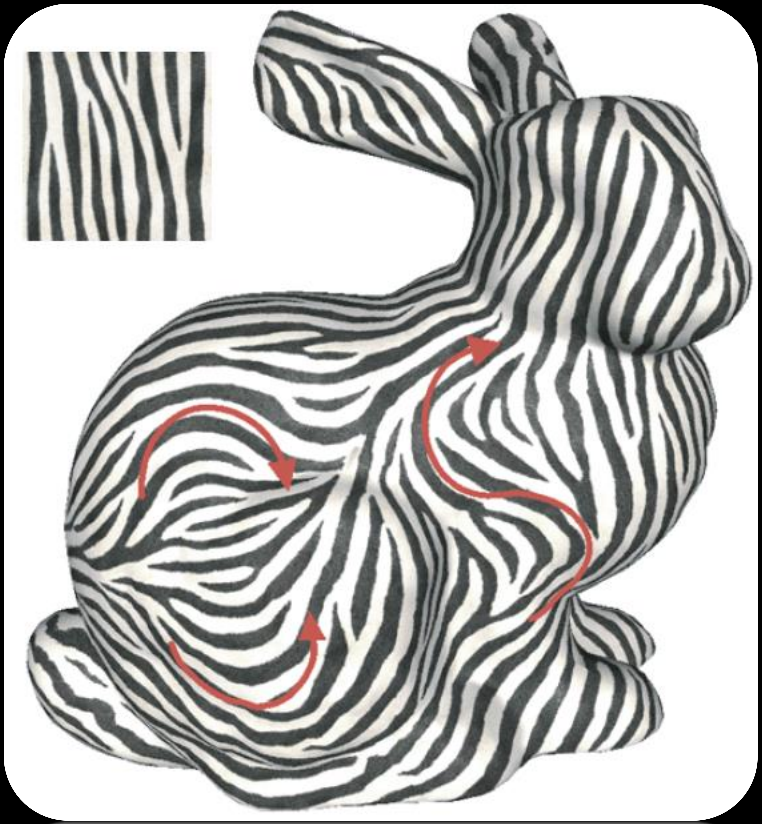
Fisher et al. "Design of Tangent Vector Fields" (SIGGRAPH 2007)



Discrete Exterior Calculus

Justin Solomon

MIT, Spring 2019

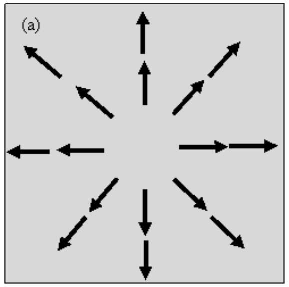
Original version from Stanford CS 468, spring 2013 (Butscher & Solomon)

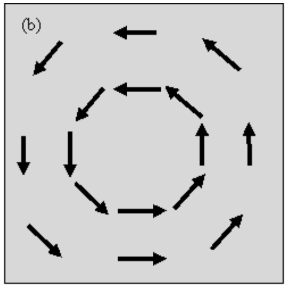
Vector Calculus

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} := \sum_i \frac{\partial v_i}{\partial x_i}$$

$$\operatorname{curl} \vec{v} = \nabla \times \vec{v} := \cdots$$

$$\Delta f = \nabla \cdot \nabla f := \sum_i \frac{\partial^2 f}{\partial x_i^2}$$





Famous Theorems (in $\mathbb{R}^2$)

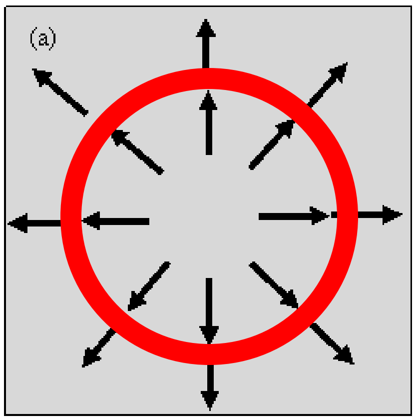
$$\int_\Omega \operatorname{div} \vec{v}\, dA = \int_{\partial\Omega} \vec{v} \cdot \vec{n}\, d\ell$$

"Divergence Theorem"

$$\int_\Omega \operatorname{curl} \vec{v}\, dA = \int_{\partial\Omega} \vec{v} \cdot \vec{t}\, d\ell$$

