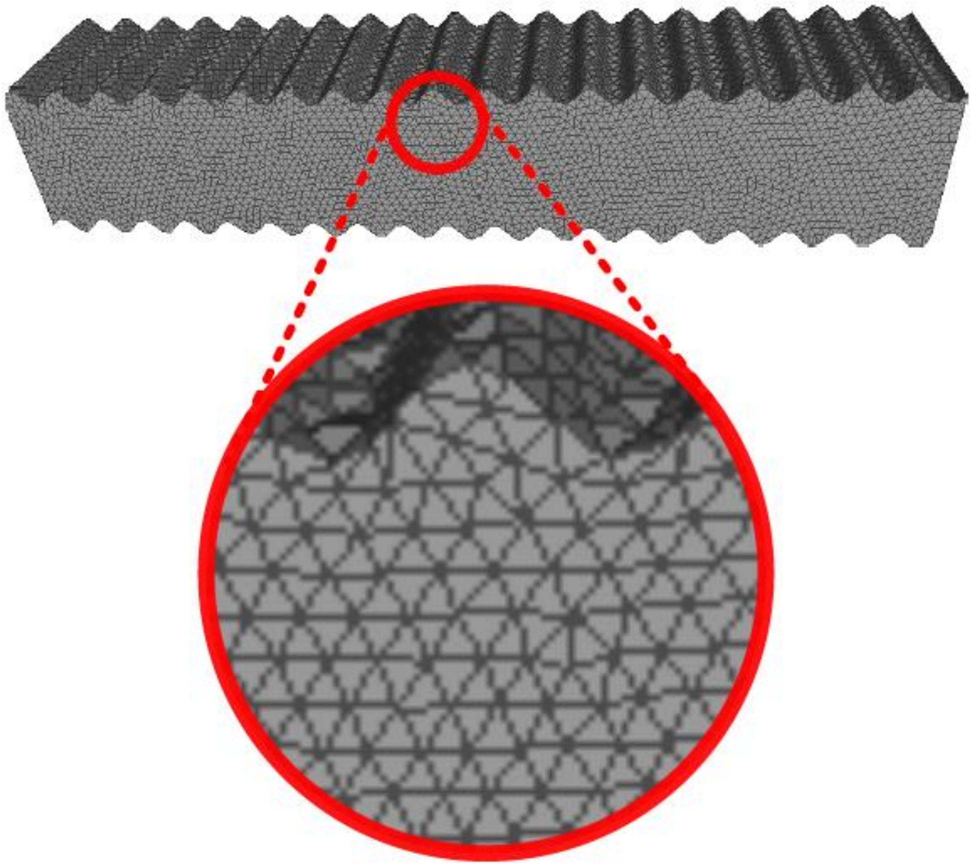
How to Interpret Geometric Data

- Collection of flat triangles
- Approximates a smooth surface



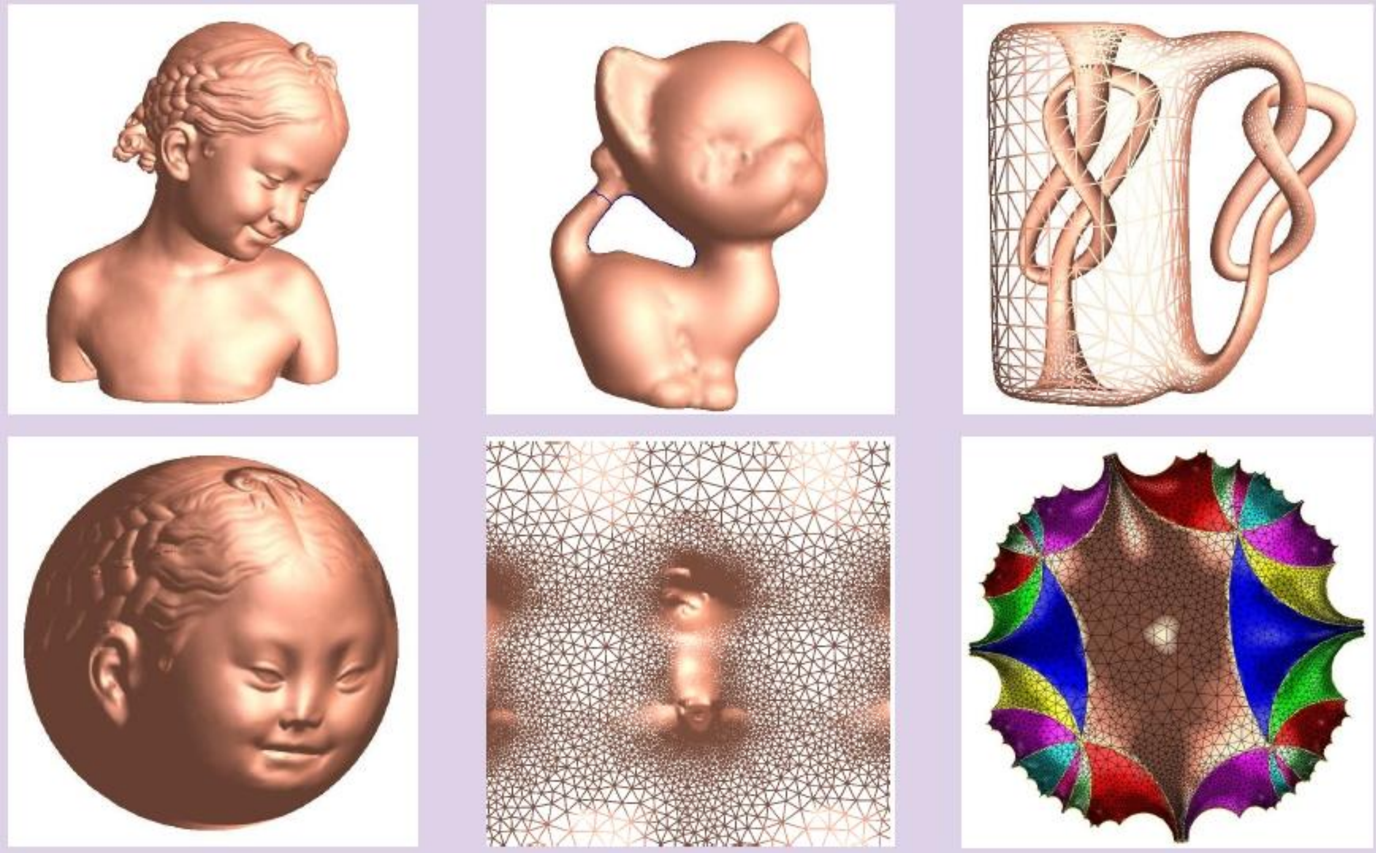
Q: Can a triangle mesh have curvature?

Jack of All Trades



Combine smooth and discrete

Example:
Discrete Differential Geometry



http://www.geomtop.org/teaching/conformal_2012Summer/title_image.jpg

Modern Approach

Discrete

vs.

Discretized

Discrete Differential Geometry

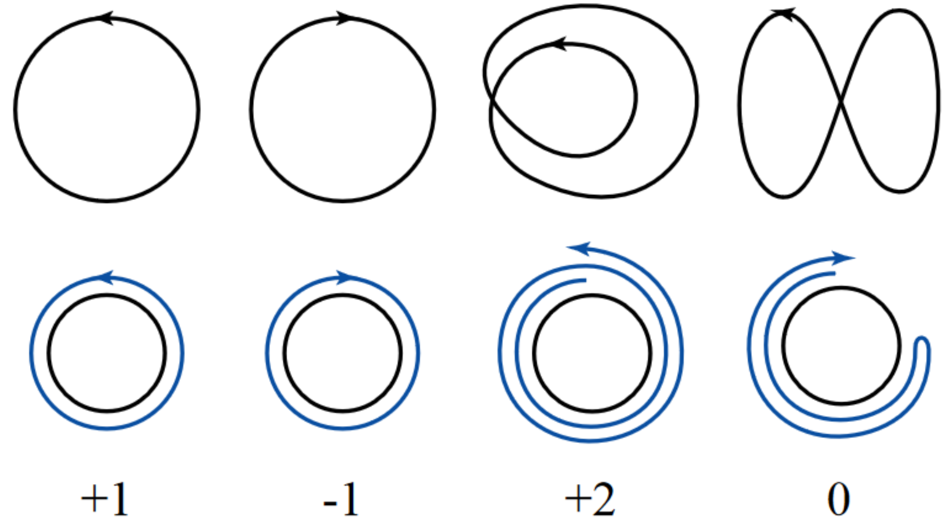
Discrete theory *paralleling* differential geometry.

Structure preservation

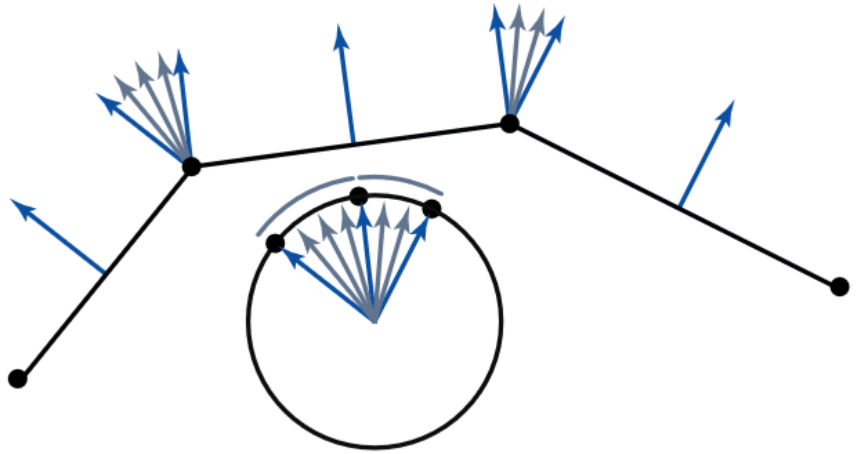
[**struhk**-cher pre-zur-**vey**-sh*uh* n]**:**

Keeping properties from the continuous abstraction exactly true in a discretization.

