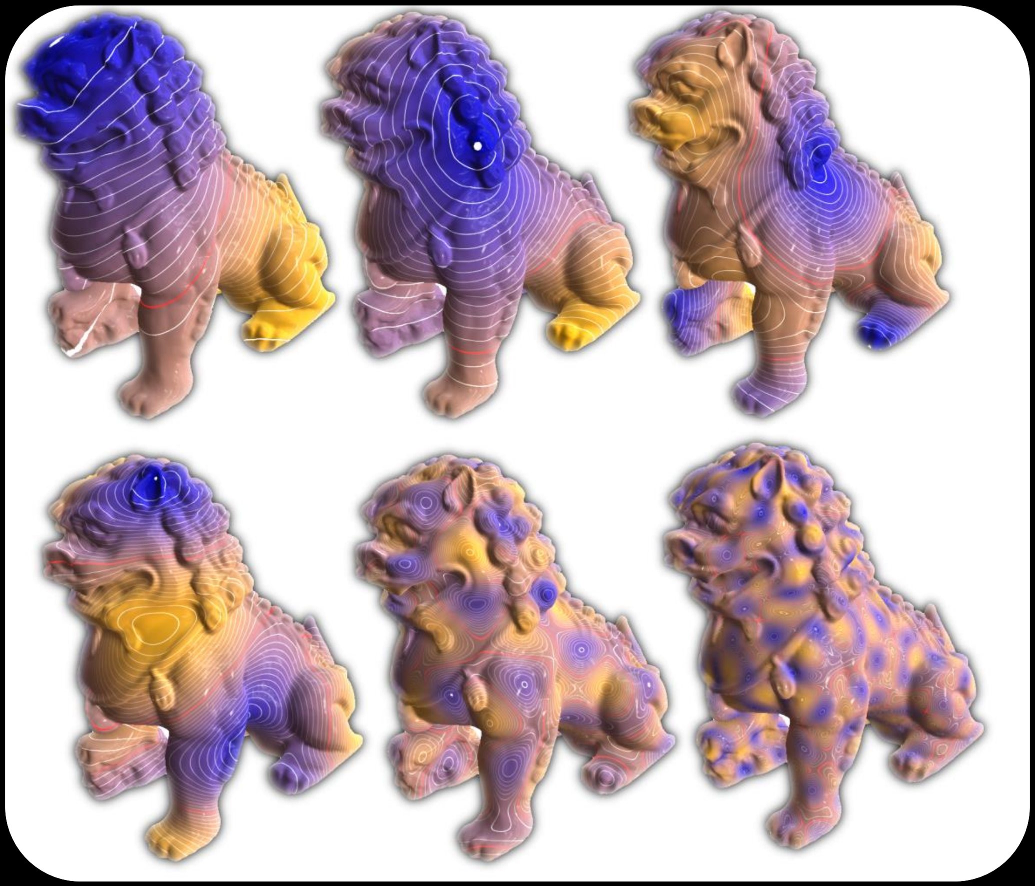
http://alice.loria.fr/publications/papers/2008/ManifoldHarmonics//photo/dragon_mhb.png



Discrete Laplacians

Justin Solomon

MIT, Spring 2019

Our Focus

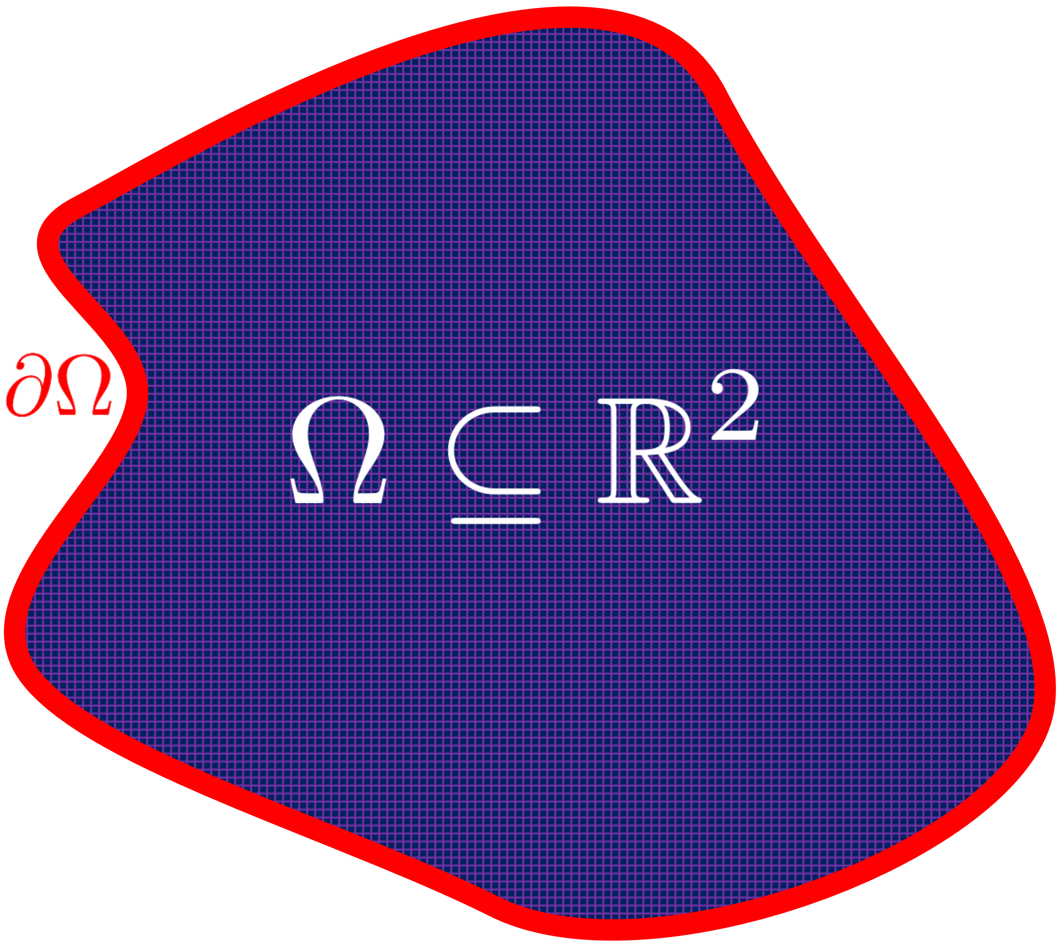
$$f \in C^\infty(M) \longrightarrow \Delta \longrightarrow \Delta f \in C^\infty(M)$$

Computational version?

The Laplacian

Recall:

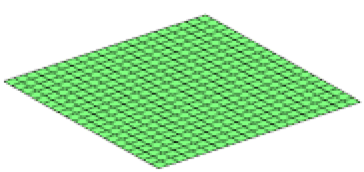
Planar Region



Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = -\Delta u$$

$$\Delta := -\sum_i \frac{\partial^2}{\partial x_i^2}$$



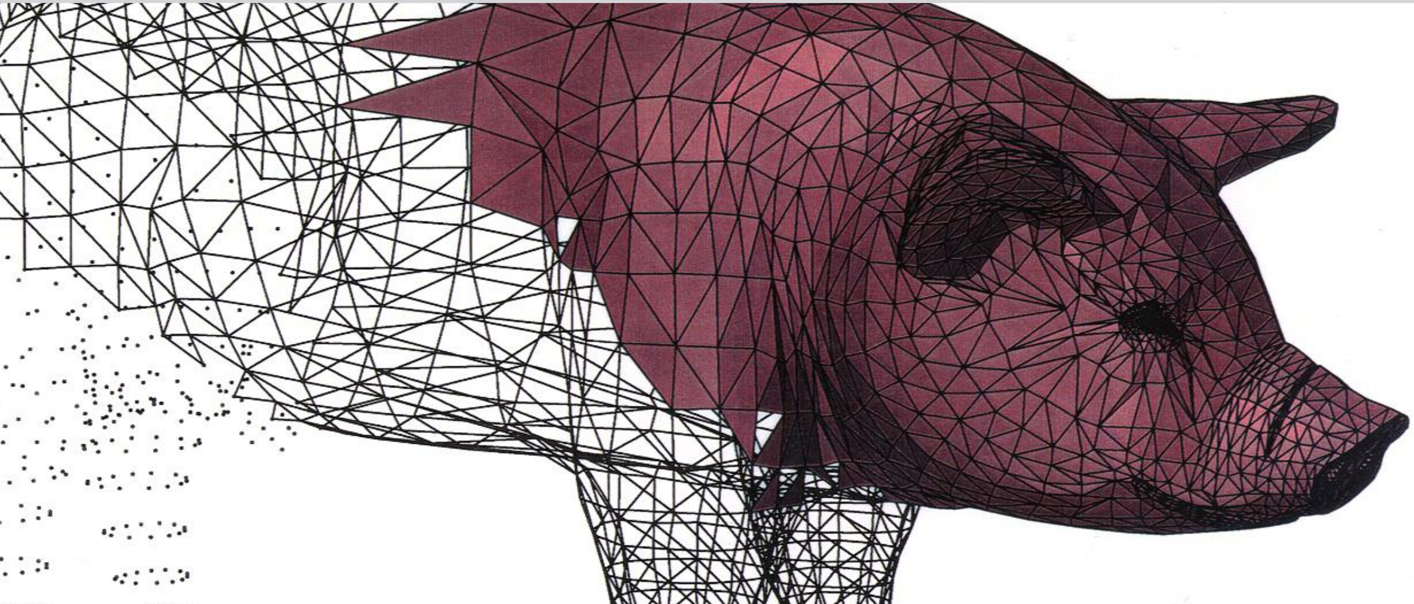
Discretizing the Laplacian

$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right)$$

?!

Problem

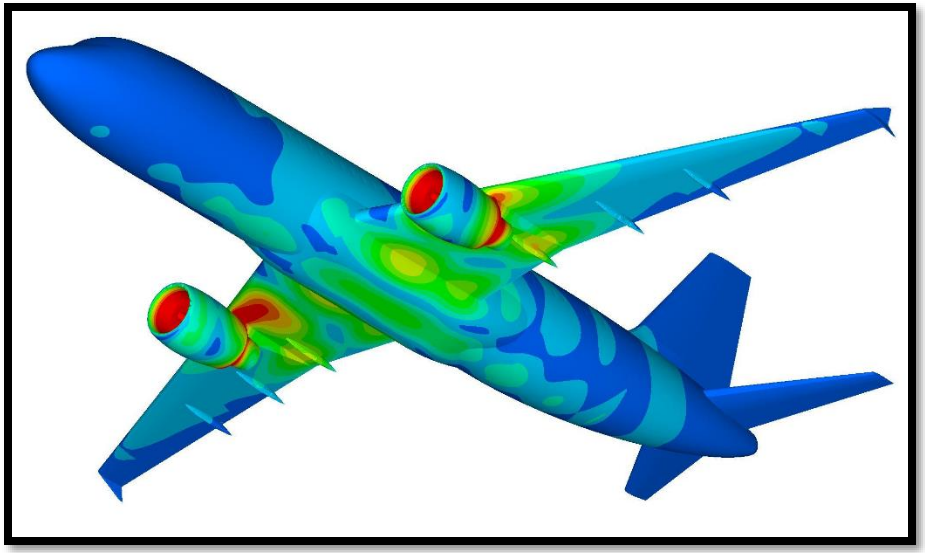
Laplacian is a *differential* operator!



