Introducing the Laplacian Operator

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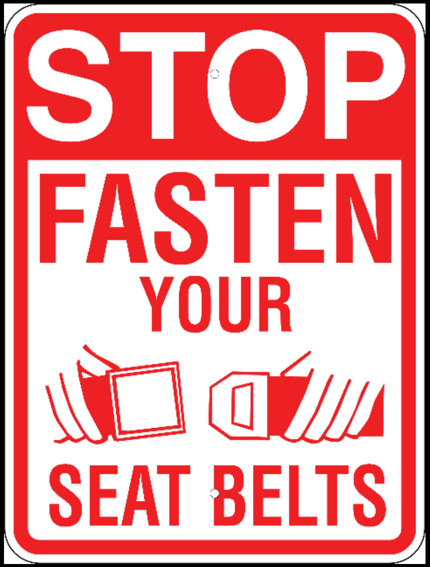
MIT, Spring 2019


WARNING
SIGN
MISTAKES
LIKELY



Lots of (sloppy) math!

http://www.wgnflag.com/xcart/images/P/G-50_StopSeatBeltsStockSign.jpg

Famous Motivation

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

> "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

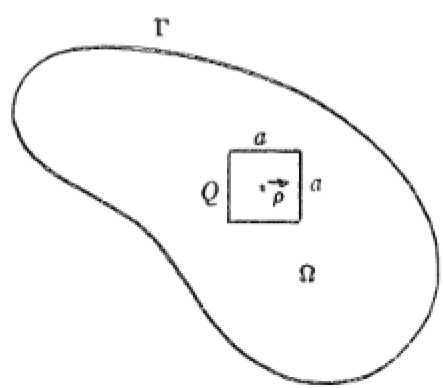
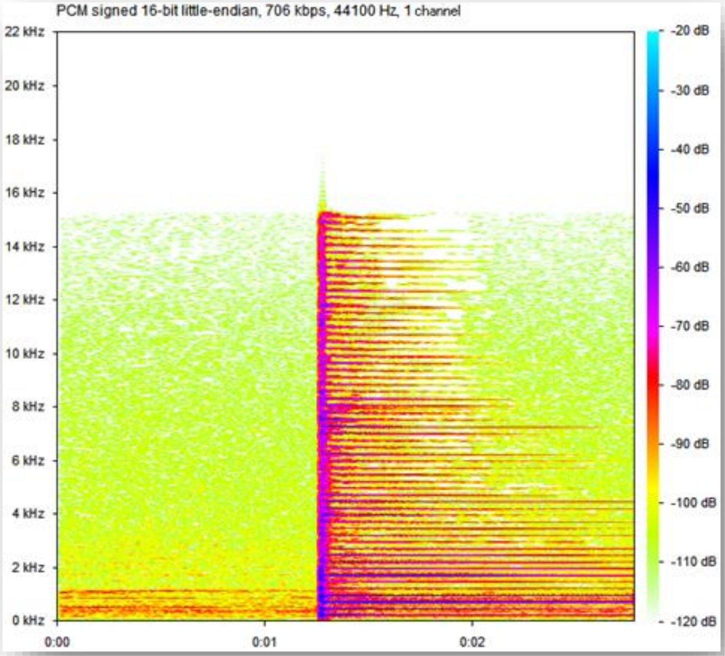


FIG. 1

An Experiment

Is this possible?

Unreasonable to Ask?

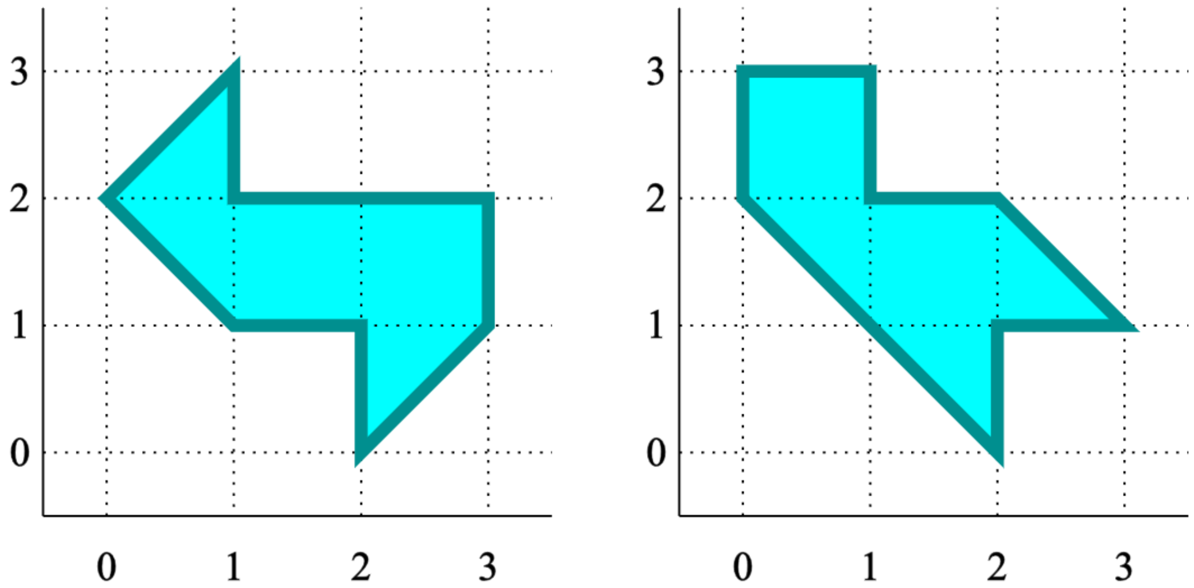


Length of string

http://www.takamine.com/templates/default/images/gclassical.png
http://firsttimeprogrammer.blogspot.com/2015/05/spectrogram-representation-of-b-note.html

Spoiler Alert

Extra credit: Make these!



"No, but…"

- Has to be a weird drum
- Spectrum tells you a lot!

Rough Intuition

http://pngimg.com/upload/hammer_PNG3886.png

You can learn a lot about a shape by **hitting it** (lightly) **with a hammer**!

Spectral Geometry

What can you learn about its shape from vibration frequencies and oscillation patterns?

$$\Delta f = \lambda f$$

Objectives

- Make “vibration modes” more precise
- Progressively more complicated domains
  - Line segments
  - Regions in $\mathbb{R}^n$
  - Graphs
  - Surfaces/manifolds
- Next time: Discretization, applications

Review:

Vector Spaces and Linear Operators

$$L[\mathbf{x} + \mathbf{y}] = L[\mathbf{x}] + L[\mathbf{y}]$$

$$L[c\mathbf{x}] = cL[\mathbf{x}]$$

$$L[\mathbf{x}] = A\mathbf{x}$$

Review:

In Finite Dimensions

$$\underbrace{A}_{\text{matrix}} \quad \underbrace{\mathbf{x}}_{\text{vector}}$$

$$\underbrace{\mathbf{x} \mapsto A\mathbf{x}}_{\text{linear operator}}$$

Recall: Spectral Theorems in $\mathbb{C}^n$

Theorem. Suppose $A \in \mathbb{C}^{n \times n}$ is Hermitian. Then, A has an orthogonal basis of n eigenvectors. If A is positive definite, the corresponding eigenvalues are nonnegative.

Our Progression

- Line segments
- Regions in $\mathbb{R}^n$
- Graphs
- Surfaces/manifolds

Transverse Wave: 1D Spring Network

(on the board)

Minus Second Derivative Operator

"Dirichlet boundary conditions"

$$\{f(\cdot) \in C^\infty([a,b]) : f(0) = f(\ell) = 0\}$$

$$\mathcal{L}[\cdot] : u \mapsto -\frac{\partial^2 u}{\partial x^2}$$

On the board: Interpretation as positive (semi-)definite operator.

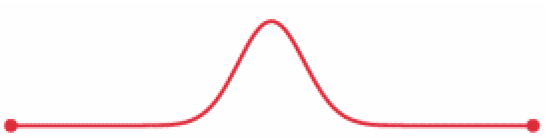
Eigenfunctions:

$$\phi_k(x) = \sqrt{\frac{2}{\ell}} \sin\left(\frac{\pi k x}{\ell}\right), \quad \lambda_k = \left(\frac{\pi k}{\ell}\right)^2$$

Physical Intuition: Wave Equation

Minus second derivative operator!

$$\frac{\partial^2 u}{\partial t^2} \underbrace{-\frac{\partial^2 u}{\partial x^2}}_{} = 0$$



Can you hear the length of an interval?

$$\lambda_k = \left(\frac{\pi k}{\ell}\right)^2$$

Yes!

Homework (?)

for a curve $\gamma(u) = (x[u], y[u]) : \mathbb{R} \to \mathbb{R}^2$

- $\Delta\gamma = (\Delta x, \Delta y)$ is gradient of arc length
- $\Delta\gamma$ is the *curvature normal* $\kappa\hat{n}$
- minimal curves are harmonic (straight lines)