Example: Turning Numbers



$$\int_{\Omega} \kappa \, ds = 2\pi k$$



$$\sum_i \alpha_i = 2\pi k$$

Images from: Grinspun and Secord, "The Geometry of Plane Curves" (SIGGRAPH 2006)

Convergence

[k*uh* n-**vur**-j*uh* ns]**:**

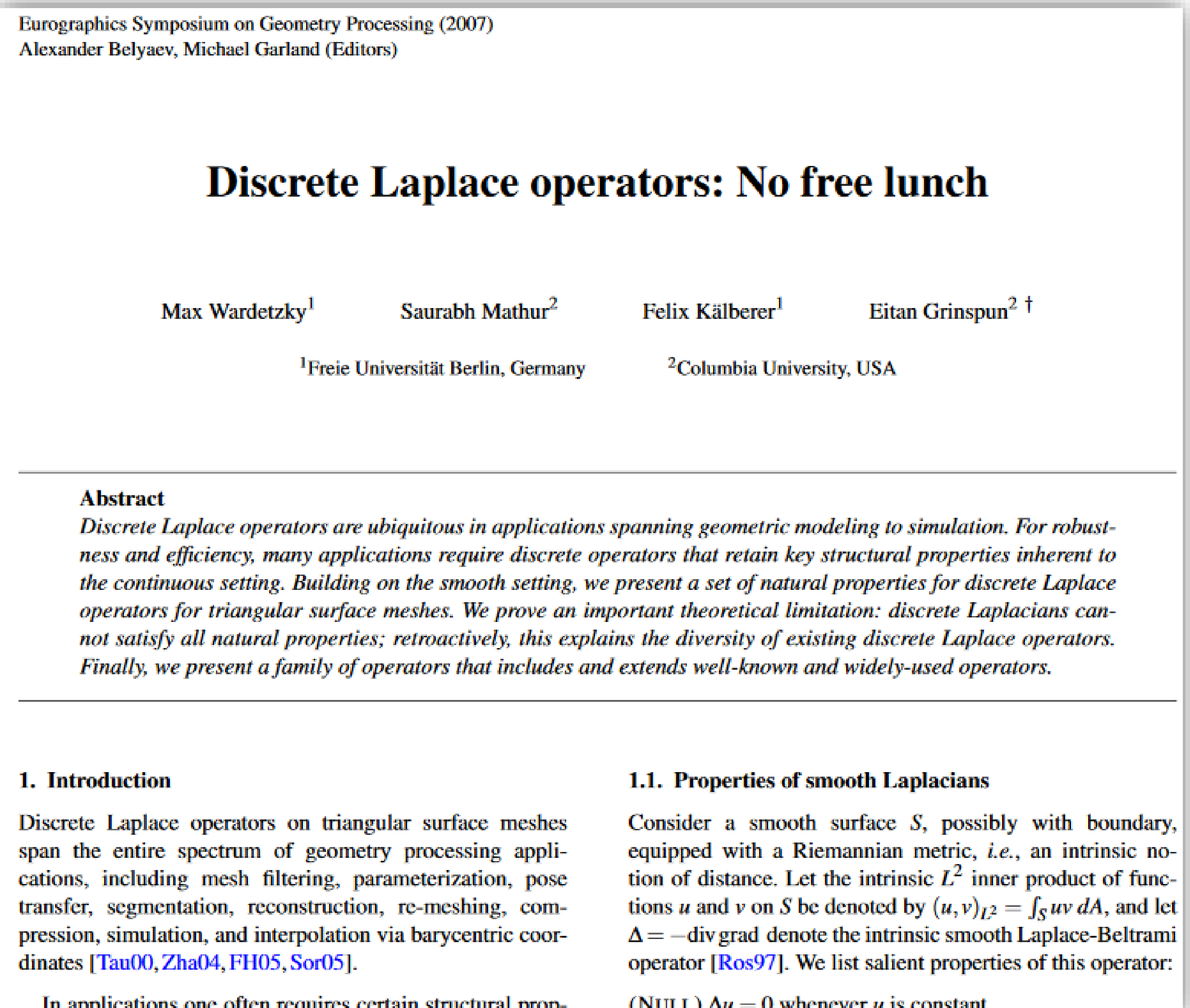
Increasing approximation quality as a discretization is refined.

Convergence *and* Structure

Can you have it all?

Fey et al. *30 Rock* (2006-2013).

Disappointing Result



Eurographics Symposium on Geometry Processing (2007)
Alexander Belyaev, Michael Garland (Editors)

Discrete Laplace operators: No free lunch

Max Wardetzky[1] Saurabh Mathur[2] Felix Kälberer[1] Eitan Grinspun[2 †]

[1]Freie Universität Berlin, Germany [2]Columbia University, USA

Abstract
Discrete Laplace operators are ubiquitous in applications spanning geometric modeling to simulation. For robustness and efficiency, many applications require discrete operators that retain key structural properties inherent to the continuous setting. Building on the smooth setting, we present a set of natural properties for discrete Laplace operators for triangular surface meshes. We prove an important theoretical limitation: discrete Laplacians cannot satisfy all natural properties; retroactively, this explains the diversity of existing discrete Laplace operators. Finally, we present a family of operators that includes and extends well-known and widely-used operators.

1. Introduction

Discrete Laplace operators on triangular surface meshes span the entire spectrum of geometry processing applications, including mesh filtering, parameterization, pose transfer, segmentation, reconstruction, re-meshing, compression, simulation, and interpolation via barycentric coordinates [Tau00, Zha04, FH05, Sor05].

In applications one often requires certain structural prop-

1.1. Properties of smooth Laplacians

Consider a smooth surface S, possibly with boundary, equipped with a Riemannian metric, *i.e.*, an intrinsic notion of distance. Let the intrinsic L^2 inner product of functions u and v on S be denoted by $(u,v)_{L^2} = \int_S uv\, dA$, and let $\Delta = -\operatorname{div}\operatorname{grad}$ denote the intrinsic smooth Laplace-Beltrami operator [Ros97]. We list salient properties of this operator:

(NULL) $\Delta u = 0$ whenever u is constant

Disappointing Result

Eurographics Symposium on Geometry Processing (2007)
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Discrete Laplace operators: No free lunch

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(NULL) $\Delta u = 0$ whenever u is constant

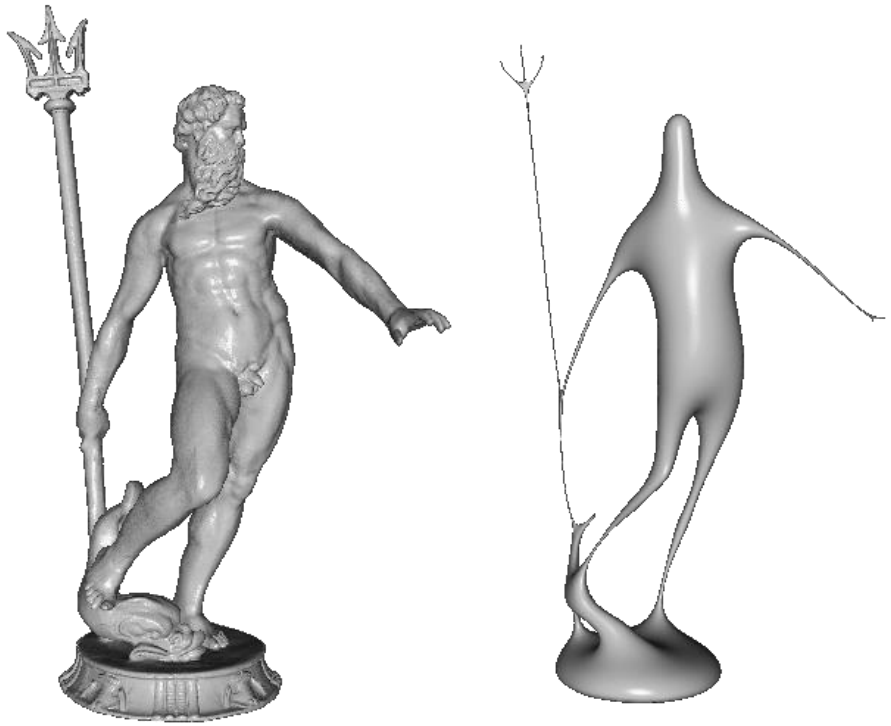
Theme

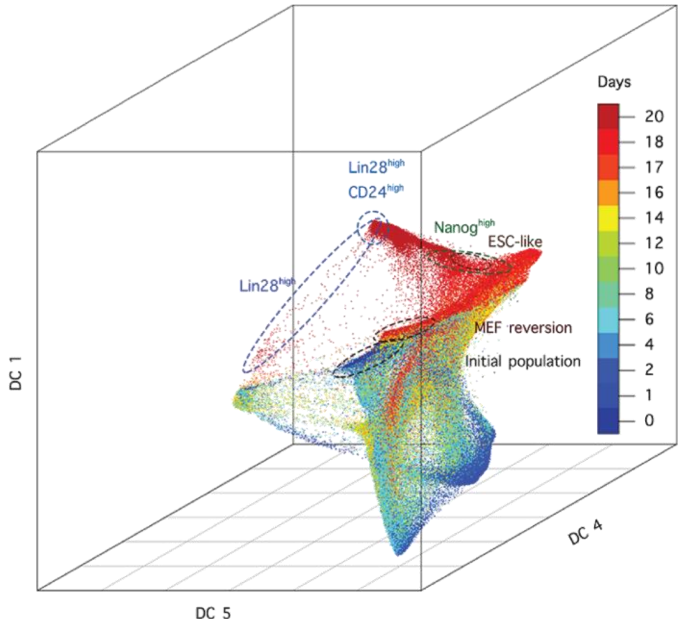
Pick and choose

which properties you need.