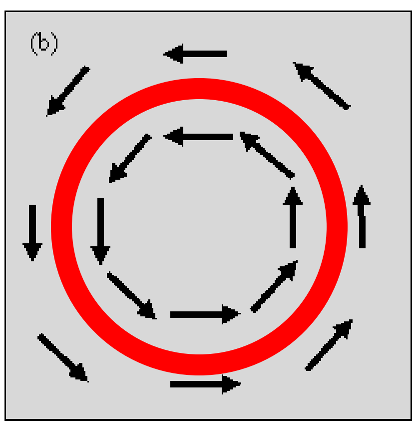
"Green's Theorem"

Famous Theorems (in $\mathbb{R}^2$)



$$\int_\Omega \operatorname{div} \vec{v}\, dA = \int_{\partial\Omega} \vec{v}\cdot\vec{n}\, d\ell$$

"Divergence Theorem"



$$\int_\Omega \operatorname{curl} \vec{v}\, dA = \int_{\partial\Omega} \vec{v}\cdot\vec{t}\, d\ell$$

"Green's Theorem"

Even Simpler Example…

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Fundamental Theorem of Calculus

Pattern?

$$\int_{\text{region}} [\text{derivative}]\, dV = \int_{\text{boundary}} [\text{quantity}]\, dA$$

One equation, all of calculus

Exterior Calculus

Extension of vector calculus
to surfaces (and manifolds).

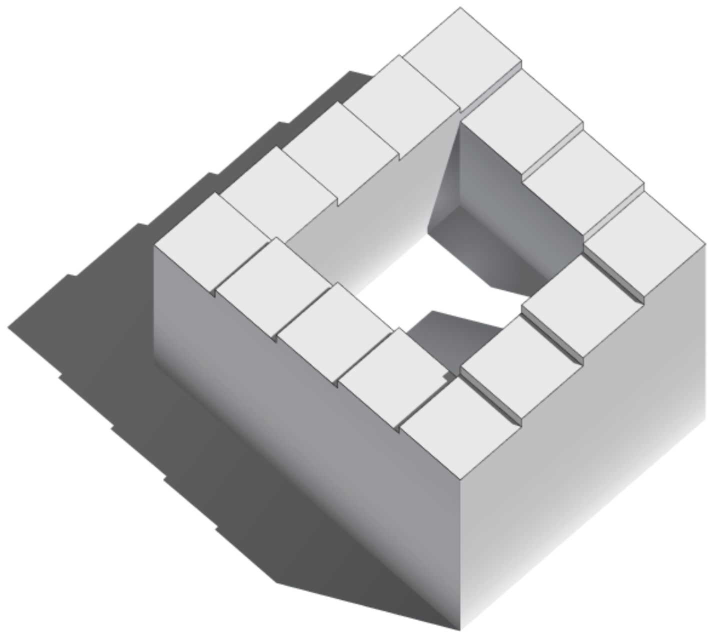
Rough Outline

1. **Exterior calculus**
   Alternating k-forms, derivatives, and integration

2. **Discrete exterior calculus**
   All that, on a simplicial complex

Many Illustrations Borrowed From…

DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION



Keenan Crane

Last updated: November 19, 2015

https://www.cs.cmu.edu/~kmcrane/Projects/DGPDEC/

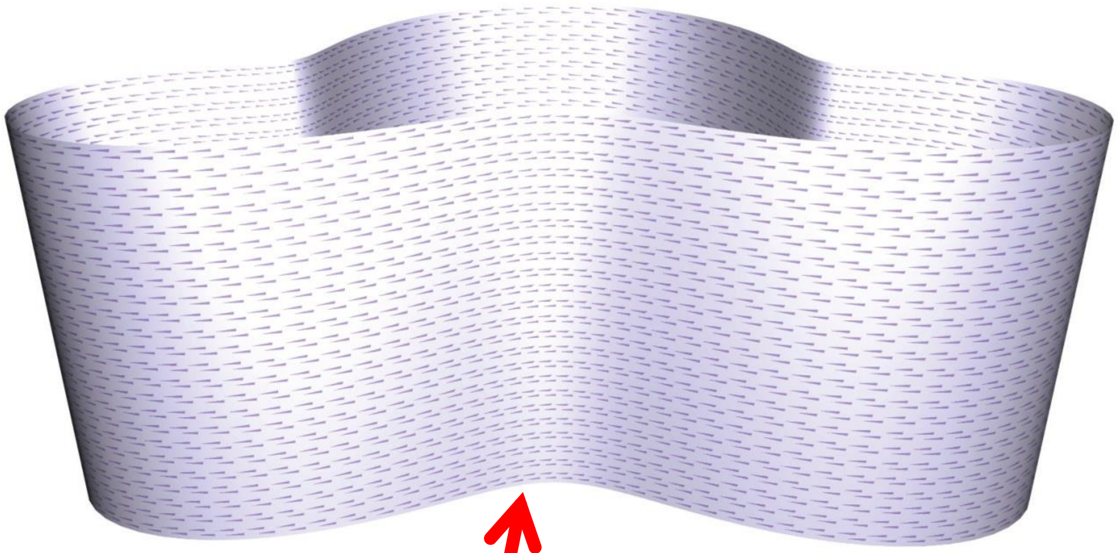
Our goal: Semester course in 2.5 lectures…