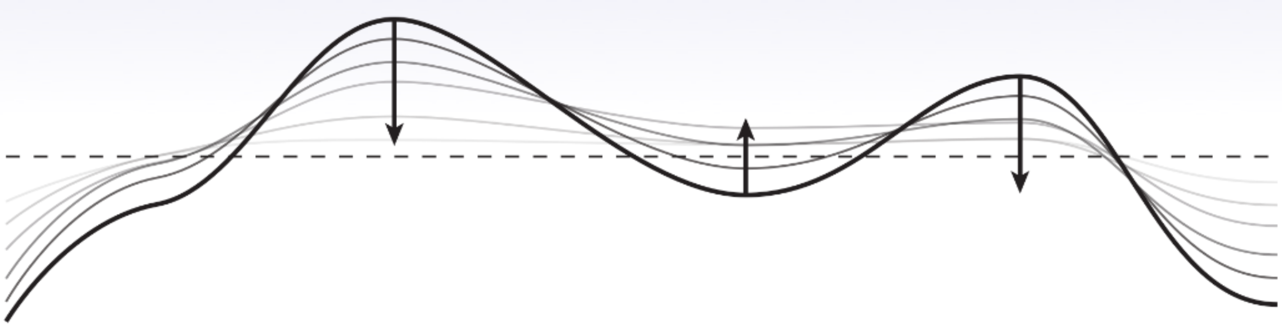
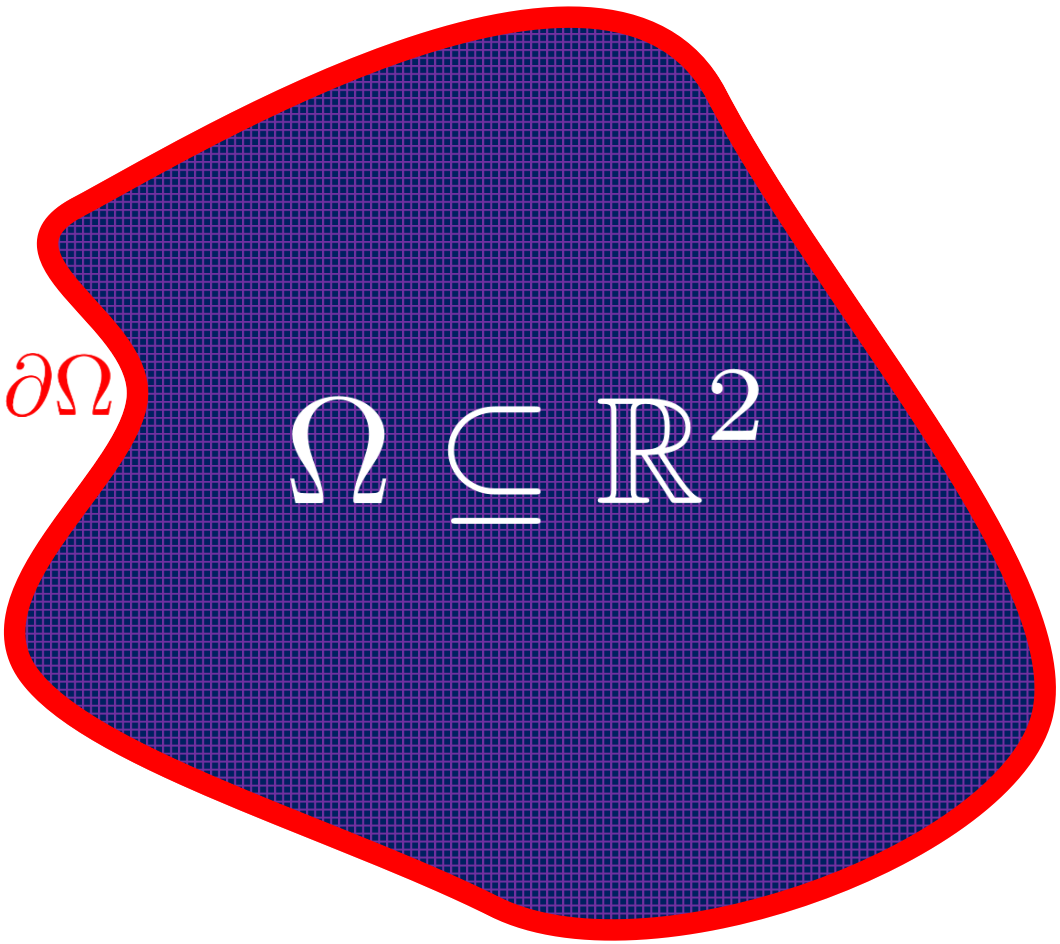


Image by E. Vouga

Our Progression

- Line segments
- Regions in $\mathbb{R}^n$
- Graphs
- Surfaces/manifolds

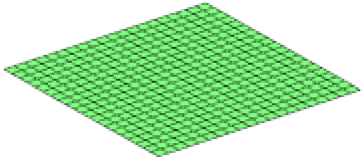
Planar Region



Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = -\Delta u$$

$$\Delta := -\sum_i \frac{\partial^2}{\partial x_i^2}$$

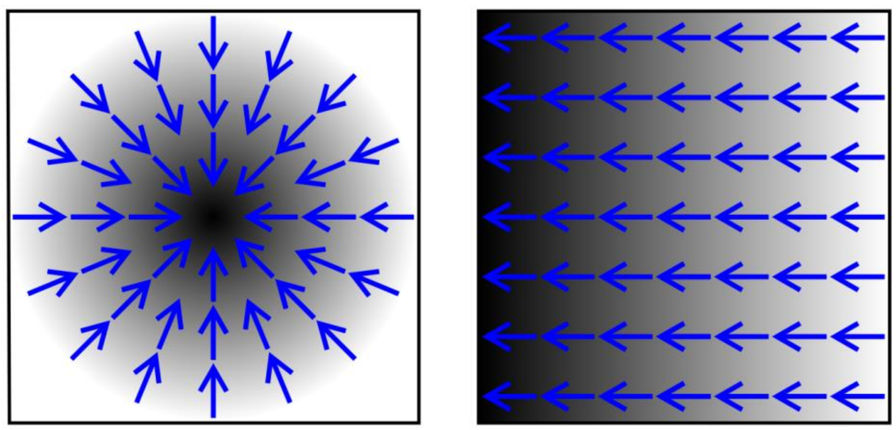


Typical Notation

$$\text{“}\Delta = -\nabla \cdot \nabla\text{”}$$

divergence

gradient

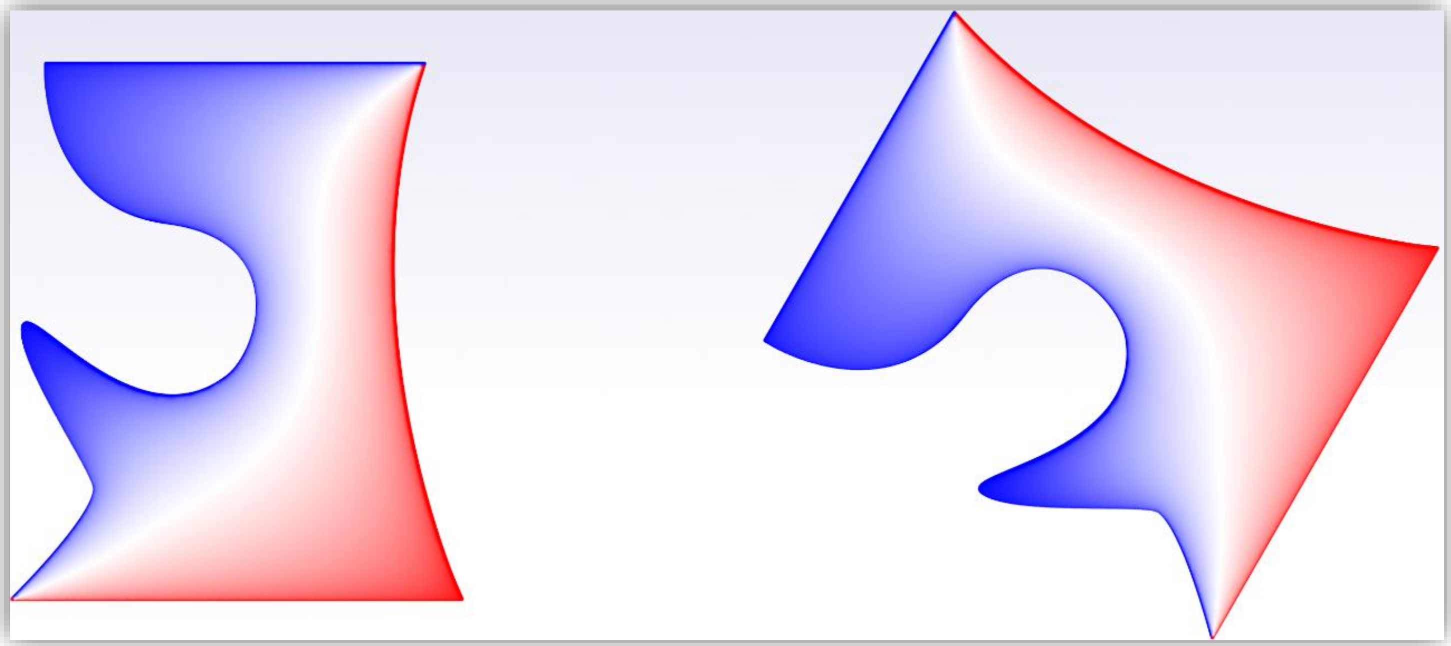


Gradient operator:

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right)$$

http://www.gamasutra.com/db_area/images/feature/4164/figy.png, https://en.wikipedia.org/wiki/Gradient

Intrinsic Operator

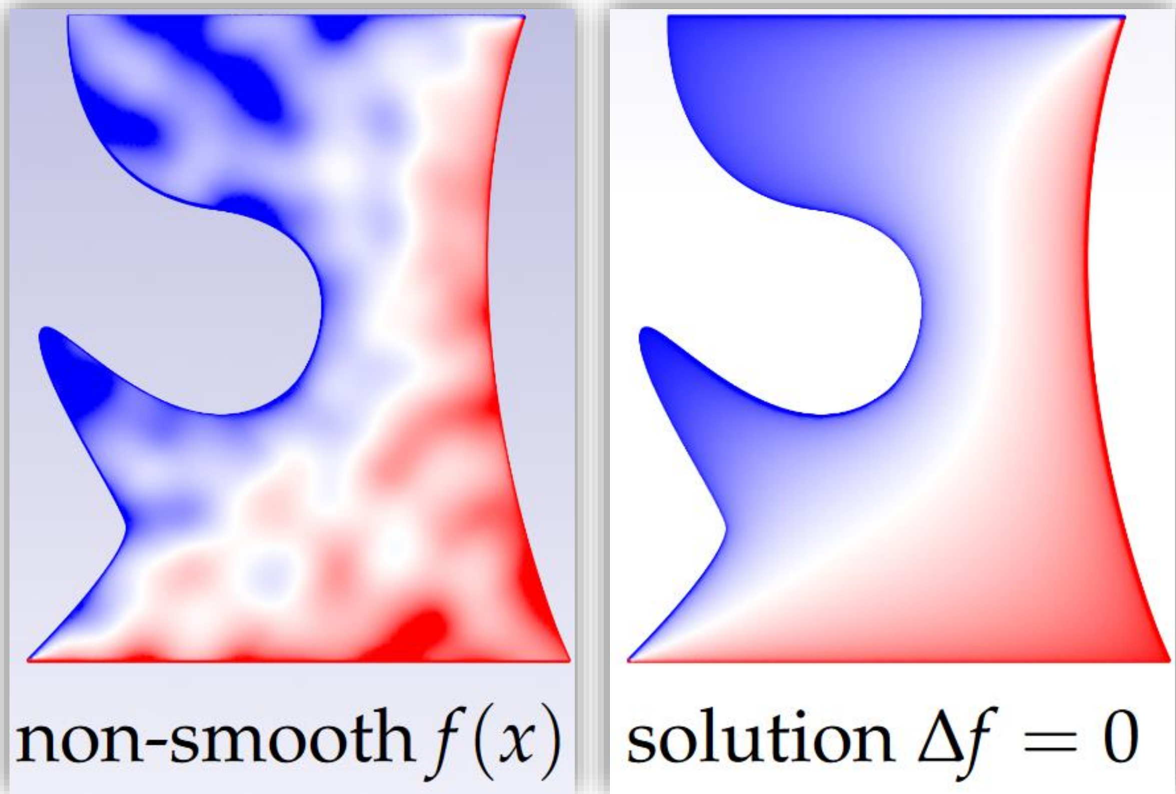


Images made by E. Vouga

Coordinate-independent (important!)