But there is a huge toolbox of algorithms to draw from!

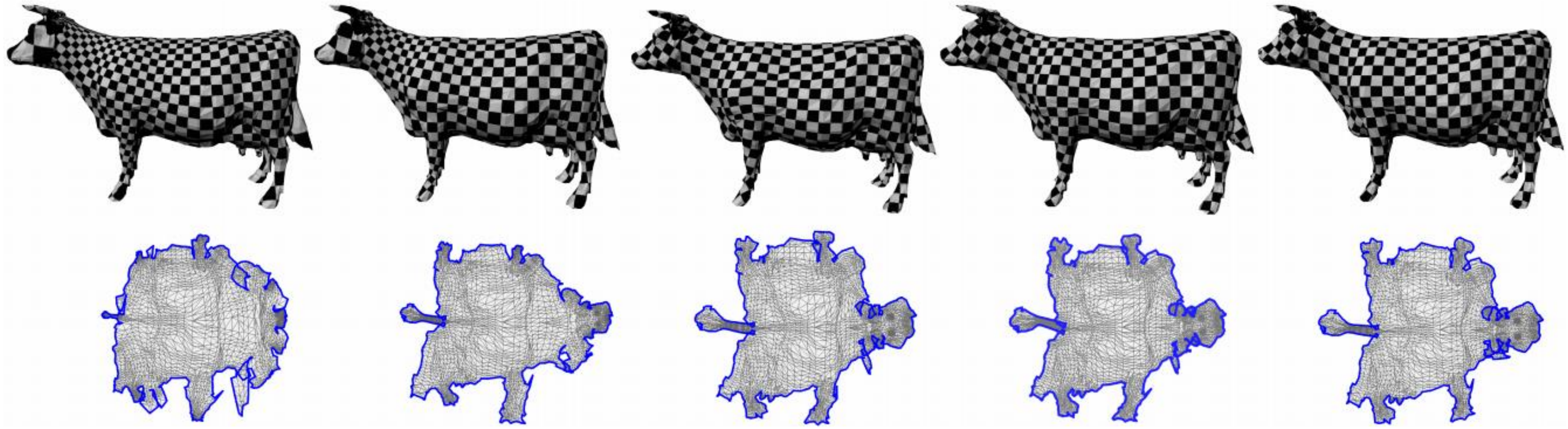
Numerical PDE





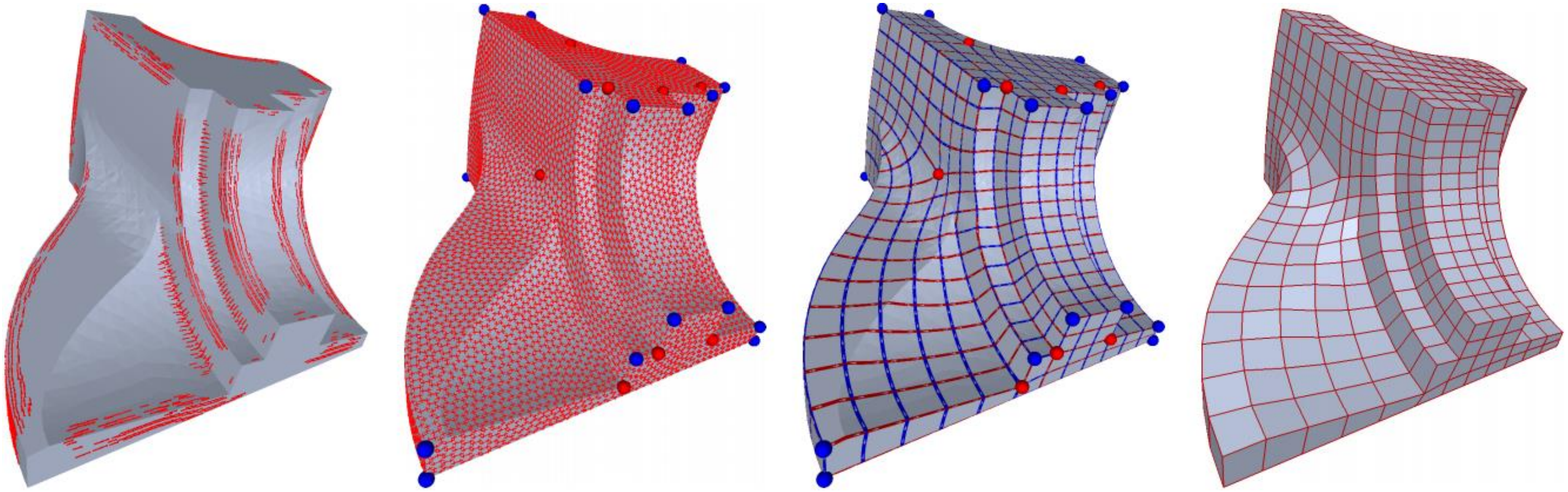
Chuang and Kazhdan. *Fast Mean-Curvature Flow via Finite-Elements Tracking*. CGF 2011.
Coifman & Lafon. *Diffusion Maps*. ACHA 2006.

Large-Scale Smooth Optimization



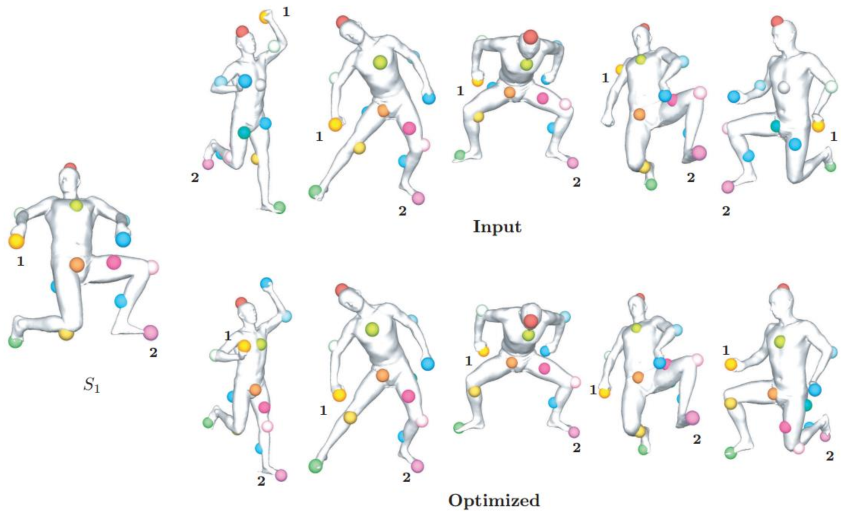
Smith and Schaefer. *Bijective parameterization with free boundaries*. SIGGRAPH 2015.

Discrete Optimization

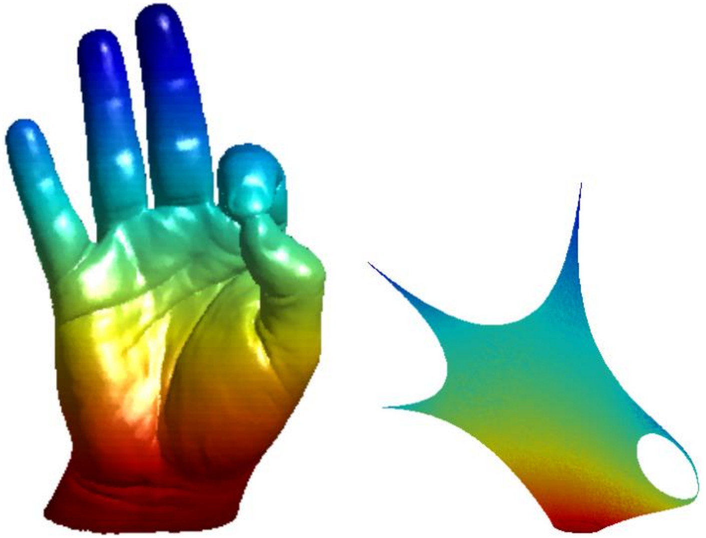


Bommes, Zimmer, Kobbelt. *Mixed-integer quadrangulation*. SIGGRAPH 2009.

Linear Algebra



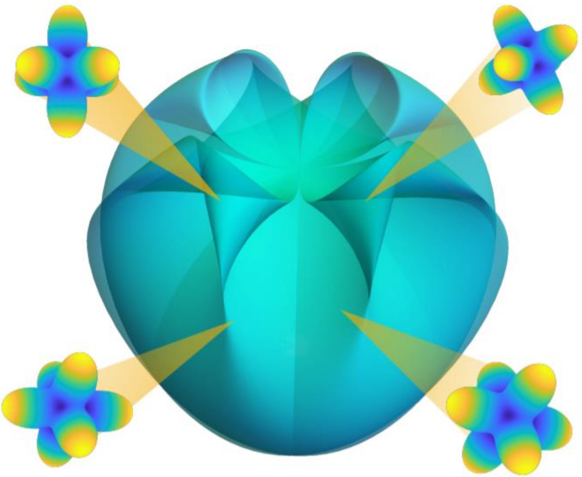
Huang, Guibas. *Consistent shape maps via semidefinite programming*. SGP 2013.