Today's Approach

First-order Galerkin

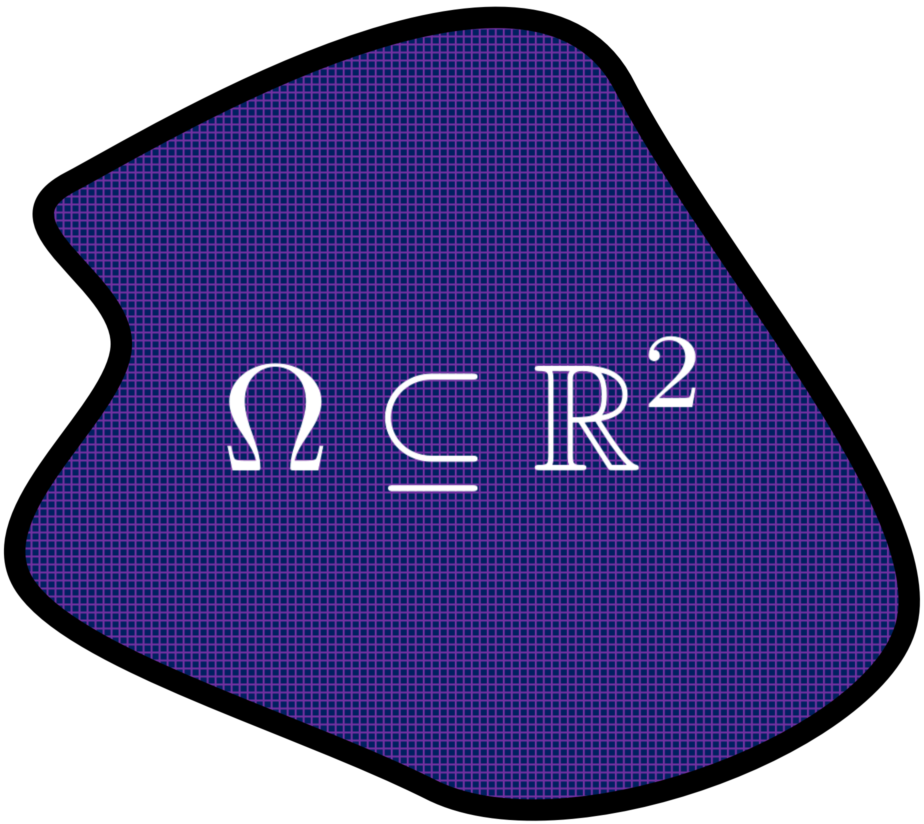
Finite element method (FEM)

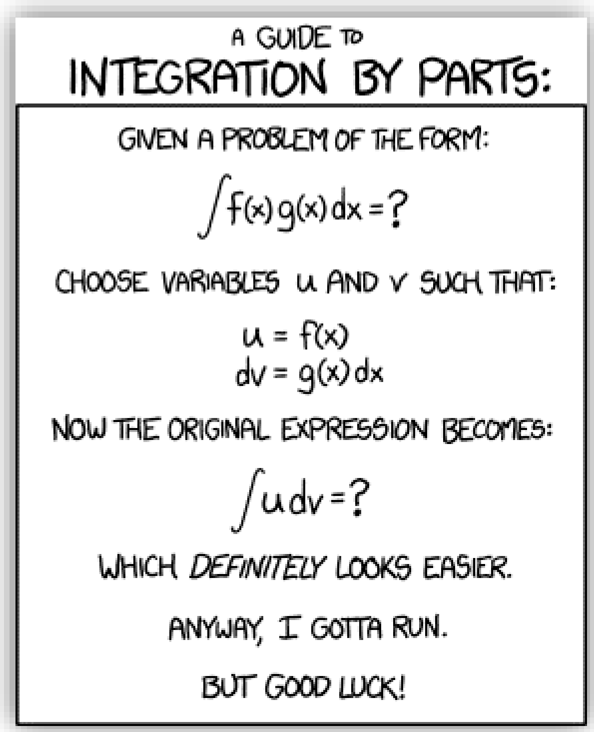


http://www.stressebook.com/wp-content/uploads/2014/08/Airbus_A320_k.jpg

Integration by Parts to the Rescue

$$\int_\Omega f \Delta g \, dA = \text{boundary terms} + \int_\Omega \nabla f \cdot \nabla g \, dA$$





https://xkcd.com/1201/

Slightly Easier?

$$\int_\Omega f \Delta g \, dA = \text{boundary terms} + \int_\Omega \nabla f \cdot \nabla g \, dA$$

Laplacian (second derivative)

Gradient (first derivative)

Slightly Easier?

$$\int_\Omega f \Delta g \, dA = \text{boundary terms} + \int_\Omega \nabla f \cdot \nabla g \, dA$$

One derivative,
one integral

Gradient
(first derivative)

Kinda-sorta cancels out?

Overview: Galerkin FEM Approach

$$g = \Delta f$$

$$\Longrightarrow \int \psi g \, dA = \int \psi \Delta f \, dA = \int (\nabla \psi \cdot \nabla f) \, dA$$

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

$$\Longrightarrow \int \psi g \, dA = \int \psi \Delta f \, dA = \int (\nabla \psi \cdot \nabla f) \, dA$$

$$\text{Approximate } f \approx \sum_i a_i \psi_i \text{ and } g \approx \sum_i b_i \psi_i$$

$$\Longrightarrow \text{Linear system } \sum_i b_i \langle \psi_i, \psi_j \rangle = \sum_i a_i \langle \nabla \psi_i, \nabla \psi_j \rangle$$

Overview:

Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = \int (\nabla \psi \cdot \nabla f) \, dA$$

$$\text{Approximate } f \approx \sum_i a_i \psi_i \text{ and } g \approx \sum_i b_i \psi_i$$

$$\implies \text{Linear system } \sum_i b_i \langle \psi_i, \psi_j \rangle = \sum_i a_i \langle \nabla \psi_i, \nabla \psi_j \rangle$$

$$\text{Mass matrix: } M_{ij} := \langle \psi_i, \psi_j \rangle$$

$$\text{Stiffness matrix: } L_{ij} := \langle \nabla \psi_i, \nabla \psi_j \rangle$$

$$\implies Mb = La$$

Which basis?

Important to Note

Not the only way

to approximate the Laplacian operator.

- Divided differences
- Higher-order elements
- Boundary element methods
- Discrete exterior calculus
- …

But this method is worth knowing, so we'll do it in detail!

L^2 Dual of a Function

Function $f : M \to \mathbb{R}$

$$\downarrow$$

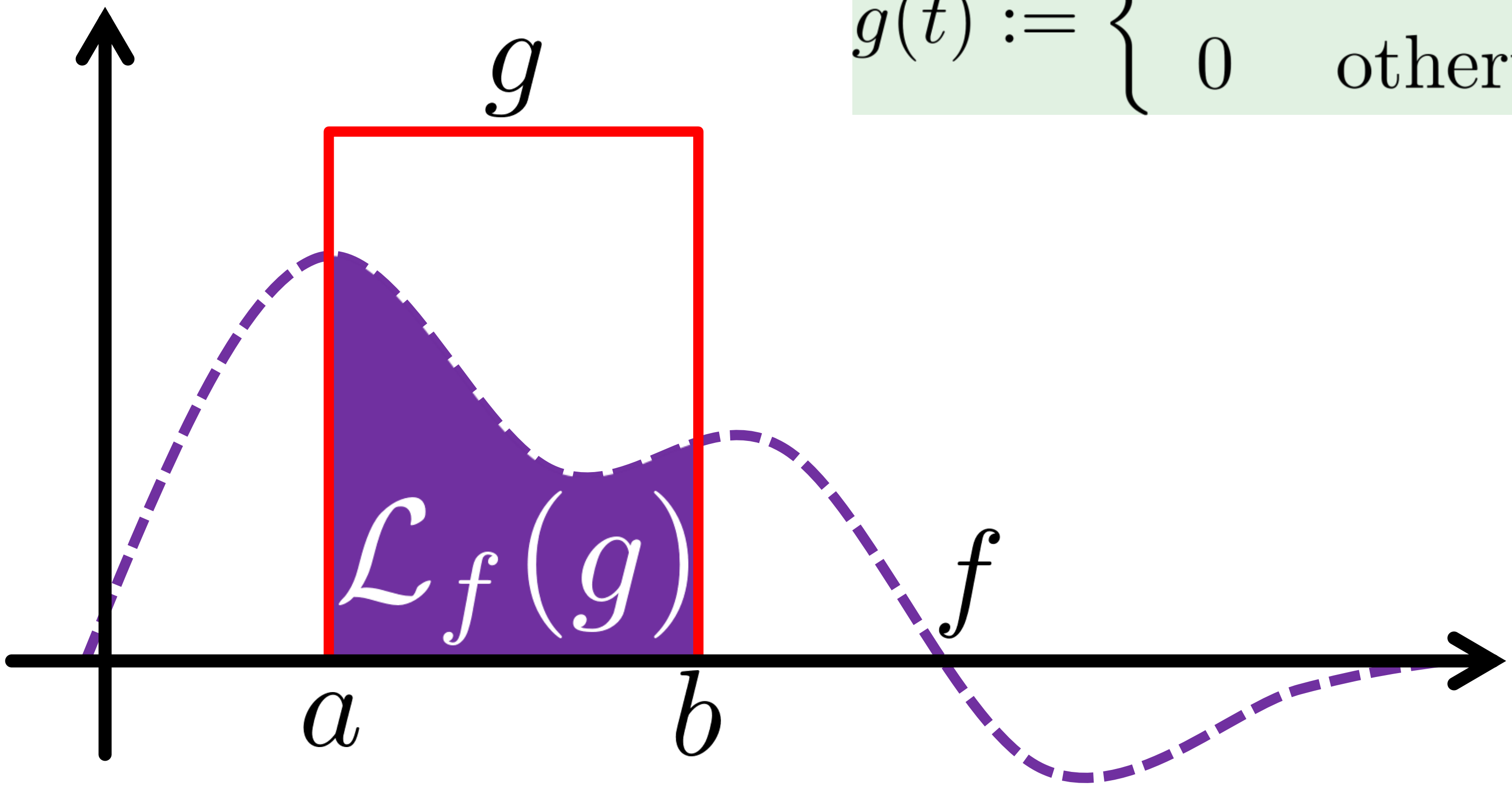
Operator $\mathcal{L}_f : L^2(M) \to \mathbb{R}$

$$\mathcal{L}_f[g] := \int_M f(x)g(x)\,dA$$

"Test function"

Observation

$$g(t) := \begin{cases} 1 & \text{if } t \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$



Can recover function from dual

Dual of Laplacian

Space of test functions (no boundary!):

$$\{g \in C^\infty(M) : g|_{\partial M} \equiv 0\}$$

$$\begin{aligned}\mathcal{L}_{\Delta f}[g] &= \int_M g \Delta f \, dA \\ &= \int_M \nabla g \cdot \nabla f \, dA\end{aligned}$$

Use Laplacian without evaluating it!

Galerkin's Approach

Choose one of each:

- Function space
- Test functions

Often the same!

