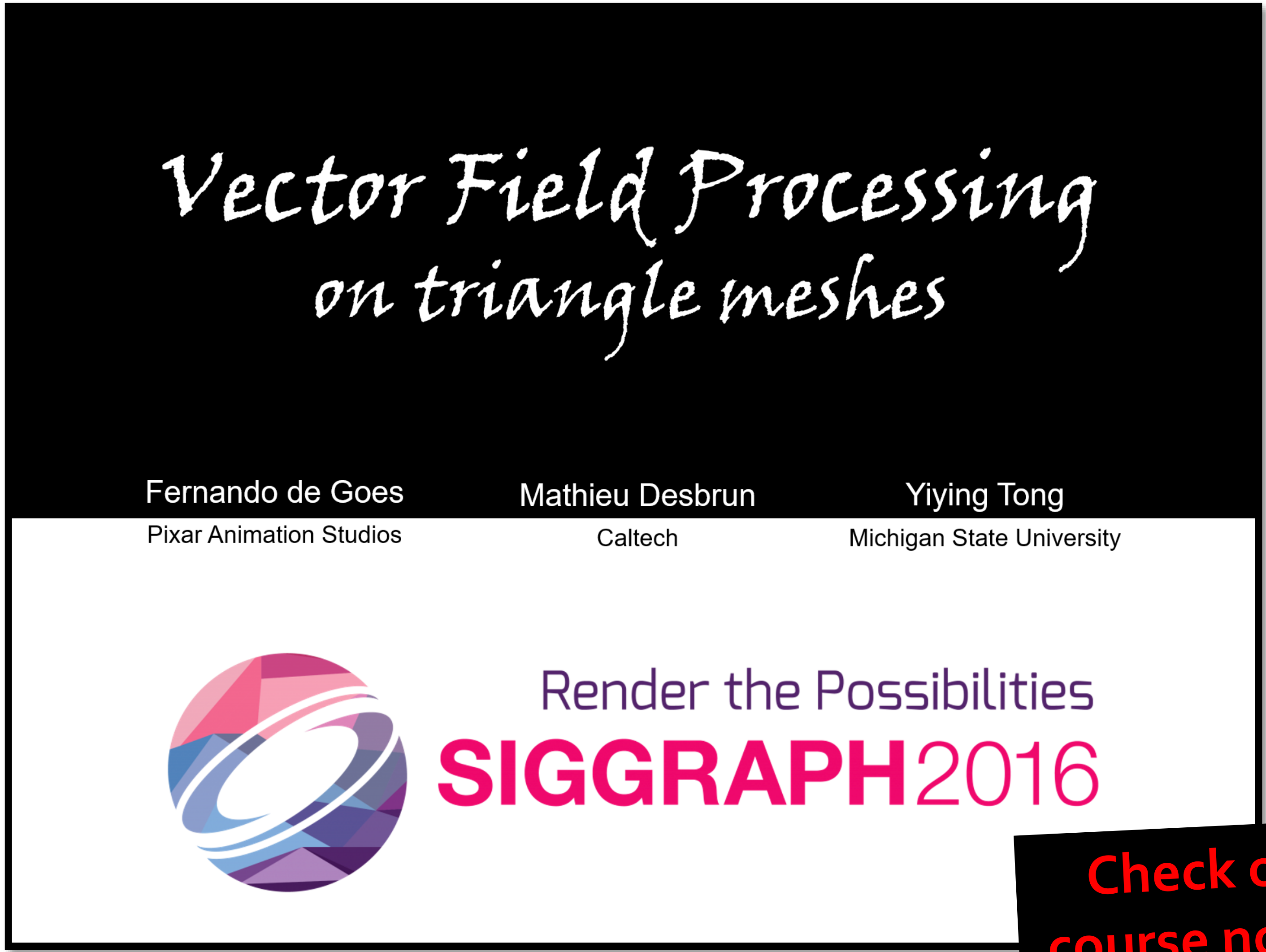
Vector Fields: Introduction

Justin Solomon

6.838: Shape Analysis

Spring 2021

Lots of Material/Slides From…



Check out course notes!

Additional Nice Reference

EUROGRAPHICS 2016
J. Madeira and G. Patow
(Guest Editors)

Volume 35 (2016), Number 2
STAR – State of The Art Report

Directional Field Synthesis, Design, and Processing

Amir Vaxman[1] Marcel Campen[2] Olga Diamanti[3] Daniele Panozzo[2,3] David Bommes[4] Klaus Hildebrandt[5] Mirela Ben-Chen[6]

[1]Utrecht University [2]New York University [3]ETH Zurich [4]RWTH Aachen University [5]Delft University of Technology [6]Technion

Abstract

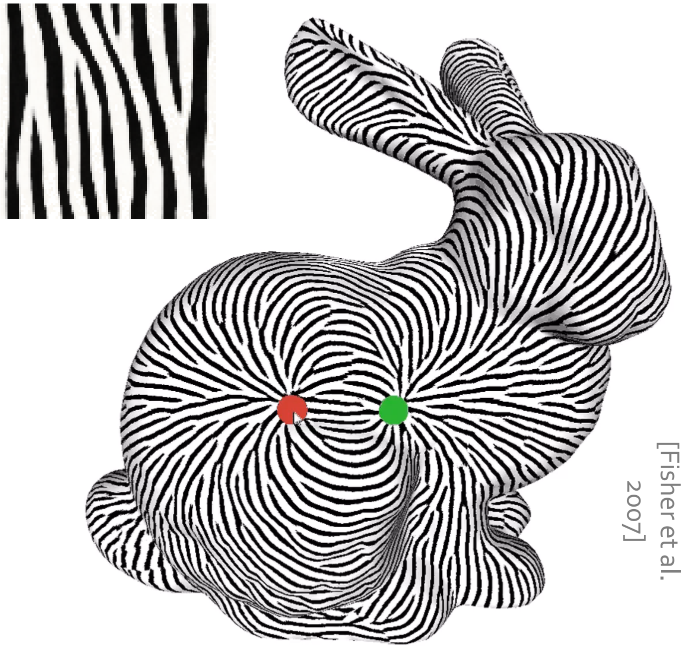
Direction fields and vector fields play an increasingly important role in computer graphics and geometry processing. The synthesis of directional fields on surfaces, or other spatial domains, is a fundamental step in numerous applications, such as mesh generation, deformation, texture mapping, and many more. The wide range of applications resulted in definitions for many types of directional fields: from vector and tensor fields, over line and cross fields, to frame and vector-set fields. Depending on the application at hand, researchers have used various notions of objectives and constraints to synthesize such fields. These notions are defined in terms of fairness, feature alignment, symmetry, or field topology, to mention just a few. To facilitate these objectives, various representations, discretizations, and optimization strategies have been developed. These choices come with varying strengths and weaknesses. This report provides a systematic overview of directional field synthesis for graphics applications, the challenges it poses, and the methods developed in recent years to address these challenges.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—

1. Introduction

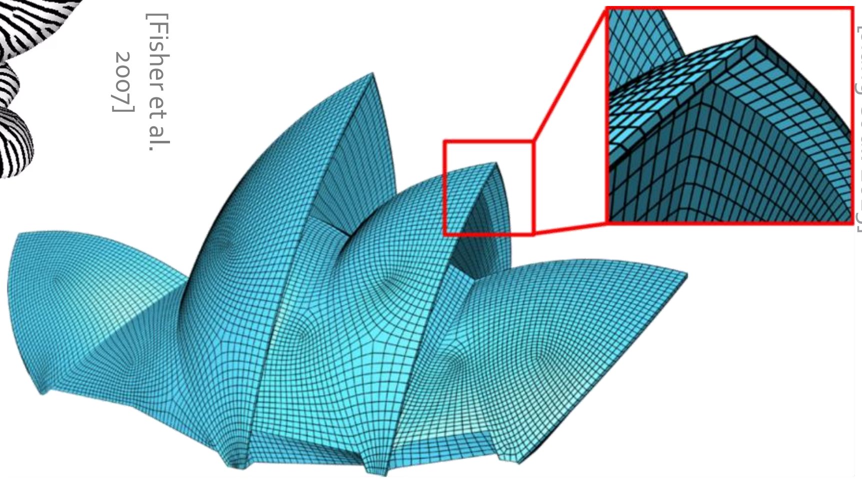
There have been significant developments in directional field synthesis over the past decade. These developments have been driven

Why Vector Fields?



[Fisher et al. 2007]

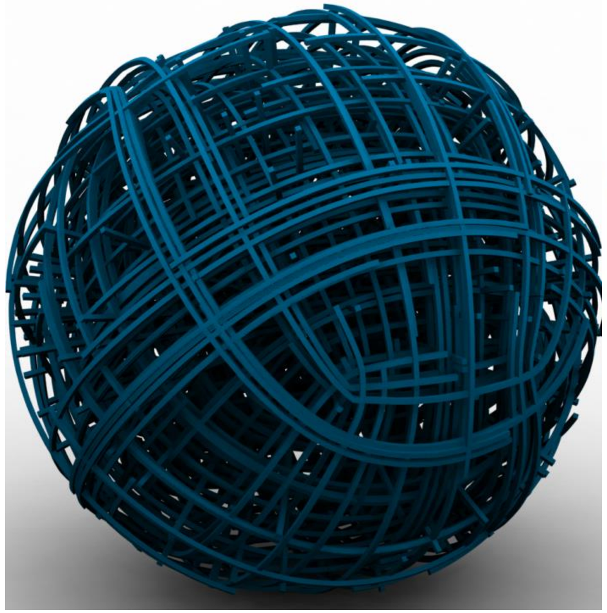
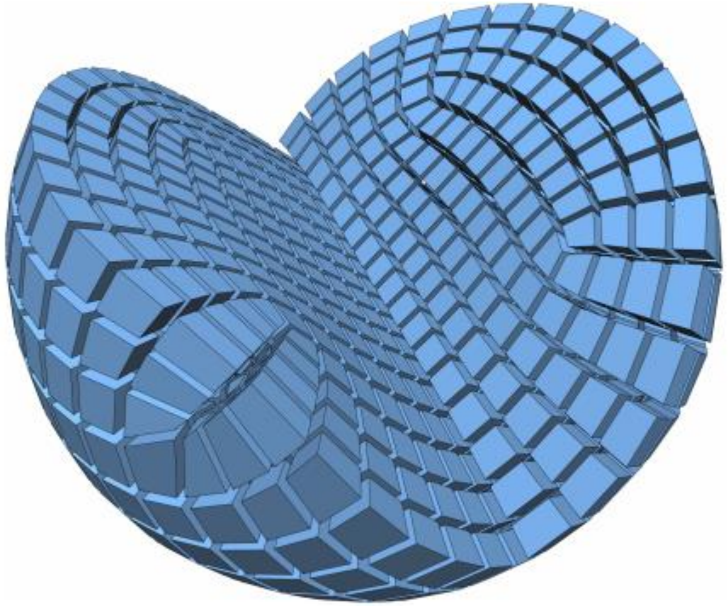
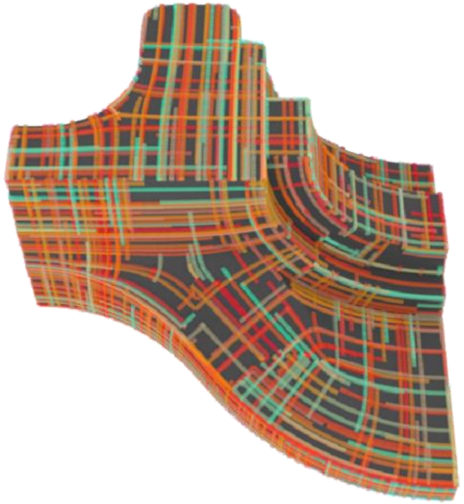
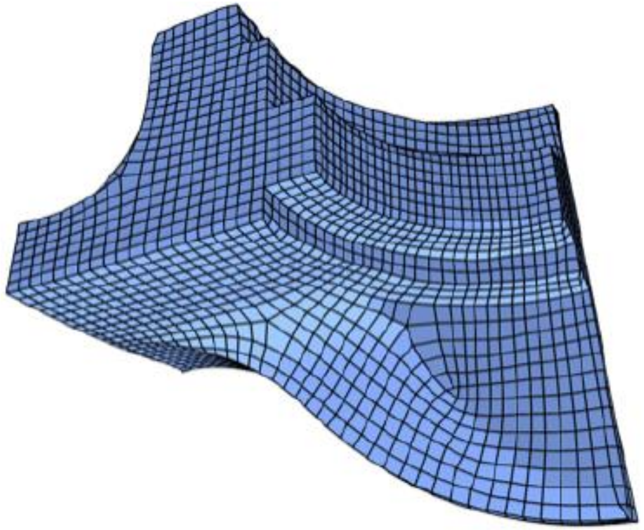
© Disney/Pixar



[Jiang et al. 2015]

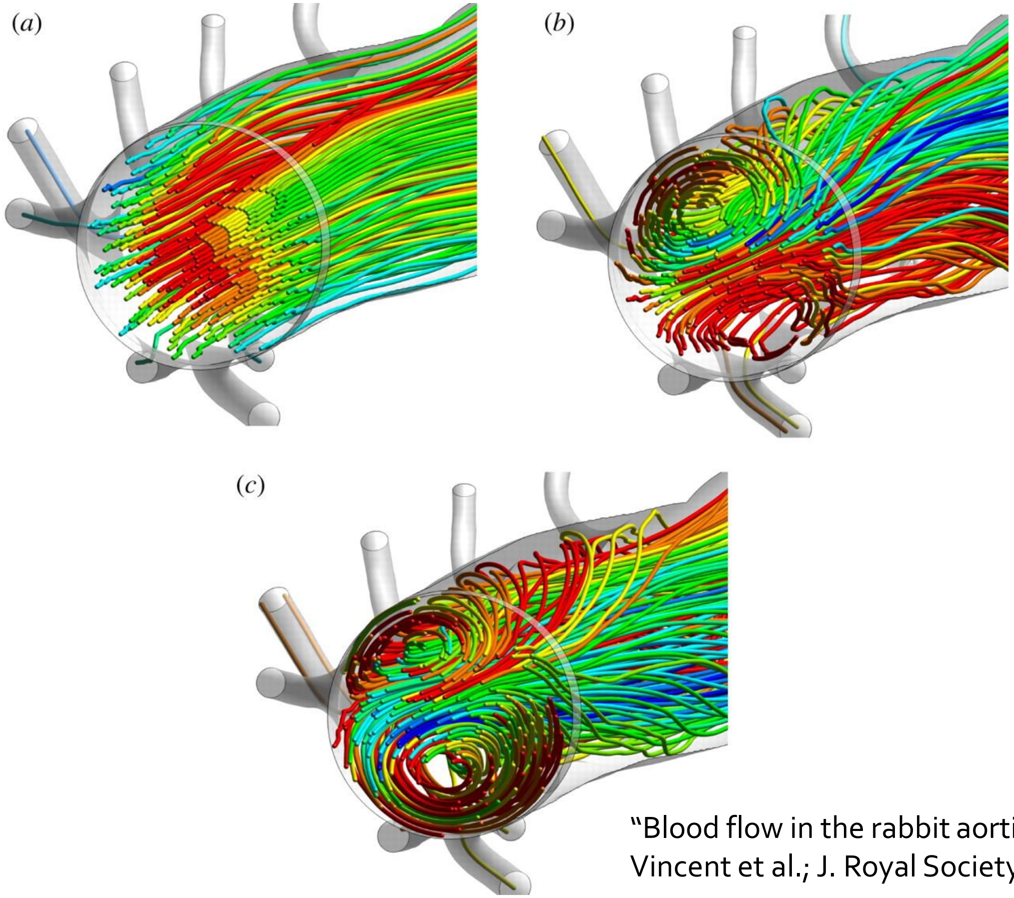
Graphics

Why Vector Fields?

Simulation and PDE

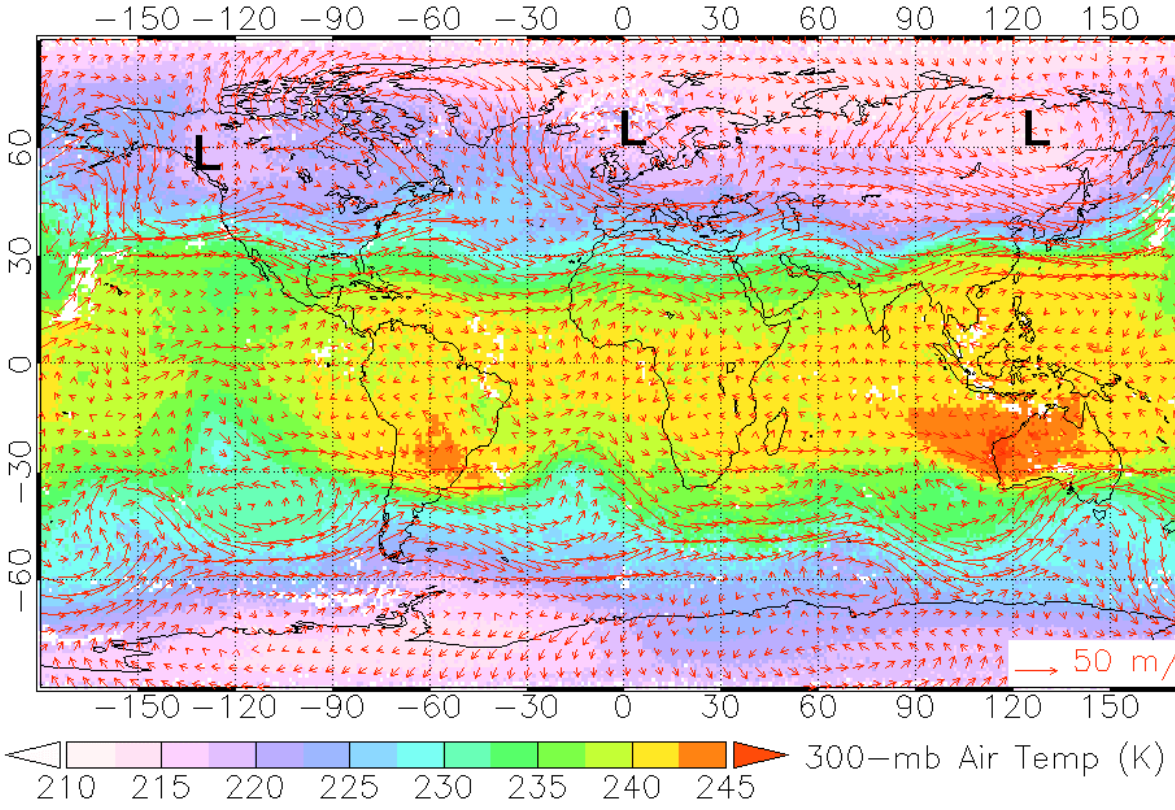
Why Vector Fields?



"Blood flow in the rabbit aortic arch and descending thoracic aorta"
Vincent et al.; J. Royal Society 2011

Biological science and imaging

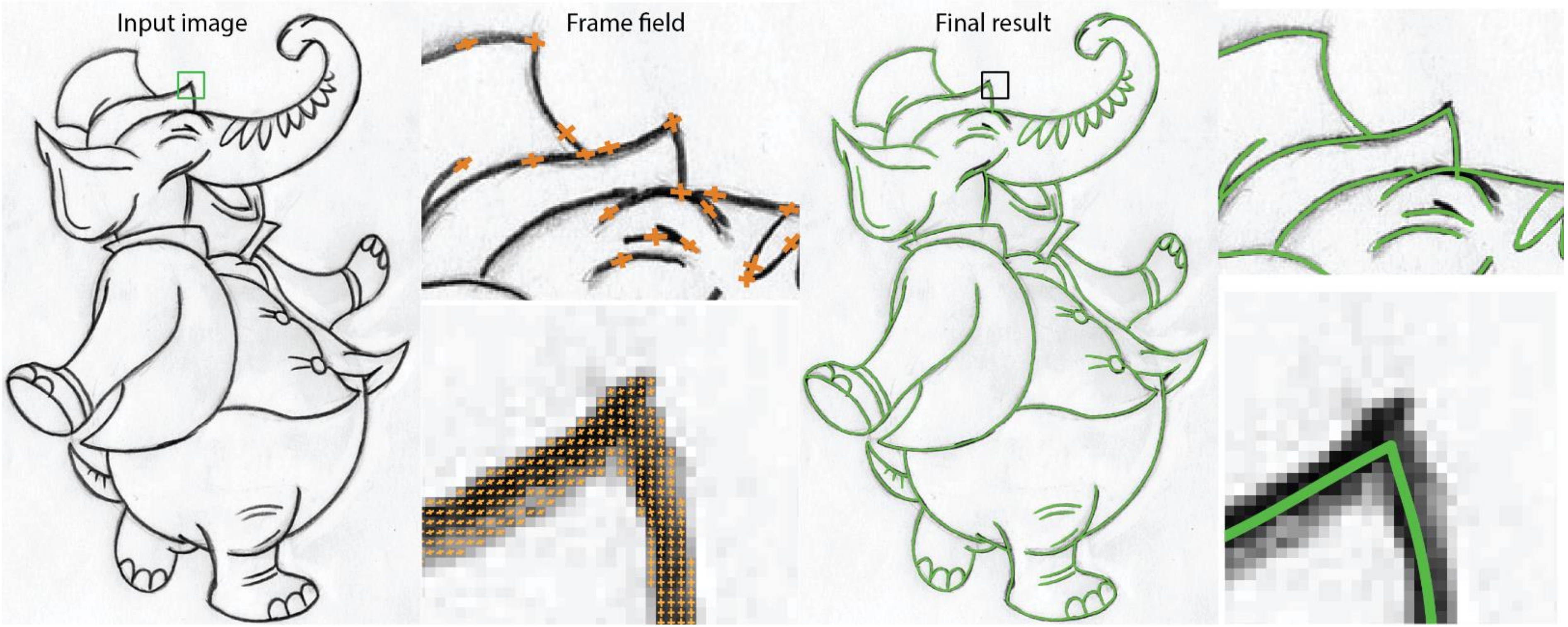
Why Vector Fields?



https://disc.gsfc.nasa.gov/featured-items/airs-monitors-cold-weather

Weather modeling

Why Vector Fields?