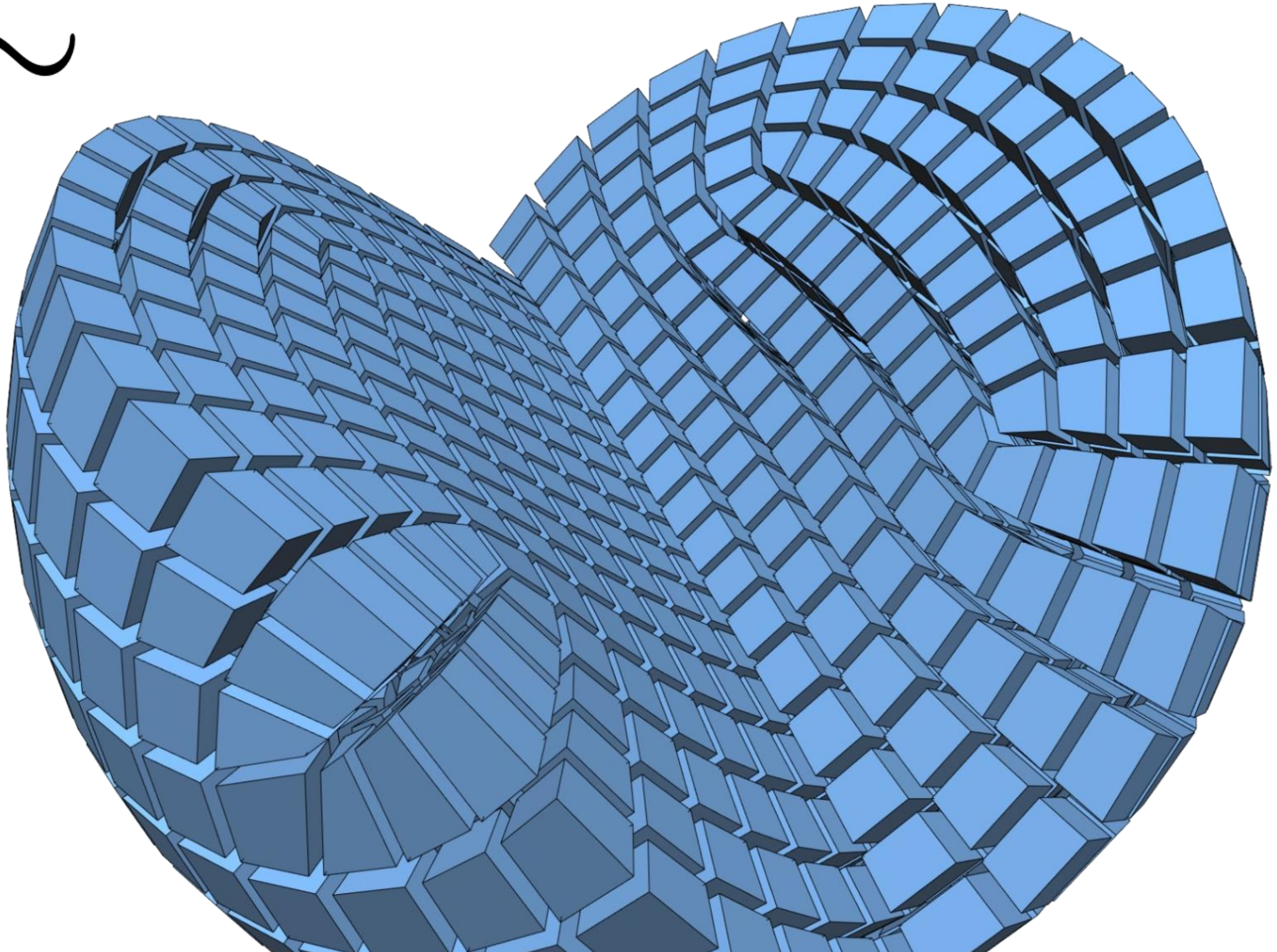


Krishnan, Fattal, Szeliski. *Efficient preconditioning of Laplacian matrices for computer graphics*. SIGGRAPH 2013.

Algebra & Representation Theory

$$\mathrm{SO}(3)/\sim$$





Palmer, Bommes, & Solomon.
Algebraic Representations for Volumetric Frame Fields.
TOG 2020.

Plan for Today

I. Theoretical toolbox

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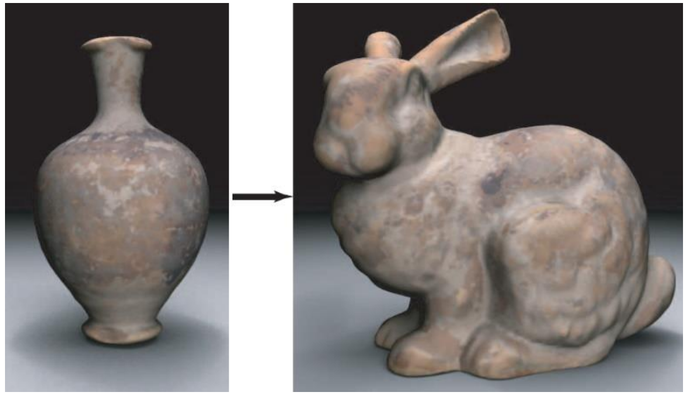
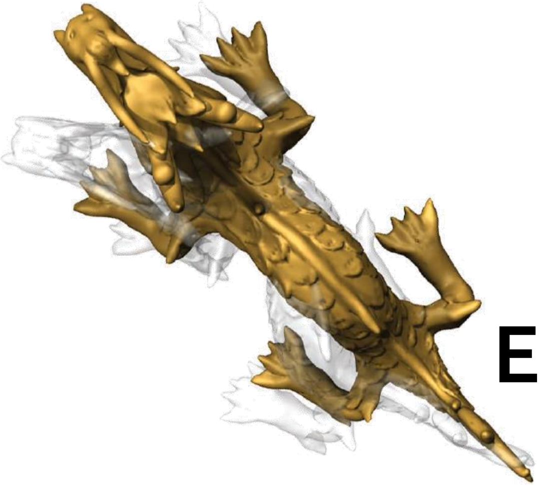
Applications

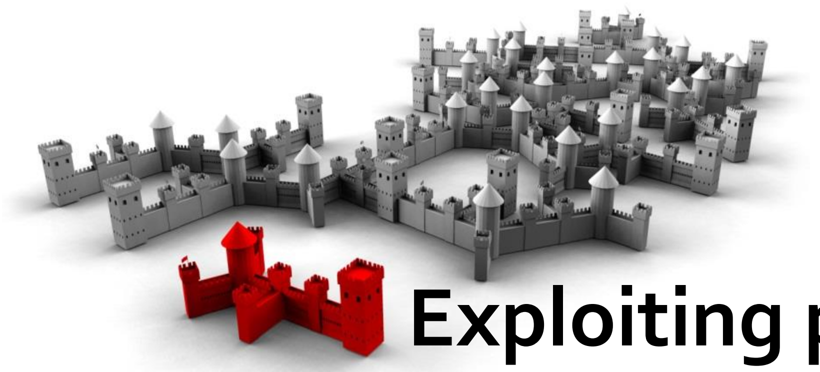


Transfer

Retrieval



Editing

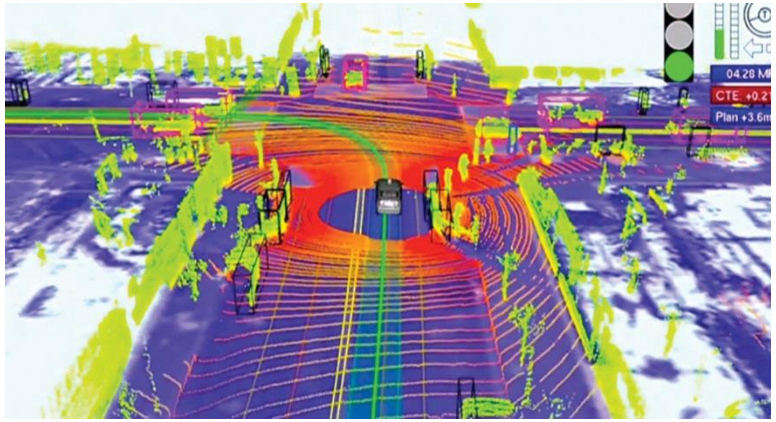


Exploiting patterns

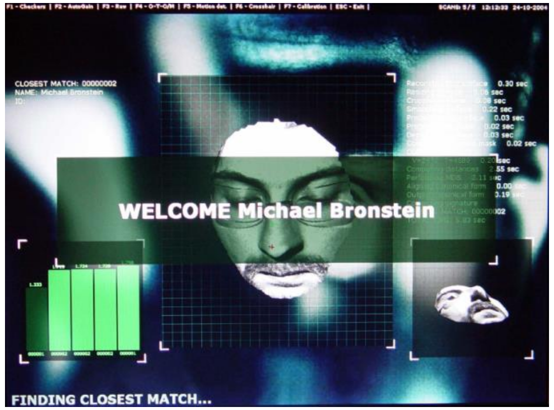
Mertens et al. "Texture Transfer Using Geometry Correlation."
Fisher et al. "Context-Based Search for 3D Models."
Mitra et al. "Symmetrization."
Bokeloh et al. "A connection between partial symmetry and inverse procedural modeling."