Dirichlet Energy

$$E[u] := \frac{1}{2}\int_\Omega \|\nabla u(\mathbf{x})\|_2^2 \, dA(\mathbf{x})$$



Images made by E. Vouga

On board:

$$\min_{u(\mathbf{x}):\Omega\to\mathbb{R}} E[u]$$

$$\text{s.t. } u|_{\partial\Omega} \text{ prescribed}$$

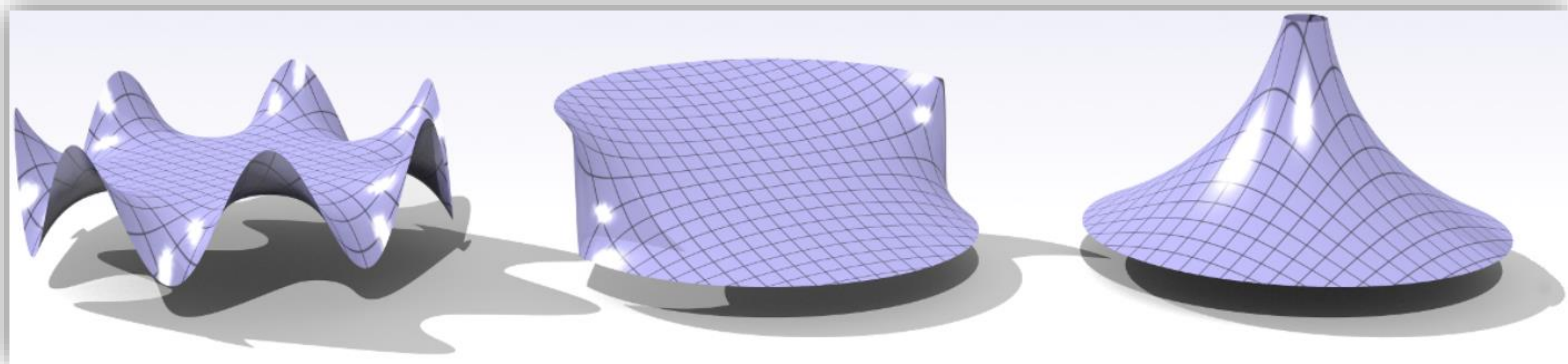
$$\Downarrow$$

$$\Delta u \equiv 0$$

"Laplace equation"
"Harmonic function"

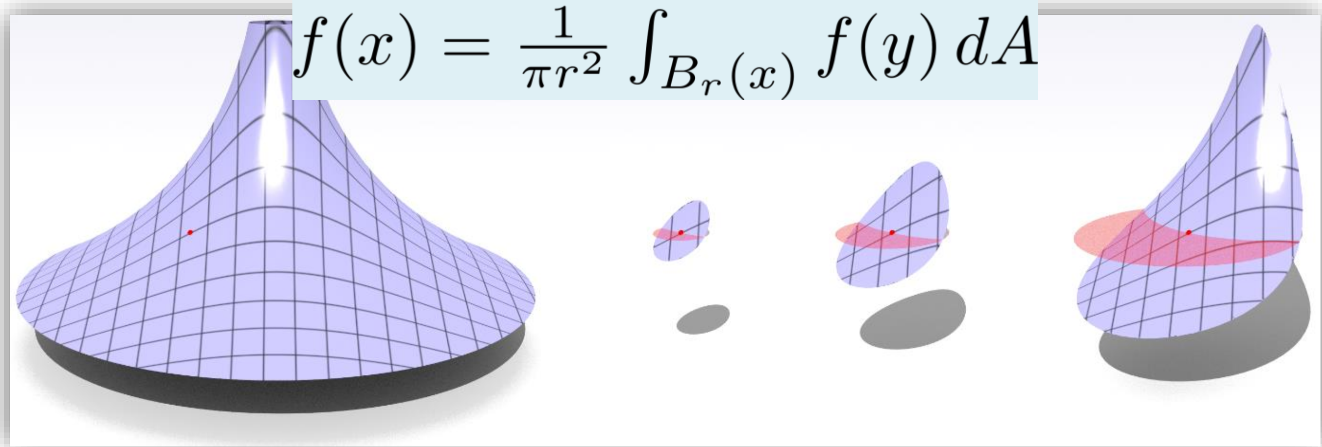
Harmonic Functions

$$\Delta f \equiv 0$$



Mean value property:

$$f(x) = \frac{1}{\pi r^2} \int_{B_r(x)} f(y)\, dA$$



Images made by E. Vouga

Application

Harmonic Coordinates

Tony DeRose Mark Meyer
Pixar Technical Memo #06-02
Pixar Animation Studios

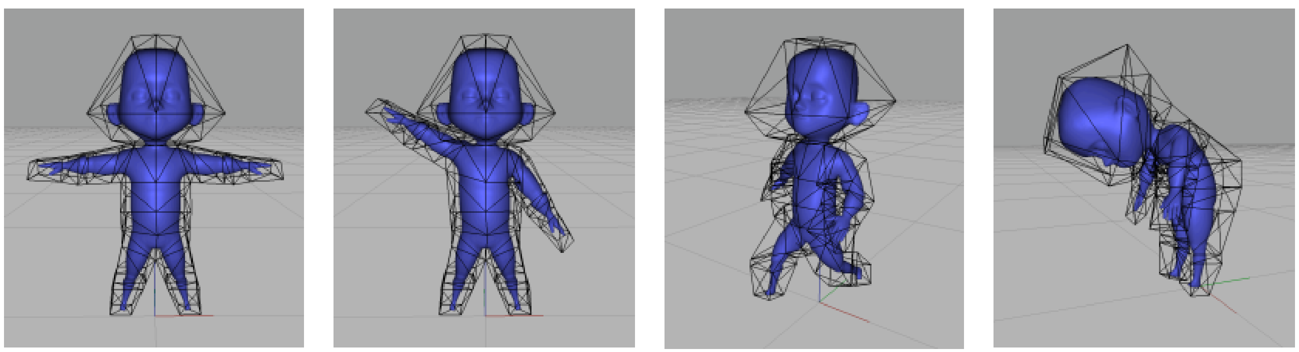
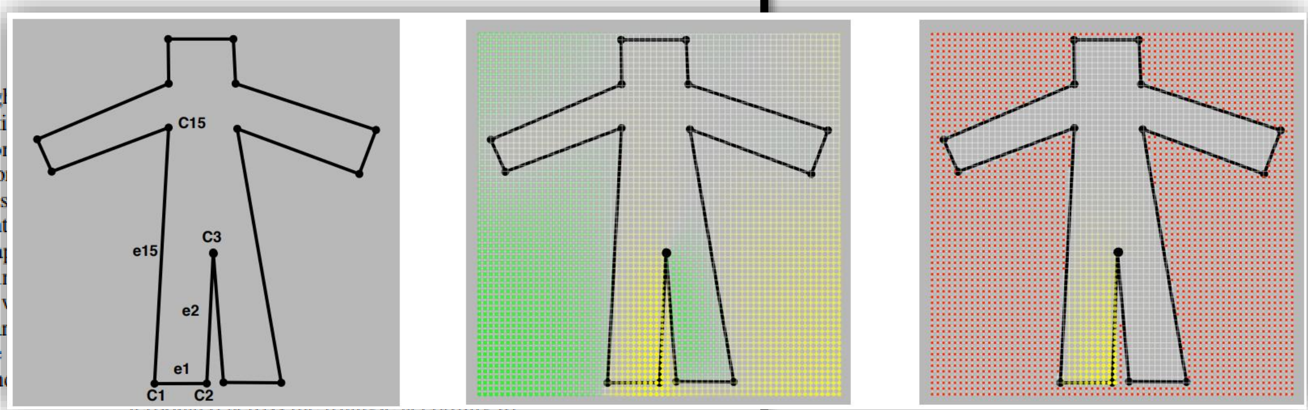


Figure 1: A character (shown in blue) being deformed by a cage (shown in black) using harmonic coordinates. (a) The character and cage at bind-time; (b) - (d) the deformed character corresponding to three different poses of the cage.