Vectorization of Line Drawings via Polyvector Fields (Bessmeltsev & Solomon; TOG 2019)

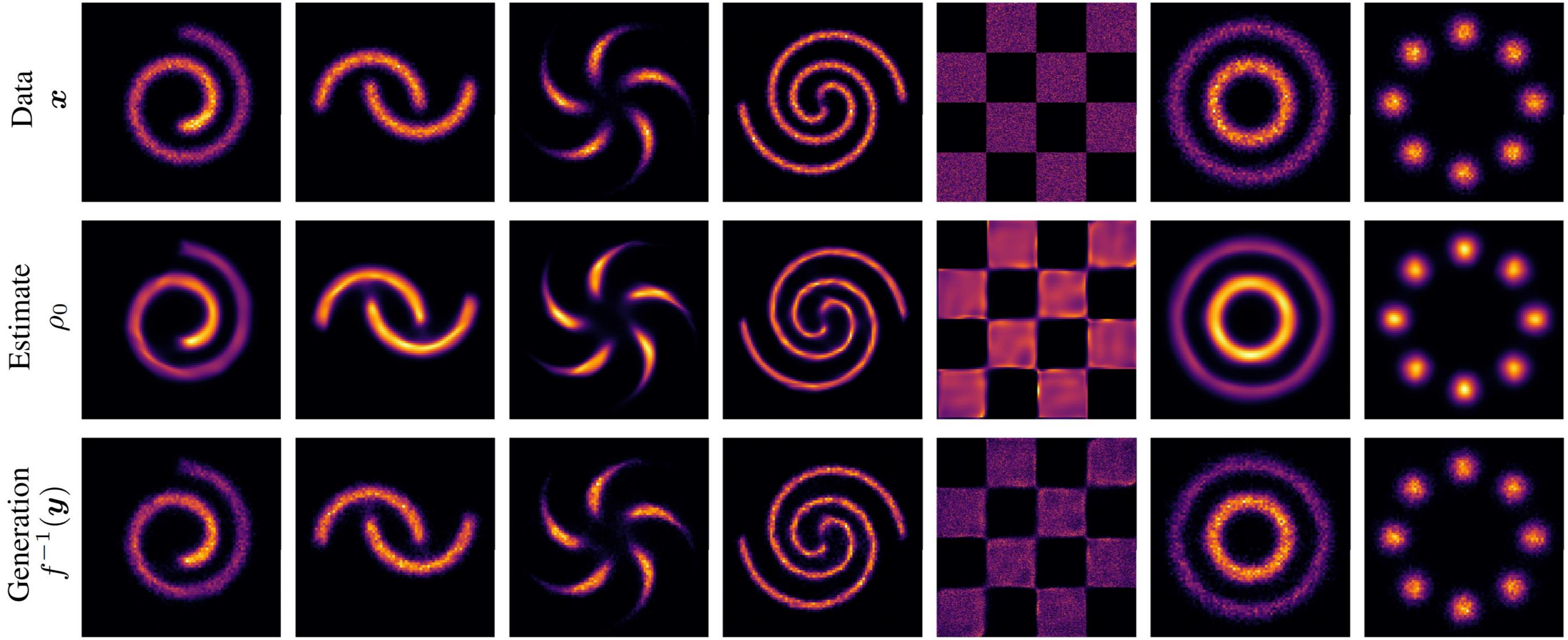
Vectorization

Why Vector Fields?

https://forum.unity3d.com/threads/megaflow-vector-fields-fluid-flows-released.278000/

Simulation and engineering

Why Vector Fields?



"OT-Flow: Fast and Accurate Continuous Normalizing Flows via Optimal Transport" (Onken et al.)

Continuous normalizing flows

Why Vector Fields?

Pastry design

Many Challenges

- Directional derivative?
- Purely intrinsic version?
- Singularities?
- Flow lines?
- …

Theoretical

- How to discretize?
- Discrete derivatives?
- Singularity detection?
- Flow line computation?
- …

Discrete

Plan

Crash course

in theory/discretization of vector fields.

Many Challenges

- Directional derivative?
- Purely intrinsic version?
- Singularities?
- Flow lines?
- …

Theoretical

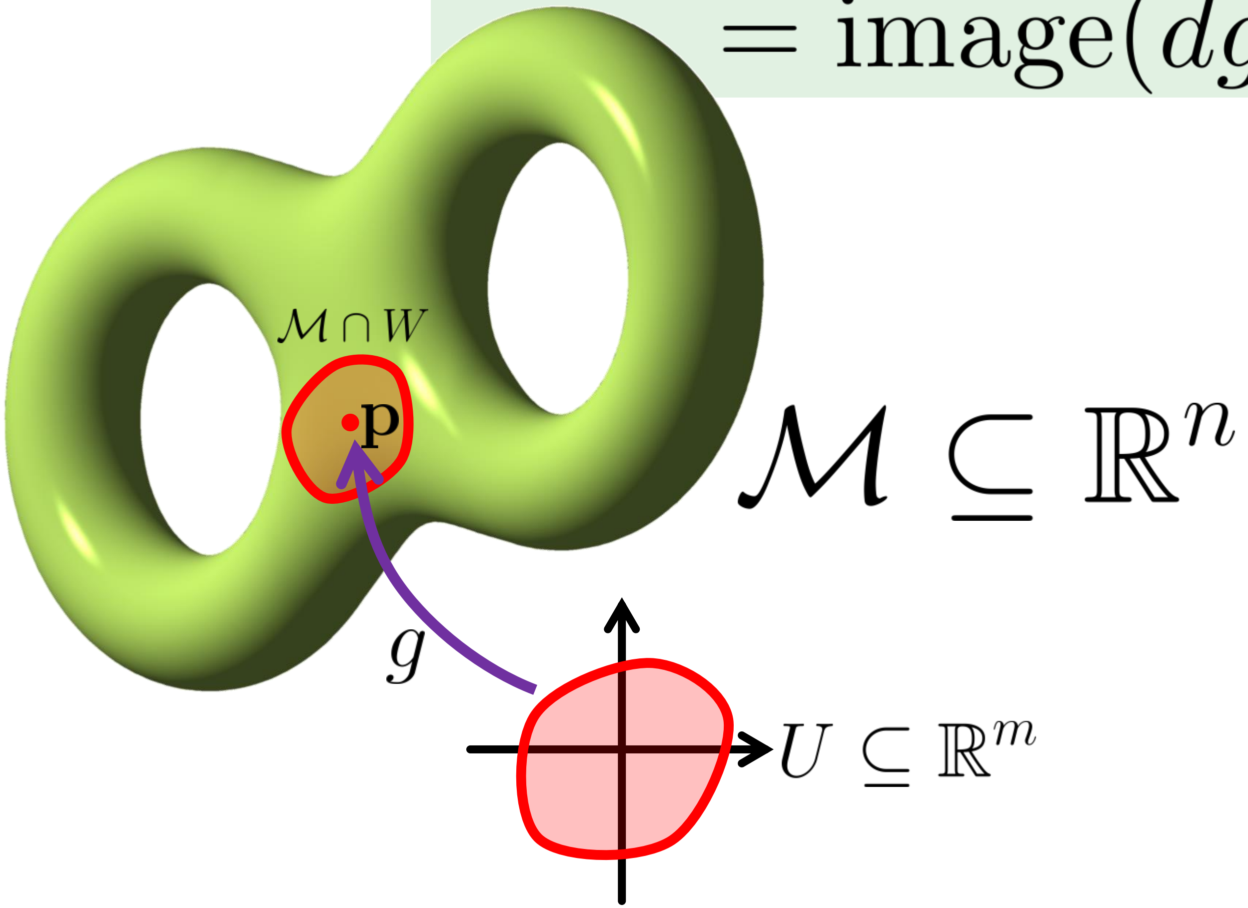
- How to discretize?
- Discrete derivatives?
- Singularity detection?
- Flow line computation?
- …

Discrete

Recall:

Tangent Space

$$T_{\mathbf{p}}\mathcal{M} = \gamma'(0), \text{ where } \gamma(0) = \mathbf{p}$$
$$= \text{image}(dg_{\mathbf{p}})$$



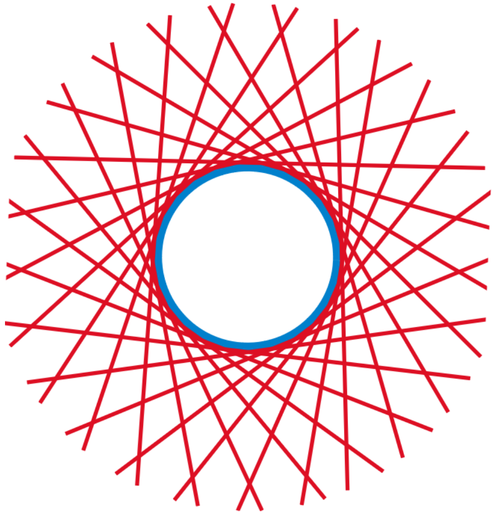
Some Definitions

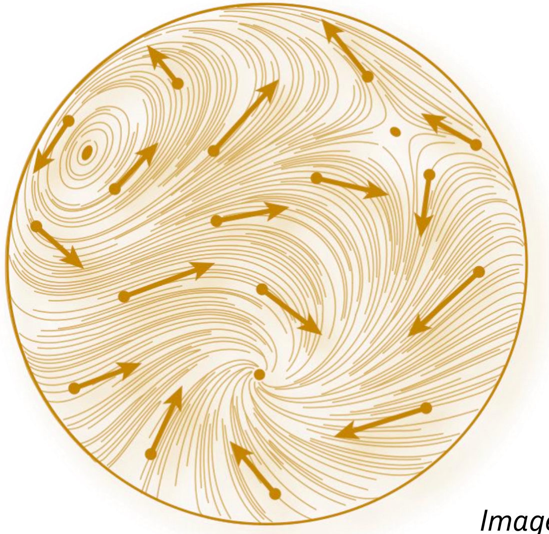
Tangent bundle:

$$TM := \{(p, v) : v \in T_pM\}$$

Vector field:

$$u : M \to TM \text{ with } u(p) = (p, v), v \in T_pM$$

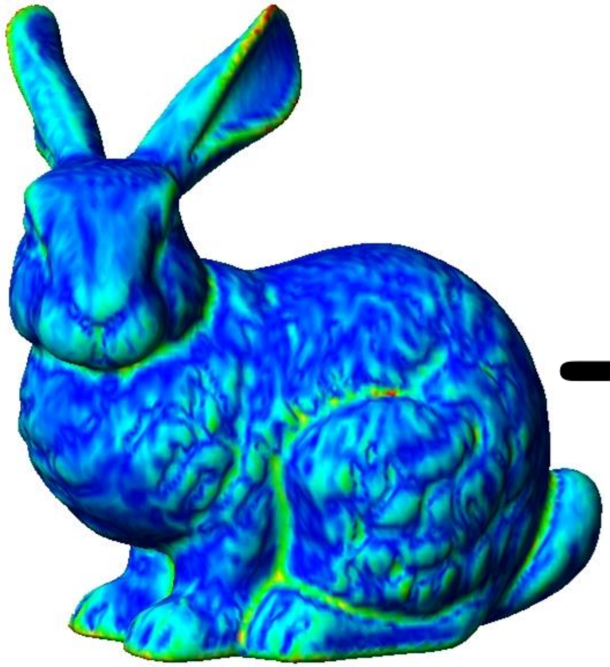




Images from Wikipedia, SIGGRAPH course

Recall:

Scalar Functions

$f :$  $\longrightarrow \mathbb{R}$

http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

Map points to real numbers

Recall:

Differential of a Map

Definition (Differential). *Suppose* $\varphi : \mathcal{M} \to \mathcal{N}$ *is a map from a submanifold* $\mathcal{M} \subseteq \mathbb{R}^k$ *into a submanifold* $\mathcal{N} \subseteq \mathbb{R}^\ell$. *Then, the* differential $d\varphi_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \to T_{\varphi(\mathbf{p})}\mathcal{N}$ *of* φ *at a point* $\mathbf{p} \in \mathcal{M}$ *is given by*

$$d\varphi_{\mathbf{p}}(\mathbf{v}) := (\varphi \circ \gamma)'(0),$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ *is any curve with* $\gamma(0) = \mathbf{p}$ *and* $\gamma'(0) = \mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$.

Linear map of tangent spaces

$$d\varphi_{\mathbf{p}}(\gamma'(0)) := (\varphi \circ \gamma)'(0)$$

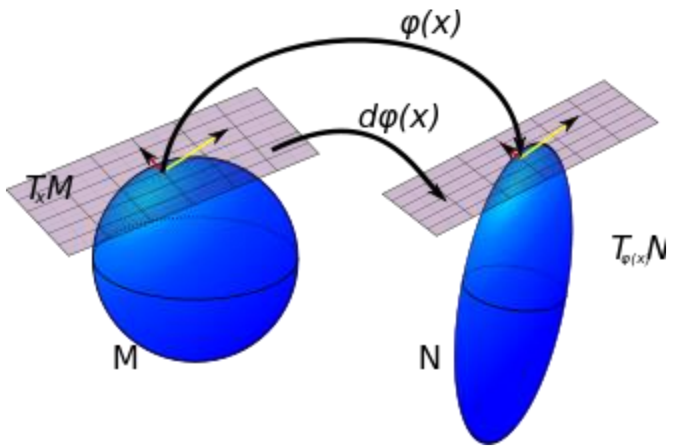
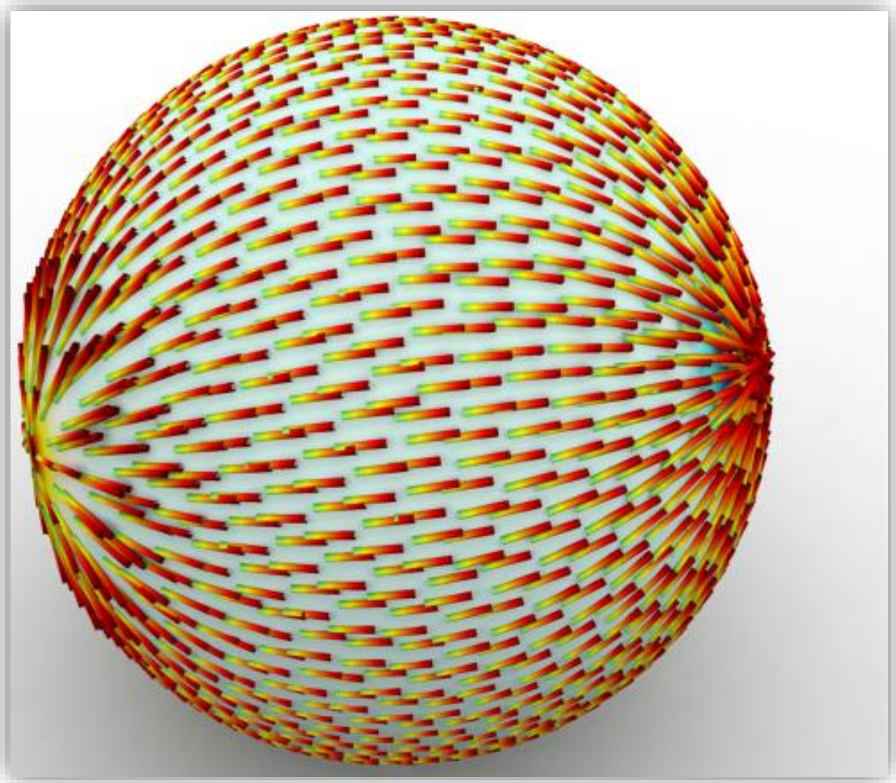


Image from Wikipedia

Recall:

Gradient Vector Field

Proposition *For each* $\mathbf{p} \in \mathcal{M}$*, there exists a unique vector* $\nabla f(\mathbf{p}) \in T_{\mathbf{p}}\mathcal{M}$ *so that* $df_{\mathbf{p}}(\mathbf{v}) = \mathbf{v} \cdot \nabla f(\mathbf{p})$ *for all* $\mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$*.*



Q:

How do you **differentiate** a vector field?

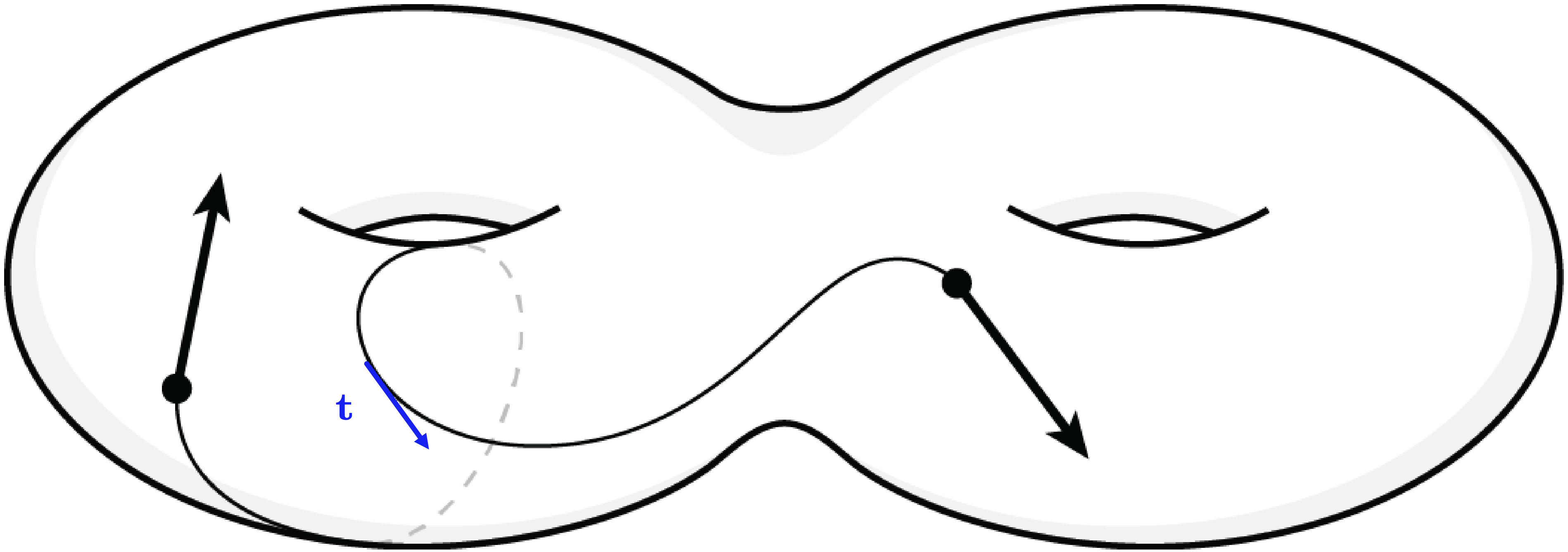
Common point of confusion.
(especially for your instructor)

http://www.wgnflag.com/xcart/images/P/G-50_StopSeatBeltsStockSign.jpg

Answer

http://www.relatably.com/m/img/complicated-memes/60260587.jpg

What's the issue?



How to identify different tangent spaces?

Many Notions of Derivative

- Differential of covector
  (defer for now) (or, forever?)

- Lie derivative
  Weak structure, purely topological

- Covariant derivative
  Strong structure, involves geometry

$$D_X Y \stackrel{?}{=} \lim_{t \to 0} \frac{Y(p + tX) - Y(p)}{t}$$

Vector Field Flows: Diffeomorphism

$$\frac{d}{dt}\psi_t = V \circ \psi_t$$

Useful property: $\psi_{t+s}(x) = \psi_t(\psi_s(x))$

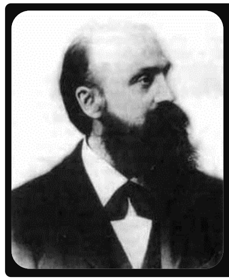
Diffeomorphism with inverse ψ_{-t}

Group structure!