Graphics

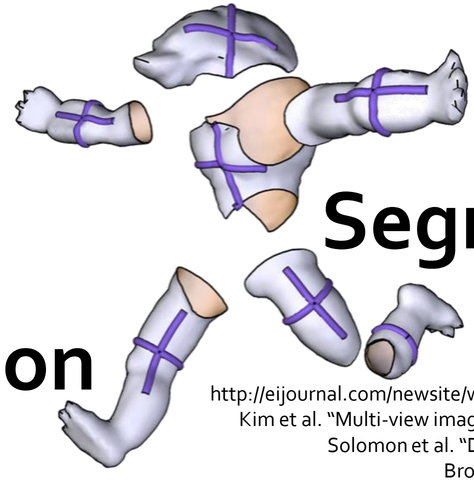
Applications



Recognition



Navigation



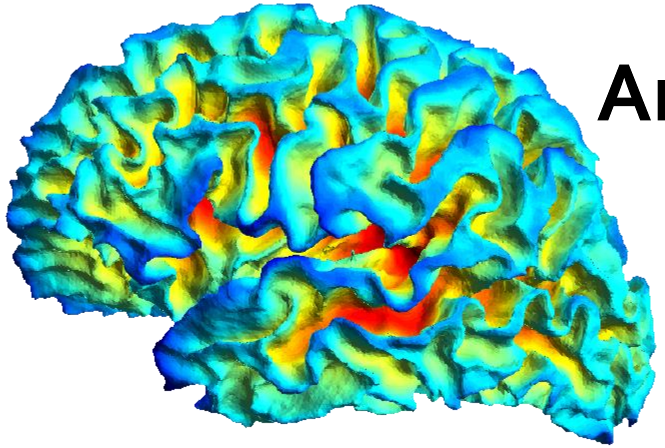
Segmentation

Reconstruction

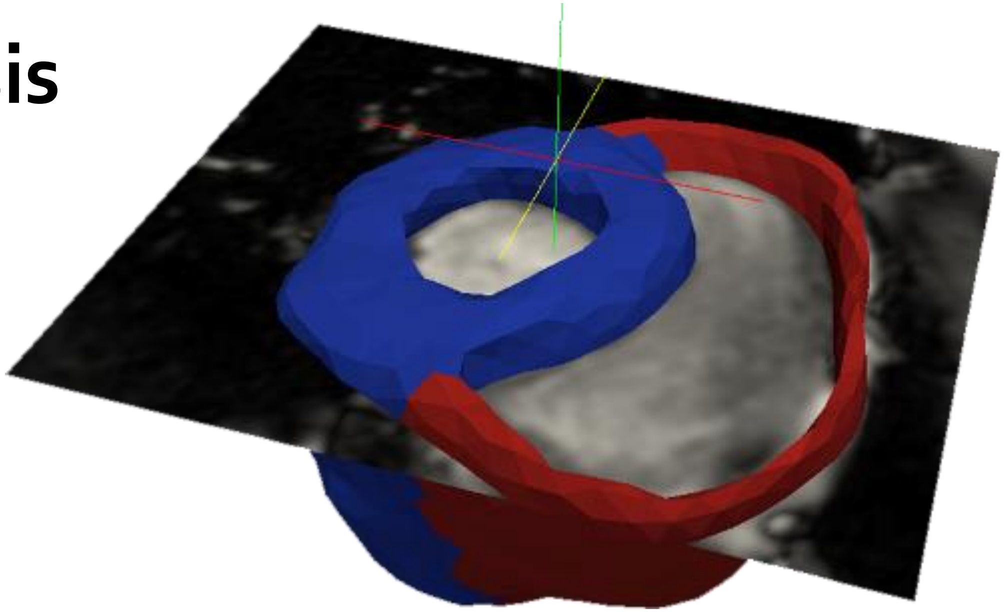
http://eijournal.com/newsite/wp-content/uploads/2012/01/VELODYNE-IMAGE.jpg
Kim et al. "Multi-view image and tof sensor fusion for dense 3d reconstruction."
Solomon et al. "Discovery of Intrinsic Primitives on Triangle Meshes."
Bronstein et al. "Three-Dimensional Face Recognition."

Vision

Applications



Analysis



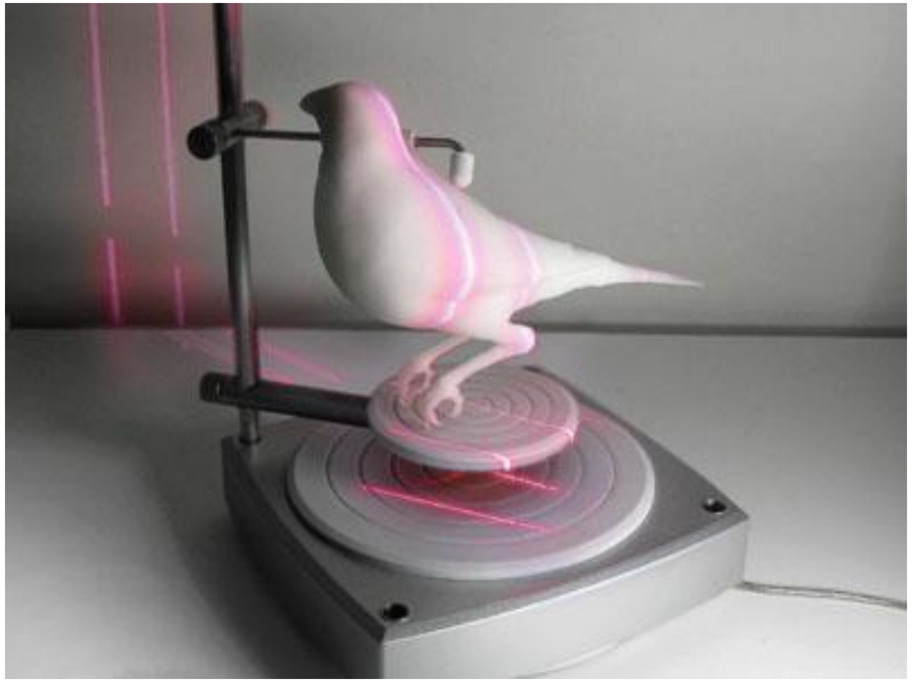
Segmentation

Registration

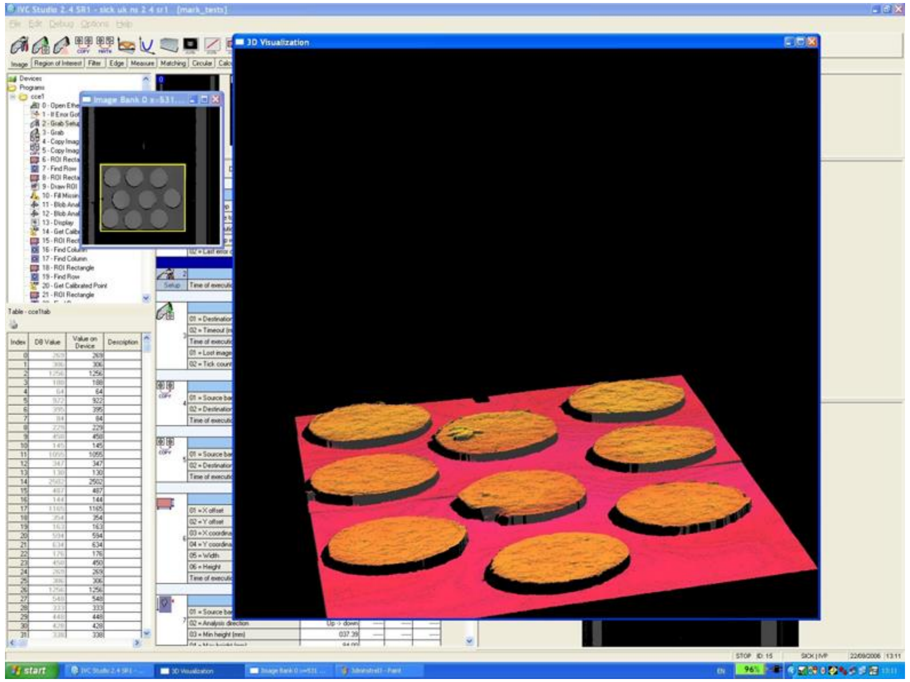
http://dmfr.birjournals.org/content/33/4/226/F3.large.jpg
http://www-sop.inria.fr/asclepios/software/inriaviz4d/SphericalImTransp.png
http://www.creatis.insa-lyon.fr/site/sites/default/files/segm2.png