Abstract

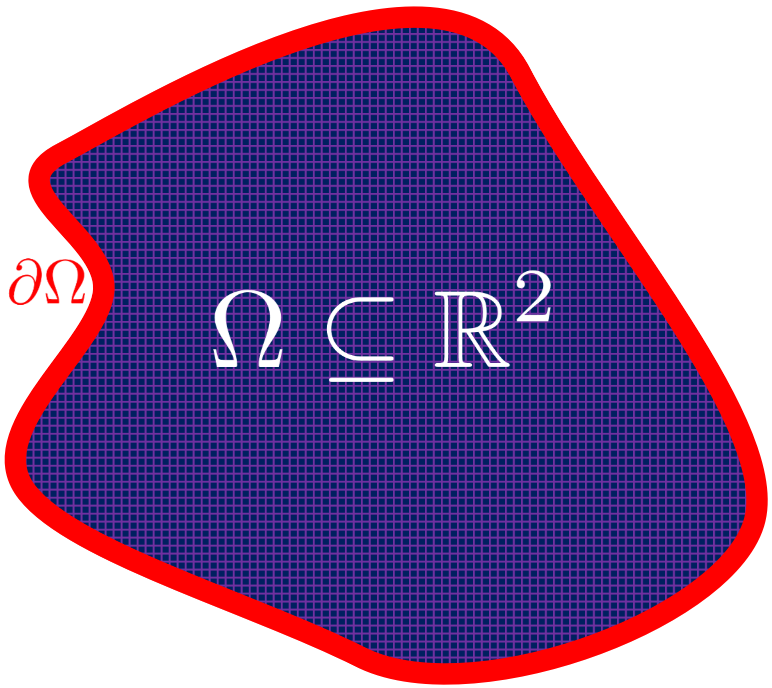
Generalizations of barycentric coordinates in two and hig
mensions have been shown to have a number of applicati
recent years, including finite element analysis, the definitio
patches (n-sided generalizations of Bézier surfaces), free-fo
formations, mesh parametrization, and interpolation. In this
we present a new form of d dimensional generalized barycent
ordinates. The new coordinates are defined as solutions to La
equation subject to carefully chosen boundary conditions. Si
lutions to Laplace's equation are called harmonic functions,
the new construction harmonic coordinates. We show that ha
coordinates possess several properties that make them more
tive than mean value coordinates when used to define two an
dimensional deformations.

terpolated across the triangle according to

Positivity, Self-Adjointness

$$\{f(\cdot) \in C^\infty(\Omega) : f|_{\partial\Omega} \equiv 0\}$$

"Dirichlet boundary conditions"



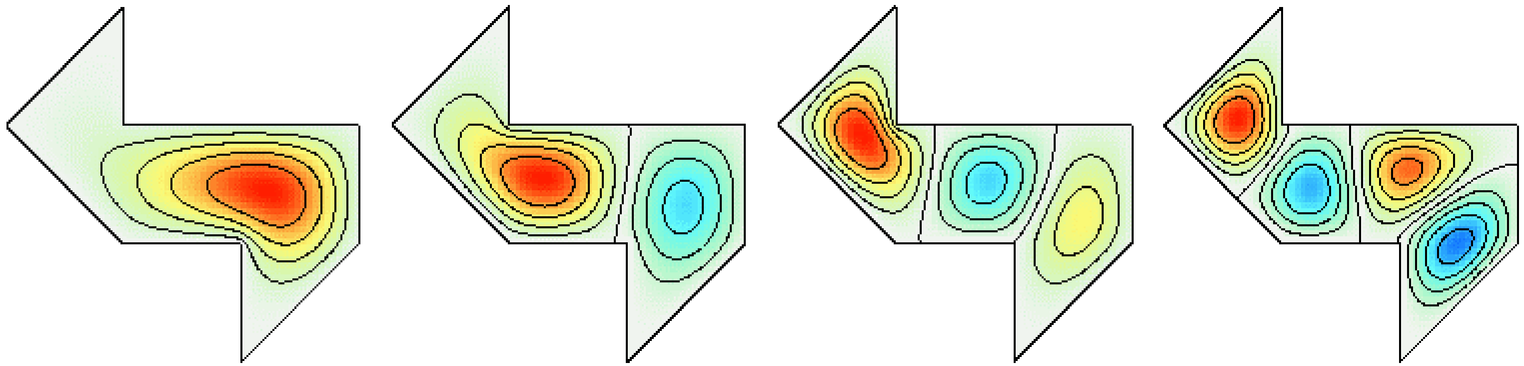
$$\mathcal{L}[f] := \Delta f$$

$$\langle f, g \rangle := \int_\Omega f(\mathbf{x}) g(\mathbf{x})\, dA(\mathbf{x})$$

On board:

1. **Positive:** $\langle f, \mathcal{L}[f] \rangle \geq 0$
2. **Self-adjoint:** $\langle f, \mathcal{L}[g] \rangle = \langle \mathcal{L}[f], g \rangle$

Laplacian Eigenfunctions



(on the board: critical points on the "unit sphere," statement of Weyl's Law)

http://www.math.udel.edu/~driscoll/research/gww1-4.gif

Small eigenvalue: Small Dirichlet Energy

Aside:

Common Misconception

$$\min_f \; E[f] \text{ s.t. } f(p) = \text{const.}$$

Point constraints are ill-advised

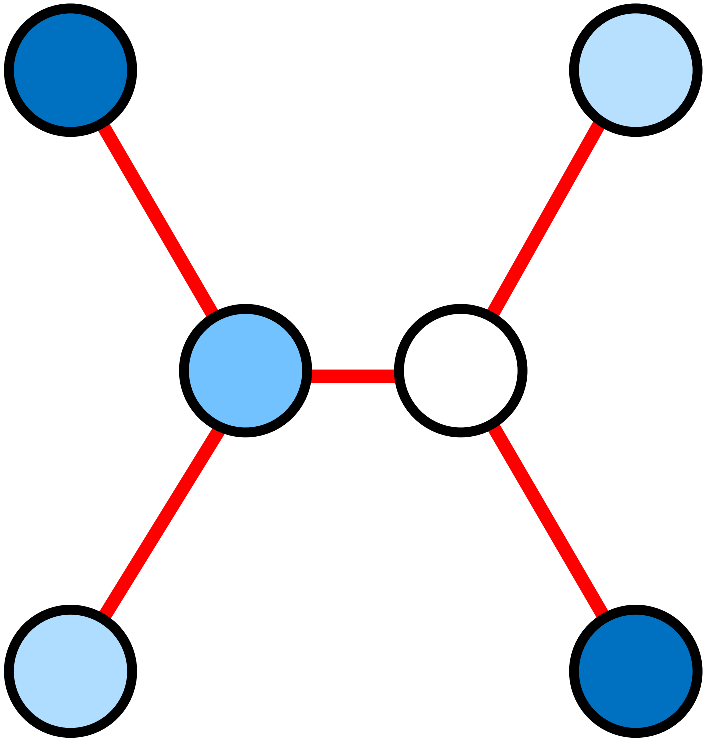
Our Progression

- Line segments
- Regions in $\mathbb{R}^n$
- Graphs
- Surfaces/manifolds

Basic Setup

- **Function:**

  One value per vertex



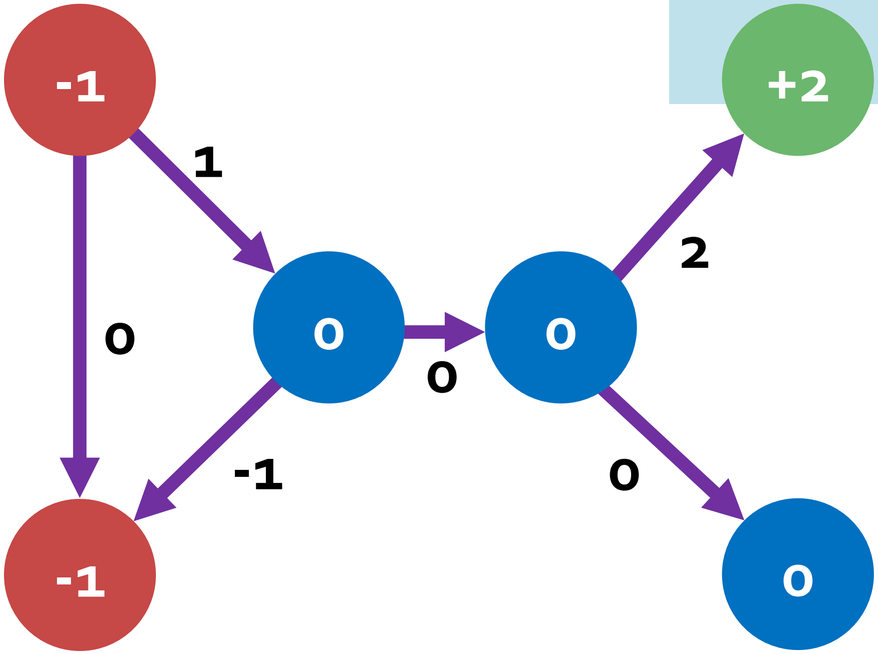
Q:

What is the **Dirichlet energy** of a function on a graph?

Differencing Operator

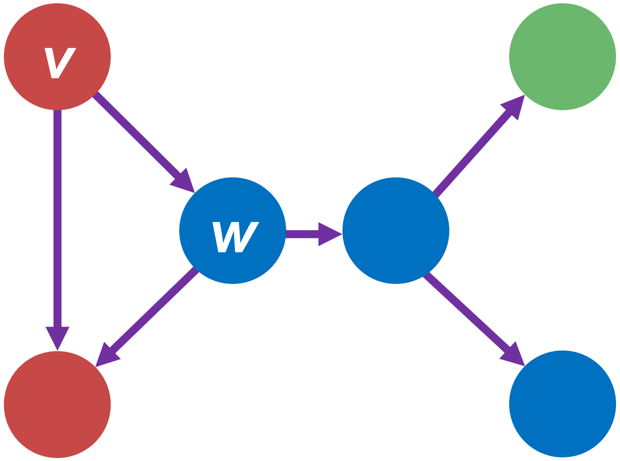
$$D_{ev} := \begin{cases} -1 & \text{if } E_{e1} = v \\ 1 & \text{if } E_{e2} = v \\ 0 & \text{otherwise} \end{cases}$$

$$D \in \{-1, 0, 1\}^{|E| \times |V|}$$



Orient edges arbitrarily

Dirichlet Energy on a Graph



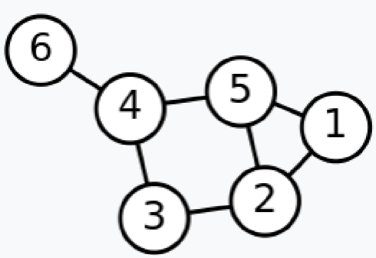
$$D_{ev} := \begin{cases} -1 & \text{if } E_{e1} = v \\ 1 & \text{if } E_{e2} = v \\ 0 & \text{otherwise} \end{cases}$$

$$E[f] := \|Df\|_2^2 = \sum_{(v,w)\in E} (f_v - f_w)^2$$

(Unweighted) Graph Laplacian

$$E[f] = \|Df\|_2^2 = f^\top (D^\top D) f := f^\top L f$$

$$L_{vw} = A - D = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

Labeled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2&0&0&0&0&0\\0&3&0&0&0&0\\0&0&2&0&0&0\\0&0&0&3&0&0\\0&0&0&0&3&0\\0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&0&1&0\\1&0&1&0&1&0\\0&1&0&1&0&0\\0&0&1&0&1&1\\1&1&0&1&0&0\\0&0&0&1&0&0 \end{pmatrix}$	$\begin{pmatrix} 2&-1&0&0&-1&0\\-1&3&-1&0&-1&0\\0&-1&2&-1&0&0\\0&0&-1&3&-1&-1\\-1&-1&0&-1&3&0\\0&0&0&-1&0&1 \end{pmatrix}$