One Derivative is Enough

$$\mathcal{L}_{\Delta f}[g] = \int_M \nabla g \cdot \nabla f \, dA$$

First Order Finite Elements

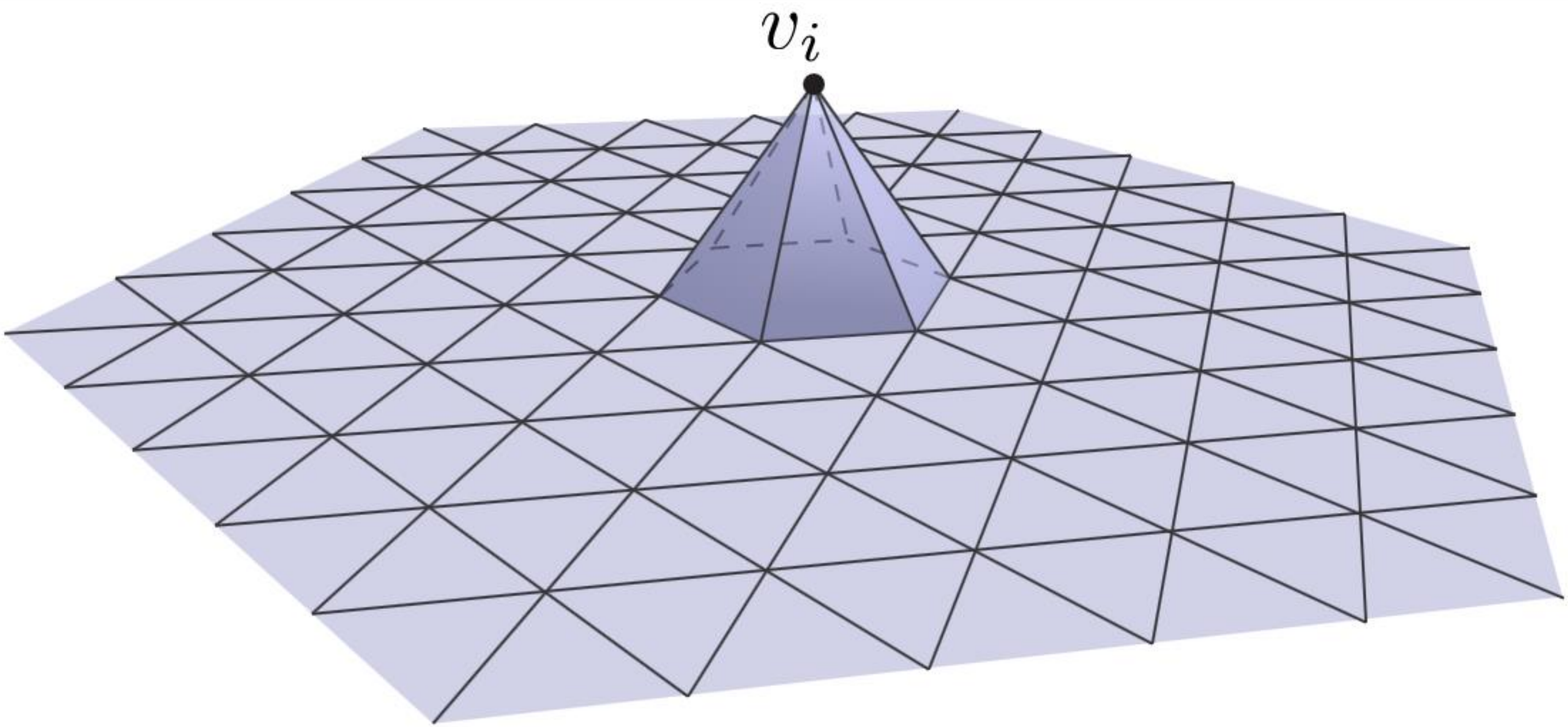
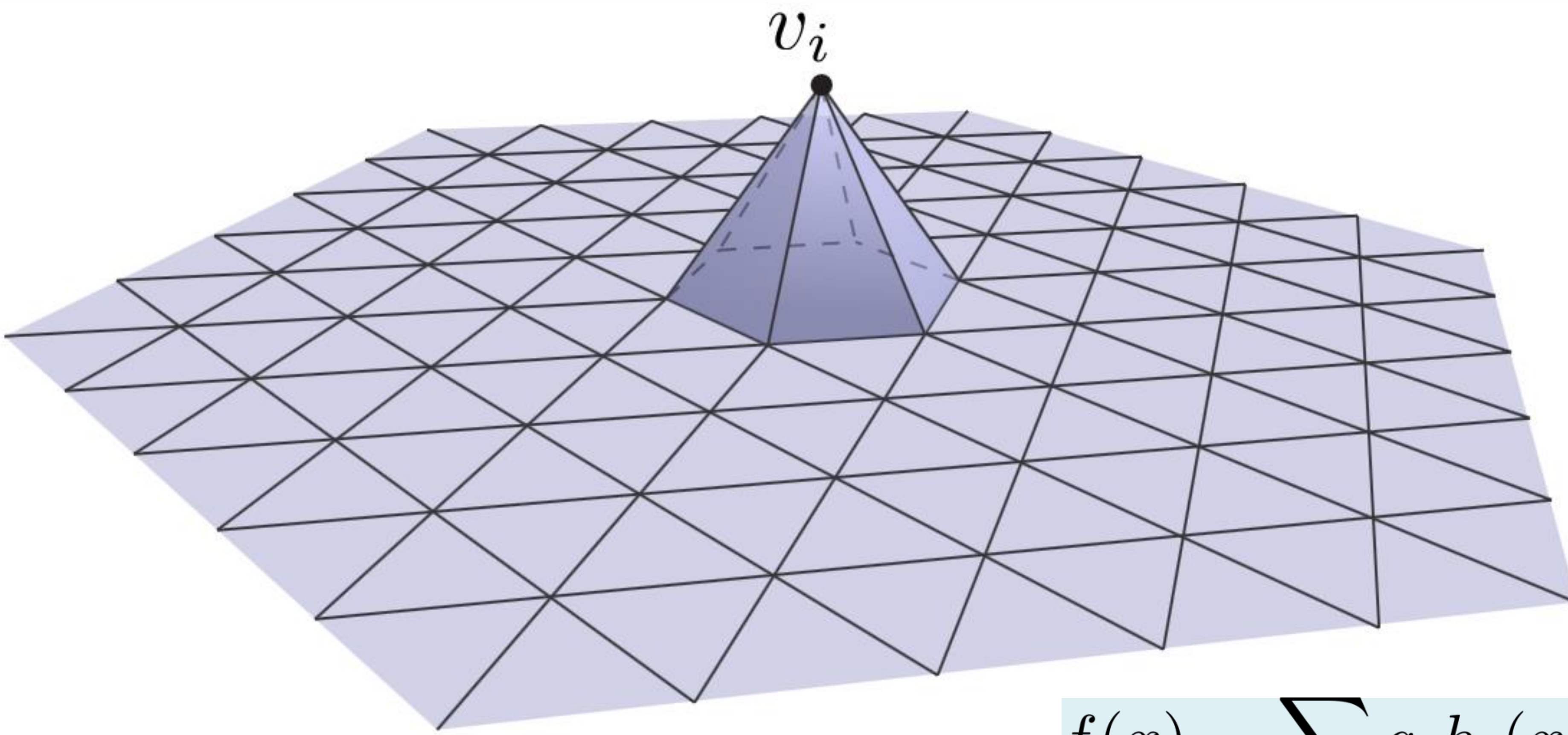


Image courtesy K. Crane, CMU

One “hat function” per vertex

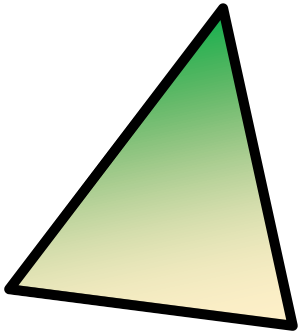
Representing Functions



$$f(x) = \sum_i a_i h_i(x)$$

$$a \in \mathbb{R}^{|V|}$$

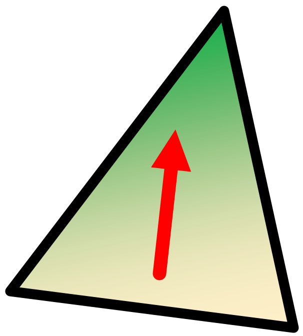
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = \int_M \nabla g \cdot \nabla f \, dA$$

Linear combination of hats
(piecewise linear)

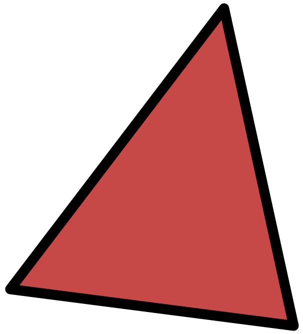
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = \int_M \nabla g \cdot \nabla f \, dA$$

One vector per face

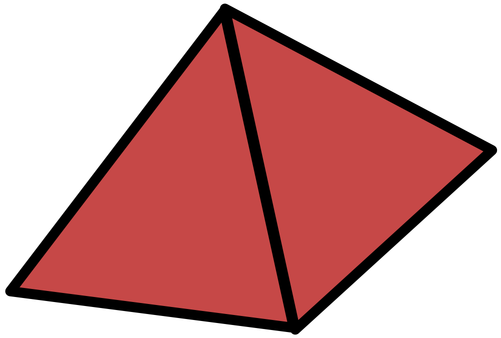
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = \int_M \nabla g \cdot \nabla f \, dA$$

One scalar per face

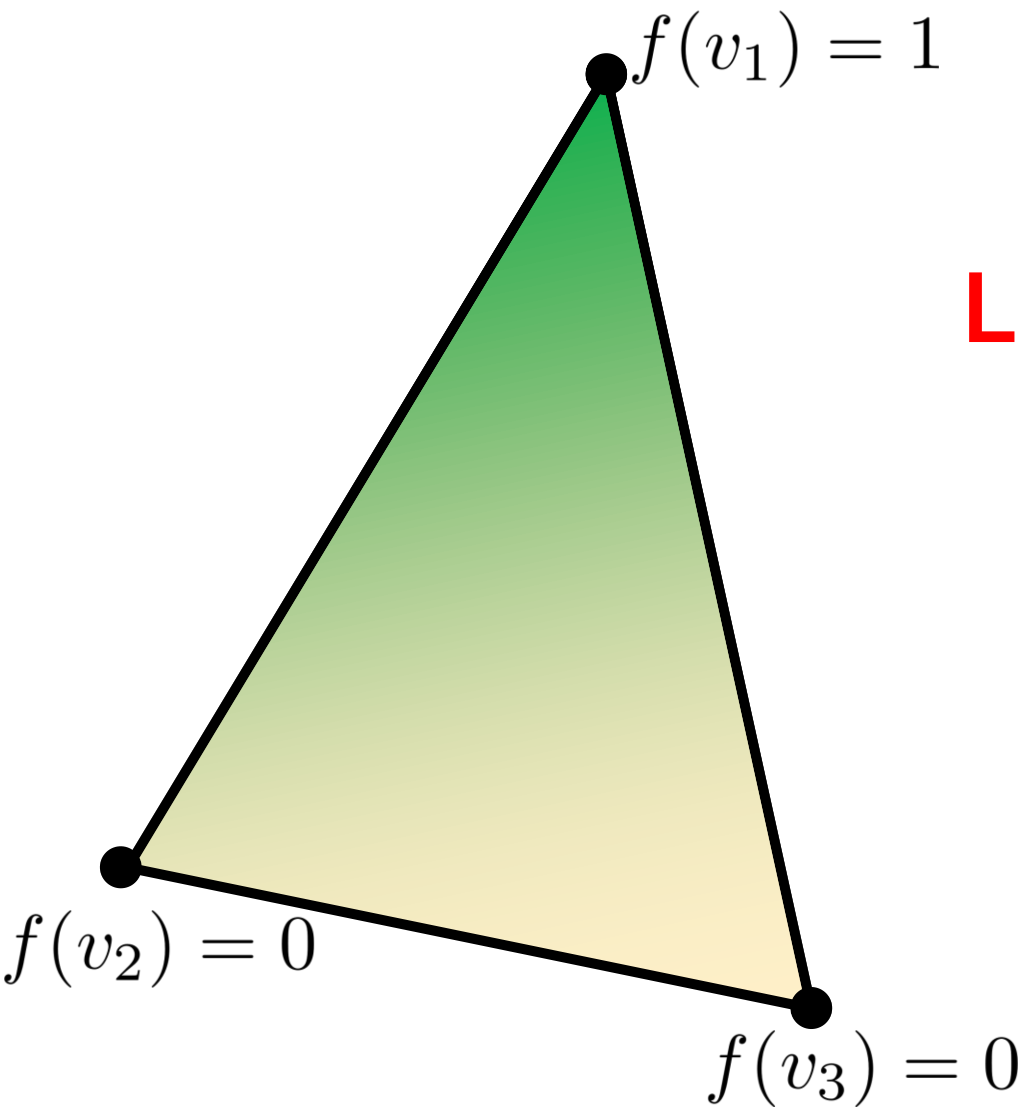
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = \int_M \nabla g \cdot \nabla f \, dA$$