Medical Imaging

Applications



Scanning

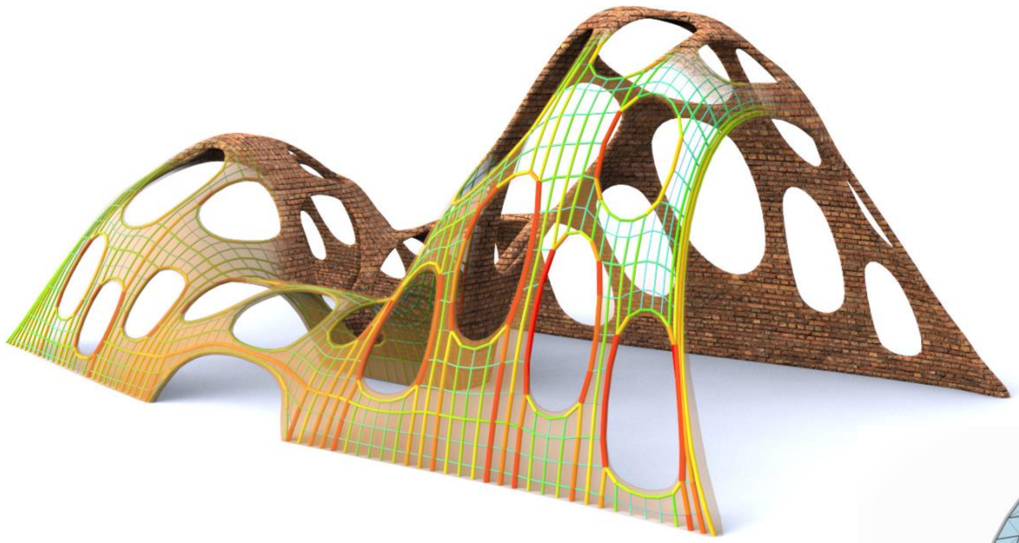


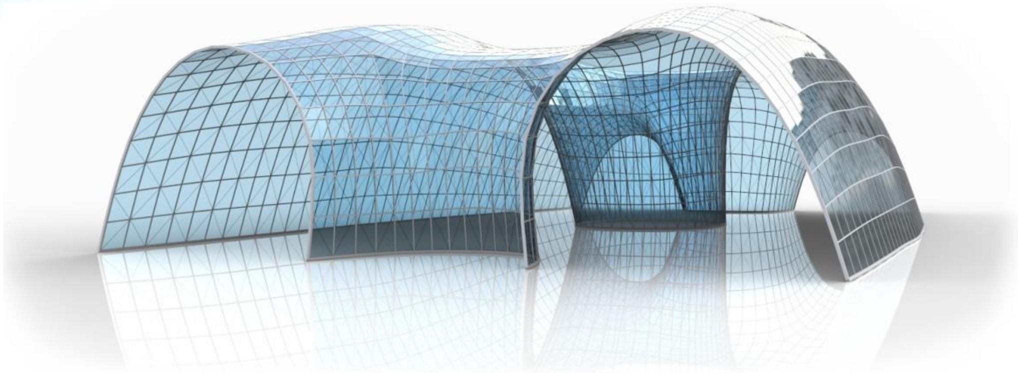
Defect detection

http://www.conduitprojects.com/php/images/scan.jpg
http://www.emeraldinsight.com/content_images/fig/0330290204005.png

Manufacturing and Fabrication

Applications



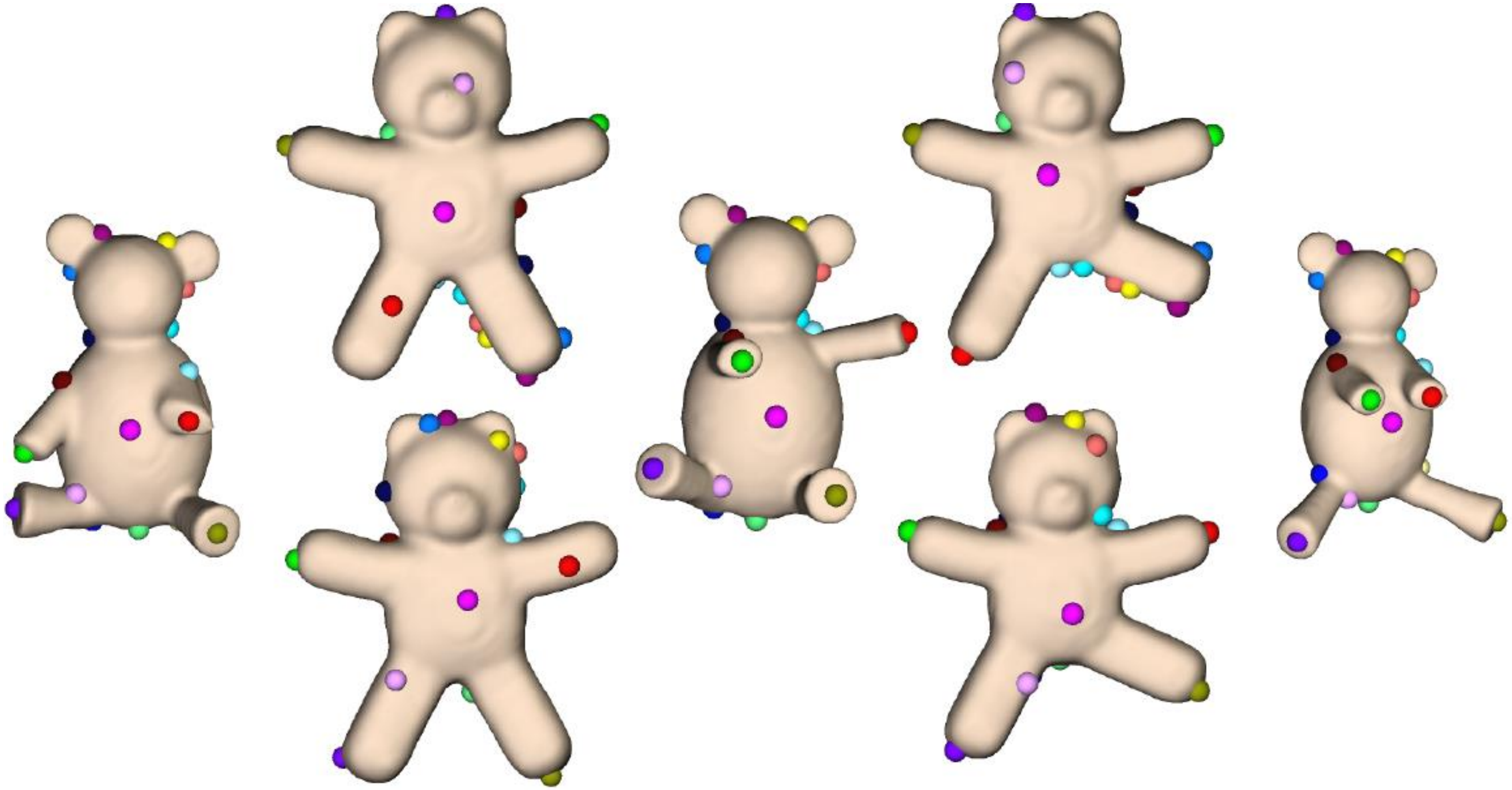


Design and analysis

Vouga et al. "Design of self-supporting surfaces."

Architecture

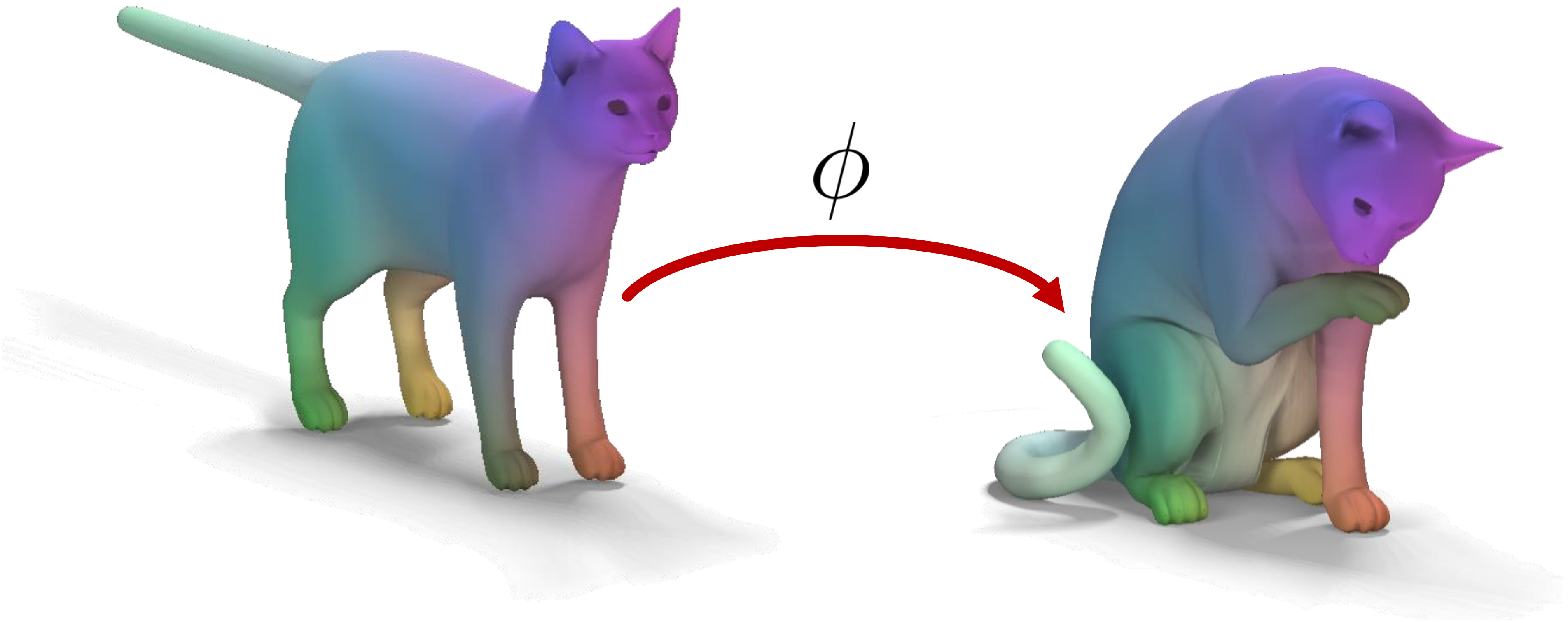
Applications



Huang et al. "Consistent shape maps via semidefinite programming."