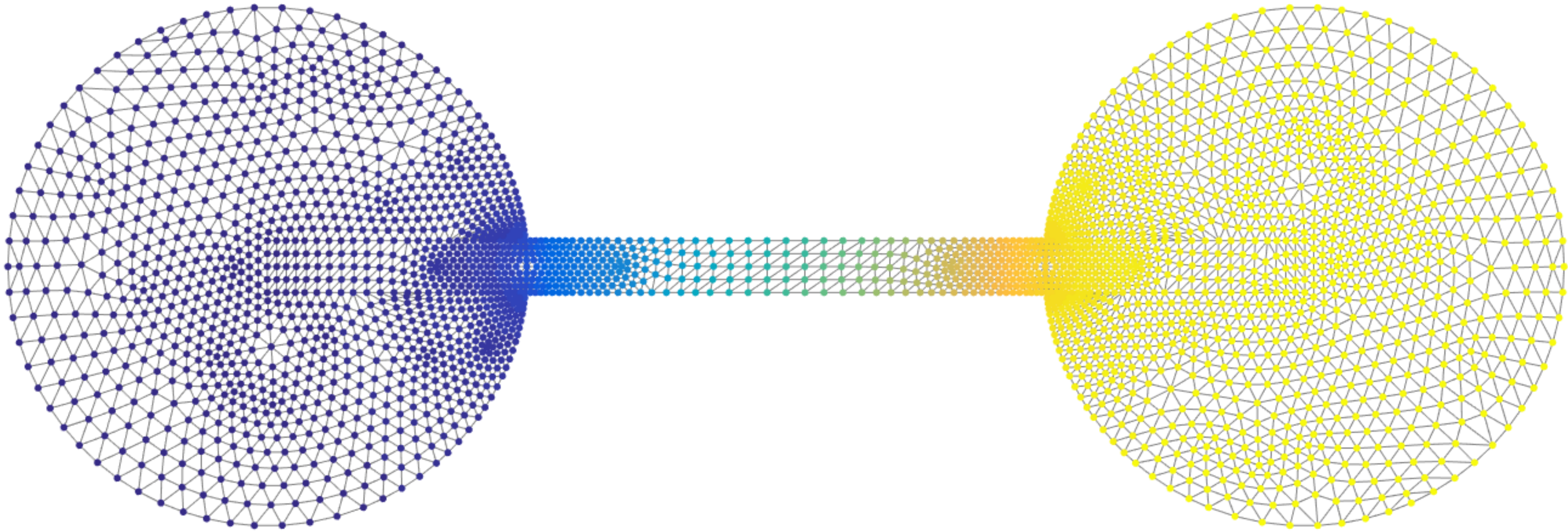
- **Symmetric**
- **Positive semidefinite**

https://en.wikipedia.org/wiki/Laplacian_matrix

Q:

What is the
smallest eigenvalue
of the graph Laplacian?

Second-Smallest Eigenvector



$$Lx = \lambda x$$

Used for graph partitioning

Fiedler vector ("algebraic connectivity")

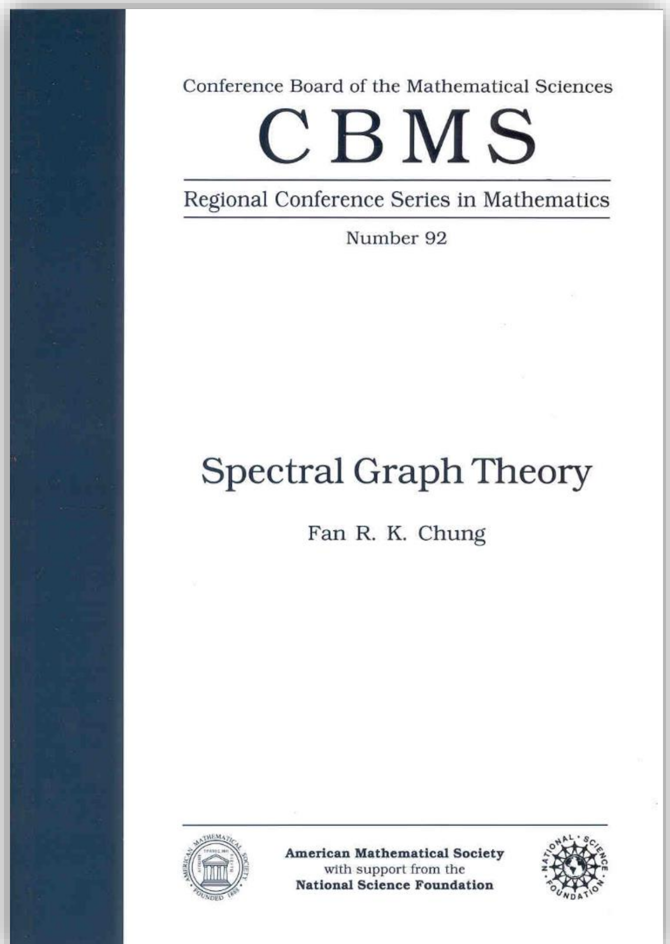
Mean Value Property

$$L_{vw} = A - D = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$$(Lx)_v = 0$$

Value at *v* is average of neighboring values

For More Information…



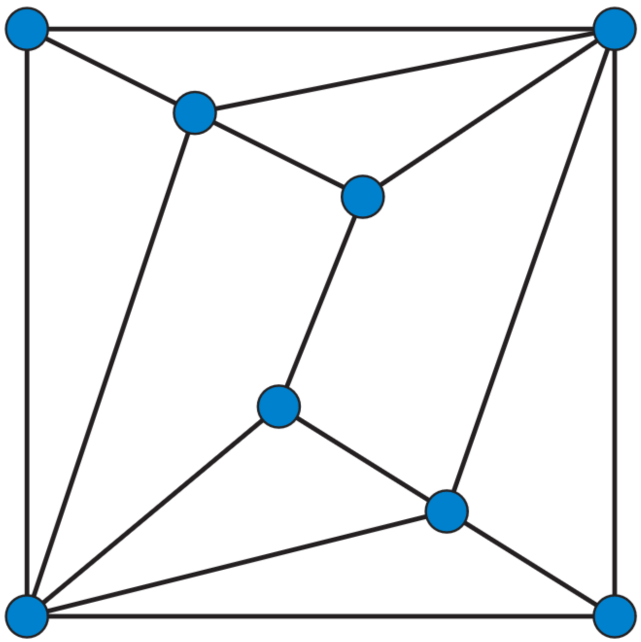
Graph Laplacian encodes lots of information!

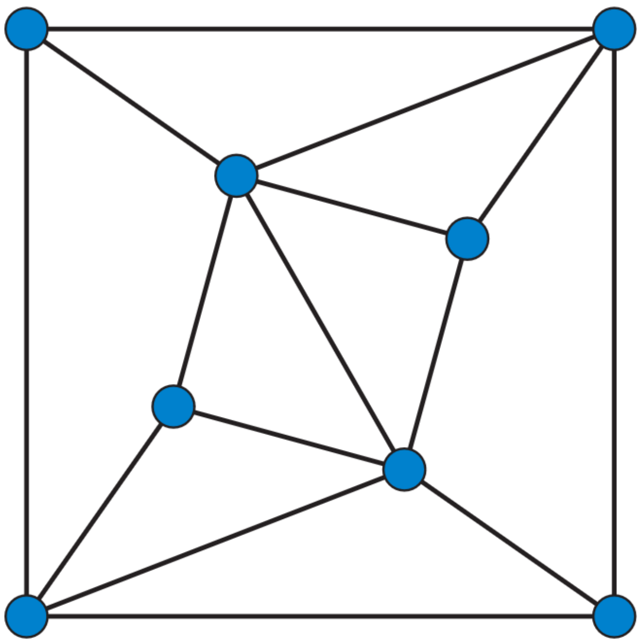
Example: Kirchoff's Theorem

Number of spanning trees equals

$$\frac{1}{n}\lambda_2\lambda_3\cdots\lambda_n$$

Hear the Shape of a Graph?





"Enneahedra"

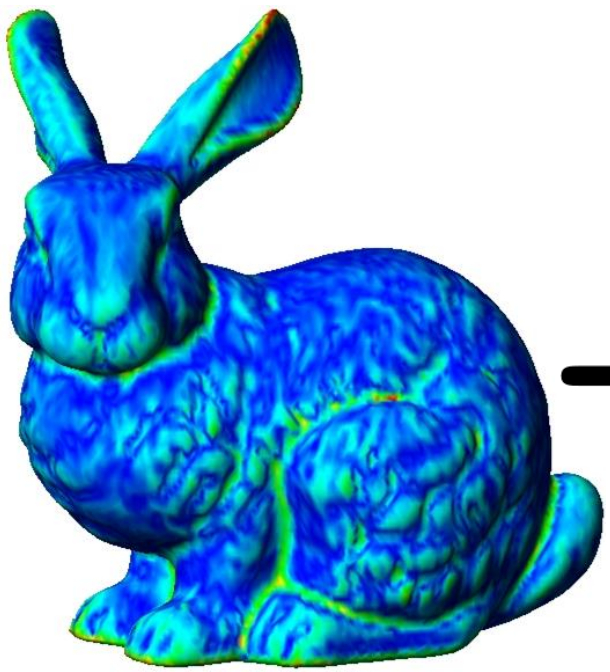
No!