Sum scalars per face
multiplied by face areas

Gradient of a Hat Function



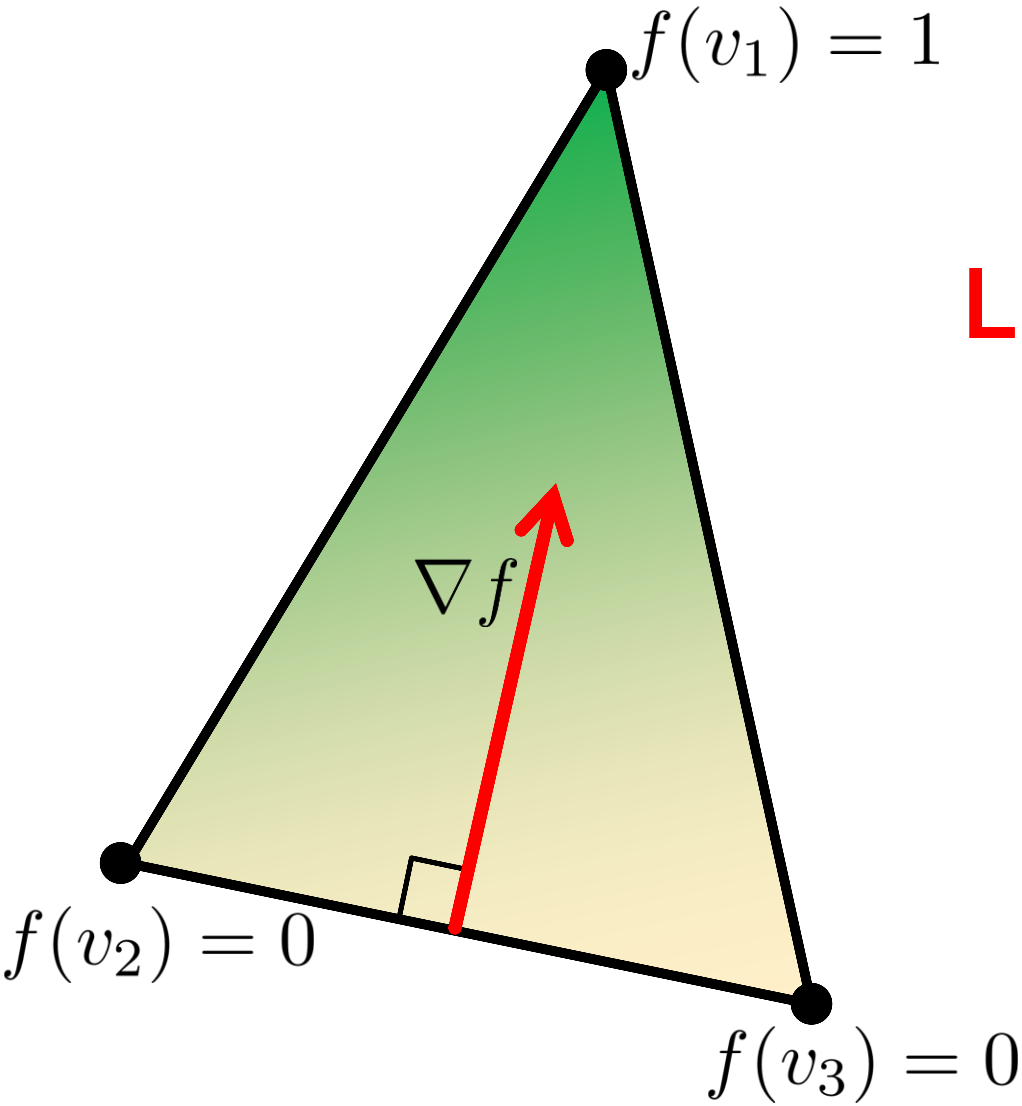
Linear along edges

$$\nabla f \cdot (v_1 - v_3) = 1$$

$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$

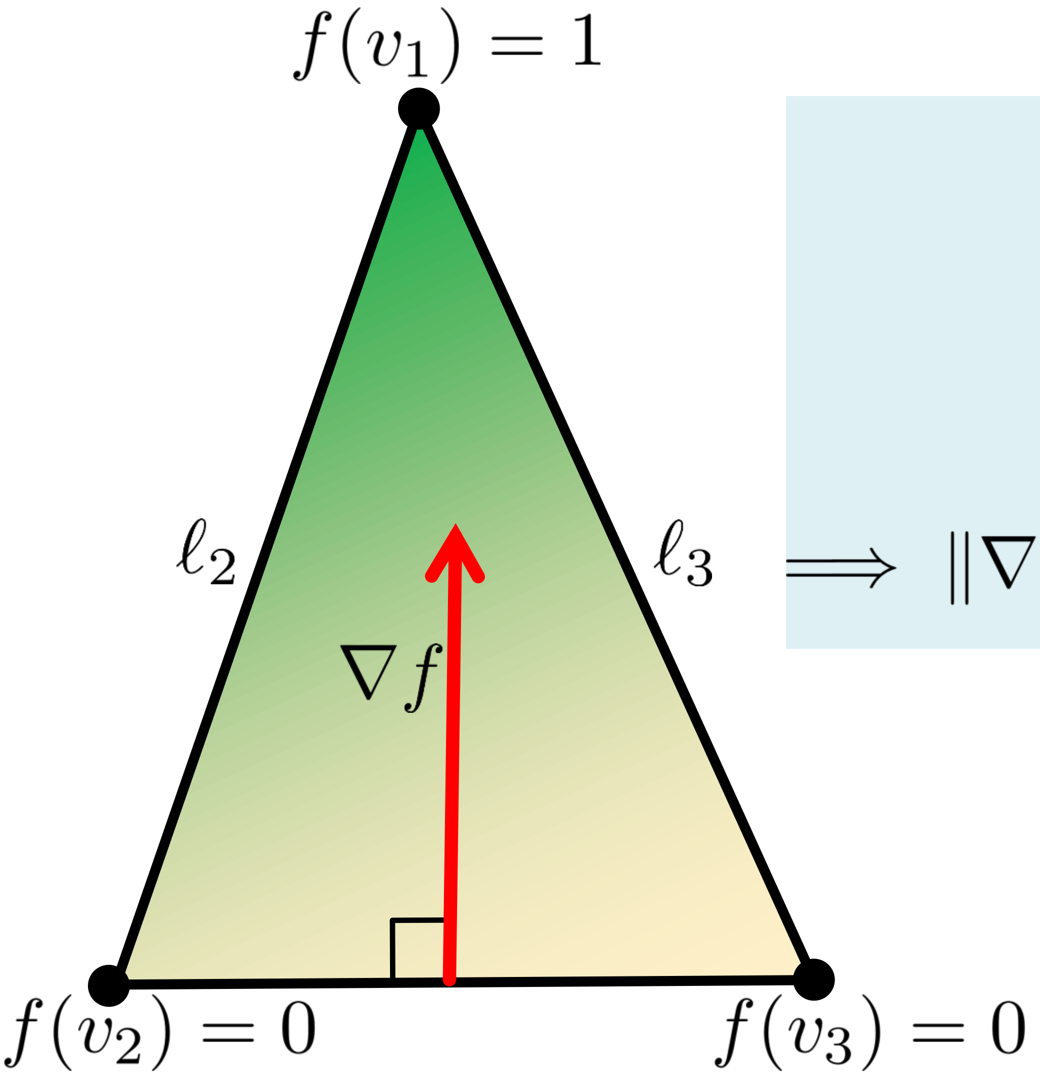
Gradient of a Hat Function



Linear along edges

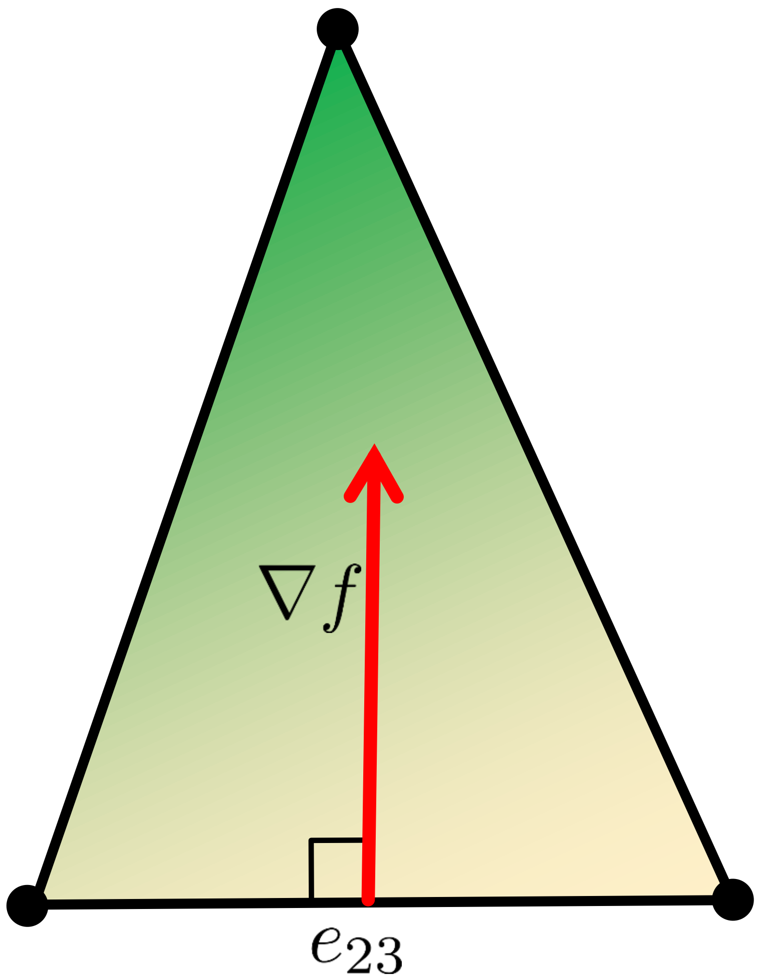
$$\nabla f \cdot (v_1 - v_3) = 1$$
$$\nabla f \cdot (v_1 - v_2) = 1$$
$$\nabla f \cdot n = 0$$
$$\Downarrow$$
$$\nabla f \cdot (v_2 - v_3) = 0$$

Gradient of a Hat Function



$$\begin{aligned} 1 &= \nabla f \cdot (v_1 - v_3) \\ &= \|\nabla f\| \ell_3 \cos\left(\frac{\pi}{2} - \theta_3\right) \\ &= \|\nabla f\| \ell_3 \sin\theta_3 \\ \implies \|\nabla f\| &= \frac{1}{\ell_3 \sin\theta_3} = \frac{1}{h} \end{aligned}$$

Gradient of a Hat Function



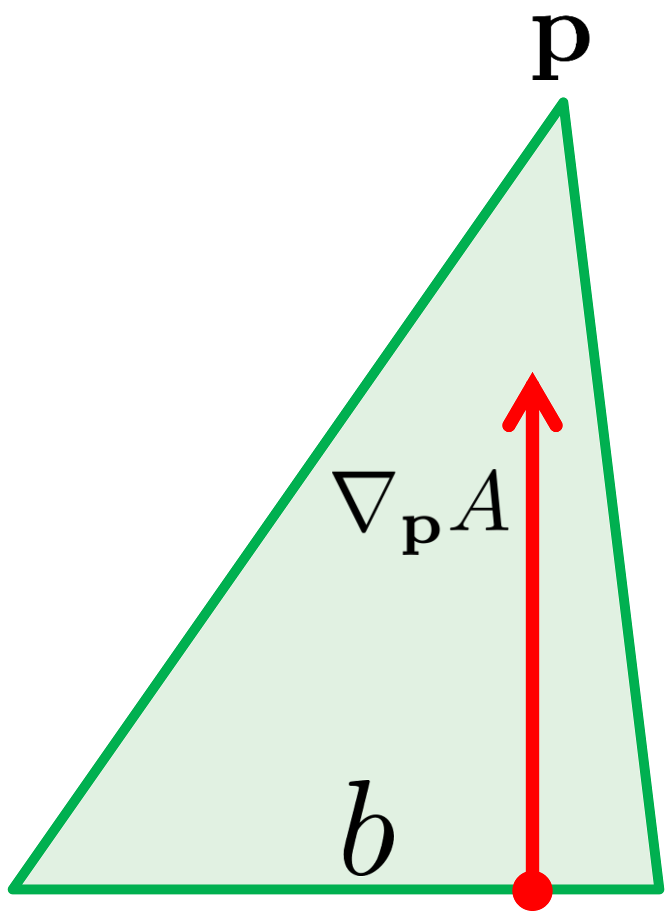
$$\|\nabla f\| = \frac{1}{\ell_3 \sin\theta_3} = \frac{1}{h}$$

$$\nabla f = \frac{e_{23}^{\perp}}{2A}$$

Length of e_{23} cancels "base" in A

Recall:

Single Triangle: Complete



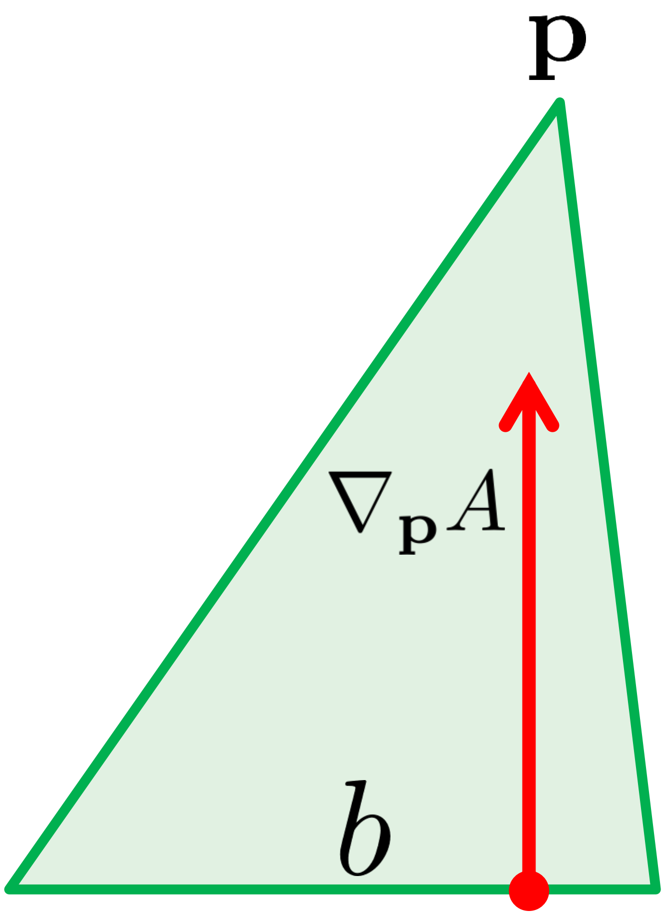
$$\mathbf{p} = p_n \mathbf{n} + p_e \mathbf{e} + p_\perp \mathbf{e}_\perp$$

$$A = \frac{1}{2} b \sqrt{p_n^2 + p_\perp^2}$$

$$\nabla_{\mathbf{p}} A = \frac{1}{2} b \mathbf{e}_\perp$$

Recall:

Single Triangle: Complete



$$\mathbf{p} = p_n \mathbf{n} + p_e \mathbf{e} + p_\perp \mathbf{e}_\perp$$

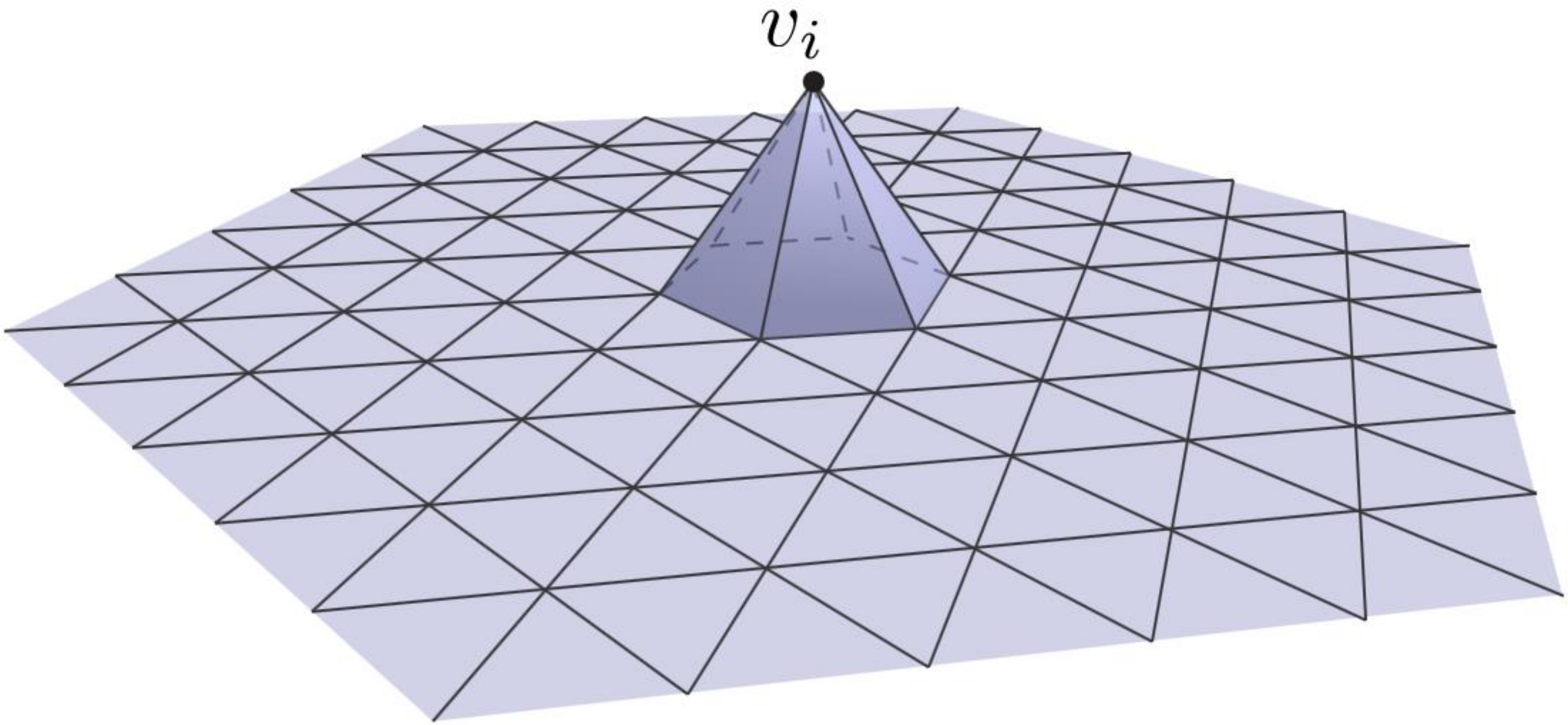
$$A = \frac{1}{2} b \sqrt{p_n^2 + p_\perp^2}$$

$$\nabla_{\mathbf{p}} A = \frac{1}{2} b \mathbf{e}_\perp$$

$$\nabla f = \frac{e_{23}^\perp}{2A} = \frac{\vec{e}_\perp}{h} = \frac{\nabla_{\vec{p}} A}{A}$$

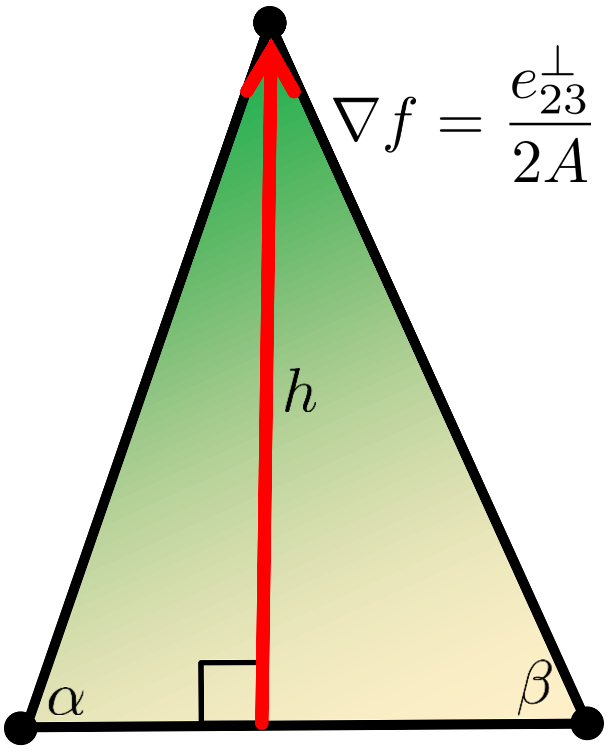
What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = \int_M \boxed{\nabla g \cdot \nabla f} \, dA$$



What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = \int_M \boxed{\nabla g \cdot \nabla f}\, dA$$



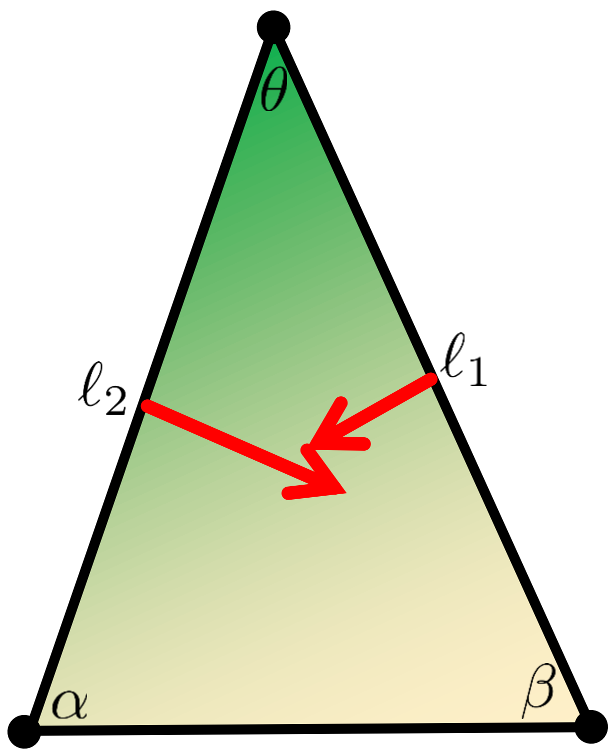
Case 1: Same vertex

$$\int_T \langle \nabla f, \nabla f \rangle \, dA = A \|\nabla f\|_2^2$$
$$= \frac{A}{h^2} = \frac{b}{2h}$$
$$= \frac{1}{2}(\cot \alpha + \cot \beta)$$

What We Actually Need

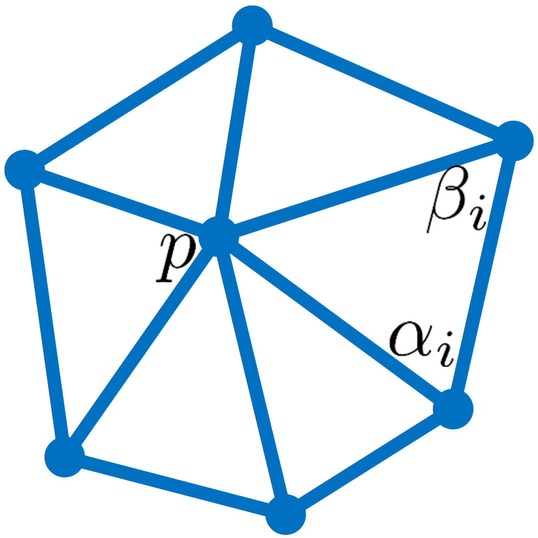
$$\mathcal{L}_{\Delta f}[g] = \int_M \boxed{\nabla g \cdot \nabla f}\, dA$$

Case 2: Different vertices

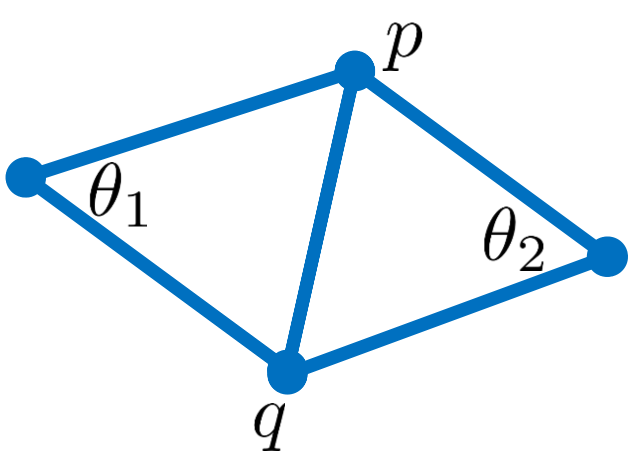


$$\begin{aligned}\int_T \langle \nabla f_\alpha, \nabla f_\beta \rangle \, dA &= A \langle \nabla f_\alpha, \nabla f_\beta \rangle \\ &= \frac{1}{4A} \langle e_{31}^\perp, e_{32}^\perp \rangle = -\frac{\ell_1 \ell_2 \cos\theta}{4A} \\ &= -\frac{1}{2h_1} \ell_2 \cos\theta = -\frac{\cos\theta}{2\sin\theta} \\ &= -\frac{1}{2} \cot\theta\end{aligned}$$

Summing Around a Vertex



$$\langle \nabla h_p, \nabla h_p \rangle = \frac{1}{2} \sum_i (\cot \alpha_i + \cot \beta_i)$$

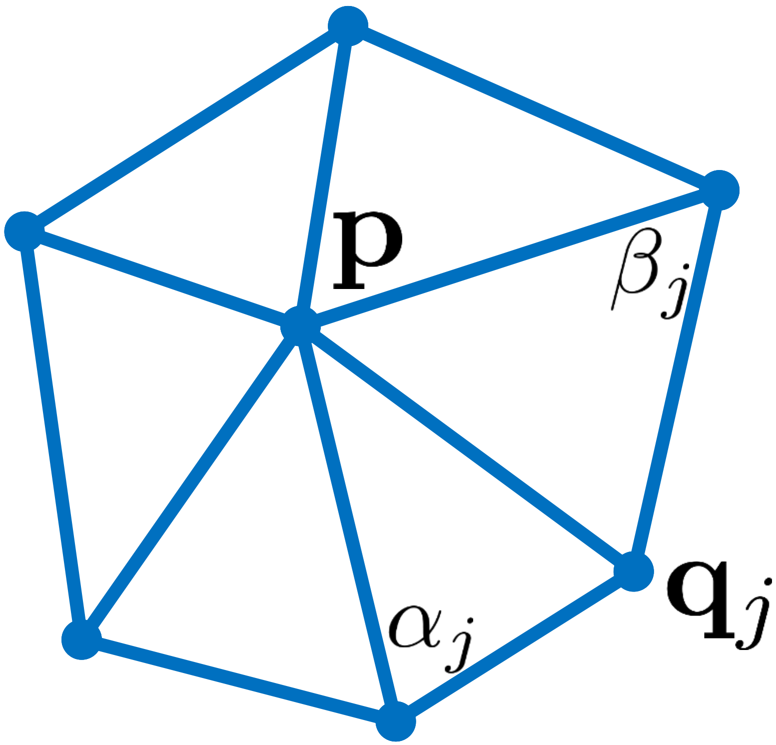


$$\langle \nabla h_p, \nabla h_q \rangle = -\frac{1}{2} (\cot \theta_1 + \cot \theta_2)$$

Recall:

Summing Around a Vertex

$$\nabla_{\mathbf{p}} A = \frac{1}{2} \sum_j (\cot \alpha_j + \cot \beta_j)(\mathbf{p} - \mathbf{q}_j)$$

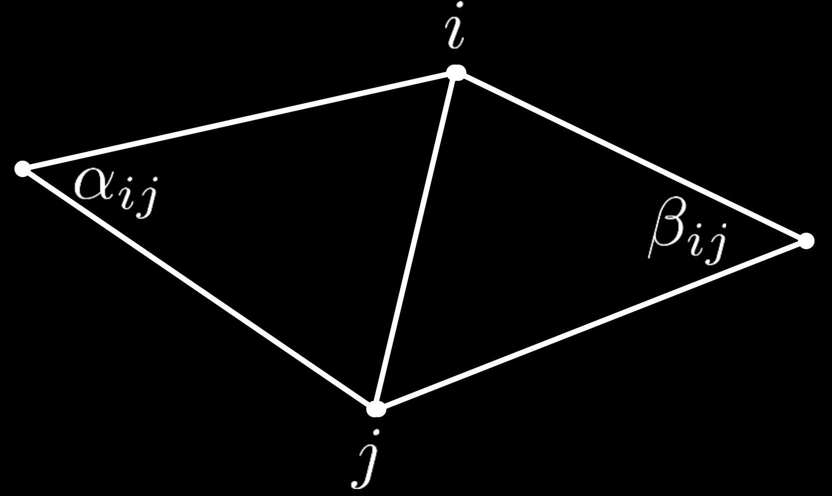


$$\nabla_{\mathbf{p}} A = \frac{1}{2}((\mathbf{p} - \mathbf{r}) \cot \alpha + (\mathbf{p} - \mathbf{q}) \cot \beta)$$

Same operator!