Shape collections

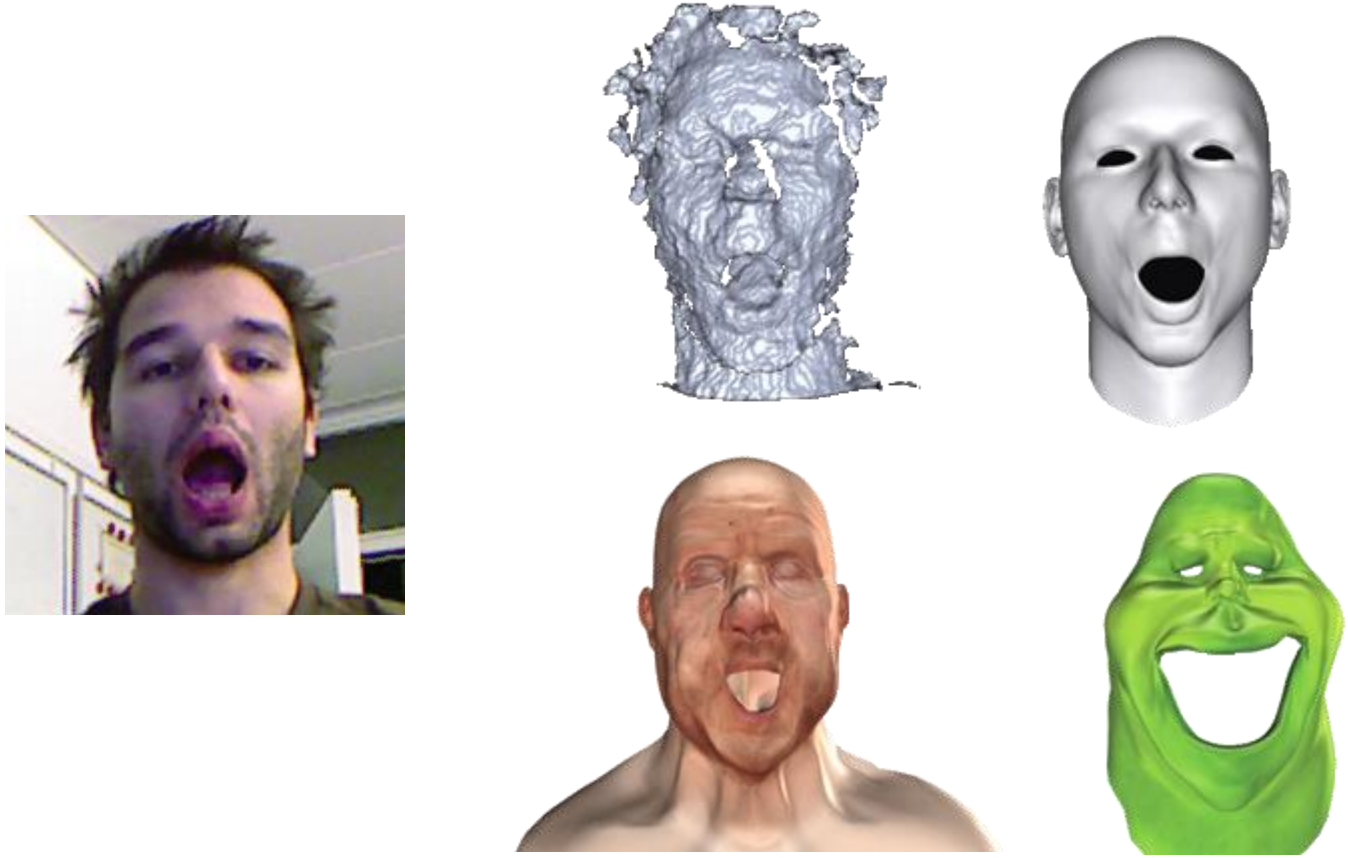
Applications



Ovsjanikov et al. "Functional maps."

Correspondence

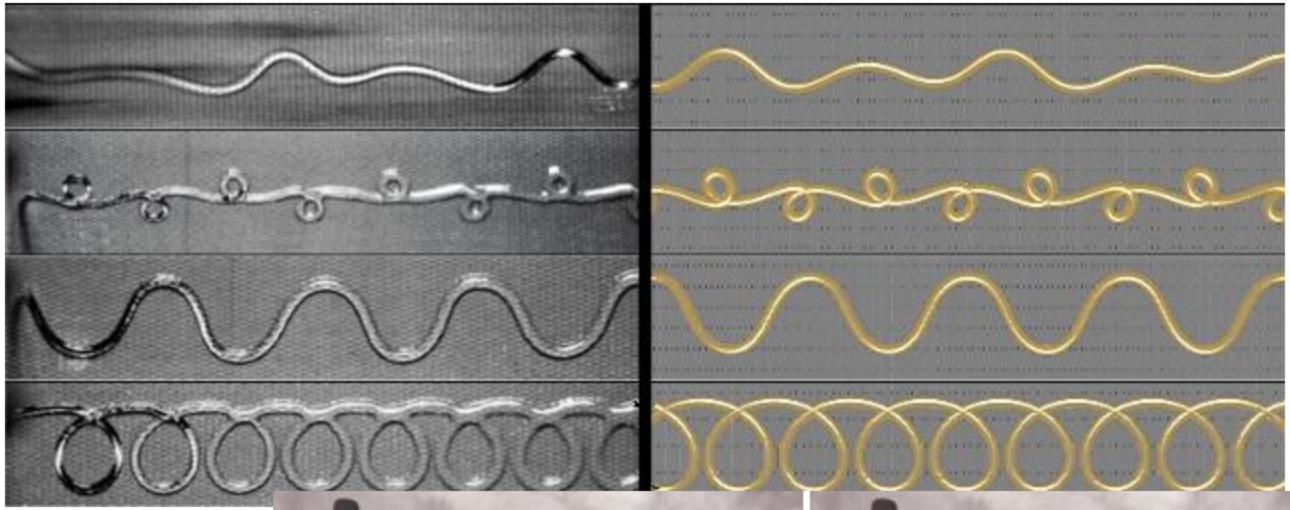
Applications



Weise et al. "Realtime performance-based facial animation."

Deformation transfer

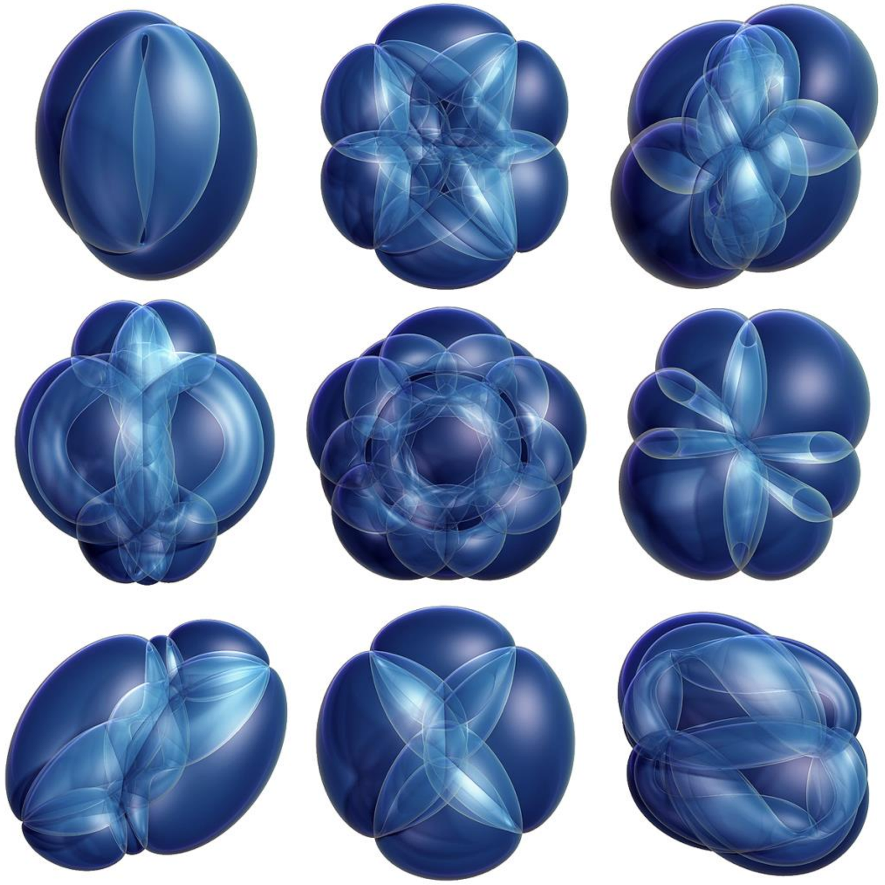
Applications



Bergou et al. "Discrete viscous threads."
Wardetzky et al. "Discrete quadratic curvature energies."