Our Progression

- **Line segments**
- **Regions in $\mathbb{R}^n$**
- **Graphs**
- **Surfaces/manifolds**

Recall:

Scalar Functions

$f:$  $\longrightarrow \mathbb{R}$

http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

Map points to real numbers

Recall:

Differential of a Map

Definition (Differential). *Suppose* $\varphi : \mathcal{M} \to \mathcal{N}$ *is a map from a submanifold* $\mathcal{M} \subseteq \mathbb{R}^k$ *into a submanifold* $\mathcal{N} \subseteq \mathbb{R}^\ell$. *Then, the* differential $d\varphi_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \to T_{\varphi(\mathbf{p})}\mathcal{N}$ *of* φ *at a point* $\mathbf{p} \in \mathcal{M}$ *is given by*

$$d\varphi_{\mathbf{p}}(\mathbf{v}) := (\varphi \circ \gamma)'(0),$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ *is any curve with* $\gamma(0) = \mathbf{p}$ *and* $\gamma'(0) = \mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$.

Linear map of tangent spaces

$$d\varphi_{\mathbf{p}}(\gamma'(0)) := (\varphi \circ \gamma)'(0)$$

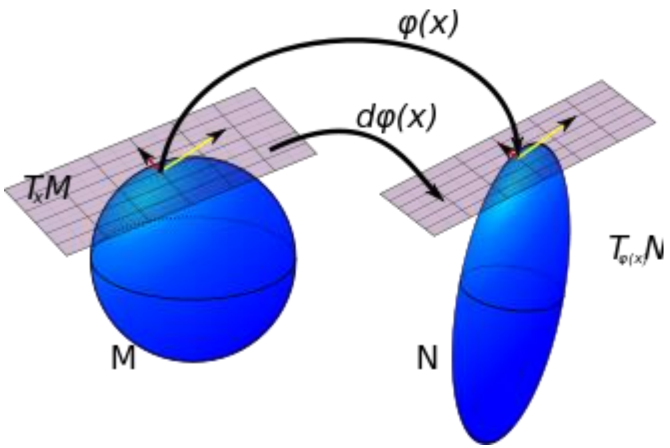
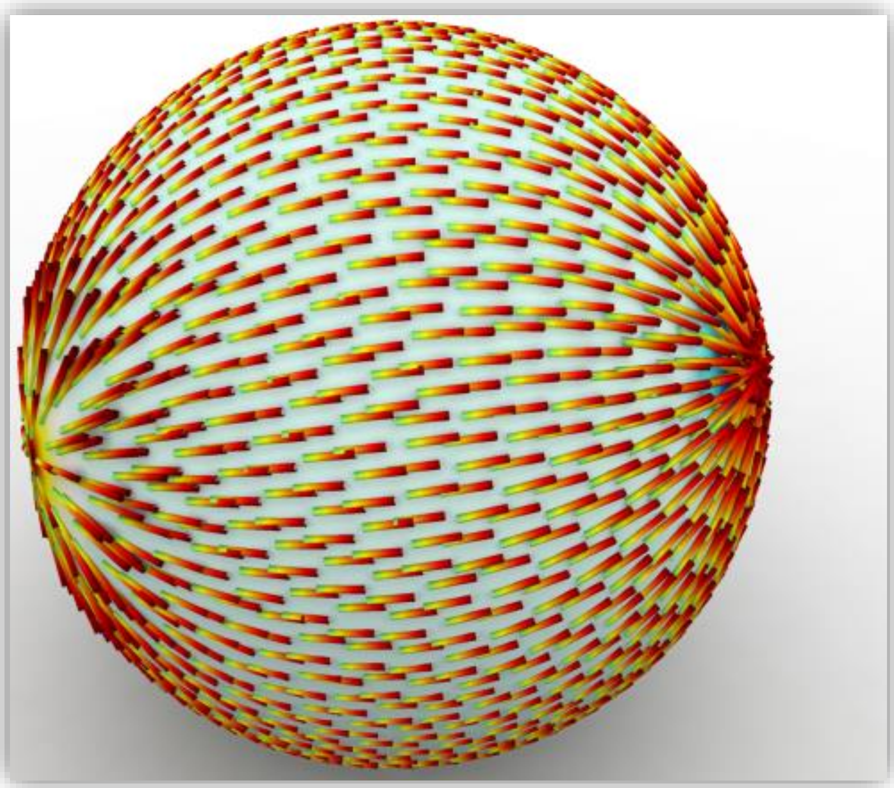


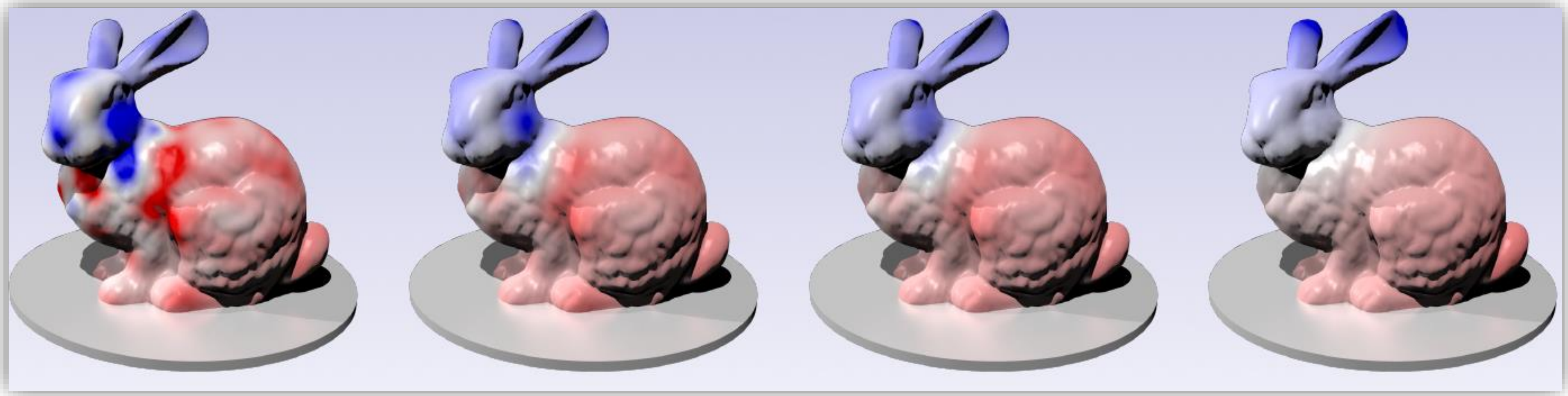
Image from Wikipedia

Gradient Vector Field

Proposition 9.2. *For each* $\mathbf{p} \in \mathcal{M}$*, there exists a unique vector* $\nabla f(\mathbf{p}) \in T_{\mathbf{p}}\mathcal{M}$ *so that* $df_{\mathbf{p}}(\mathbf{v}) = \mathbf{v} \cdot \nabla f(\mathbf{p})$ *for all* $\mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$*.*



Dirichlet Energy



Decreasing ***E***

$$E[f] := \int_S \|\nabla f\|_2^2 \, dA$$

Images made by E. Vouga

From Inner Product to Operator

$$\langle f, g \rangle_\Delta := \int_S \nabla f(x) \cdot \nabla g(x)\, dA$$
$$:= \langle f, \Delta g \rangle$$

Implies
$\langle f, f \rangle \geq 0$

On the board:
"Motivation" from finite-dimensional linear algebra.

Laplace-Beltrami operator

What is Divergence?

$$\mathbf{v} : \mathcal{M} \to \mathbb{R}^3 \text{ where } \mathbf{v}(\mathbf{p}) \in T_{\mathbf{p}}\mathcal{M}$$
$$d\mathbf{v}_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \to \mathbb{R}^3$$
$$\{\mathbf{e}_1, \mathbf{e}_2\} \subset T_{\mathbf{p}}\mathcal{M} \text{ orthonormal basis}$$

$$(\nabla \cdot \mathbf{v})_{\mathbf{p}} := \sum_{i=1}^{2} \langle \mathbf{e}_i, d\mathbf{v}(\mathbf{e}_i) \rangle_{\mathbf{p}}$$

Things we should check (but probably won't):

- **Independent of choice of basis**
- $\boldsymbol{\Delta} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$

Flux Density: Backward Definition

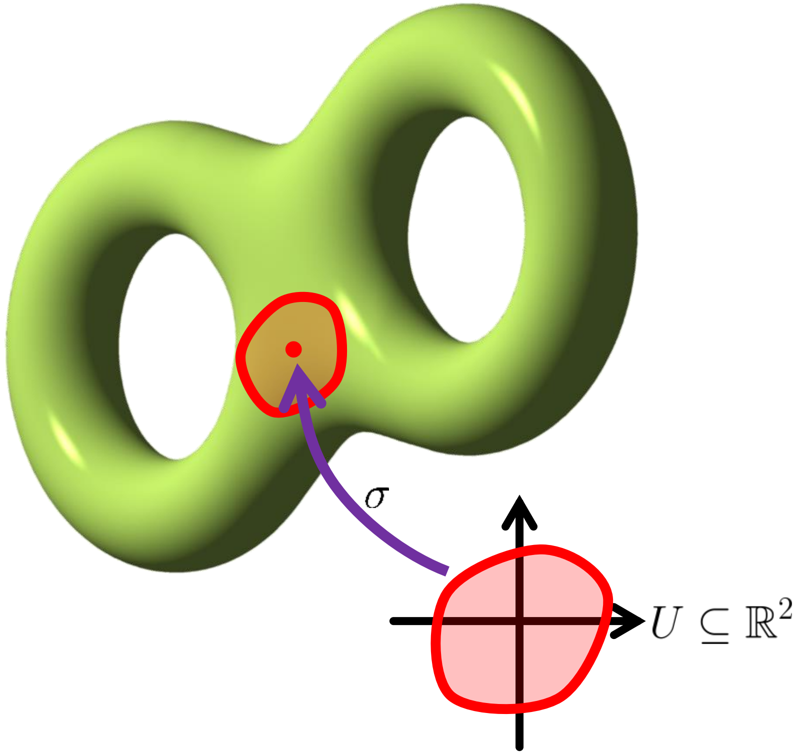
$$\nabla \cdot \mathbf{v}(\mathbf{p}) := \lim_{r \to 0} \frac{\oint_{\partial B_r(\mathbf{p})} \mathbf{v} \cdot \mathbf{n}_{\text{tangent}} \, d\ell}{\text{vol}(B_r(\mathbf{p}))}$$

On board: Draw schematic
Challenge: Short derivation?