Fun example:

Killing Vector Fields (KVFs)

Preserves distances infinitesimally



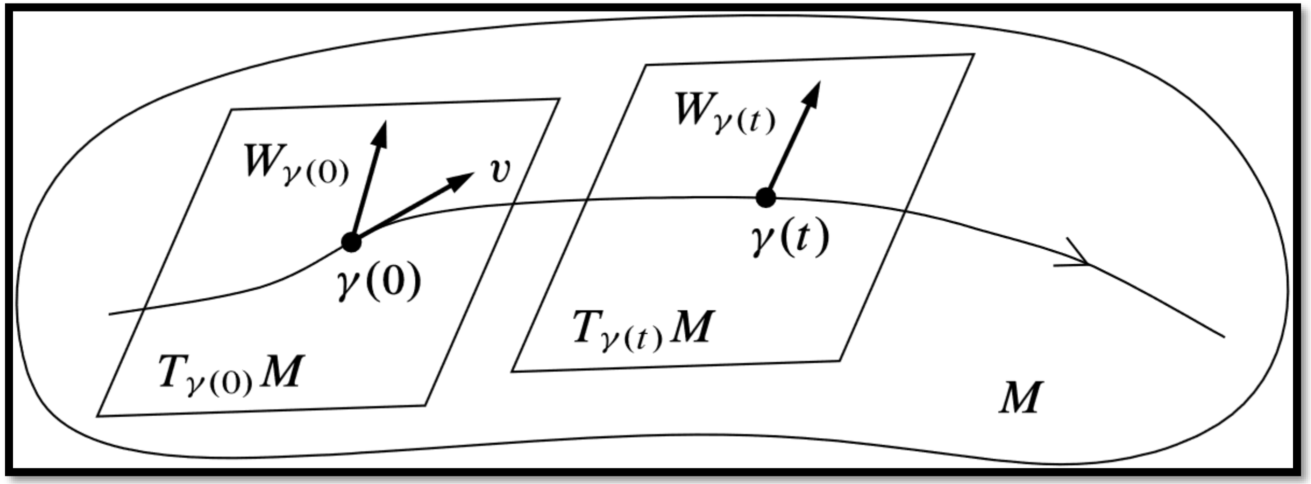
Wilhelm Killing

1847-1923

Germany

http://www.bradleycorp.com/image/985/9184b_highres.jpg

Differential of Vector Field Flow



$$d\psi_t(p) : T_pM \to T_{\psi_t(p)}M$$

Image from *Smooth Manifolds*, Lee

Lie Derivative

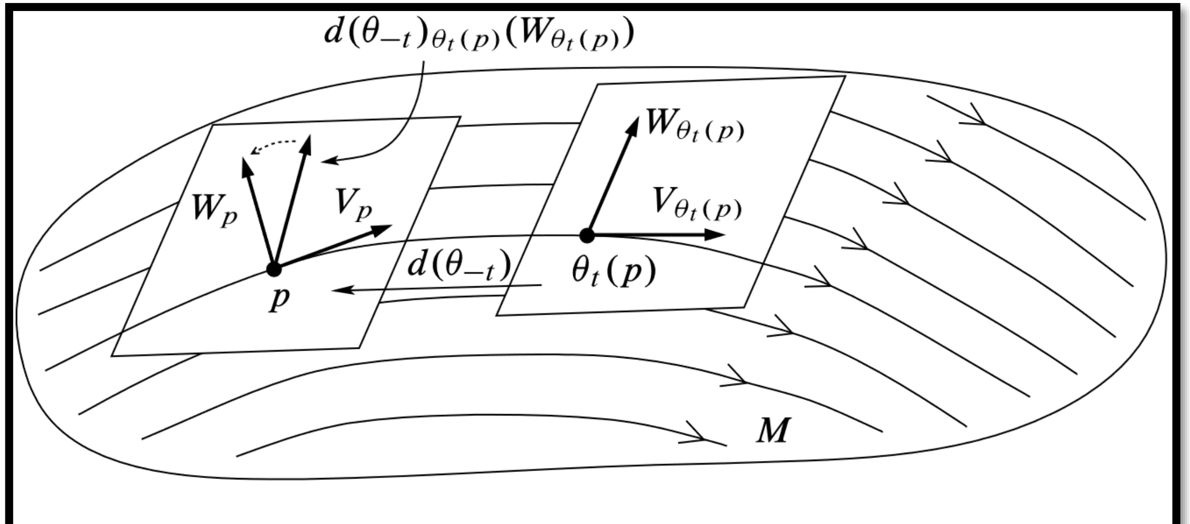


Fig. 9.13 The Lie derivative of a vector field

$$(\mathcal{L}_V W)_p := \lim_{t \to 0} \frac{1}{t} \left[(d\psi_{-t})_{\psi_t(p)}(W_{\psi_t(p)}) - W_p \right]$$

Image from *Smooth Manifolds*, Lee

It’s pronounced

“Lee”

Not “Lahy” or “Lye”

(BTW: It’s “oiler,” not “you-ler”)

Counterintuitive Property of Lie Derivative

$$(\mathcal{L}_V W)_p := \lim_{t \to 0} \frac{1}{t} \left[(d\psi_{-t})_{\psi_t(p)} (W_{\psi_t(p)}) - W_p \right]$$

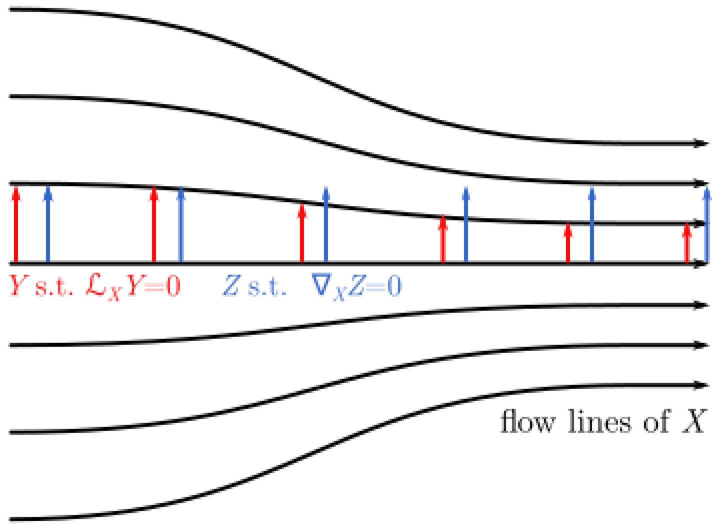
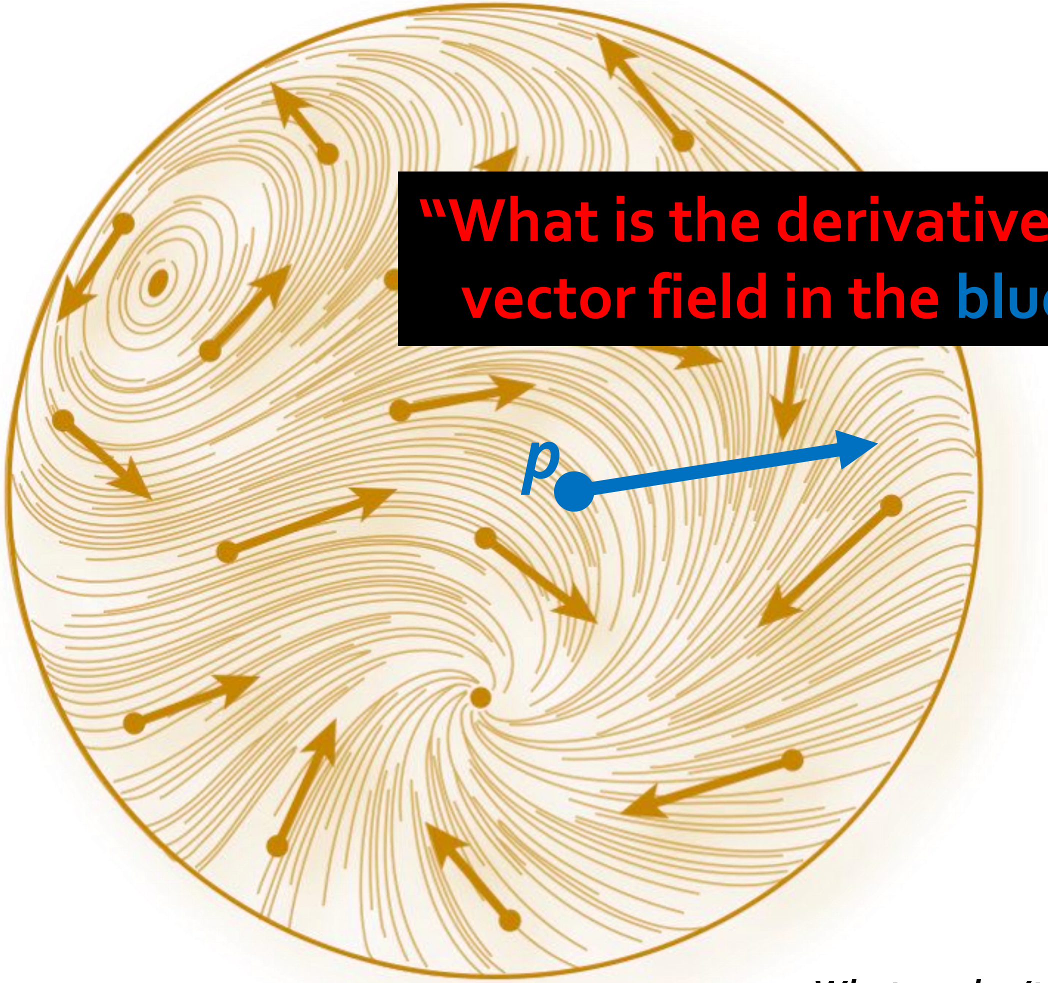


Image courtesy A. Carapetis

Depends on structure of *V*

What We Want

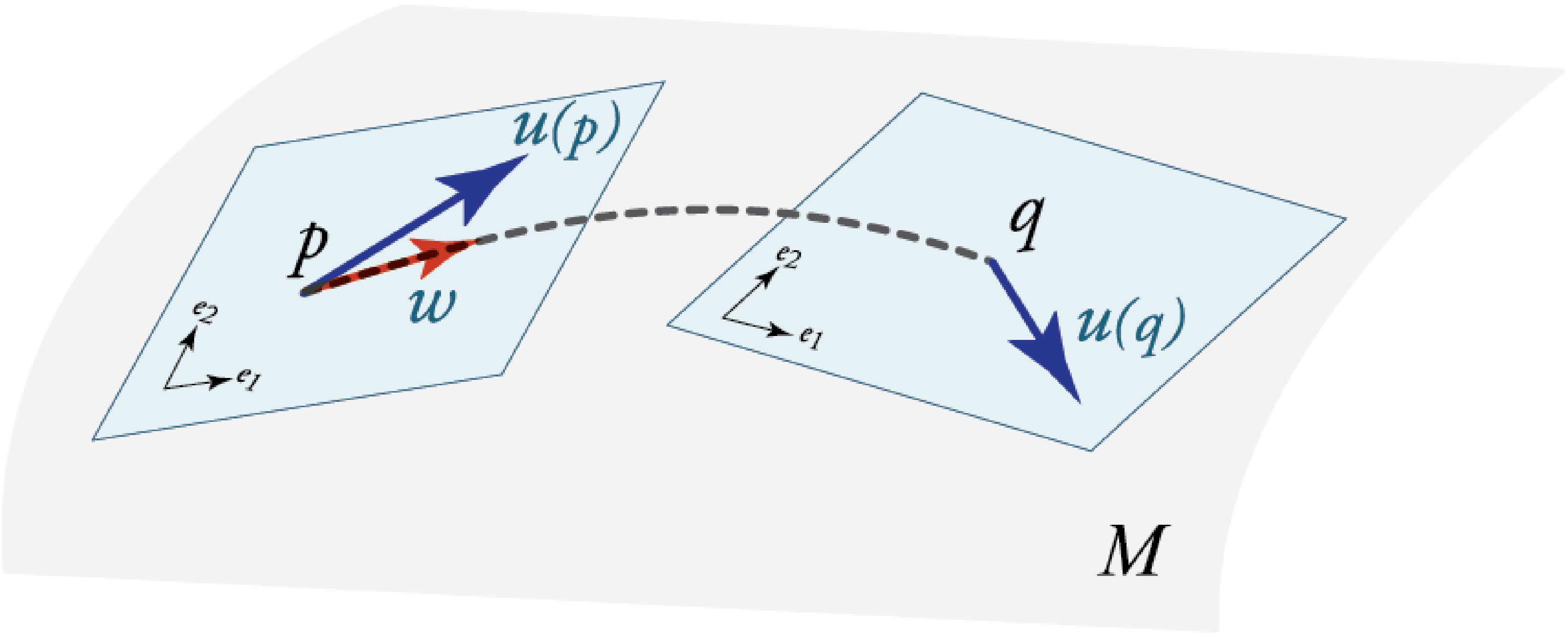


"What is the derivative of the orange vector field in the blue direction?"

What we don't want:

Specify blue direction anywhere but at *p*.

Parallel Transport



Canonical identification of tangent spaces

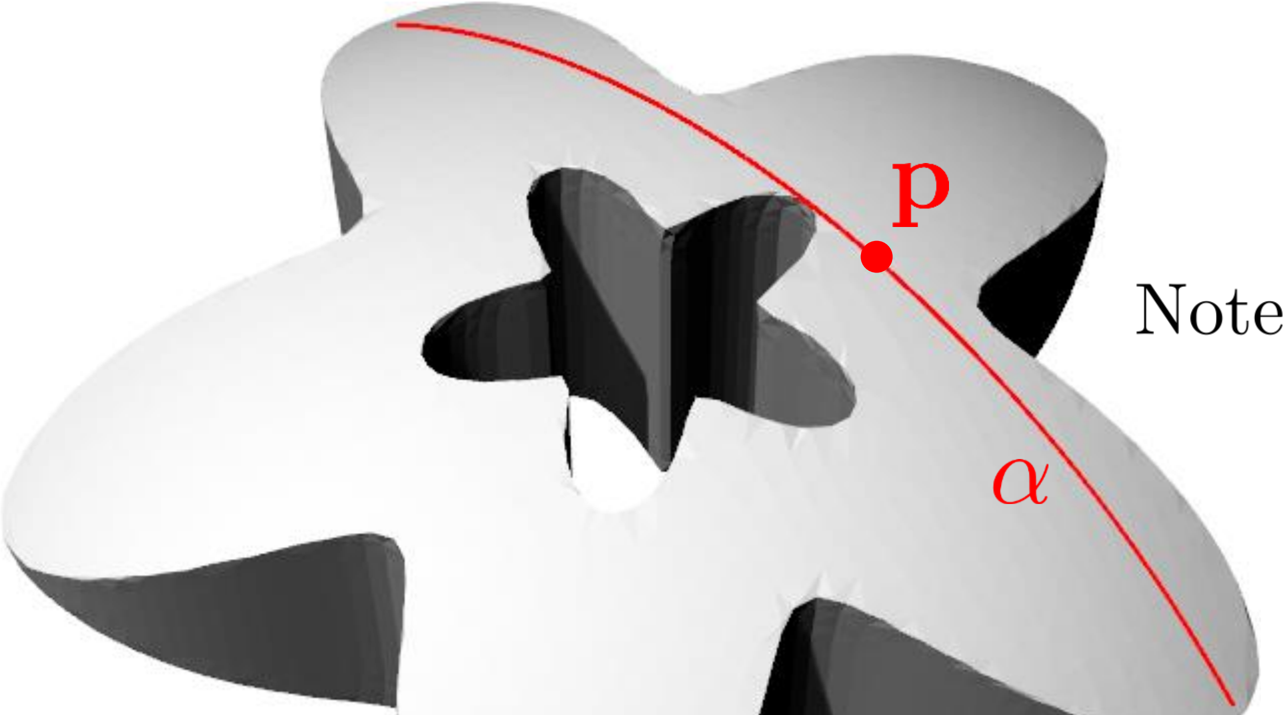
Homework 3

Covariant Derivative (Embedded)

$$\nabla_{\mathbf{v}}\mathbf{w} := [d\mathbf{w}(\mathbf{v})]^{\parallel} = \text{proj}_{T_{\mathbf{p}}\mathcal{M}}(\mathbf{w} \circ \alpha)'(0)$$

Integral curve of v through p

Synonym: (Levi-Civita) Connection



Note: $[d\mathbf{w}(\mathbf{v})]^{\perp} = \mathrm{I\!I}(\mathbf{v}, \mathbf{w})\mathbf{n}$

Some Properties

Properties of the Covariant Derivative

As defined, $\nabla_V Y$ depends only on V_p and Y to first order along c.

Also, we have the Five Properties:

1. C^∞-linearity in the V-slot:
   $\nabla_{V_1+fV_2} Y = \nabla_{V_1} Y + f\,\nabla_{V_2} Y$ where $f : S \to \mathbb{R}$
2. $\mathbb{R}$-linearity in the Y-slot:
   $\nabla_V(Y_1 + aY_2) = \nabla_V Y_1 + a\nabla_V Y_2$ where $a \in \mathbb{R}$
3. Product rule in the Y-slot:
   $\nabla_V(f\,Y) = f \cdot \nabla_V Y + (\nabla_V f) \cdot Y$ where $f : S \to \mathbb{R}$
4. The metric compatibility property:
   $\nabla_V \langle Y, Z \rangle = \langle \nabla_V Y, Z \rangle + \langle Y, \nabla_V Z \rangle$
5. The "torsion-free" property:
   $\nabla_{V_1} V_2 - \nabla_{V_2} V_1 = [V_1, V_2]$

The Lie bracket

$$[V_1, V_2](f) := D_{V_1} D_{V_2}(f) - D_{V_2} D_{V_1}(f)$$

Defines a vector field, which is tangent to S if V_1, V_2 are!

Slide by A. Butscher, Stanford CS 468

Challenge Problem

4-3. In your study of differentiable manifolds, you have already seen another way of taking "directional derivatives of vector fields," the Lie derivative $\mathcal{L}_X Y$.

(a) Show that the map $\mathcal{L}\colon \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ is not a connection.

(b) Show that there is a vector field on $\mathbf{R}^2$ that vanishes along the x^1-axis, but whose Lie derivative with respect to ∂_1 does not vanish on the x^1-axis. [This shows that Lie differentiation does not give a well-defined way to take directional derivatives of vector fields along curves.]

Riemannian Manifolds, Lee

Recall:

Geodesic Equation

$$\mathrm{proj}_{T_{\gamma(s)}\mathcal{M}}\left[\gamma''(s)\right] \equiv 0$$

- The only acceleration is out of the surface
  - No steering wheel!

Intrinsic Geodesic Equation

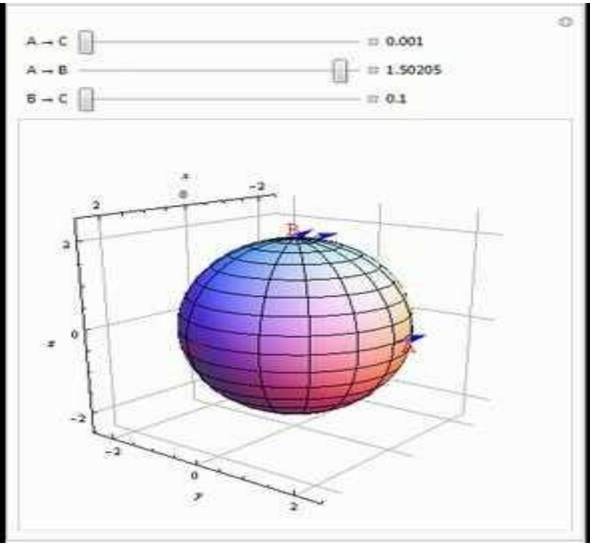
$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$$

- No stepping on the accelerator
- No steering wheel!

Parallel Transport

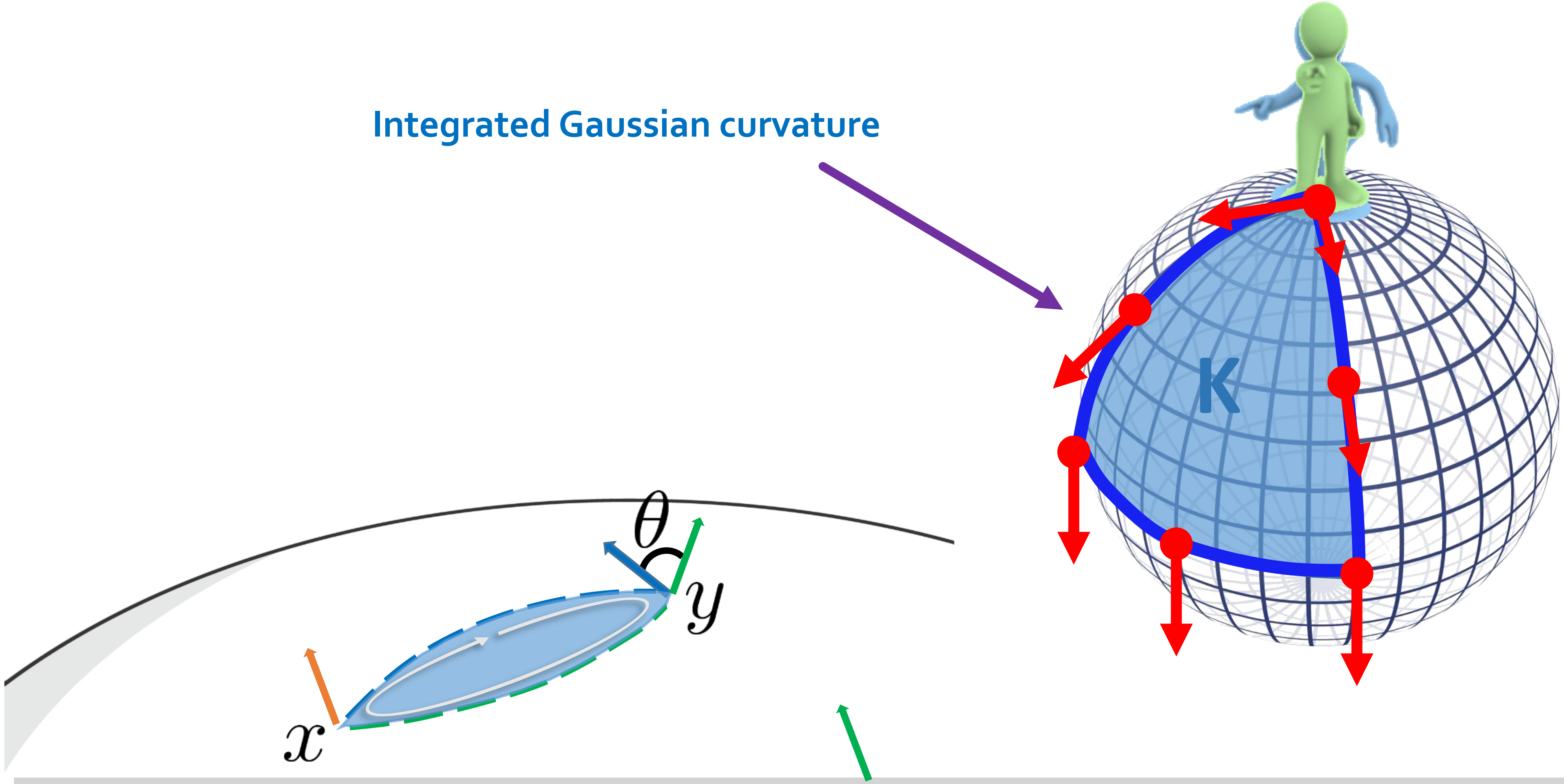
Only path-independent if domain is flat.

$$0 = \nabla_{\dot{\gamma}(t)} \mathbf{v}$$



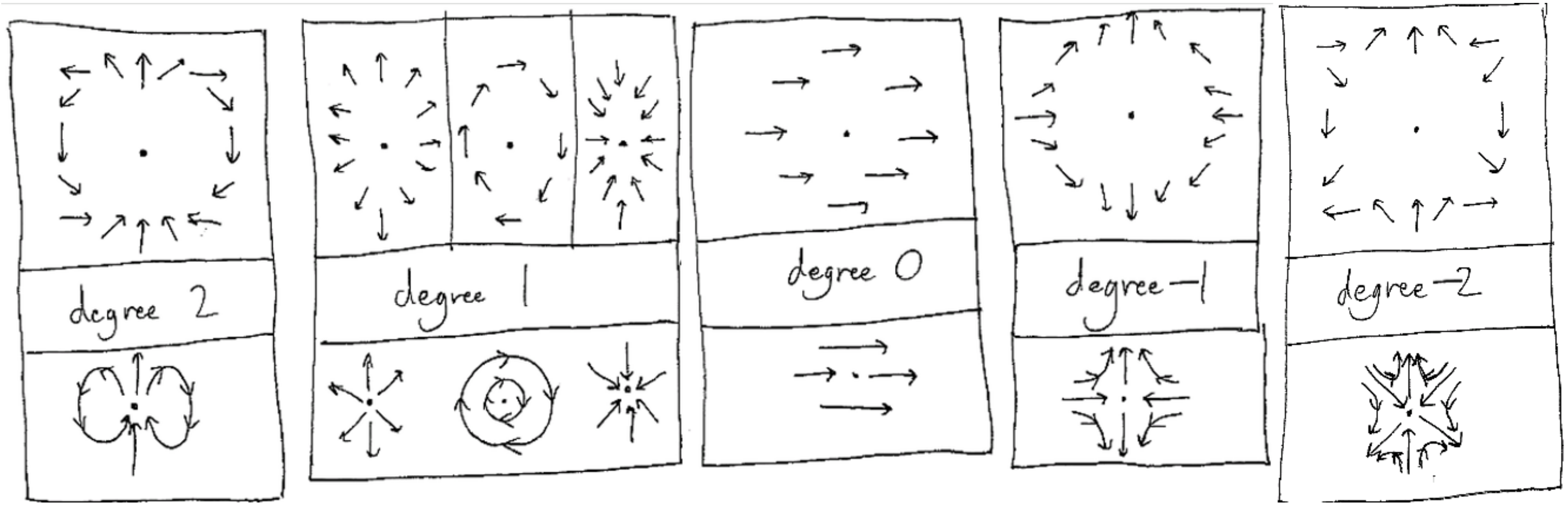
Preserves length, inner product
(can be used to *define* covariant derivative)

Holonomy



Path dependence of parallel transport

2D Vector Field Topology

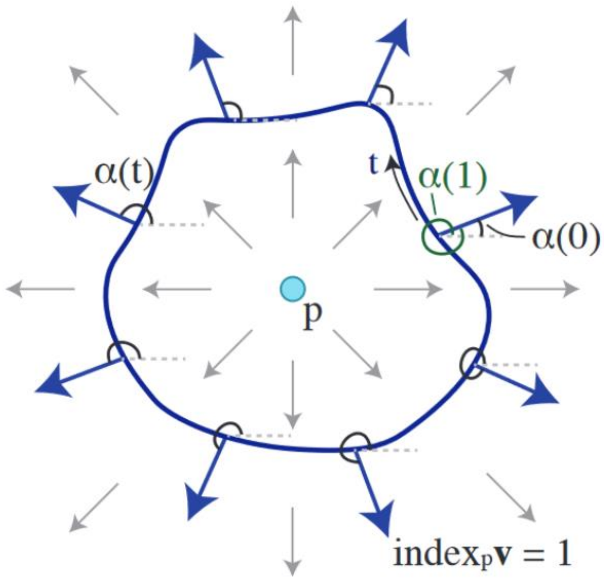


Drawings by Jonas Kibelbek

Poincaré-Hopf Theorem

$$\sum_i \text{index}_{x_i}(v) = \chi(M)$$

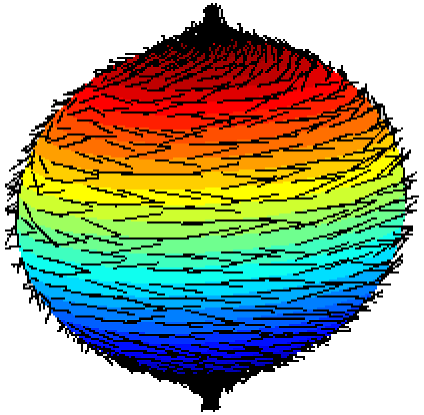
where vector field v has isolated singularities $\{x_i\}$.