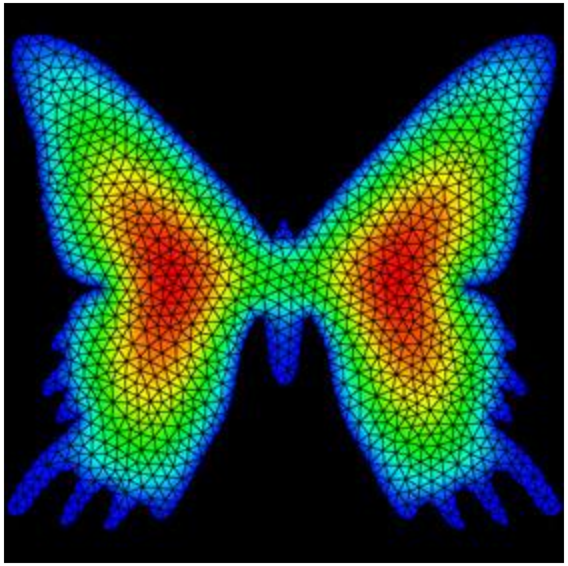
The Cotangent Laplacian

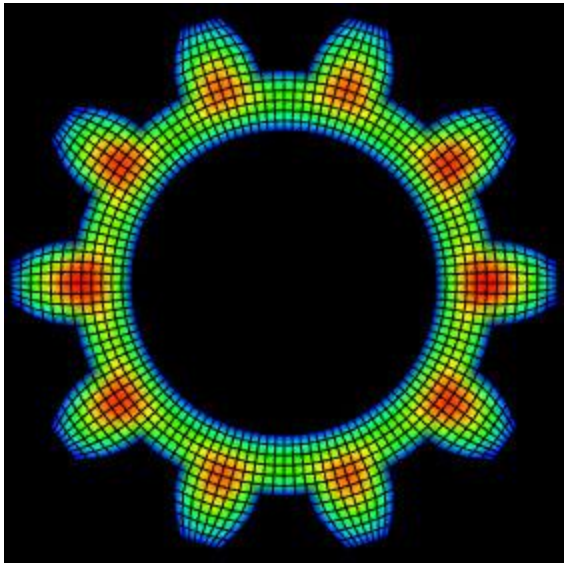
$$L_{ij} = \begin{cases} \frac{1}{2}\sum_{i\sim k}(\cot\alpha_{ik} + \cot\beta_{ik}) & \text{if } i = j \\ -\frac{1}{2}(\cot\alpha_{ij} + \cot\beta_{ij}) & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$



Poisson Equation

$$\Delta f = g$$

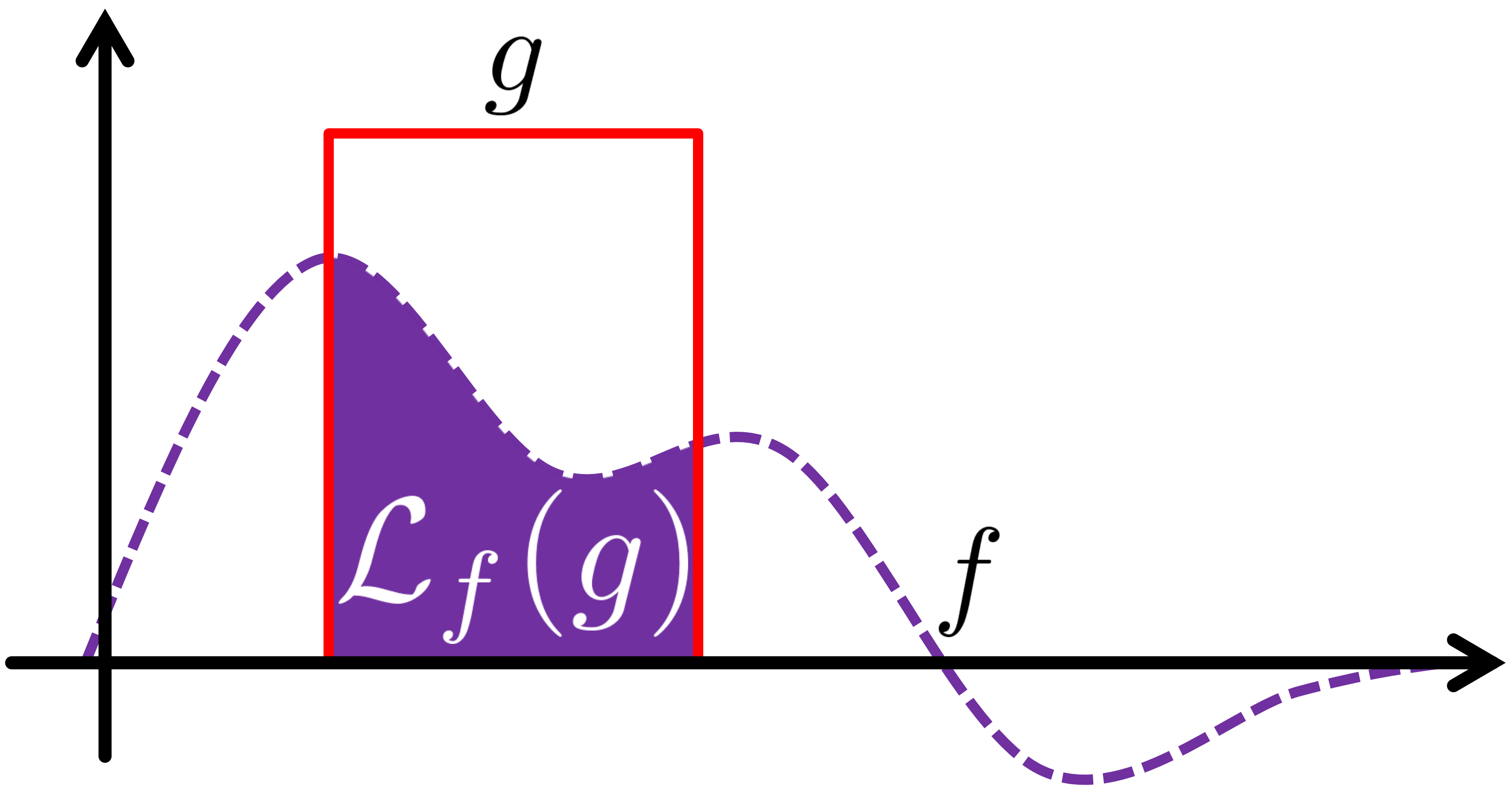




http://nylander.wordpress.com/2006/05/24/finite-element-method-fem-solution-to-poisson%E2%80%99s-equation-on-triangular-mesh/

Weak Solutions

$$\int_M \phi \Delta f \, dA = \int_M \phi g \, dA \ \forall \text{ test functions } \phi$$



FEM Hat Weak Solutions

$$\int_M h_i \Delta f \, dA = \int_M h_i g \, dA \ \forall \text{ hat functions } h_i$$

$$\begin{aligned}
\int_M h_i \Delta f \, dA &= \int_M \nabla h_i \cdot \nabla f \, dA \\
&= \int_M \nabla h_i \cdot \nabla \sum_j a_j h_j \, dA \\
&= \sum_j a_j \int_M \nabla h_i \cdot \nabla h_j \, dA \\
&= \sum_j L_{ij} a_j
\end{aligned}$$

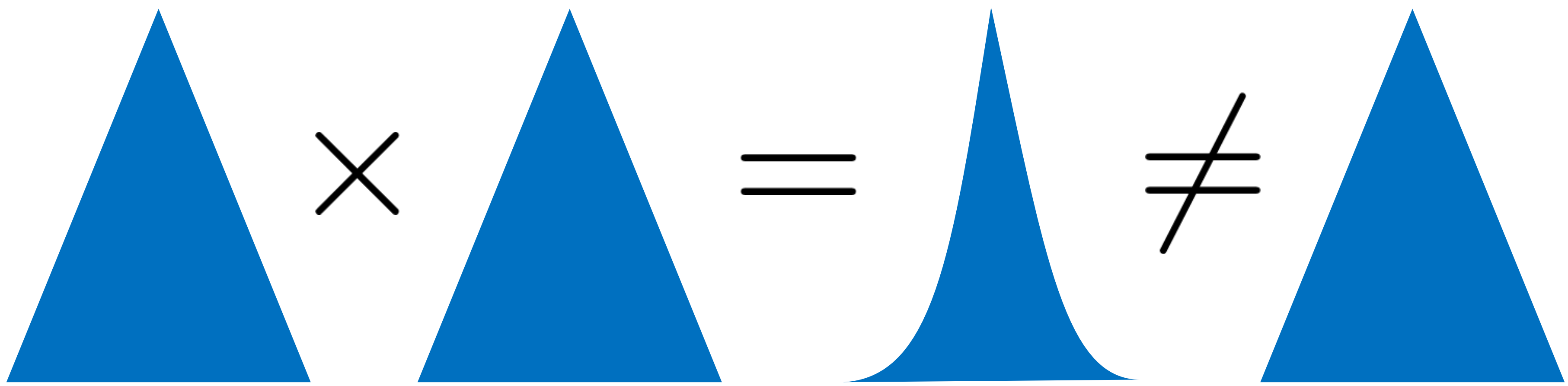
Stacking Integrated Products

$$\begin{pmatrix} \int_M h_1 \Delta f \, dA \\ \int_M h_2 \Delta f \, dA \\ \vdots \\ \int_M h_{|V|} \Delta f \, dA \end{pmatrix} = \begin{pmatrix} \sum_j L_{1j} a_j \\ \sum_j L_{2j} a_j \\ \vdots \\ \sum_j L_{|V|j} a_j \end{pmatrix} = L\mathbf{a}$$

Multiply by Laplacian matrix!