Sanity Check: Local Version

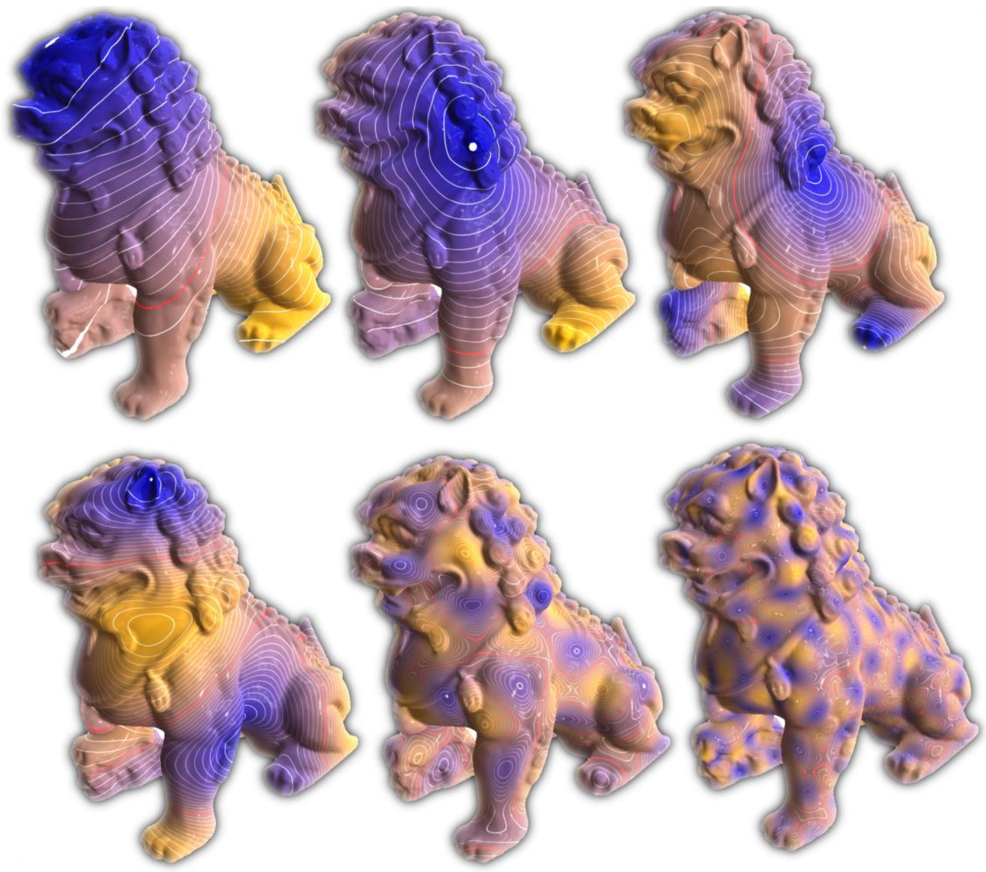
$$f : \mathcal{M} \to \mathbb{R}$$

$$\text{Pullback: } \sigma^* f := f \circ \sigma : U \to \mathbb{R}$$



Laplace-Beltrami **coincides with Laplacian** on $\mathbb{R}^2$ when σ takes x, y axes to orthonormal vectors.

Eigenfunctions



$$\Delta\psi_i = \lambda_i\psi_i$$

Vibration modes of surface (not volume!)

http://alice.loria.fr/publications/papers/2008/ManifoldHarmonics//photo/dragon_mhb.png

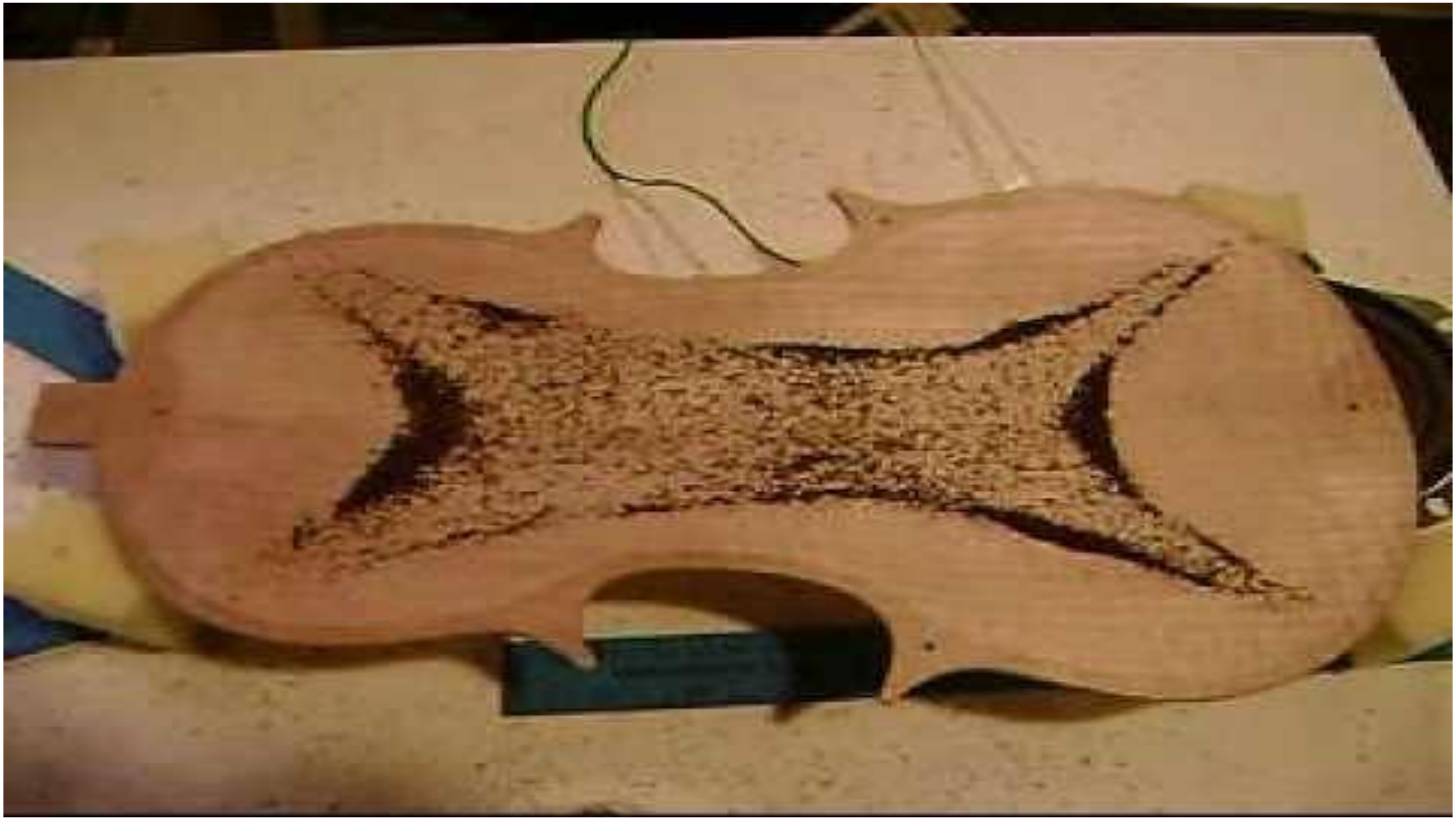
Chladni Plates

https://www.youtube.com/watch?v=CGiiSlMFFlI

Performance Art?