Simulation

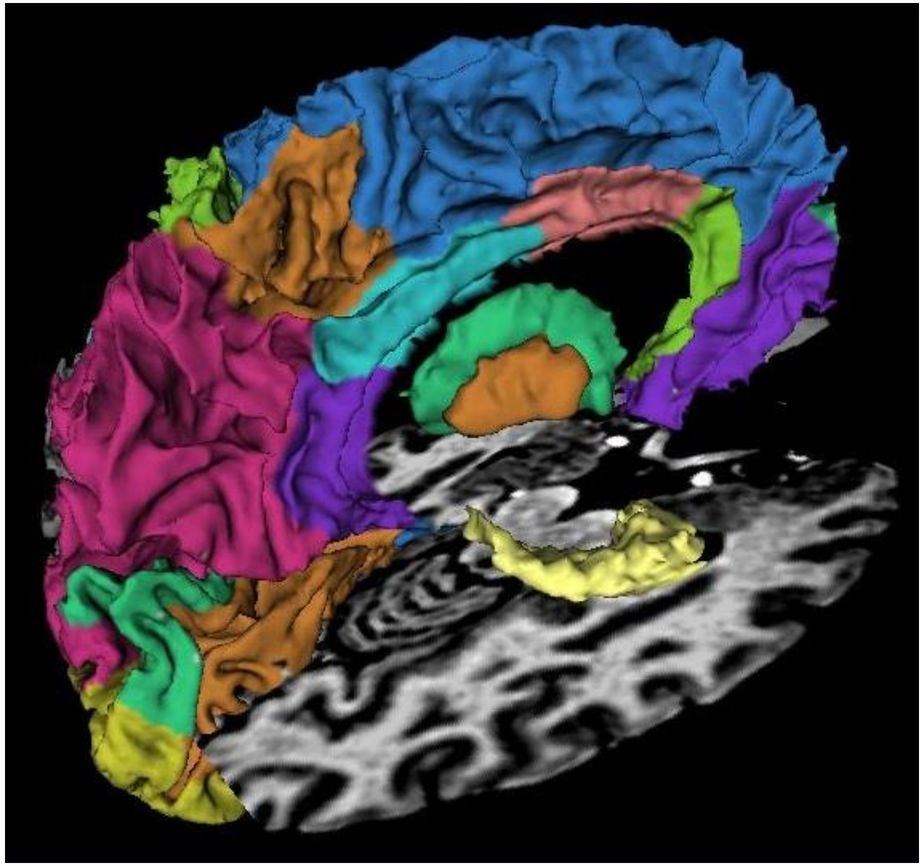
Applications

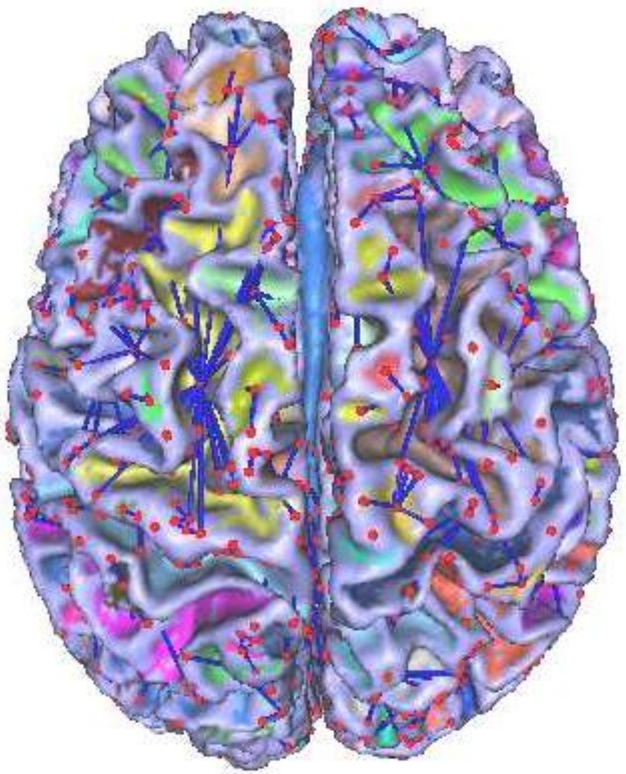


Crane et al. "Spin Transformations of Discrete Surfaces."

Scientific visualization

Applications

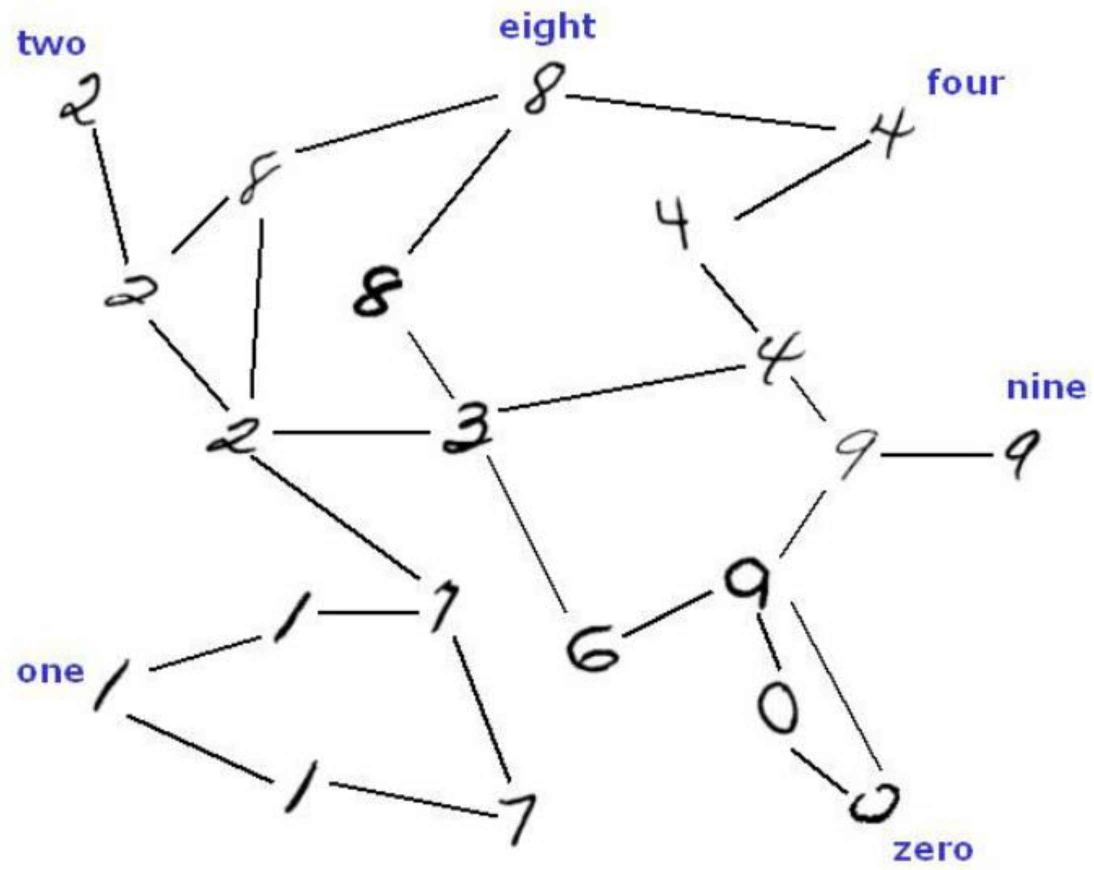




http://www.bioinformaticslaboratory.nl/twiki/pub/EBioScience/News/freesurfer-3d.jpg
http://hal.inria.fr/docs/00/40/21/30/IMG/vivodtzev_et_al-Dagstuhl03.jpg

Segmentation

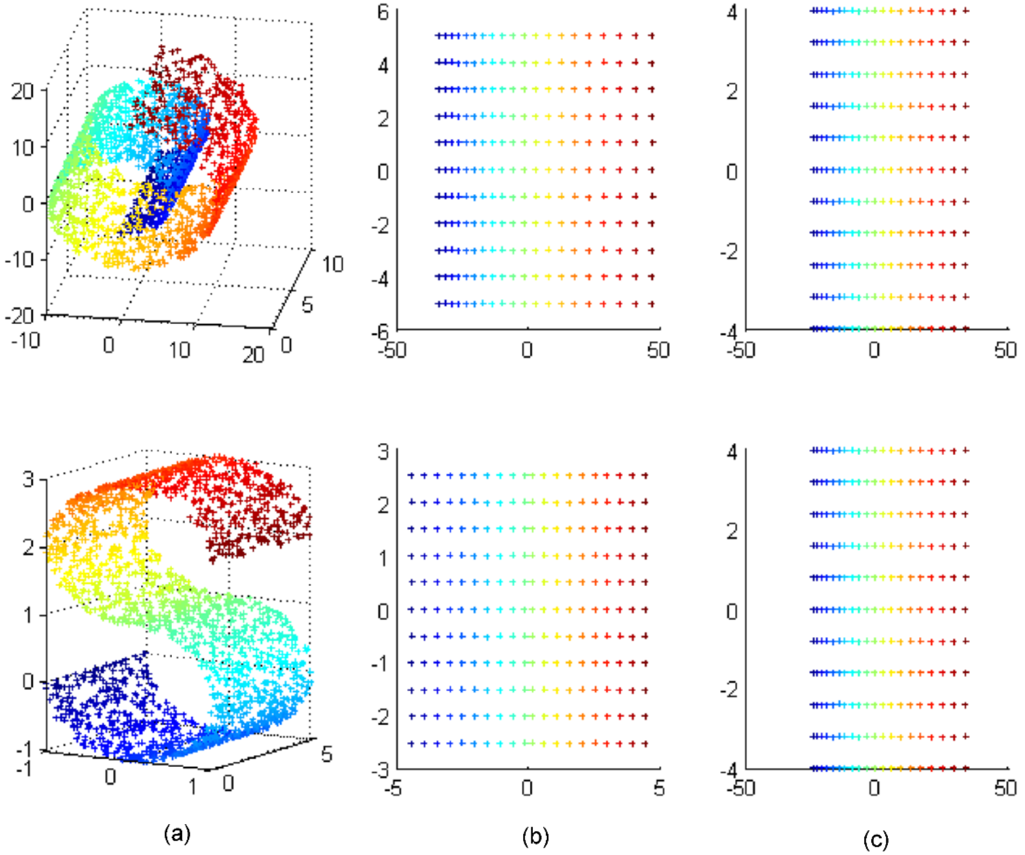
Applications



Zhu et al. *Semi-Supervised Learning Using Gaussian Fields and Harmonic Functions*. ICML 2003.

Machine learning

Applications



Hou et al. *Novel semisupervised high-dimensional correspondences learning method*. Opt. Eng. 2008.

Statistics

6.838:

Shape Analysis

Justin Solomon

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