Problematic Right Hand Side

$$\int_M h_i \Delta f \, dA = \int_M h_i g \, dA \; \forall \text{ hat functions } h_i$$



Product of hats is quadratic

A Few Ways Out

- Just do the integral
  "Consistent" approach
- Approximate some more
- Redefine g

A Few Ways Out

- Just do the integral

  "Consistent" approach

- Approximate some more

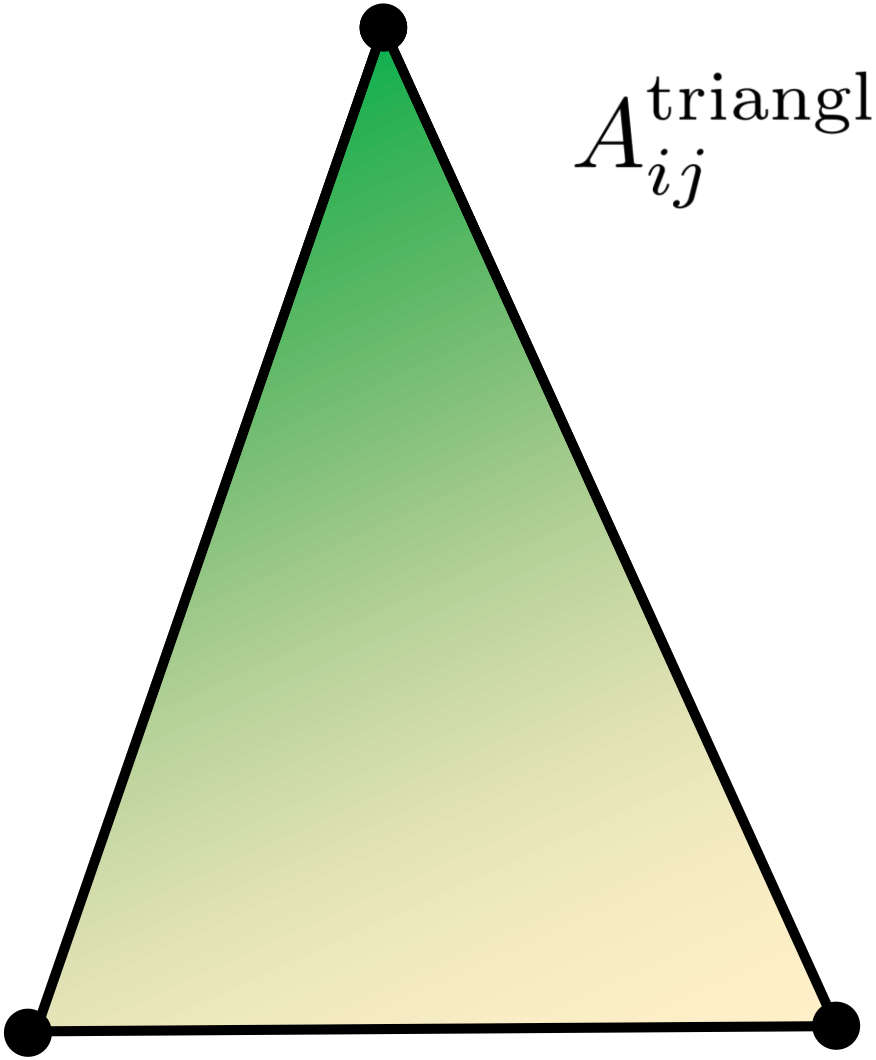
- Redefine *g*

The Mass Matrix

$$A_{ij} := \int_M h_i h_j \, dA$$

- **Diagonal** elements: Norm of h_i
- **Off-diagonal** elements: Overlap between h_i and h_j

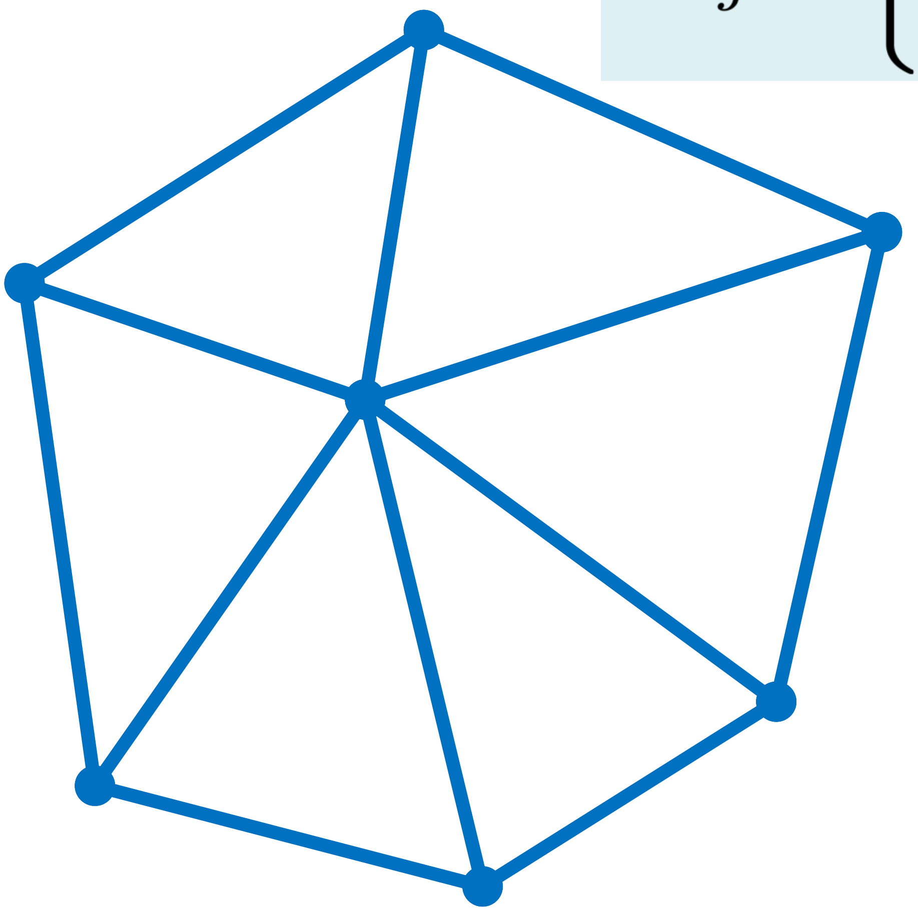
Consistent Mass Matrix



$$A_{ij}^{\text{triangle}} = \begin{cases} \dfrac{\text{area}}{6} & \text{if } i = j \\ \dfrac{\text{area}}{12} & \text{if } i \neq j \end{cases}$$

Non-Diagonal Mass Matrix

$$M_{ij} = \begin{cases} \dfrac{\text{one-ring area}}{6} & \text{if } i = j \\ \dfrac{\text{adjacent area}}{12} & \text{if } i \neq j \end{cases}$$



Properties of Mass Matrix

- Rows sum to one ring area / 3
- Involves only vertex and its neighbors
- Partitions surface area

Issue: Not diagonal!

Use for Integration

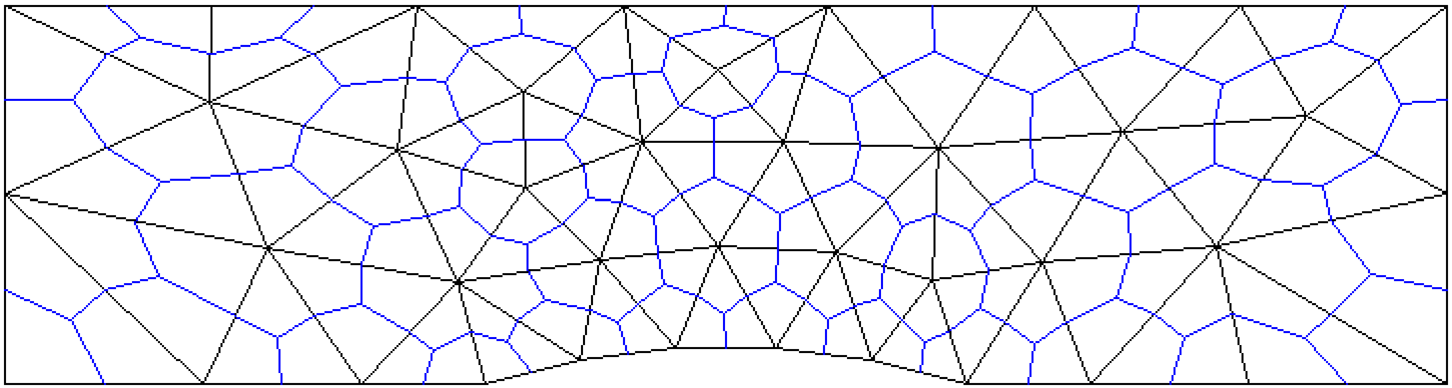
$$\int_M f = \int_M \sum_j a_j h_j (\cdot 1)$$

$$= \int_M \sum_j a_j h_j \sum_i h_i$$

$$= \sum_{ij} A_{ij} a_j$$

$$= \mathbf{1}^\top A \mathbf{a}$$

Lumped Mass Matrix



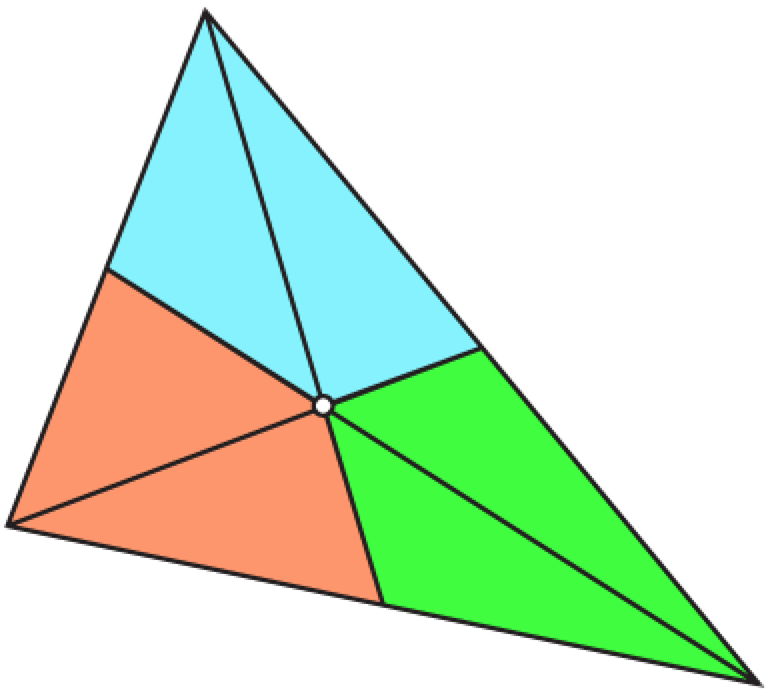
$$\tilde{a}_{ii} := \text{Area}(\text{cell } i)$$

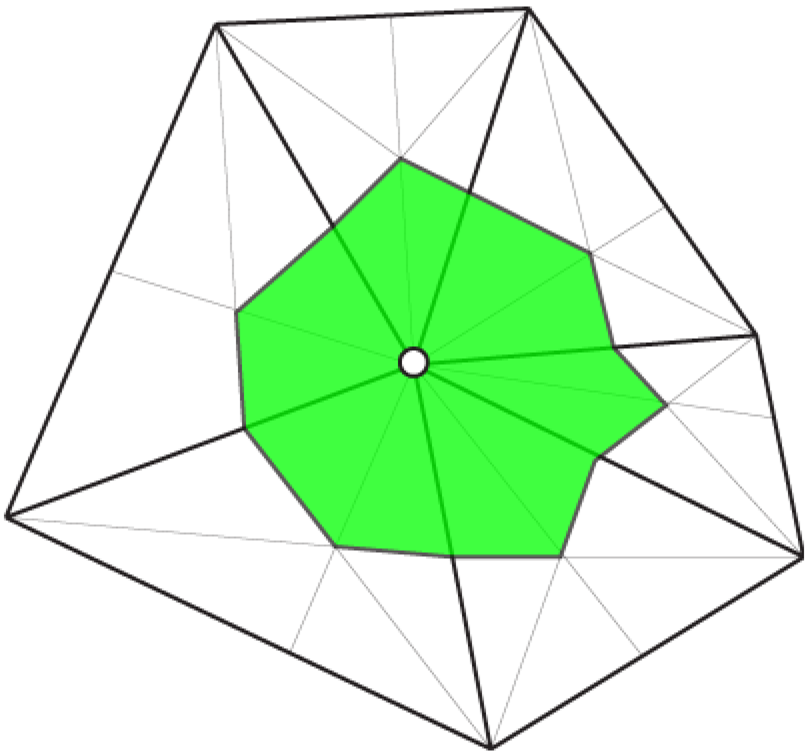
Won't make big difference for smooth functions

http://users.led-inc.eu/~phk/mesh-dualmesh.html

Approximate with diagonal matrix

Simplest: Barycentric Lumped Mass





http://www.alecjacobson.com/weblog/?p=1146

Area/3 to each vertex

Ingredients

- **Cotangent Laplacian L**

  Per-vertex function to integral of its Laplacian against each hat

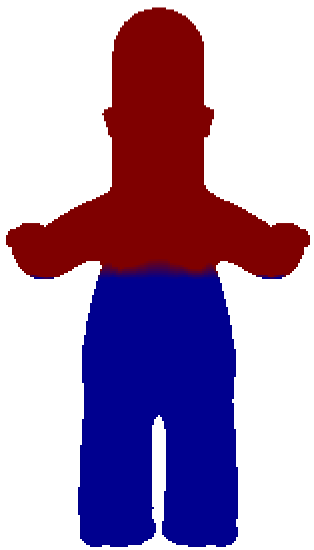
- **Area weights A**

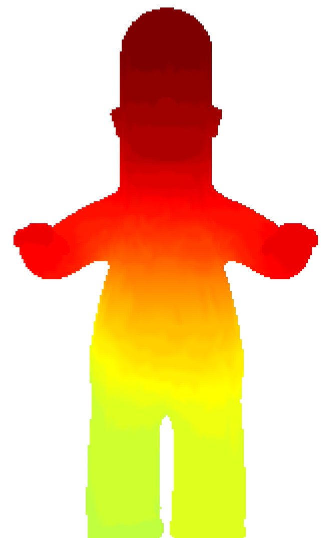
  Integrals of pairwise products of hats (or approximation thereof)