https://www.youtube.com/watch?v=Fyzqd2_T09Q

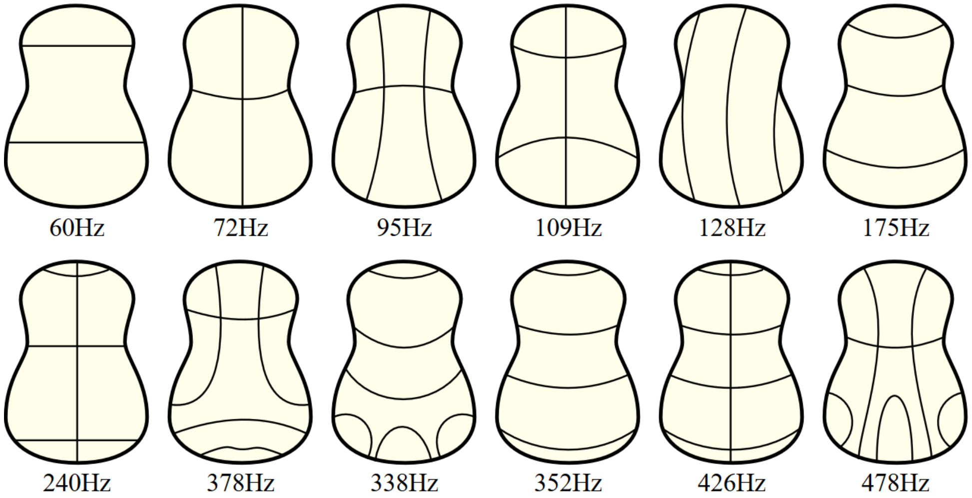
Practical Application



https://www.youtube.com/watch?v=3uMZzVvnSiU

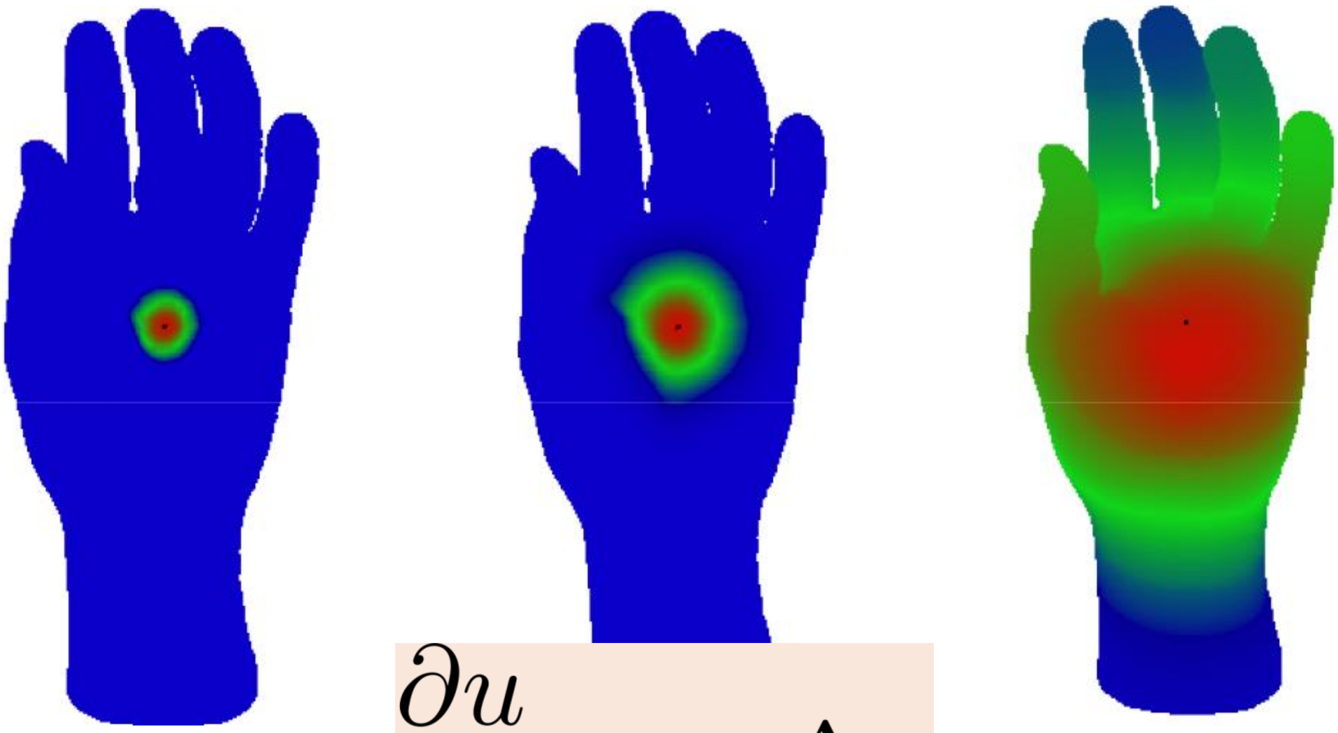
Nodal Domains

Theorem (Courant). **The *n*-th eigenfunction of the Dirichlet boundary value problem has at most *n* nodal domains.**



https://en.wikipedia.org/wiki/Ernst_Chladni

Additional Connection to Physics

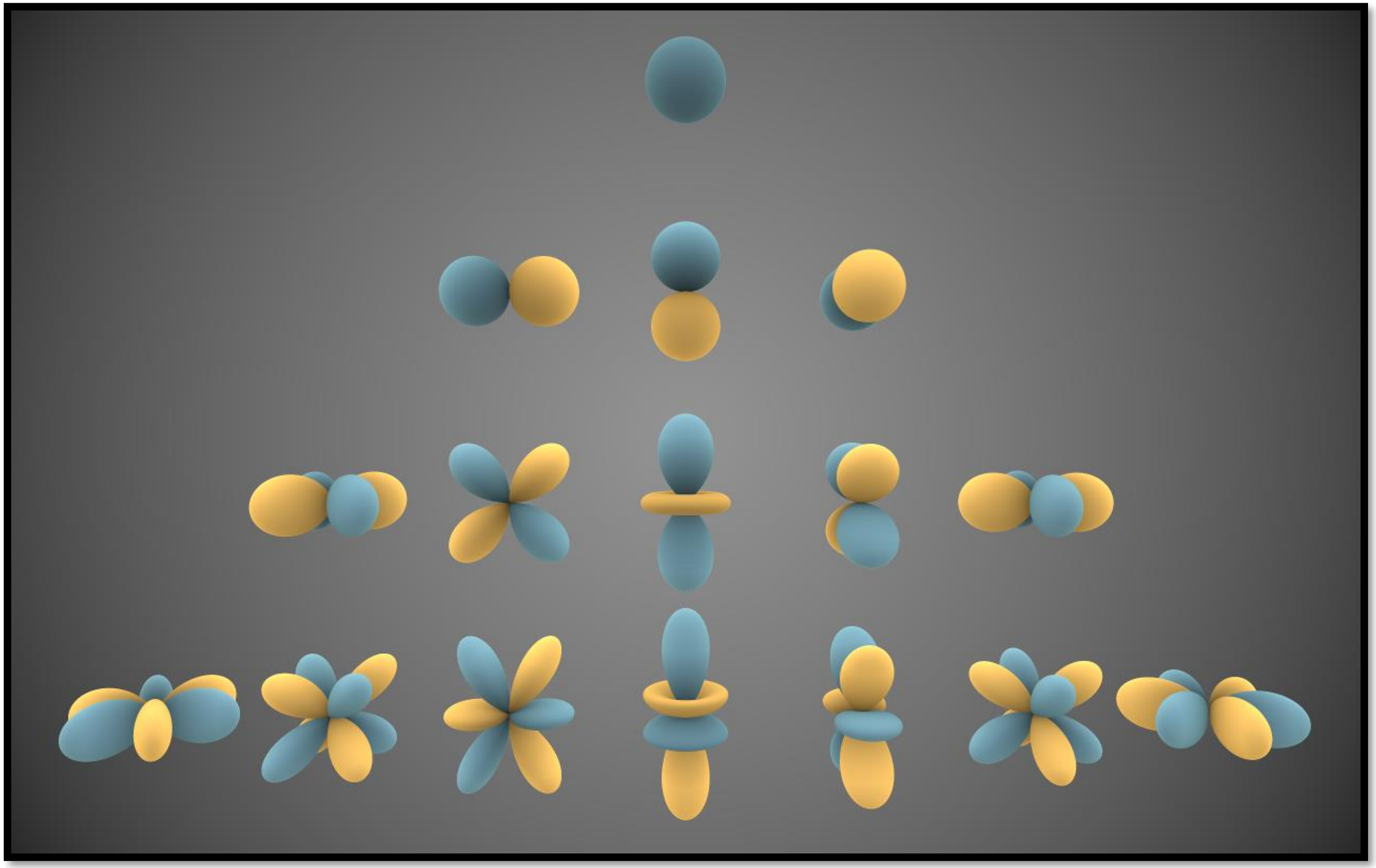


$$\frac{\partial u}{\partial t} = -\Delta u$$

http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

Spherical Harmonics



https://en.wikipedia.org/wiki/Spherical_harmonics

Weyl's Law

$$N(\lambda) := \#\ \text{eigenfunctions} \leq \lambda$$

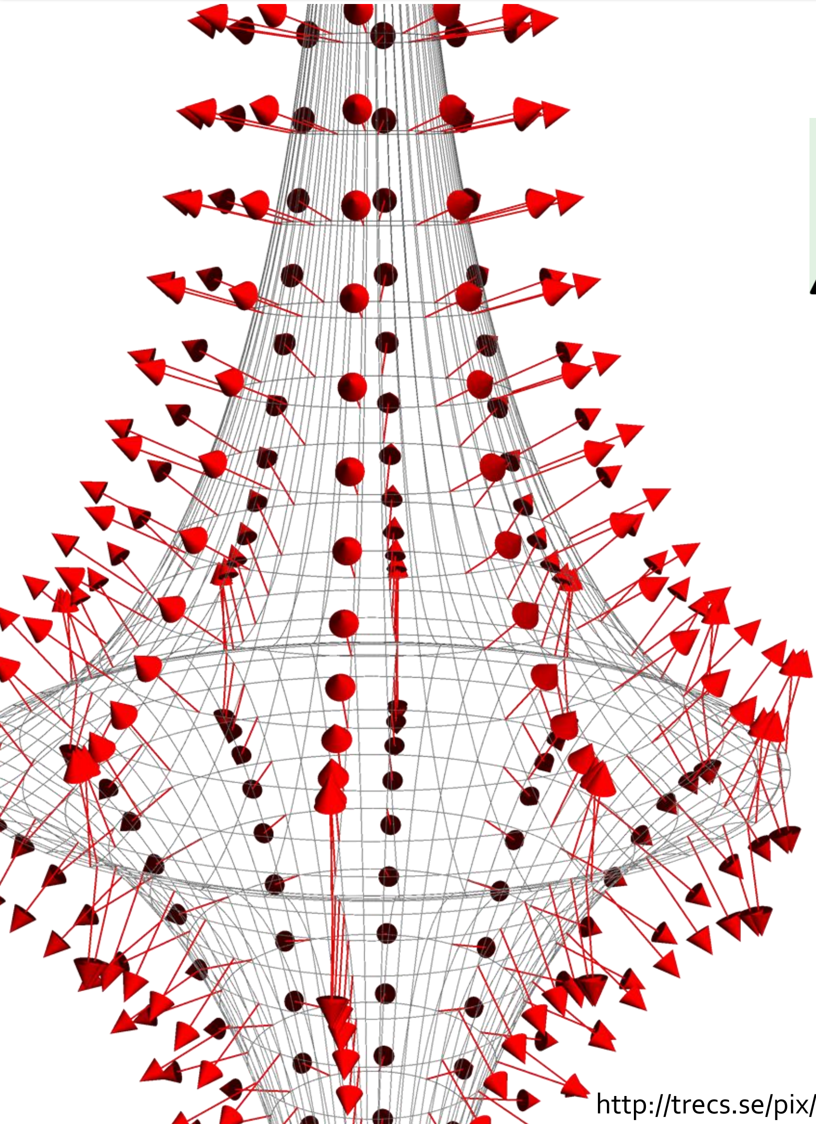
$$\omega_d := \text{volume of unit ball in } \mathbb{R}^d$$

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d/2}} = (2\pi)^{-d}\omega_d \text{vol}(\Omega)$$

Corollary: $\text{vol}(\Omega) = (2\pi)^d \lim_{R\to\infty} \frac{N(R)}{R^{d/2}}$

For surfaces: $\lambda_n \sim \dfrac{4\pi}{\text{vol}(\Omega)} n$

Laplacian of *xyz* function



$$\Delta \mathbf{x} = H\mathbf{n}$$

Intuition:

Laplacian measures difference with neighbors.

http://trecs.se/pix/pseudosphereNormalVectorField4096.png

Introducing the Laplacian Operator

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