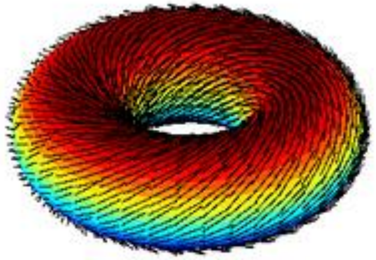


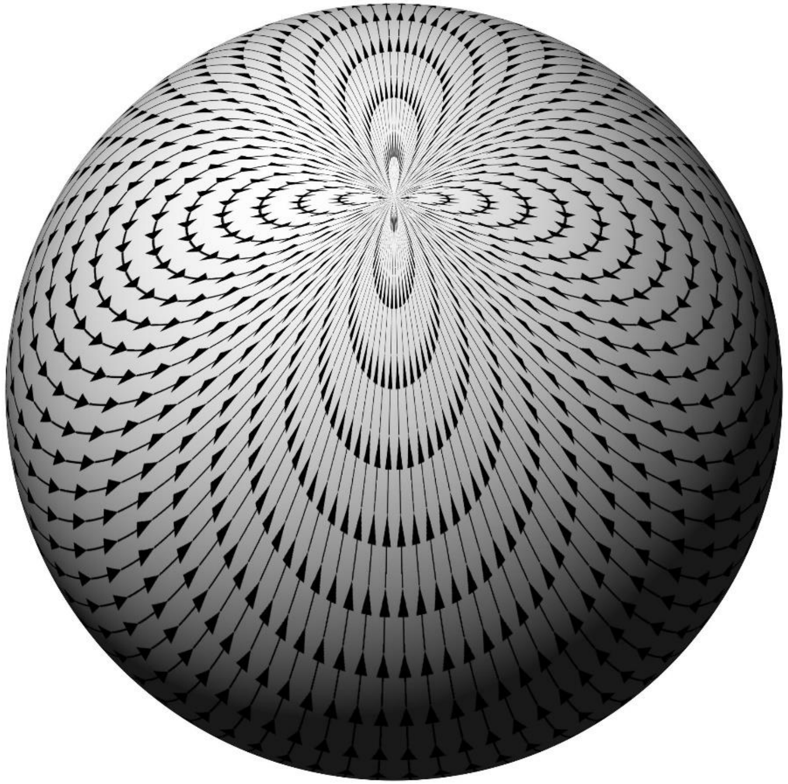
$$v(c(t)) = \|v(c(t))\| \begin{pmatrix} \cos\alpha(t) \\ \sin\alpha(t) \end{pmatrix}$$

Image from "Directional Field Synthesis, Design, and Processing" (Vaxman et al., EG STAR 2016)

Famous Corollary







Science Diagrams that Look Like Shitposts
@scienceshitpost

Those are a few of the concepts and objects studied by topology: now we'll look at a theorem.

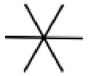
If you look at the way the hairs lie on a dog, you will find that they have a 'parting' down the dog's back, and another along the stomach. Now topologically a dog is a sphere (assuming it keeps its mouth shut and neglecting internal organs) because all we have to do is shrink its legs and fatten it up a bit (Figure 90).

Figure 90

9:30 PM · Jan 20, 2020 · Twitter Web App

Hairy ball theorem

Extension in 2D: Direction Fields

	1-vector field	One vector, classical "vector field"
	2-direction field	Two directions with π symmetry, "line field", "2-RoSy field"
	1^3-vector field	Three independent vectors, "3-polyvector field"
	4-vector field	Four vectors with $\pi/2$ symmetry, "non-unit cross field"
	4-direction field	Four directions with $\pi/2$ symmetry, "unit cross field", "4-RoSy field"
	2^2-vector field	Two pairs of vectors with π symmetry each, "frame field"
	2^2-direction field	Two pairs of directions with π symmetry each, "non-ortho. cross field"
	6-direction field	Six directions with $\pi/3$ symmetry, "6-RoSy"
	2^3-vector field	Three pairs of vectors with π symmetry each

"Directional Field Synthesis, Design, and Processing" (Vaxman et al., EG STAR 2016)

Polyvector Fields

Eurographics Symposium on Geometry Processing 2014
Thomas Funkhouser and Shi-Min Hu
(Guest Editors)

Volume 33 (2014), Number 5

Designing N-PolyVector Fields with Complex Polynomials

Olga Diamanti[1] Amir Vaxman[2] Daniele Panozzo[1] Olga Sorkine-Hornung[1]

[1]ETH Zurich, Switzerland
[2]Vienna Institute of Technology, Austria

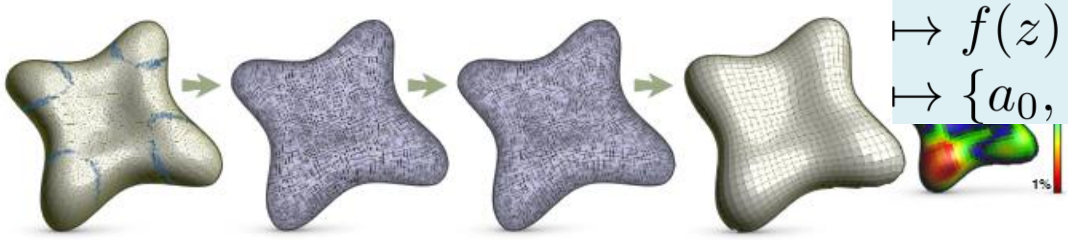


Figure 1: *A smooth 4-PolyVector field is generated from a sparse set of principal direction constraints (faces in light blue). We optimize the field for conjugacy and use it to guide the generation of a planar-quad mesh. Pseudocolor represents planarity.*

Abstract

We introduce N-PolyVector fields, a generalization of N-RoSy fields for which the vectors are neither necessarily orthogonal nor rotationally symmetric. We formally define a novel representation for N-PolyVectors as the root sets of complex polynomials and analyze their topological and geometric properties. A smooth N-PolyVector field can be efficiently generated by solving a sparse linear system without integer variables. We exploit the flexibility of N-PolyVector fields to design conjugate vector fields, offering an intuitive tool to generate planar quadrilateral meshes.

$$\{u_0, u_1, \ldots, u_k\}$$
$$\mapsto f(z) := (z - u_0) \cdot \cdots \cdot (z - u_k)$$
$$\mapsto f(z) = z^{k+1} + a_k z^k + \cdots + a_1 z + a_0$$
$$\mapsto \{a_0, \ldots, a_k\}$$

One encoding of direction fields

Vector Fields: Introduction

Justin Solomon

6.838: Shape Analysis

Spring 2021

Extra: Volumetric Frame Fields

Justin Solomon

6.838: Shape Analysis

Spring 2021

Volumetric Challenge: Hex Meshing Problem

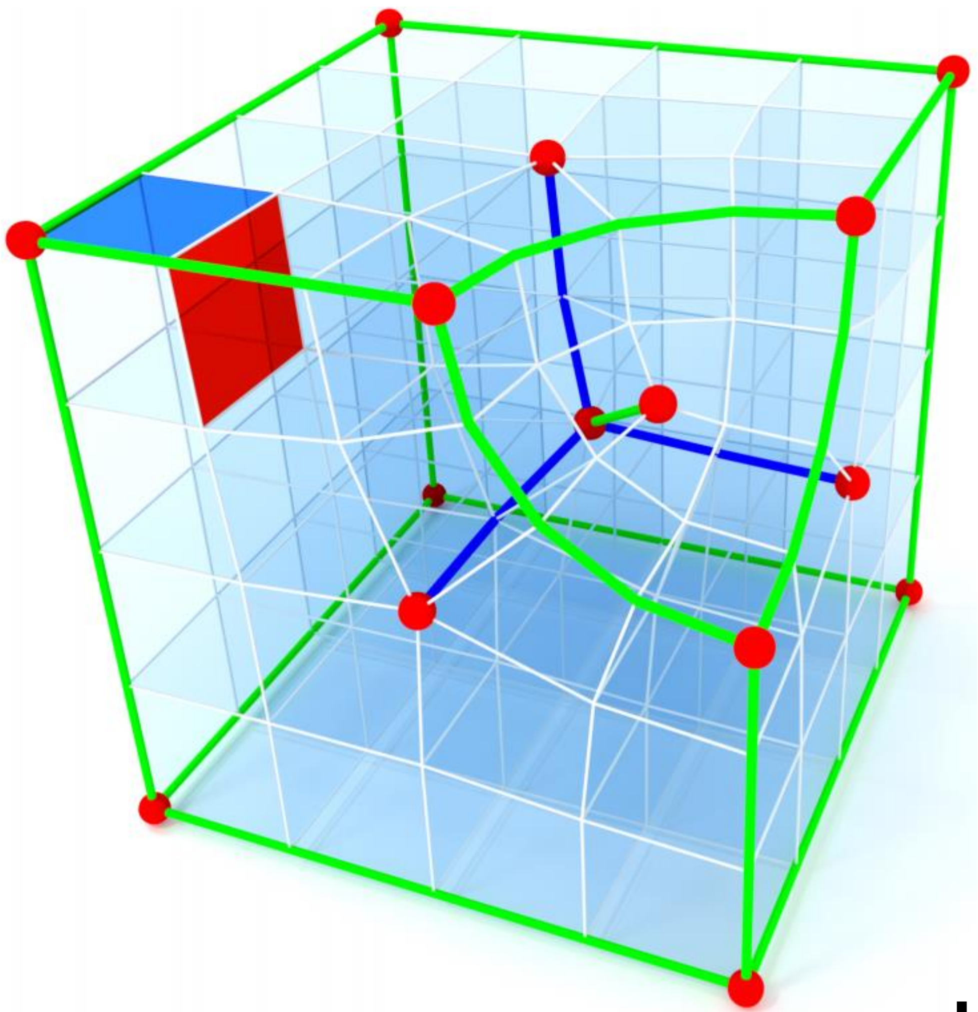
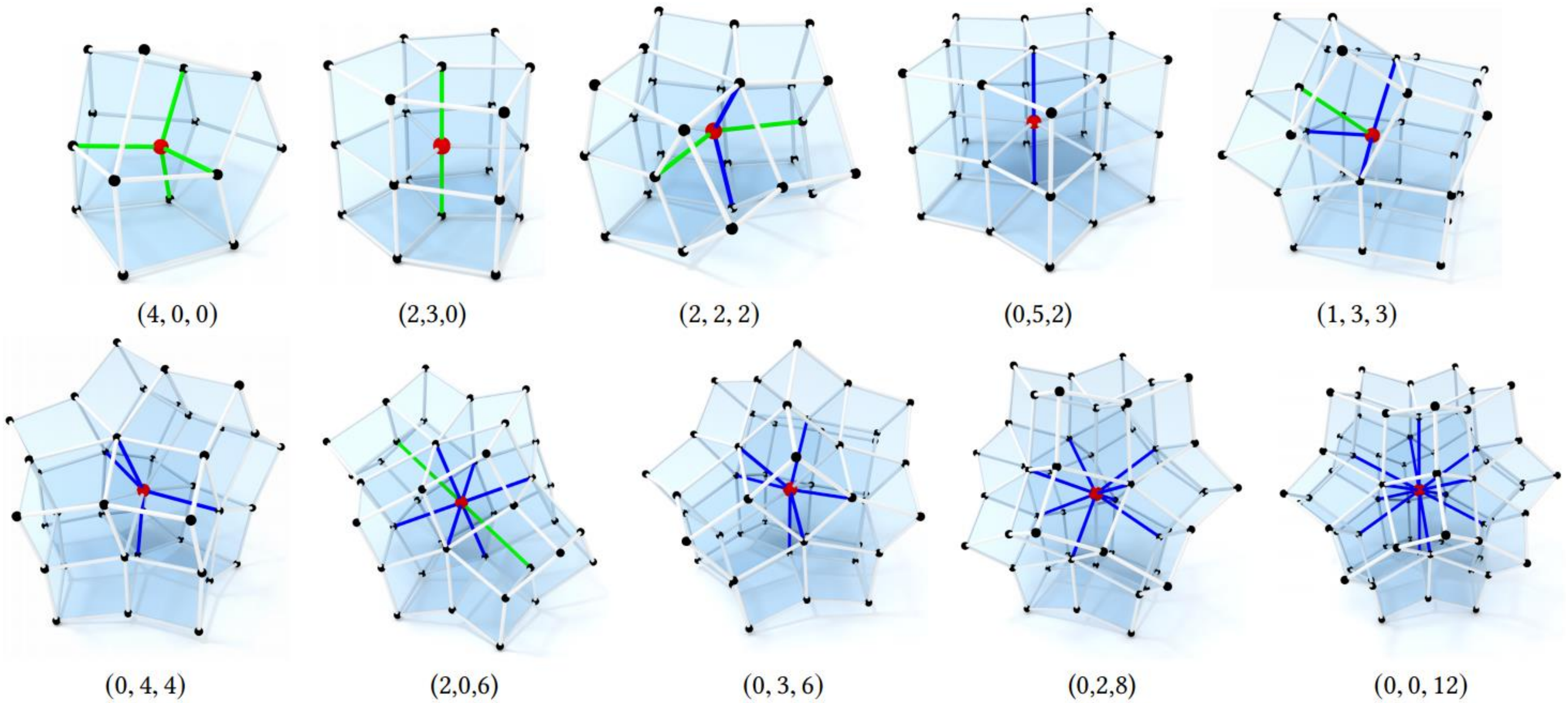


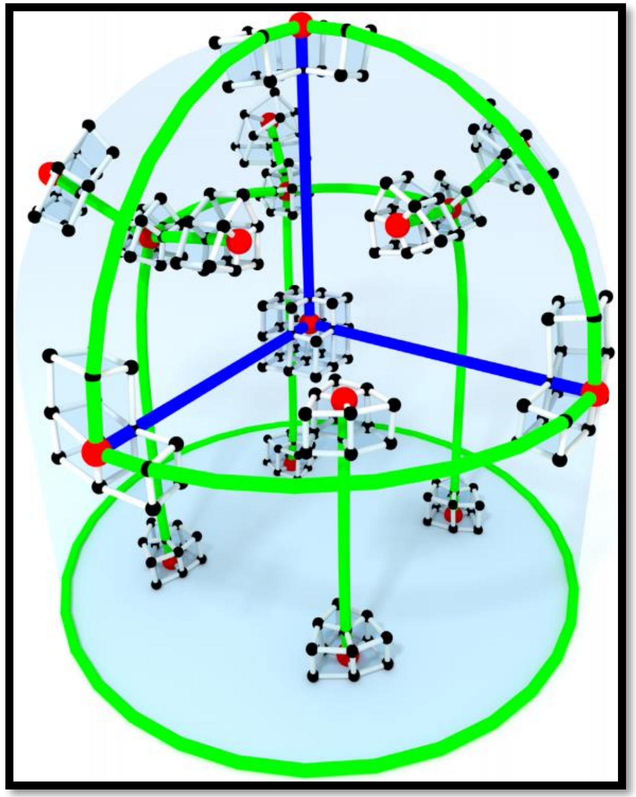
Image from:
Liu et al. "Singularity-Constrained Octahedral Fields for Hexahedral Meshing." SIGGRAPH 2018, Vancouver.

What singular structures are possible?

What is the relationship between meshes and fields?

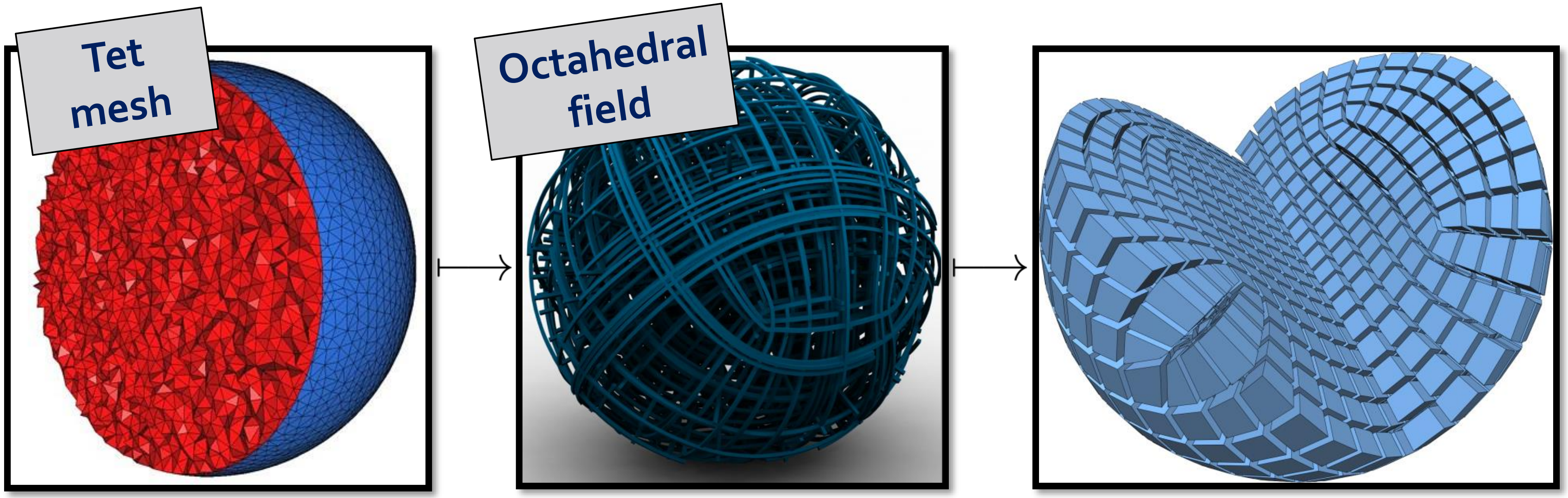
Hex Mesh Singular Structures





Images from:
Liu et al. "Singularity-Constrained Octahedral Fields for Hexahedral Meshing." SIGGRAPH 2018, Vancouver.