Solving the Poisson Equation

$$\Delta f = g \longrightarrow L\vec{f} = A\vec{g}$$

g





f

Must integrate to zero

Determined up to constant

Important Detail: Boundary Conditions

$$\begin{aligned} \Delta f(x) &= g(x)\ \forall x \in \Omega \\ f(x) &= u(x)\ \forall x \in \Gamma_D \\ \nabla f \cdot n &= v(x)\ \forall x \in \Gamma_N \end{aligned}$$

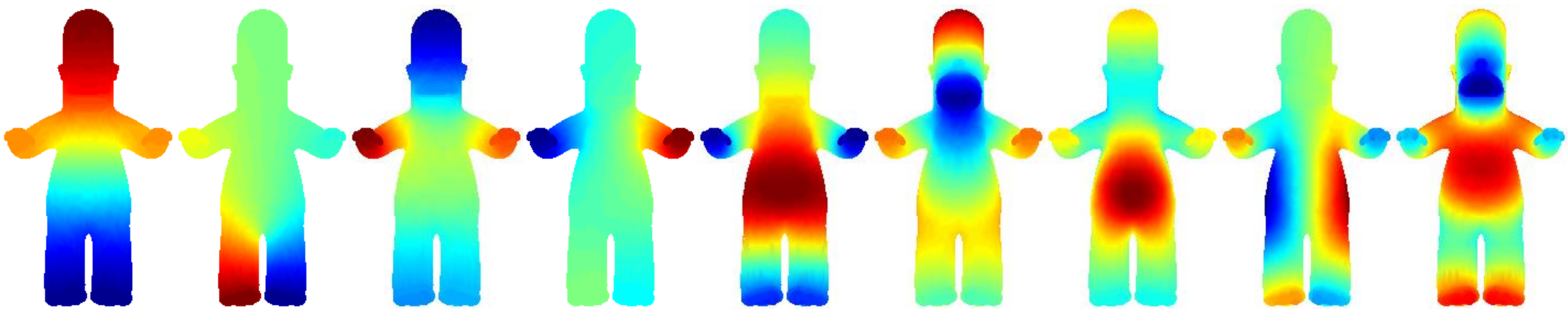
Strong form

$$\int_\Omega \nabla f \cdot \nabla \phi = \int_{\Gamma_N} v(x)\phi(x)\, d\Gamma - \int_\Omega f(x)\phi(x)\, d\Omega$$

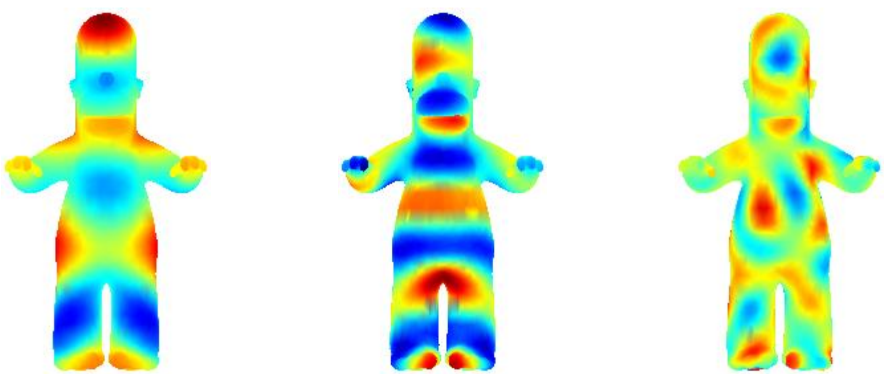
$$f(x) = u(x)\ \forall x \in \Gamma_D$$

Weak form

Eigenhomers



2 3 4 5 6 7 8 9 10

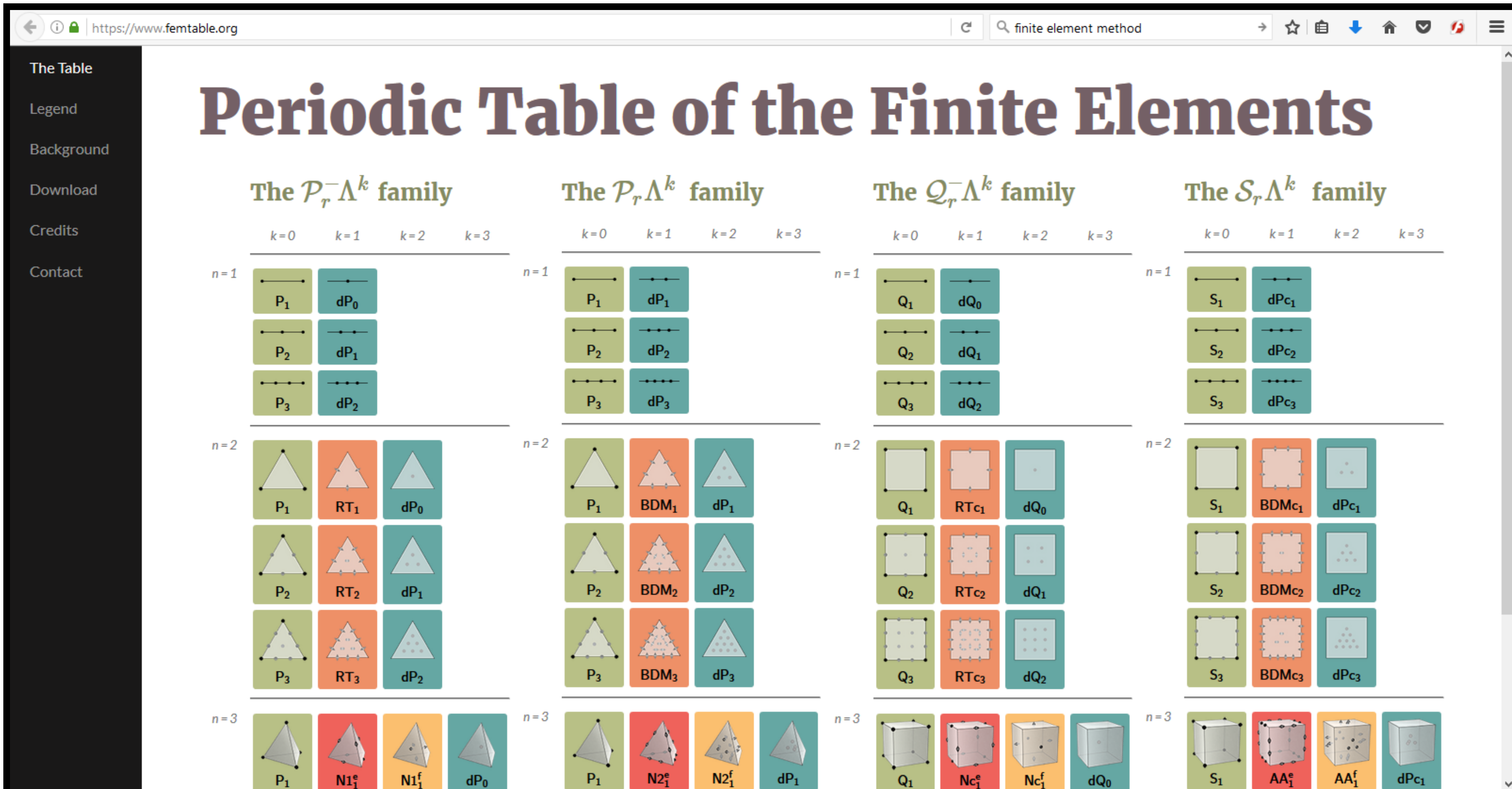


25 50 100

What is smallest eigenvalue?

On board: FEM approach?

Higher-Order Elements



https://www.femtable.org/

Point Cloud Laplace: Easiest Option

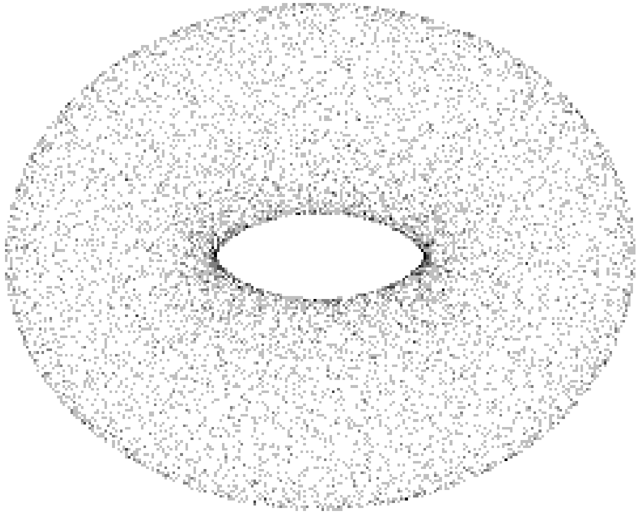
$$W_{ij} = \exp\left(-\frac{\|x_i - x_j\|^2}{t}\right)$$

Tricky parameter to choose

$$D_{ii} = \sum_j W_{ji}$$

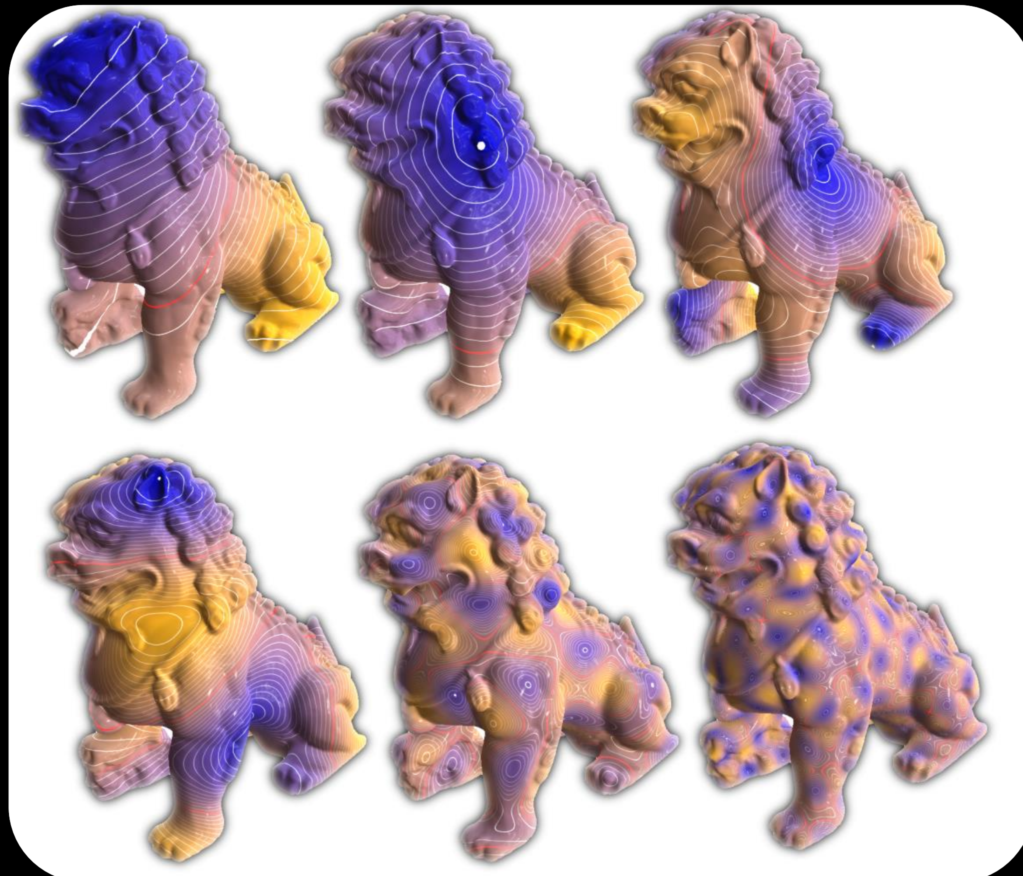
$$L = D - W$$

$$Lf = \lambda D f$$



"Laplacian Eigenmaps for Dimensionality Reduction and Data Representation"
Belkin & Niyogi 2003

http://alice.loria.fr/publications/papers/2008/ManifoldHarmonics//photo/dragon_mhb.png



Discrete Laplacians

Justin Solomon

MIT, Spring 2019