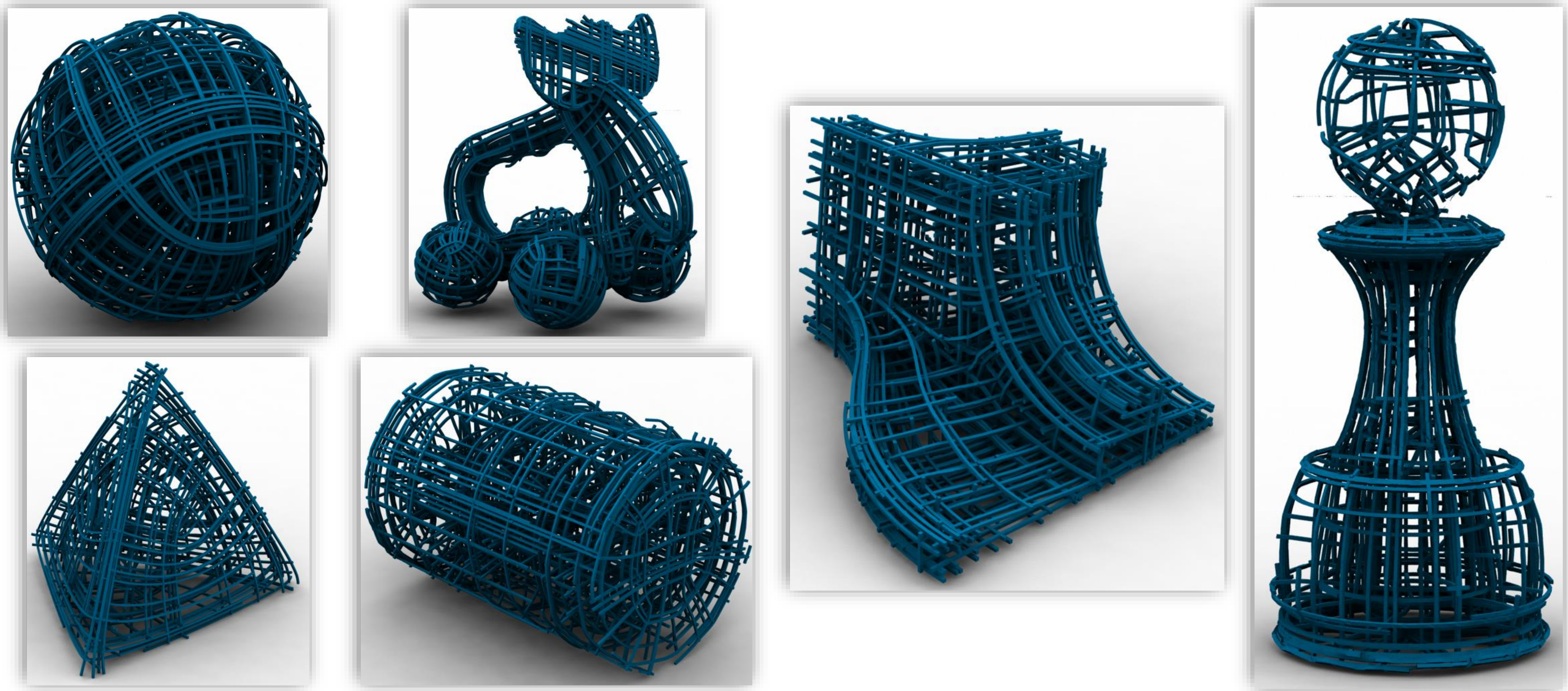
Field-Guided Meshing Pipeline



Sphere tet mesh from http://doc.cgal.org/latest/Mesh_3/index.html

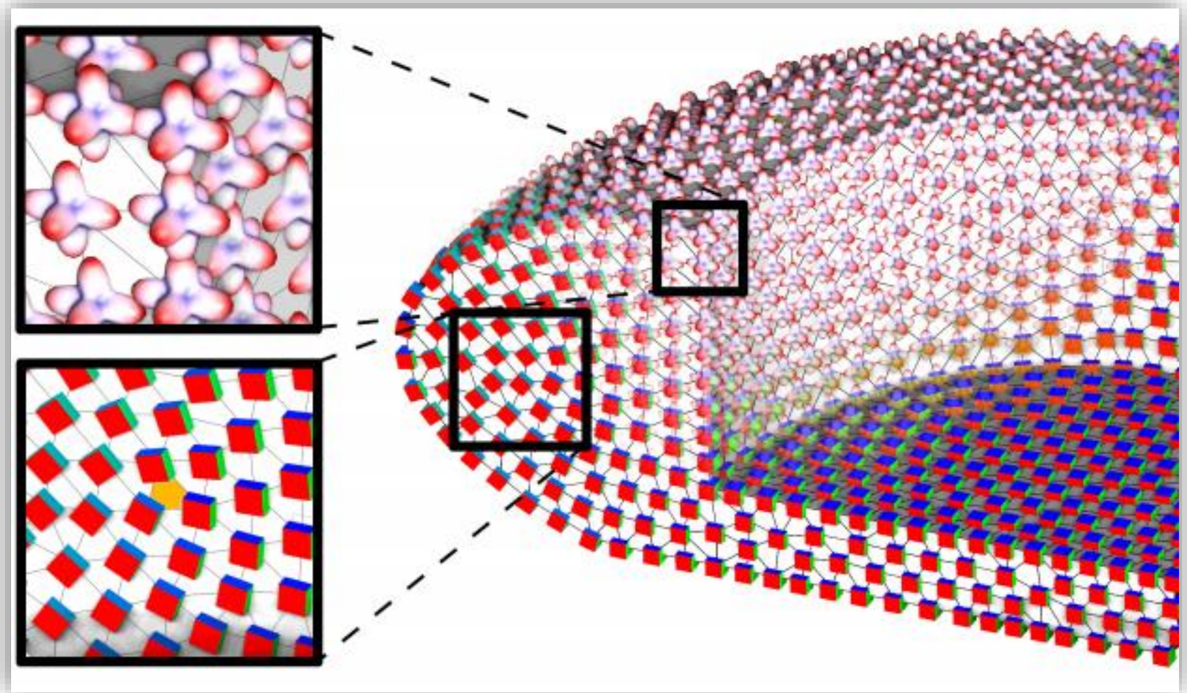
Frame per element on a tet mesh

Example Frame Fields



Images from:
Solomon, et al. "Boundary Element Octahedral Fields in Volumes." ACM Transactions on Graphics (TOG) 36.3, 2017.

Frame Field Representation



Nine spherical harmonic coefficients **per point**

Original idea in [Huang et al. 2011]
Visualization from [Ray, Sokolov, and Lévy 2016]

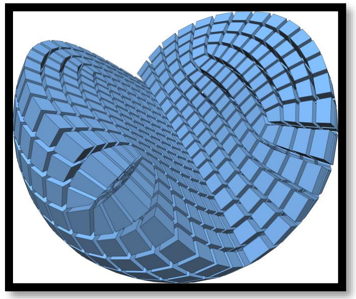
$$f(x, y, z) = x^4 + y^4 + z^4$$

Issue

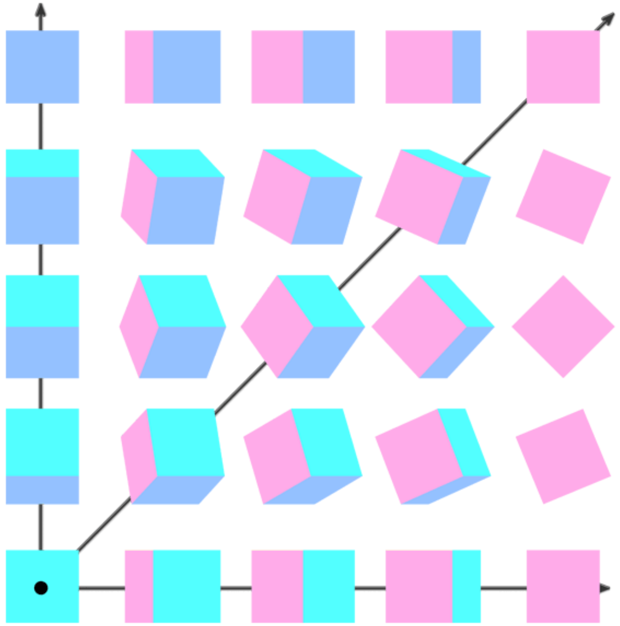
$$f(x, y, z) = x^4 + y^4 + z^4$$

$$\{\text{rotations of } f(x, y, z)\} \not\supseteq \{\text{degree-4 polynomials}\}$$

More Careful Characterization



https://design.tutsplus.com/



$$\{\text{Rotations of a cube}\} \cong SO(3)/O$$

Rotations

Octahedral group

Algebraic variety!

Palmer et al. "Algebraic Representations for Volumetric Frame Fields." ACM Transactions on Graphics (TOG) 39.2, 2020.

Octahedral variety

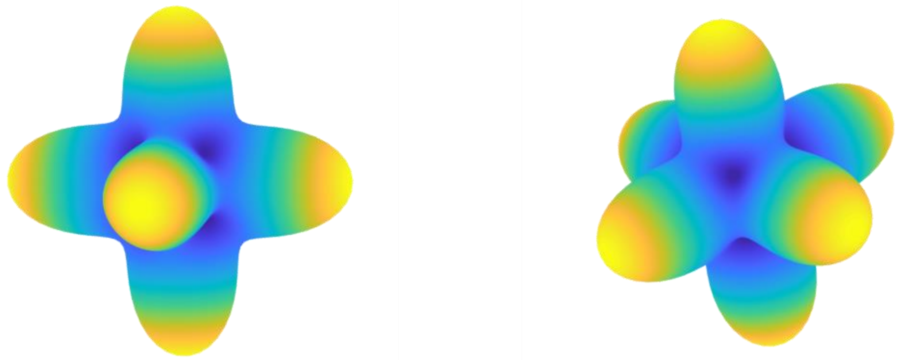
Representation Theory Perspective

Space of rotations — Wigner d-matrices

$$\begin{array}{ccc} SO(3) & \xrightarrow{\rho} & GL(9) \\ \downarrow & & \downarrow \cdot q_0 \\ \mathcal{F} & \overset{\phi}{\dashrightarrow} & F \end{array}$$

Octahedral variety — Orbit of f

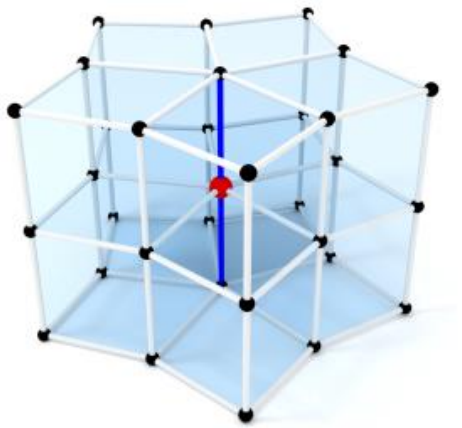
Isometry (up to scale)



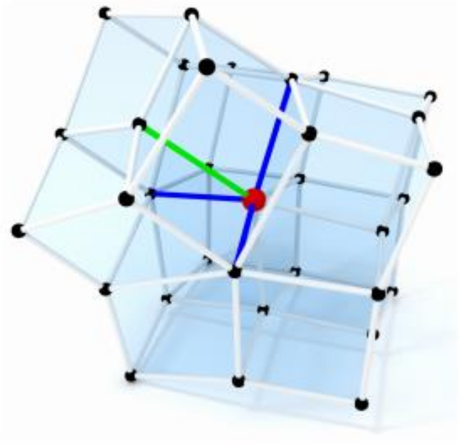
Roughly: Coefficients of $f(R^\top \mathbf{x})$

Extension: Odeco Frames

$$\sum_i \lambda_i (v_i^\top x)^d$$



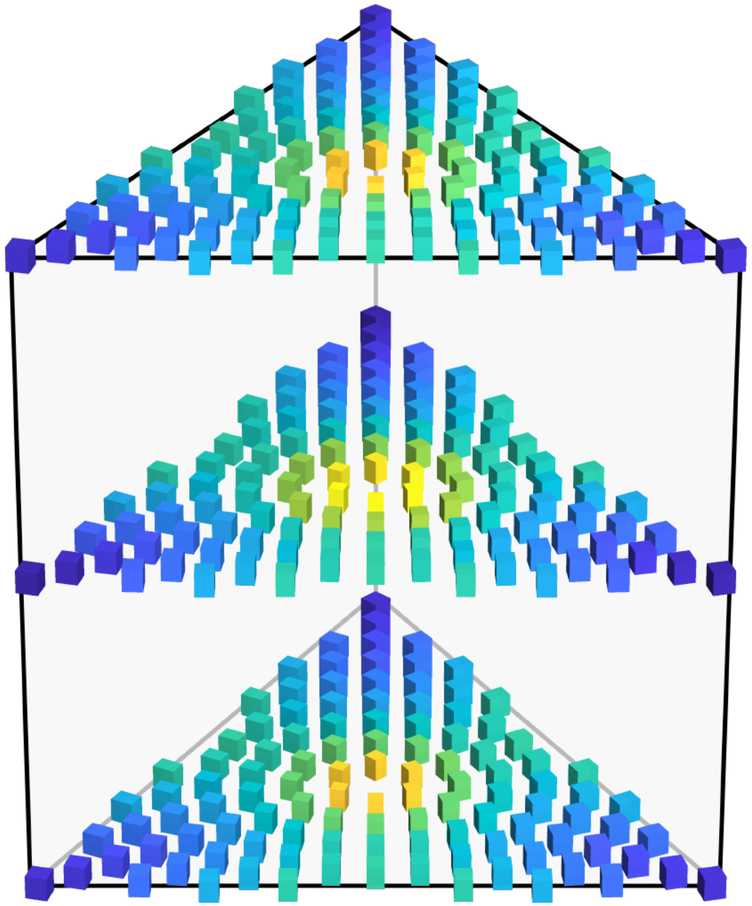
(0,5,2)

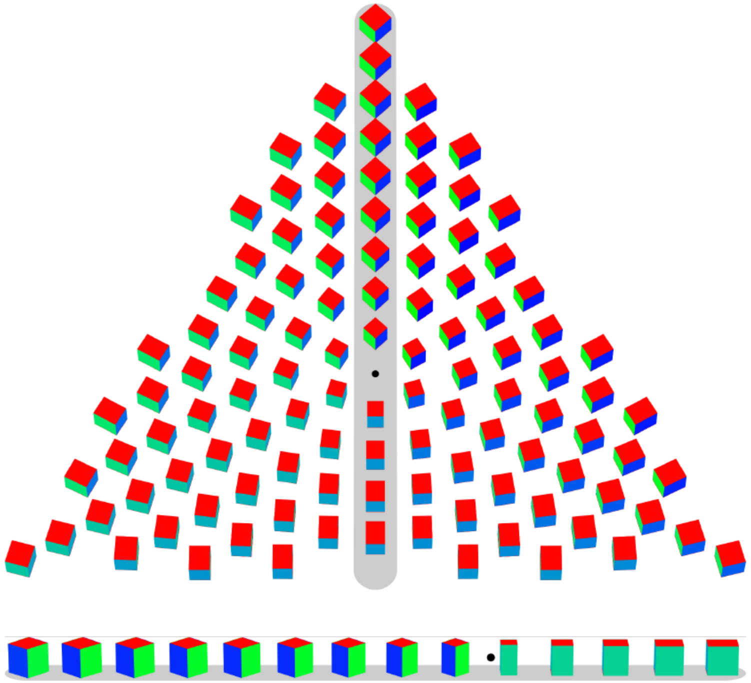


(1, 3, 3)

Orthogonally-decomposable tensors

Why Odeco?

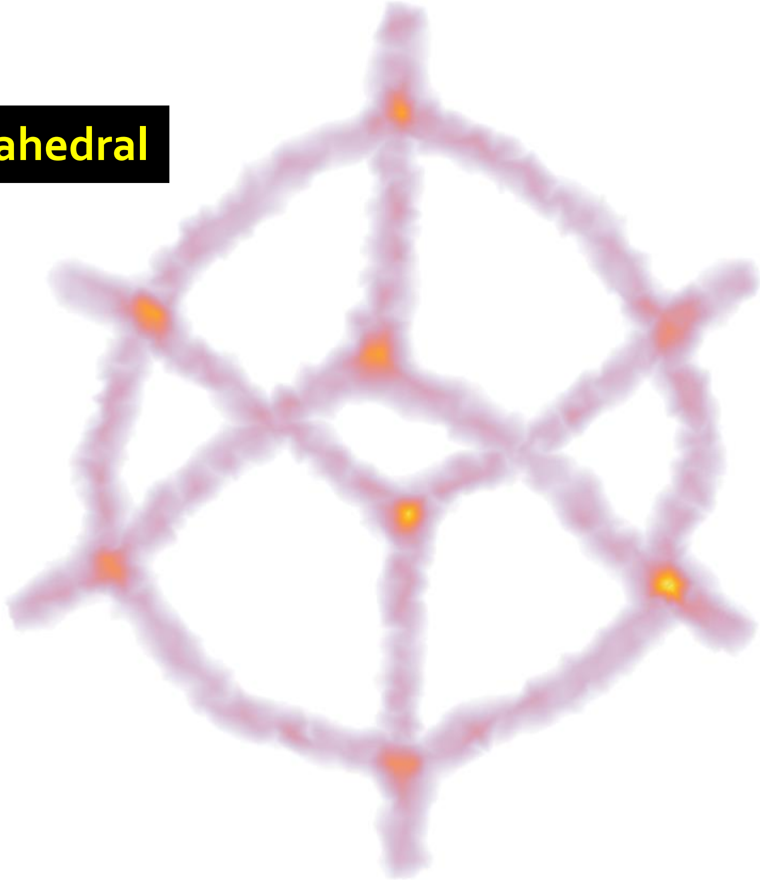




Vanishing near singular curves

Why Odeco?

Octahedral



Odeco

Energy density

Extra: Volumetric Frame Fields

Justin Solomon

6.838: Shape Analysis

Spring 2021

MIT EECS

Vector Fields: Discretization

Justin Solomon

6.838: Shape Analysis

Spring 2021

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Many Challenges

- Directional derivative?
- Purely intrinsic version?
- Singularities?
- Flow lines?
- …

Theoretical

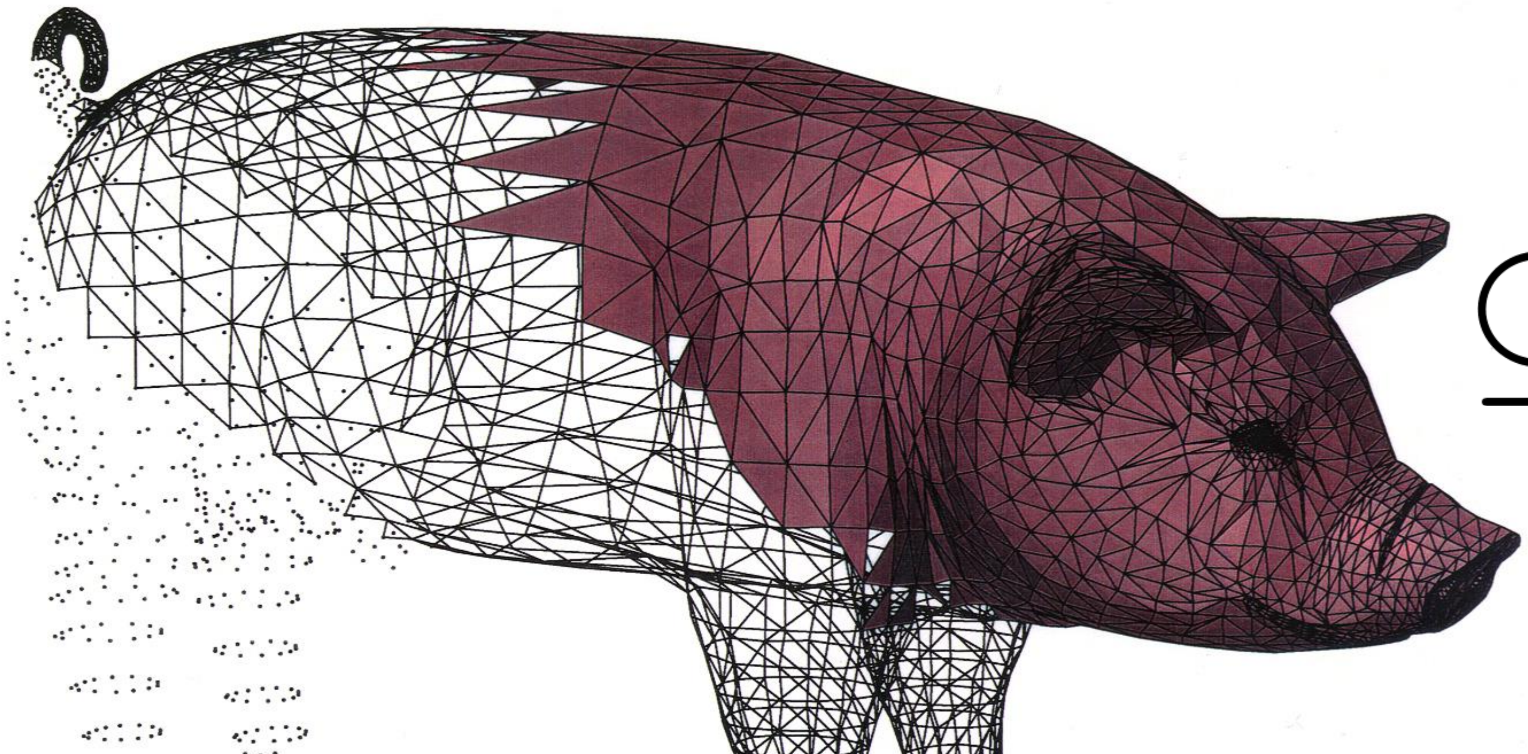
- How to discretize?
- Discrete derivatives?
- Singularity detection?
- Flow line computation?
- …

Discrete

Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based



$\subseteq \mathbb{R}^3$

Vector Fields on Triangle Meshes

No consensus:

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- Edge-based
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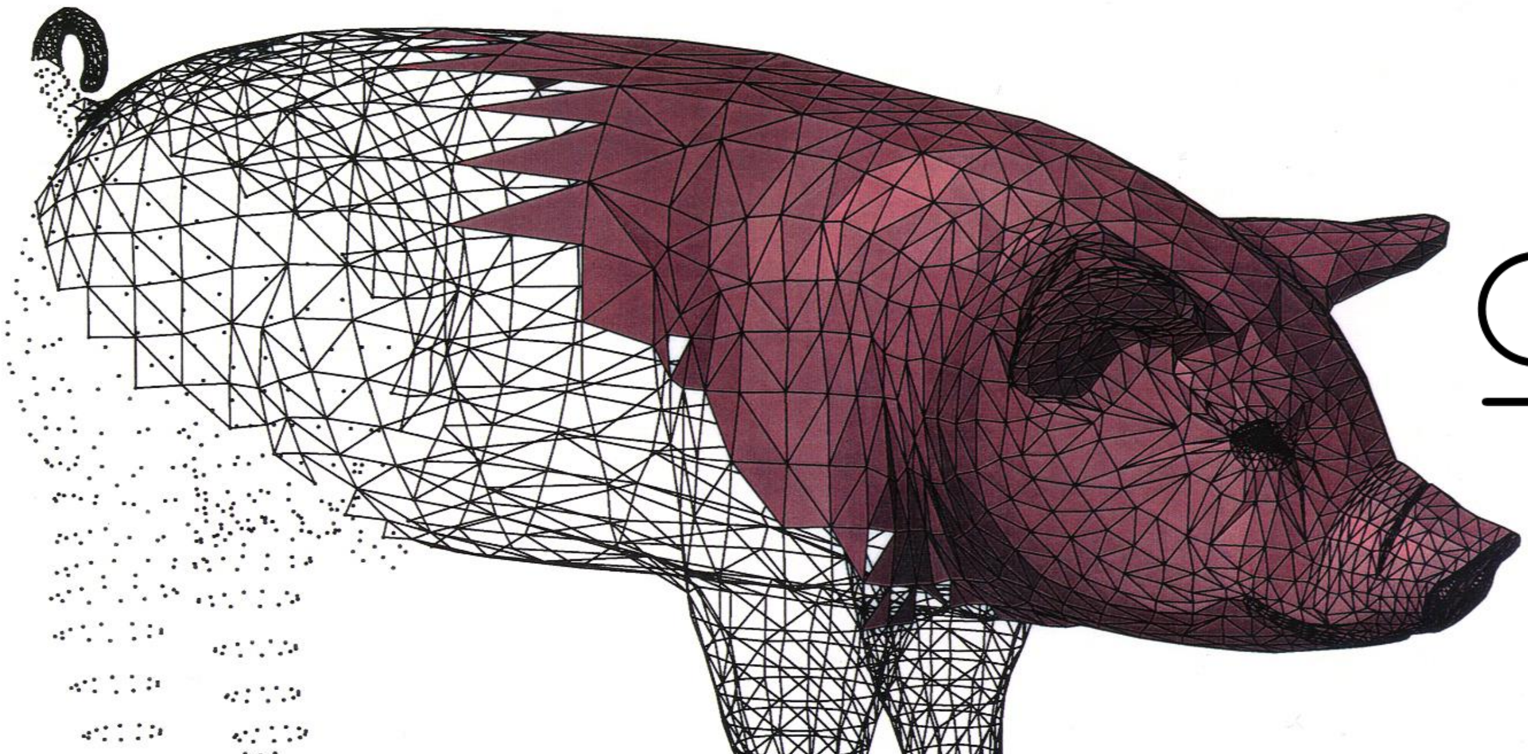
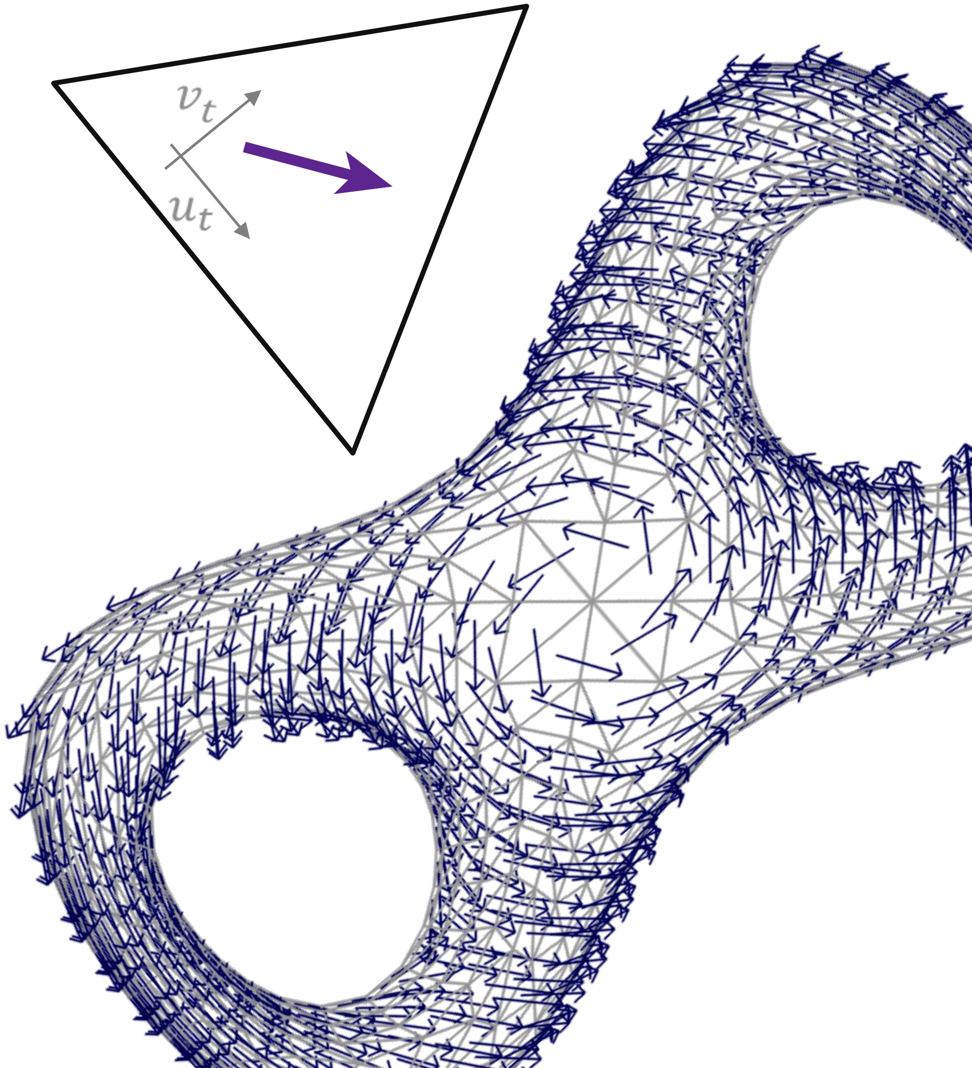
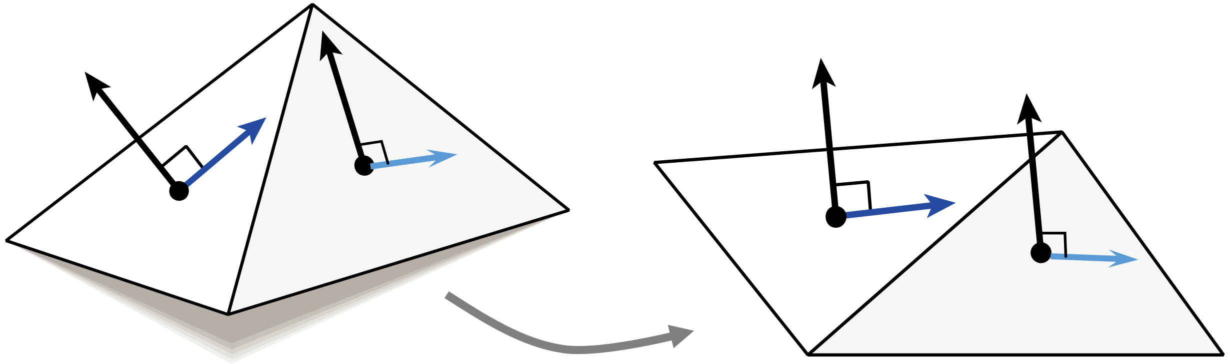


$\subseteq \mathbb{R}^3$

Triangle-Based

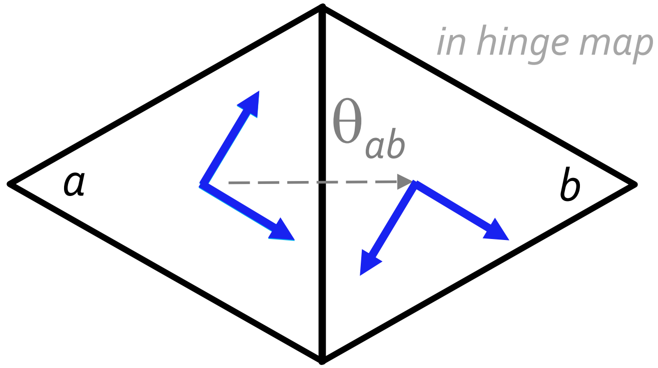
- Triangle as its **own tangent plane**
- One vector per triangle
  - Piecewise constant
  - Discontinuous at edges/vertices
- Easy to unfold/hinge

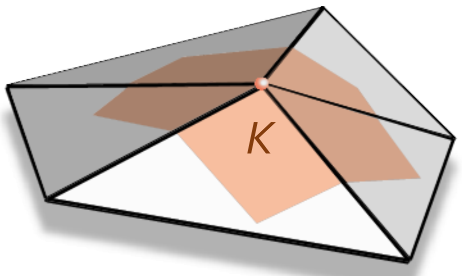




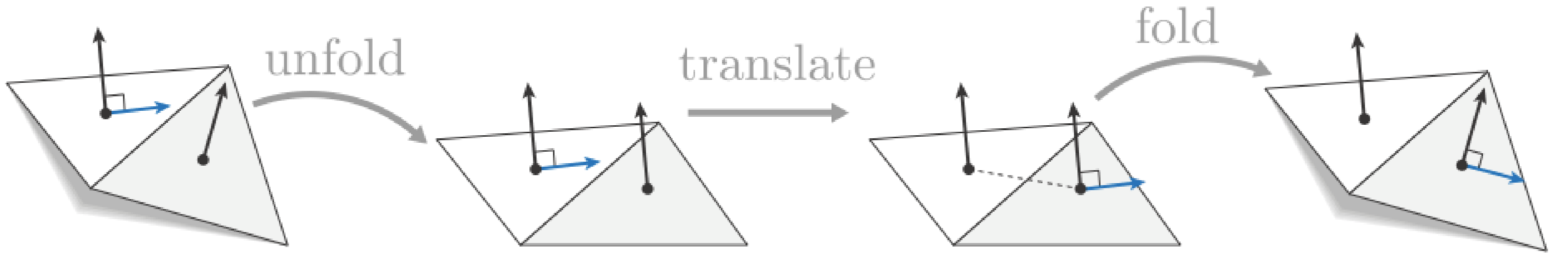
Discrete Levi-Civita Connection

- Simple notion of parallel transport
- Transport around vertex:
  Excess angle is (integrated)
  Gaussian curvature (holonomy!)

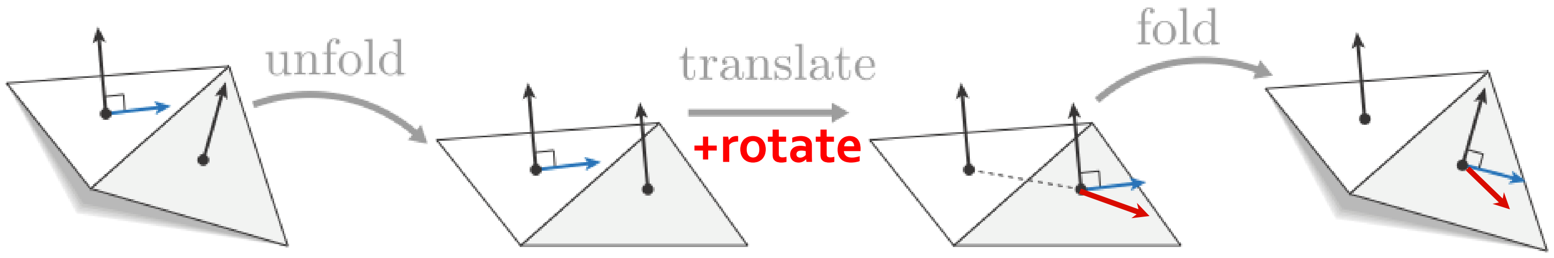




$$K = 2\pi - \sum_a \gamma_a$$



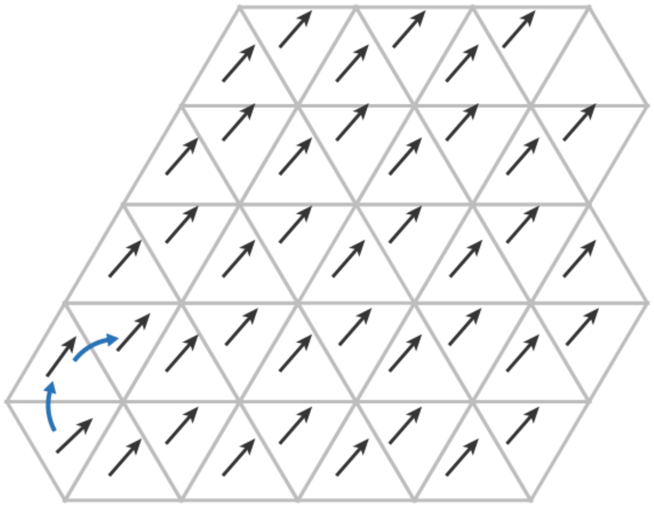
Arbitrary Connection

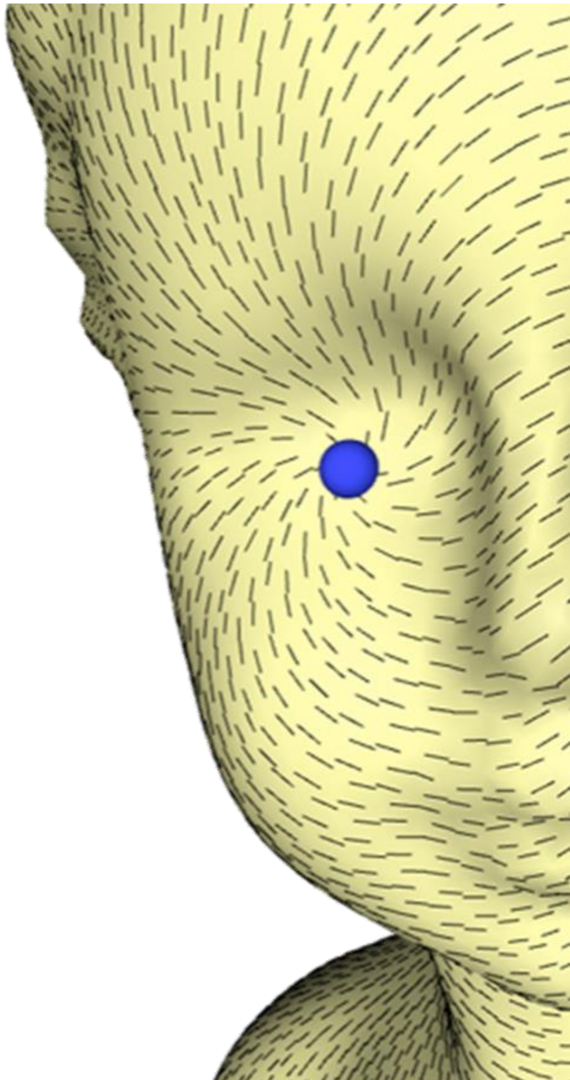


Represent using angle θ_{edge} of extra rotation.

Trivial Connections

- Vector field design
- Zero holonomy on discrete cycles
  - Except for a few singularities
- Path-independent away from singularities





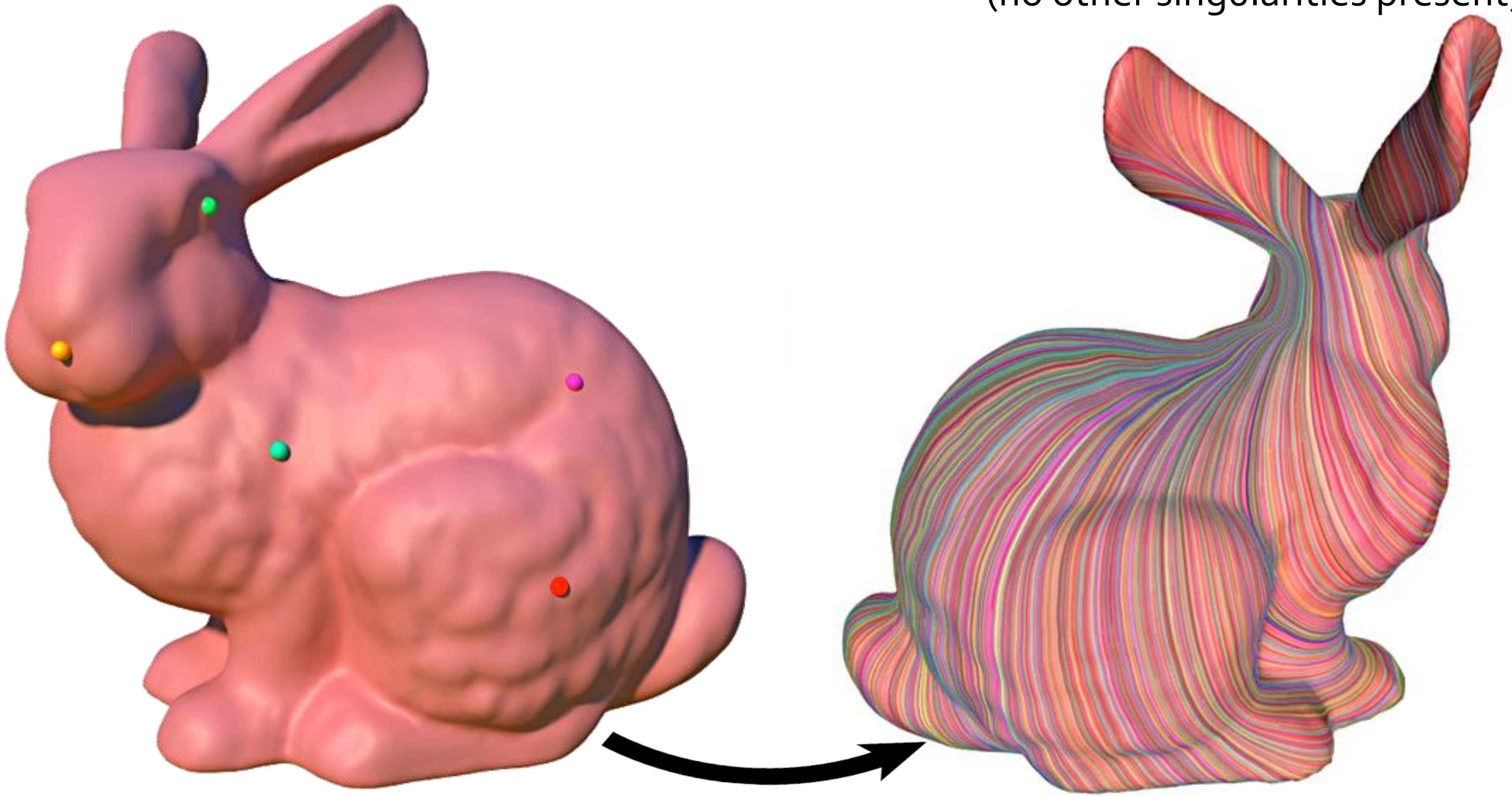
"Trivial Connections on Discrete Surfaces." Crane et al., SGP 2010.

Trivial Connections: Details

- Solve θ_{edge} of extra rotation per edge
- Linear constraint:
  Zero holonomy on basis cycles
  - V+2g constraints: Vertex cycles plus harmonic
  - Fix curvature at chosen singularities
- Underconstrained: Minimize $||\vec{\theta}||$
  - "Best approximation" of Levi-Civita

Result

Resulting trivial connection
(no other singularities present)



Linear system

Extension:

Helmholtz-Hodge Decomposition

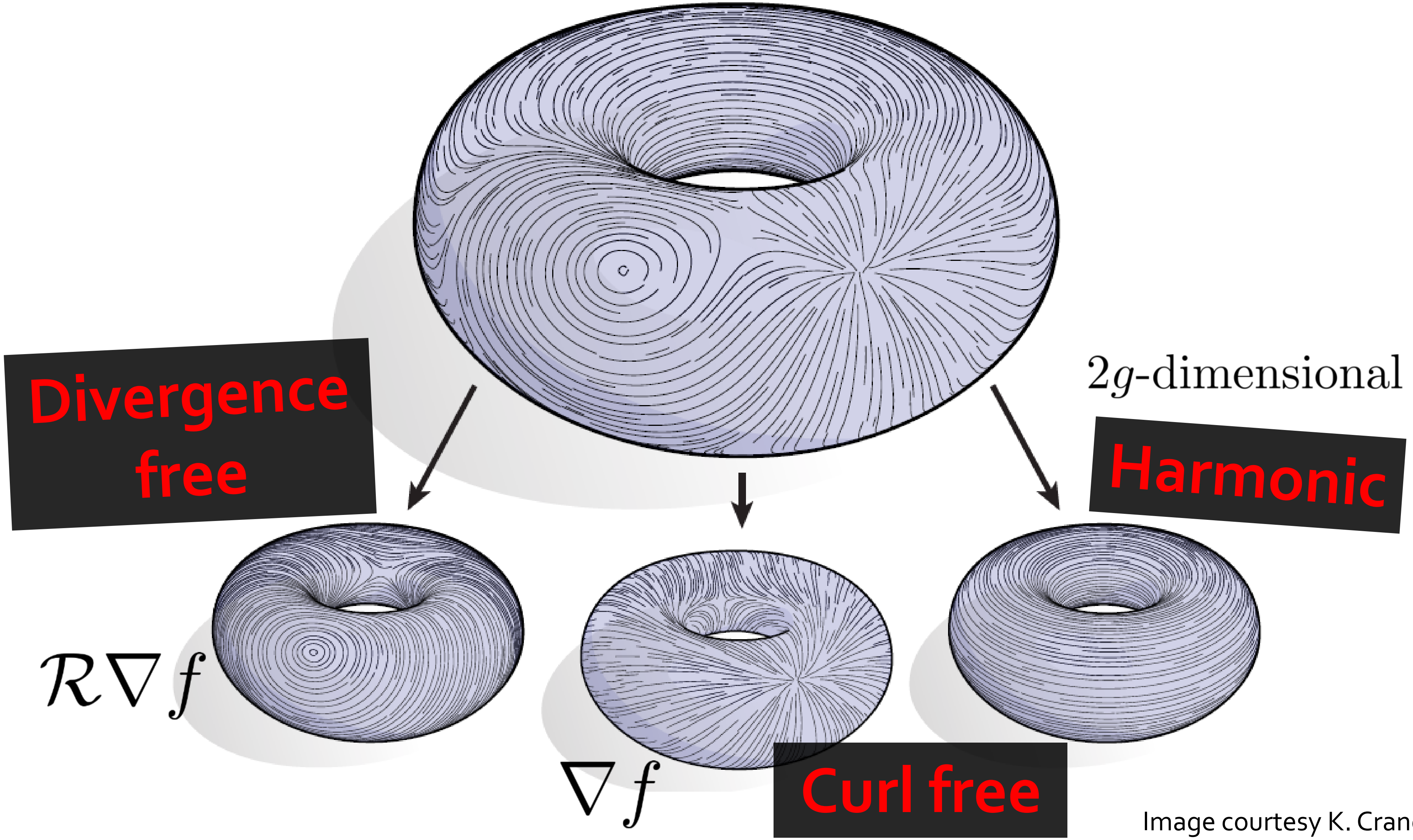
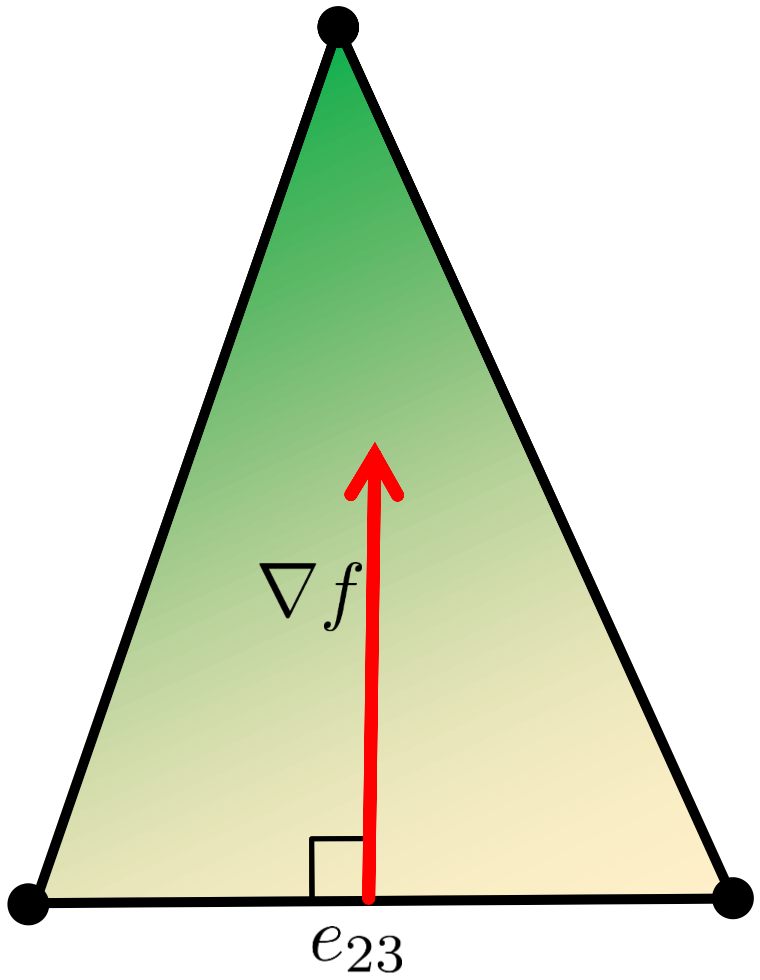


Image courtesy K. Crane

Recall:

Gradient of a Hat Function



$$\|\nabla f\| = \frac{1}{\ell_3 \sin \theta_3} = \frac{1}{h}$$

$$\nabla f = \frac{e_{23}^{\perp}}{2A}$$

Length of e_{23} cancels "base" in A

Recall:

Euler Characteristic

$$V - E + F := \chi$$

$$\chi = 2 - 2g$$

$g = 0$

$g = 1$

$g = 2$

Discrete Helmholtz-Hodge

$$2 - 2g = V - E + F$$
$$\implies 2F = (V - 1) + (E - 1) + 2g$$

Dimensionality works out perfectly!

Either

- Vertex-based gradients
- Edge-based rotated gradients

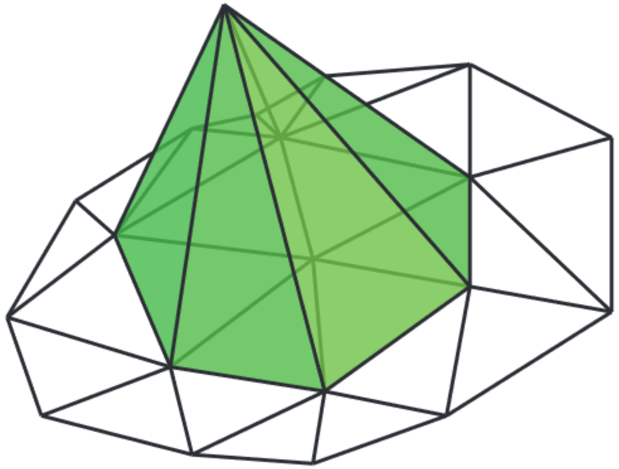
or

Can work out div/grad/curl

- Edge-based gradients
- Vertex-based rotated gradients

“Mixed” finite elements

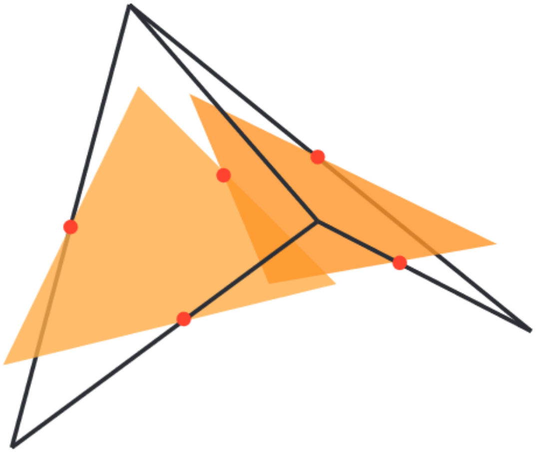
Face-Based Calculus



Vertex-based

"Conforming"

Already did this in 6.838



Edge-based

"Nonconforming"

[Wardetzky 2006]

Relationship: $\psi_{ij} = \phi_i + \phi_j - \phi_k$

Gradient Vector Field

Volumetric Extension?

3D Hodge Decompositions of Edge- and Face-based Vector Fields

RUNDONG ZHAO, Michigan State University
MATHIEU DESBRUN, California Institute of Technology
GUO-WEI WEI and YIYING TONG, Michigan State University

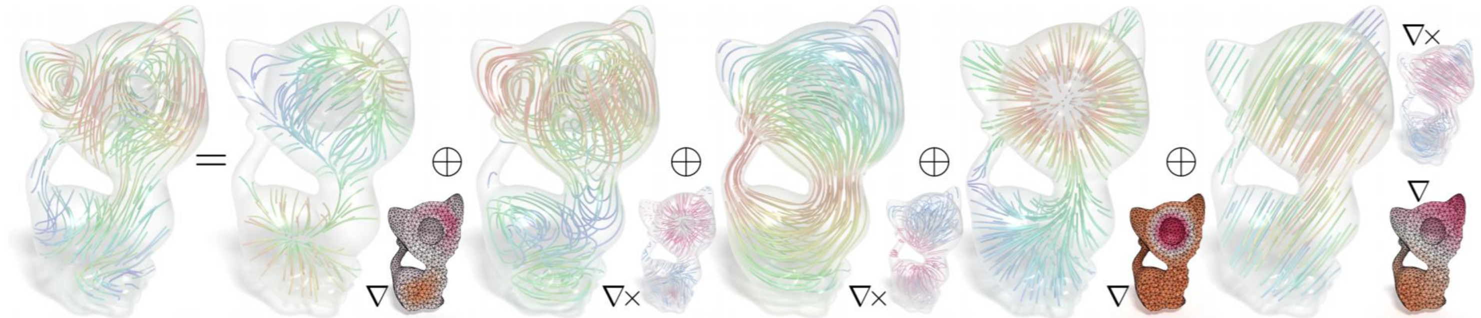


Fig. 1. **Five-Component Vector Field Decomposition.** On a tetrahedral mesh of the kitten with a spherical cavity, a vector field is decomposed into a gradient field with zero potential on the boundary, a curl field with its vector potential orthogonal to the boundary, a pair of tangential and normal harmonic fields, and a harmonic field that is both a gradient and a curl field. Potential fields are shown in the corners of their corresponding components.

We present a compendium of Hodge decompositions of vector fields on tetrahedral meshes embedded in the 3D Euclidean space. After describing the foundations of the Hodge decomposition in the continuous setting, we describe how to implement a five-component orthogonal decomposition that generically splits, for a variety of boundary conditions, any given discrete vector field expressed as discrete differential forms into two potential fields, as well as three additional harmonic components that arise from the topology or boundary of the domain. The resulting decomposition is proper and mimetic, in the sense that the theoretical dualities on the kernel spaces of vector Laplacians valid in the continuous case (including correspondences to cohomology and homology groups) are exactly preserved in the discrete realm. Such a decomposition only involves simple linear algebra with symmetric matrices, and can thus serve as a basic computational tool for vector

static and dynamical problems — for instance, fluid simulation to enforce incompressibility. The mathematical foundations behind such decompositions were developed using the theory of differential forms for any finite-dimensional compact manifold without boundary early on [Hodge 1941], but were fully extended to manifolds with boundaries much more recently [Shonkwiler 2009].

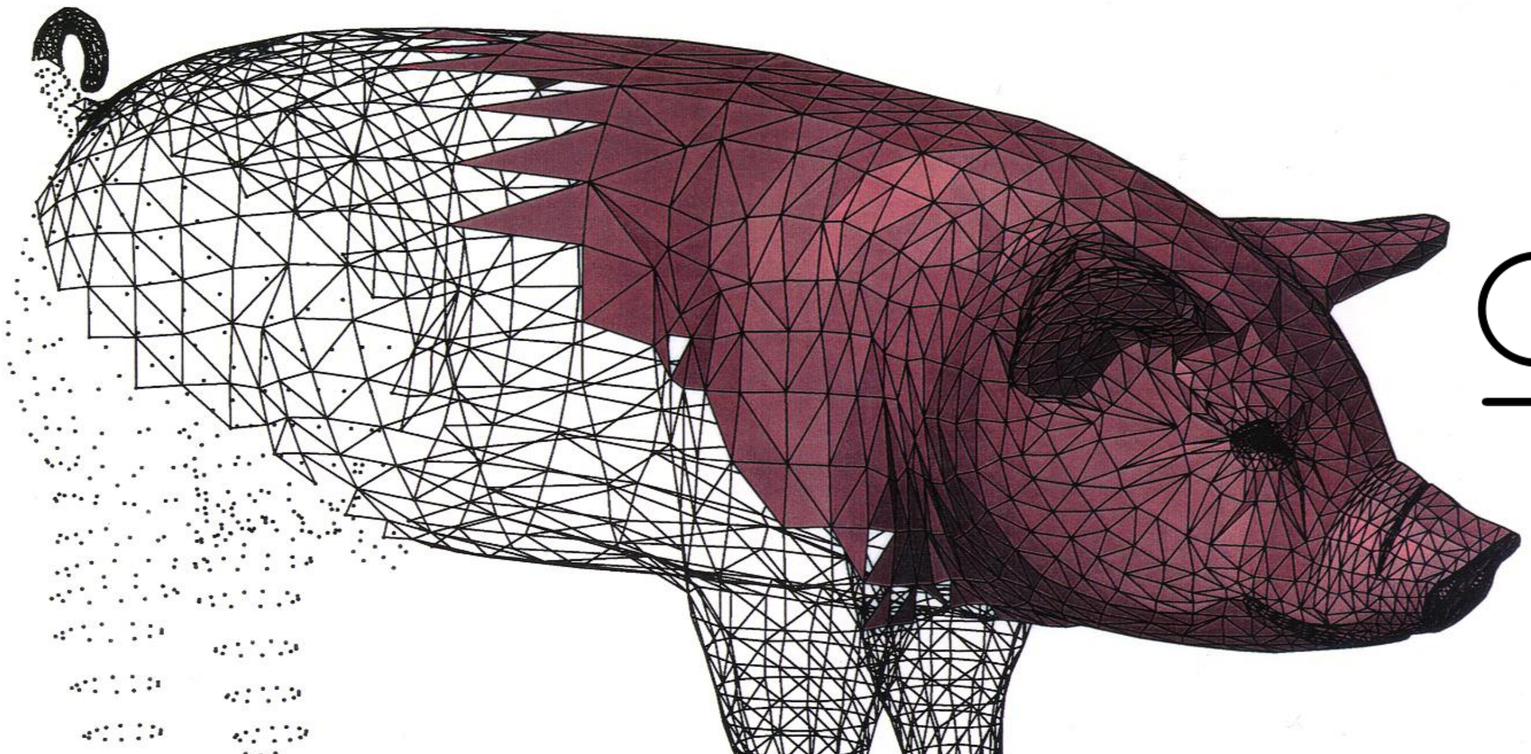
In computer graphics, the analysis and processing of vector fields over *surfaces* have received plenty of attention in recent years. Consequently, the resulting computational tools needed to achieve a Hodge decomposition have been well documented and tested on various applications; see, e.g., recent surveys on surface vector field analysis [Vaxman et al. 2016; de Goes et al. 2016a]. For the case

Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based

Defer to DEC!

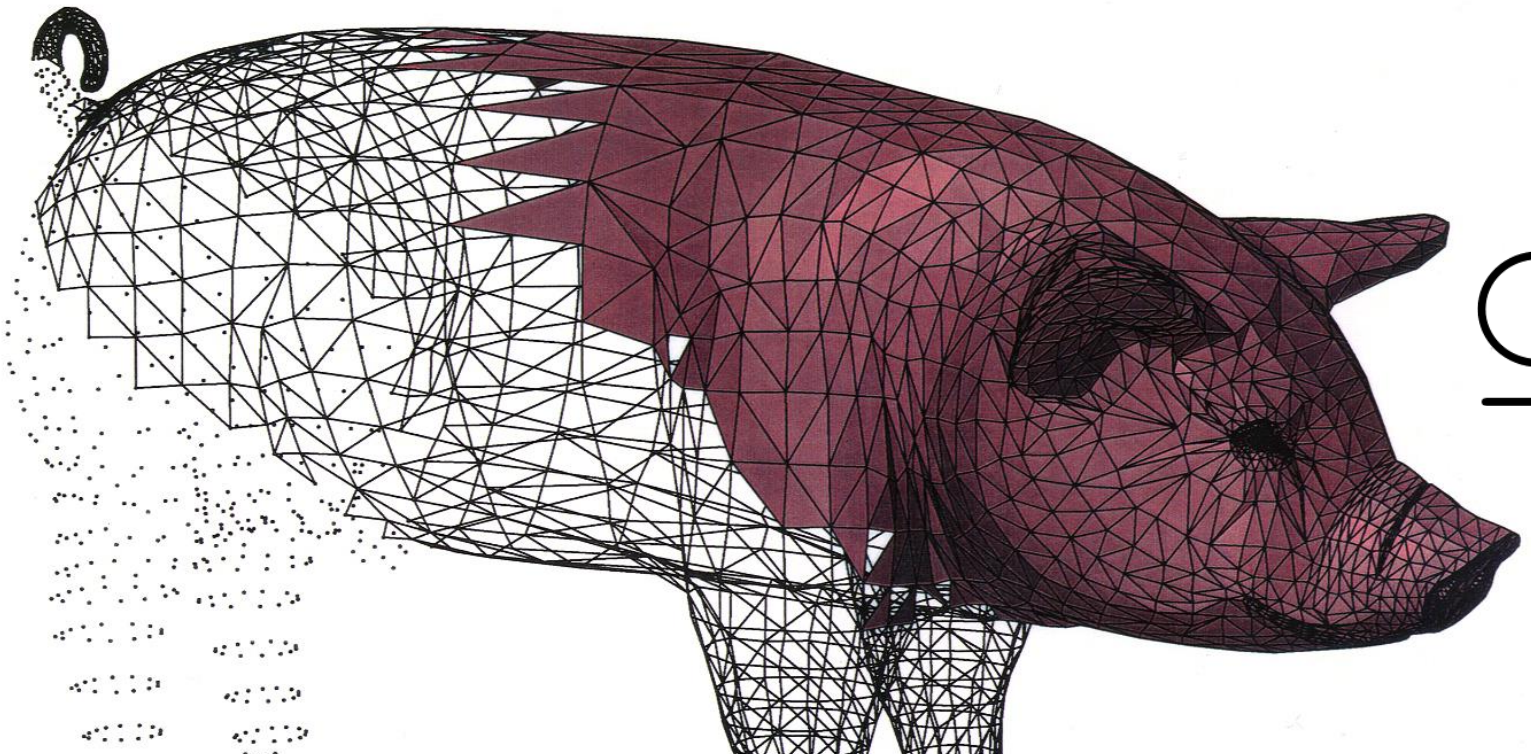


$$\subseteq \mathbb{R}^3$$

Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based

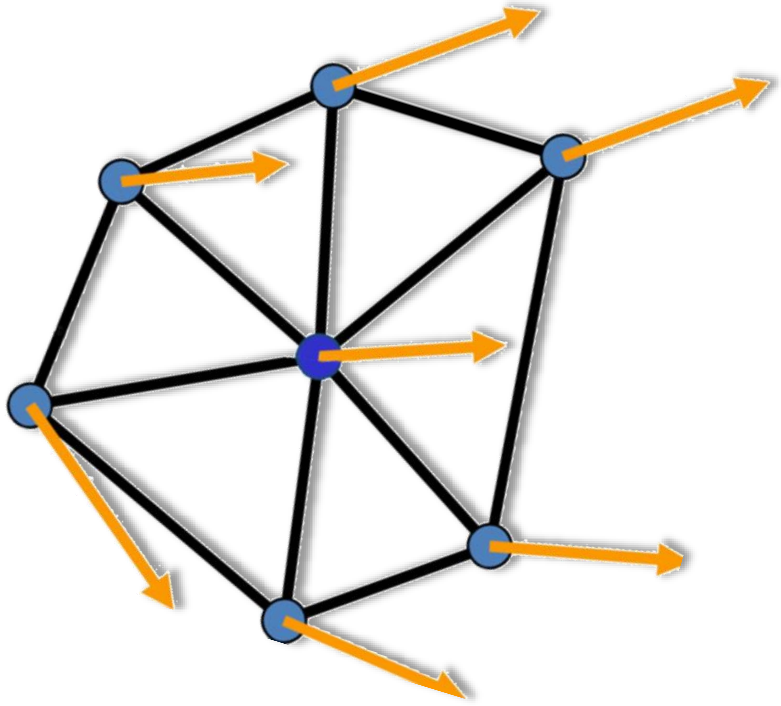


$$\subseteq \mathbb{R}^3$$

Vertex-Based Fields

- Pros
  - Possibility of higher-order differentiation
- Cons
  - Vertices don't have natural tangent spaces
  - Gaussian curvature concentrated

2D (Planar) Case: Easy



Piecewise-linear (*x*,*y*) components

Example: Killing Energy

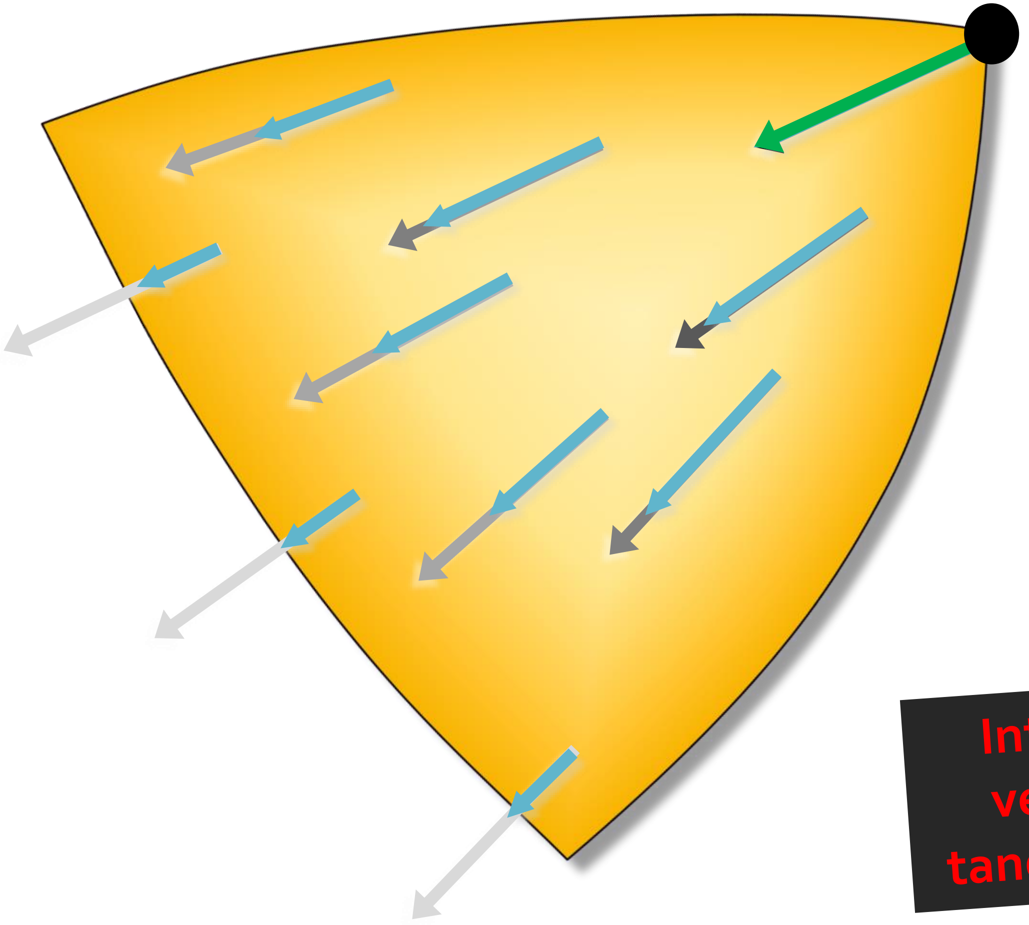
"AKVF:" Approximate Killing Vector Field

$$\|\mathbf{d}\|^2_{\text{KVF}} := \frac{1}{2}\int_{\Omega(\mathbf{x})} \|J_\mathbf{d} + J_\mathbf{d}^\top\|^2_{\text{Fro}}\, dA = \mathbf{d}^\top K(\mathbf{x})\mathbf{d}$$

Killing operator

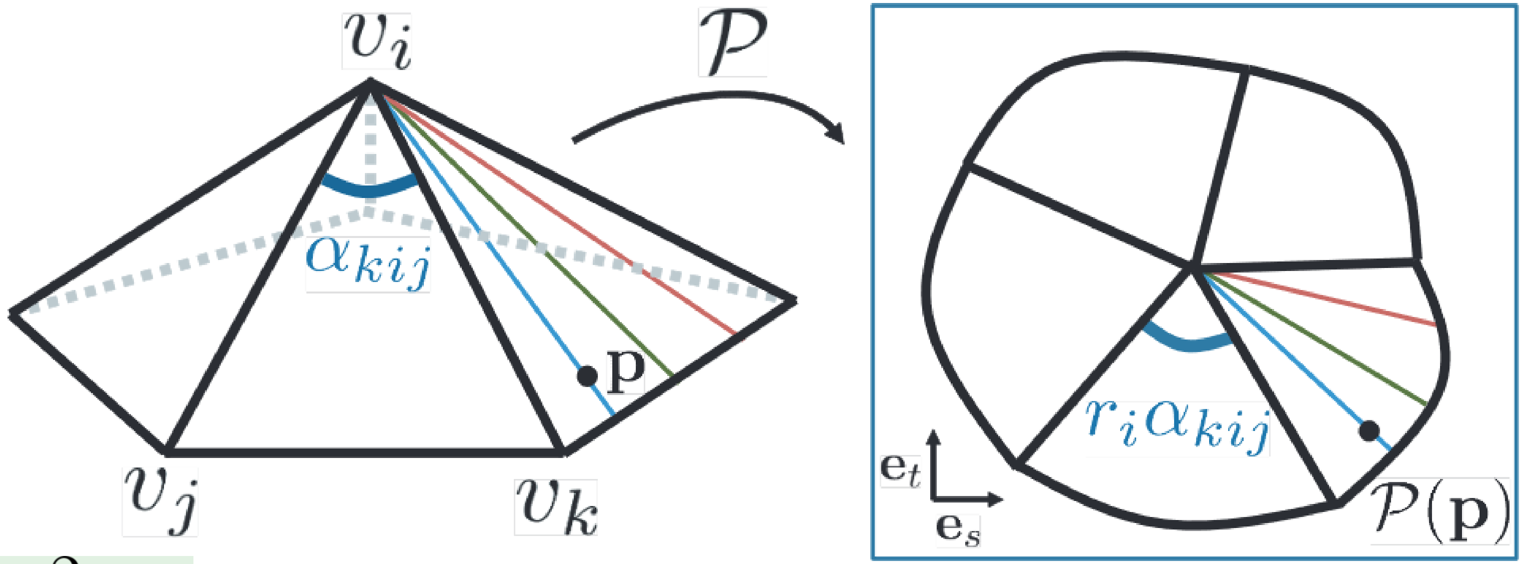
"As-Killing-As-Possible Vector Fields for Planar Deformation" (SGP 2011) *Solomon, Ben-Chen, Butscher, Guibas*

3D Case: Ambiguous



Interpolate vector and tangent space!

Geodesic Polar Map



$$r_i := \frac{2\pi}{2\pi - \kappa_i}$$

Preserve radial lines
(change their spacing)

Provides notion of tangency, but continuity issues

"Vector Field Design on Surfaces," Zhang et al., TOG 2006

Parallel transport radially from vertex

Recent Method

Discrete Connection and Covariant Derivative for Vector Field Analysis and Design

Beibei Liu and Yiying Tong
Michigan State University
and
Fernando de Goes and Mathieu Desbrun
California Institute of Technology

Includes basis, derivative operators

In this paper, we introduce a discrete definition of connection on simplicial manifolds, involving closed-form continuous expressions within simplices and finite rotations across simplices. The finite-dimensional parameters of this connection are optimally computed by minimizing a quadratic measure of the deviation to the (discontinuous) Levi-Civita connection induced by the embedding of the input triangle mesh, or to any metric connection with arbitrary cone singularities at vertices. From this discrete connection, a covariant derivative is constructed through exact differentiation, leading to explicit expressions for local integrals of first-order derivatives (such as divergence, curl and the Cauchy-Riemann operator), and for L_2-based energies (such as the Dirichlet energy). We finally demonstrate the utility, flexibility, and accuracy of our discrete formulations for the design and analysis of vector, n-vector, and n-direction fields.

Categories and Subject Descriptors: I.3.5 [**Computer Graphics**]: Computational Geometry & Object Modeling—*Curve & surface representations.*

CCS Concepts: •**Computing methodologies** → **Mesh models;**

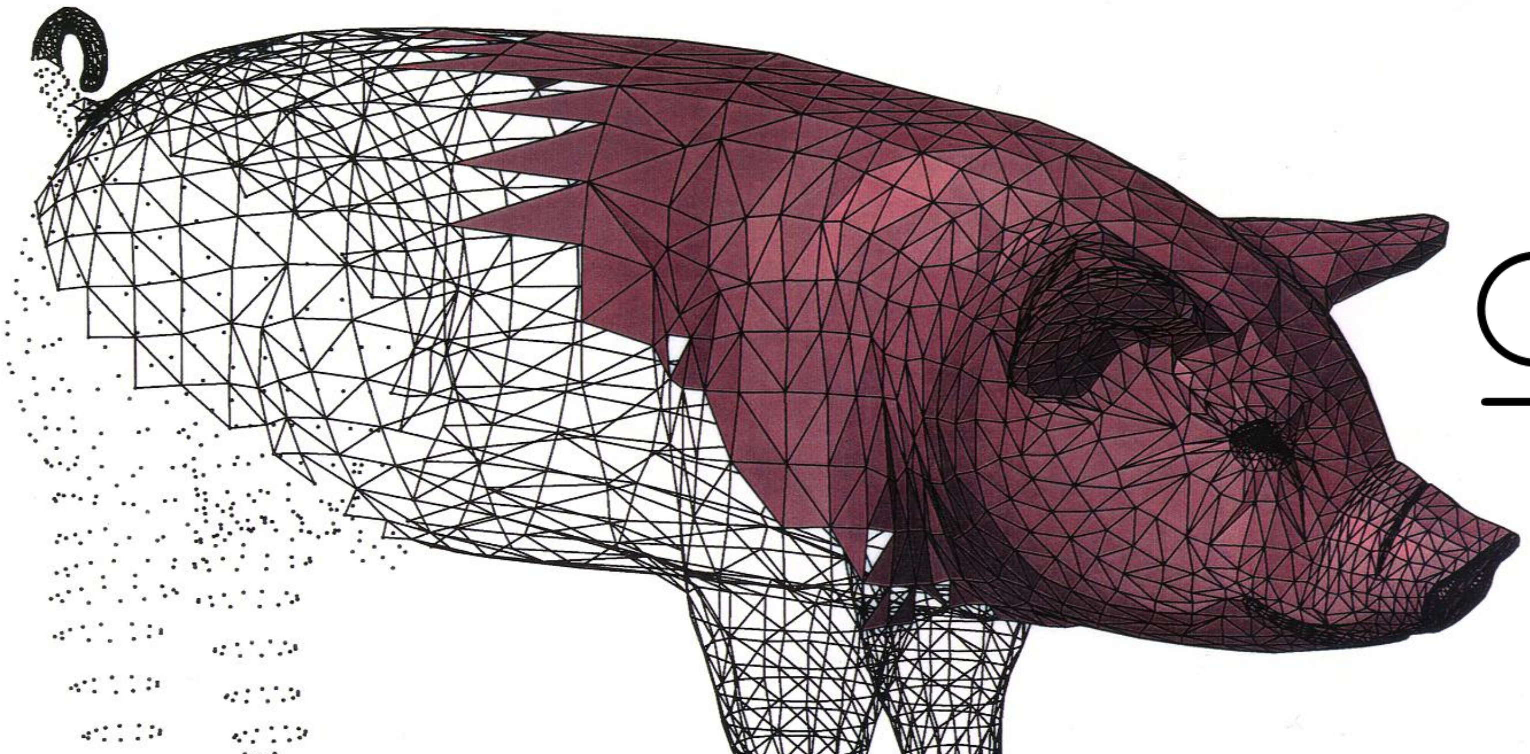
digital geometry processing, with applications ranging from texture synthesis to shape analysis, meshing, and simulation. However, existing discrete counterparts of such a differential operator acting on simplicial manifolds can either approximate local derivatives (such as divergence and curl) or estimate global integrals (such as the Dirichlet energy), but not both simultaneously.

In this paper, we present a unified discretization of the covariant derivative that offers closed-form expressions for both local and global first-order derivatives of vertex-based tangent vector fields on triangulations. Our approach is based on a new construction of discrete connections that provides consistent interpolation of tangent vectors within and across mesh simplices, while minimizing the deviation to the Levi-Civita connection induced by the 3D embedding of the input mesh—or more generally, to any metric connection with arbitrary cone singularities at vertices. We demonstrate the relevance of our contributions by providing new computational tools to design and edit vector and n-direction fields.

Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based

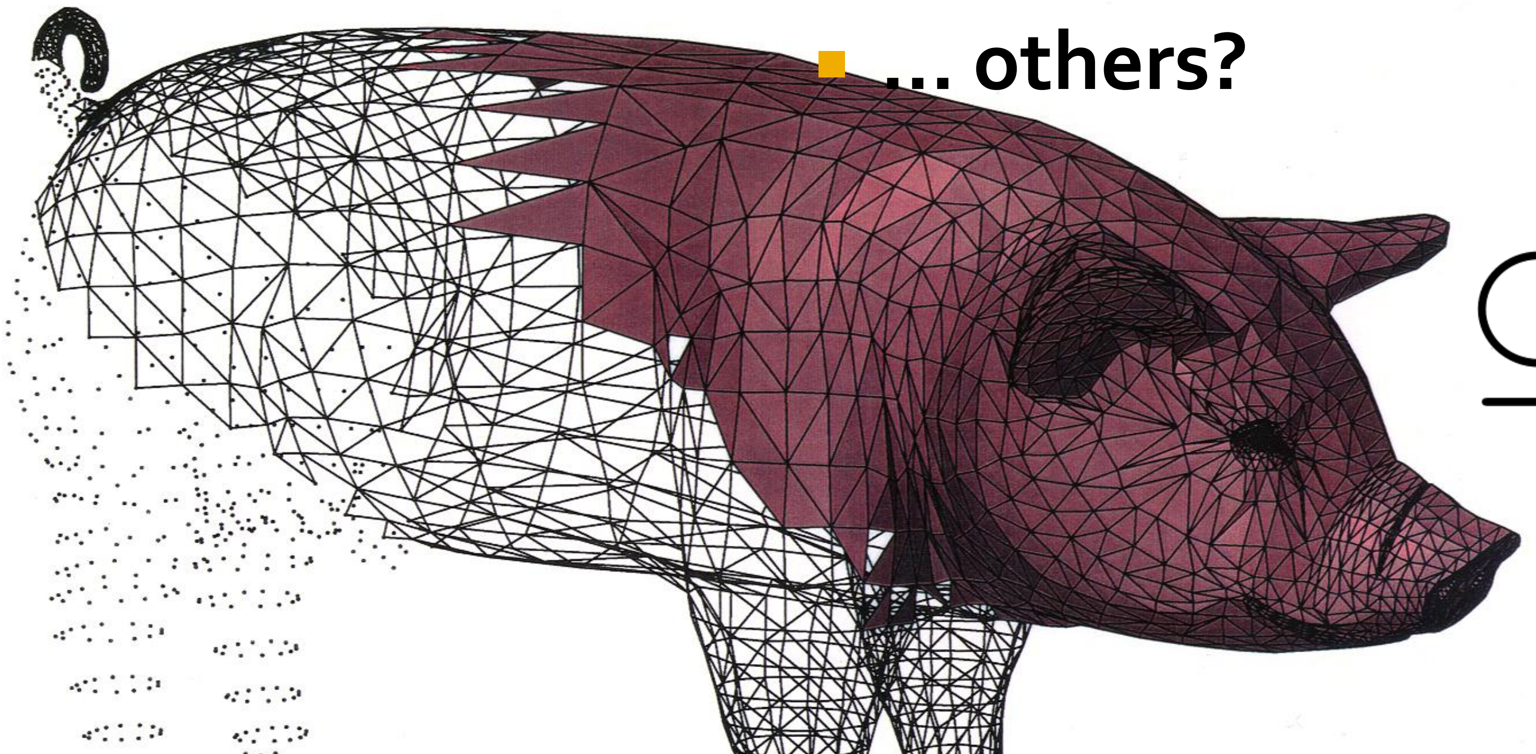


$$\subseteq \mathbb{R}^3$$

Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based
- … others?



$\subseteq \mathbb{R}^3$

More Exotic Choice

An Operator Approach to Tangent Vector Field Processing

Omri Azencot[1] and Mirela Ben-Chen[1] and Frédéric Chazal[2] and Maks Ovsjanikov[3]

[1]Technion - Israel Institute of Technology
[2] Geometrica, INRIA
[3]LIX, École Polytechnique

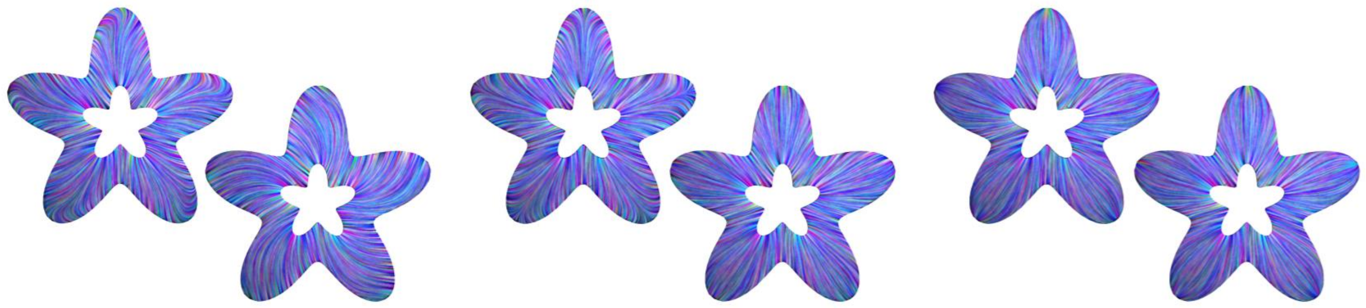


Figure 1: *Using our framework various vector field design goals can be easily posed as linear constraints. Here, given three symmetry maps: rotational (S1), bilateral (S2) and front/back (S3), we can generate a symmetric vector field using only S1 (left), S1 + S2 (center) and S1 + S2 + S3 (right). The top row shows the front of the 3D model, and the bottom row its back.*

Vector fields as derivative operators

Abstract
In this paper, we introduce a novel coordinate-free method for manipulating and analyzing vector fields on discrete surfaces. Unlike the commonly used representations of a vector field as an assignment of vectors to the faces of the mesh, or as real values on edges, we argue that vector fields can also be naturally viewed as operators whose domain and range are functions defined on the mesh. Although this point of view is common in differential geometry

Subdivision Fields

Subdivision Exterior Calculus for Geometry Processing

Fernando de Goes, Pixar Animation Studios
Mathieu Desbrun, Caltech
Mark Meyer, Pixar Animation Studios
Tony DeRose, Pixar Animation Studios

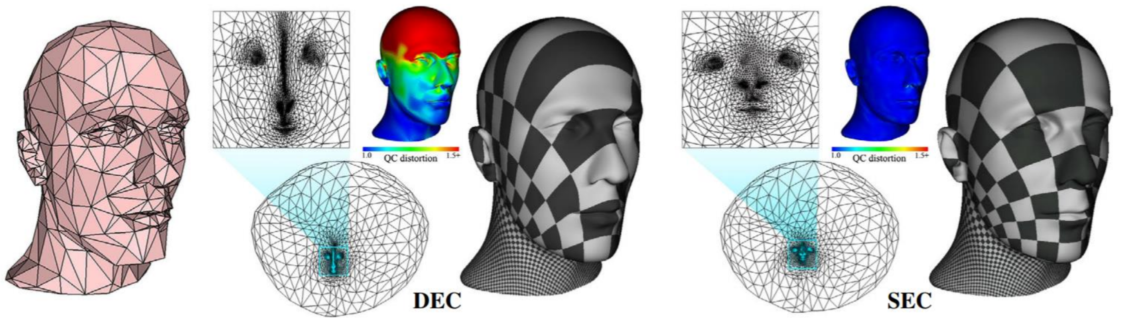


Figure 1: ***Subdivision Exterior Calculus (SEC).*** *We introduce a new technique to perform geometry processing applications on subdivision surfaces by extending Discrete Exterior Calculus (DEC) from the polygonal to the subdivision setting. With the preassemble of a few operators on the control mesh, SEC outperforms DEC in terms of numerics with only minor computational overhead. For instance, while the spectral conformal parameterization [Mullen et al. 2008] of the control mesh of the mannequin head (left) results in large quasi-conformal distortion* (mean = 1.784, max = 9.4) *after subdivision (middle), simply substituting our SEC operators for the original DEC operators significantly reduces distortion* (mean = 1.005, max = 3.0) *(right). Parameterizations, shown at level 1 for clarity, exhibit substantial differences.*

Abstract

This paper introduces a new computational method to solve differential equations on subdivision surfaces. Our approach adapts the numerical framework of Discrete Exterior Calculus (DEC) from the polygonal to the subdivision setting by exploiting the refinability of subdivision basis functions. The resulting *Subdivision Exterior Calculus* (SEC) provides significant improvements in accuracy compared to existing polygonal techniques, while offering exact finite-dimensional analogs of continuum structural identities such as Stokes' theorem and Helmholtz-Hodge decomposition. We demonstrate the versatility and efficiency of SEC on common geometry processing tasks including parameterization, geodesic distance computation, and vector field design.

Keywords: Subdivision surfaces, discrete exterior calculus, discrete differential geometry, geometry processing.

Concepts: •**Mathematics of computing → Discretization;** *Computations in finite fields;*

and Schröder 2000; Warren and Weimer 2001]. In spite of this prominence, little attention has been paid to numerically solving differential equations on subdivision surfaces. This is in sharp contrast to a large body of work in geometry processing that developed discrete differential operators for polygonal meshes [Botsch et al. 2010] serving as the foundations for several applications ranging from parameterization to fluid simulation [Crane et al. 2013a].

Among the various polygonal mesh techniques, Discrete Exterior Calculus (DEC) [Desbrun et al. 2008] is a coordinate-free formalism for solving scalar and vector valued differential equations. In particular, it reproduces, rather than merely approximates, essential properties of the differential setting such as Stokes' theorem. Given that the control mesh of a subdivision surface is a polygonal mesh, applying existing DEC methods directly to the control mesh may seem tempting. However, this approach ignores the geometry of the limit surface, thus introducing a significant loss of accuracy in the discretization process (Fig. 1). A customary workaround is to perform computations on a denser polygonal mesh generated by

Subdivision Directional Fields

BRAM CUSTERS, Utrecht University/TU Eindhoven
AMIR VAXMAN, Utrecht University

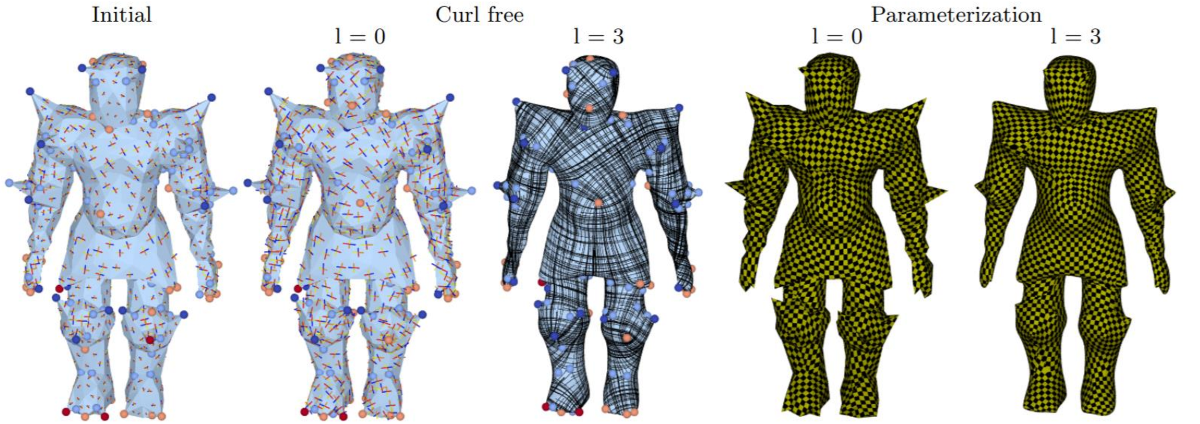


Fig. 1. Rotationally-seamless parameterization with a subdivision directional field. An initial field (left) is optimized for low curl at the coarsest level $l = 0$. We subdivide the field to fine level $l = 3$ (center), and then solve for a seamless parameterization in both levels (right). Our subdivision preserves curl, and thus results in a low integration error in both levels. The coarse-level optimization takes 7.5 secs, the subdivision 7.6 secs, and the parameterization 7.0 secs, to a total of 22.1 secs. This is a speedup of about two orders of magnitude compared to running the curl optimization directly on the fine level, taking 1438.7 secs.

We present a novel linear subdivision scheme for face-based tangent directional fields on triangle meshes. Our subdivision scheme is based on a novel coordinate-free representation of directional fields as halfedge-based scalar quantities, bridging the mixed finite-element representation with discrete exterior calculus. By commuting with differential operators, our subdivision is structure-preserving: it reproduces curl-free fields precisely, and reproduces divergence-free fields in the weak sense. Moreover, our subdivision scheme directly extends to directional fields with several vectors per face by working on the branched covering space. Finally, we demonstrate how our scheme can be applied to directional-field design, advection, and robust earth mover's distance computation, for efficient and robust computation.

CCS Concepts: •**Computing methodologies → Mesh models; Mesh geometry models;** *Shape analysis;*

Additional Key Words and Phrases: Directional Fields, Vector Fields, Subdi-

1 INTRODUCTION

Directional fields are central objects in geometry processing. They represent flows, alignments, and symmetry on discrete meshes. They are used for diverse applications such as meshing, fluid simulation, texture synthesis, architectural design, and many more. There is then great value in devising robust and reliable algorithms that design and analyze such fields. In this paper, we work with piecewise-constant tangent directional fields, defined on the faces of a triangle mesh. A directional field is the assignment of several vectors per face, where the most commonly-used fields comprise single vectors. The piecewise-constant face-based representation of directional fields is a mainstream representation within the (mixed) *finite-element method* (**FEM**), where the vectors are often gradients of piecewise-linear functions spanned by values on the vertices.

Vector Fields: Discretization

Justin Solomon

6.838: Shape Analysis
Spring 2021

Extra: Continuous Normalizing Flows

Justin Solomon

6.838: Shape Analysis

Spring 2021

MIT EECS