Rough Outline

1. **Exterior calculus**
   Alternating *k*-forms, derivatives, and integration

2. **Discrete exterior calculus**
   All that, on a simplicial complex

New Rule



Everything must be **intrinsic**!

Vector fields are **tangent**!

Dual of a Vector Space *V*

$$\mathcal{V}^* := \{\text{linear maps } \xi : \mathcal{V} \to \mathbb{R}\}$$

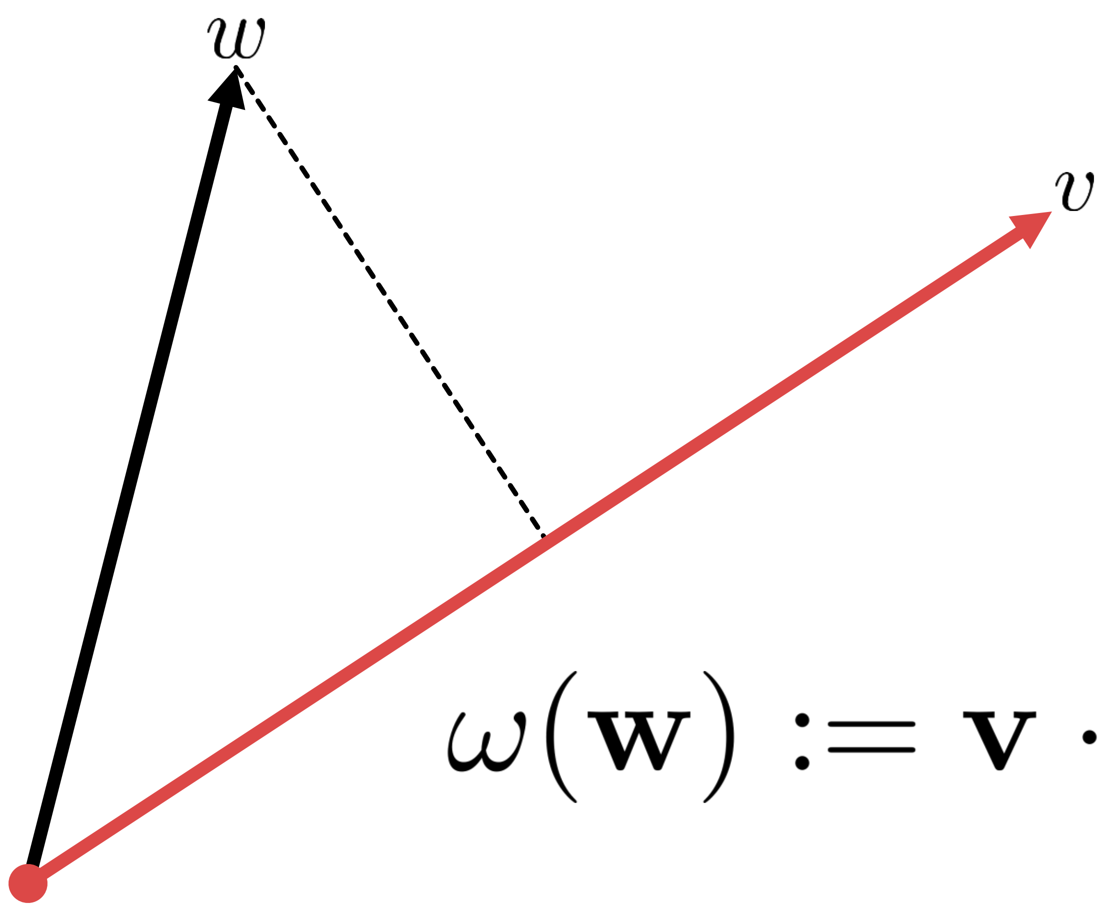
Property:

***V*, *V** have same dimension.**

$$\{\mathbf{e}_i\} \text{ basis for } \mathcal{V} \implies \{dx^i\} \text{ basis for } \mathcal{V}^*$$

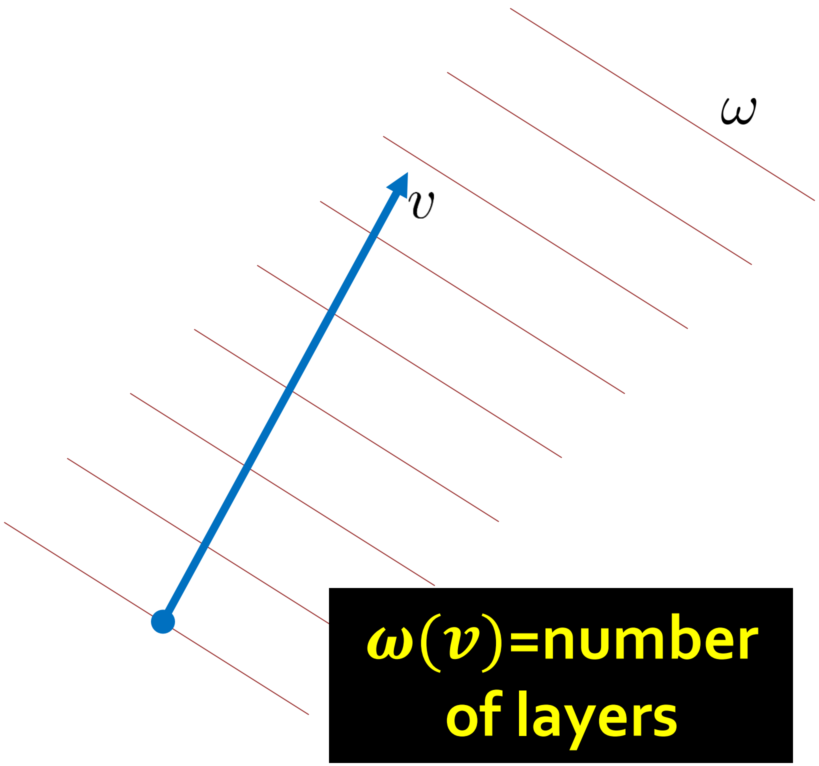
$$dx^i(\mathbf{e}_j) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

One-Form: Dual of a Vector



$$\omega(\mathbf{w}) := \mathbf{v} \cdot \mathbf{w}$$

Intuition



https://www.aliexpress.com/price/needle-shredder_price.html

Needle in a 1-form onion

More Intuition

$$\mathbf{v} \cdot \mathbf{w} = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

Row vs. column vectors

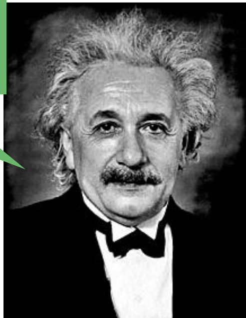
Some (Common) Terrible Notation

$$g_{ij} := \langle \mathbf{e}_i, \mathbf{e}_j \rangle, g^{ij} := (g^{-1})_{ij}$$

$$\mathbf{v} := \sum_i v^i \mathbf{e}_i \qquad \mathbf{w} := \sum_j w^j \mathbf{e}_j$$

$$\begin{aligned} \langle \mathbf{v}, \mathbf{w} \rangle &= \sum_{ij} v^i w^j \langle \mathbf{e}_i, \mathbf{e}_j \rangle \\ &:= \sum_{ij} v^i w^j g_{ij} \\ &= v^i w^j g_{ij} \end{aligned}$$

Sum over repeated indices!



Musical Isomorphisms: Flat

$$\mathbf{v}^\flat(\mathbf{w}) := \langle \mathbf{v}, \mathbf{w} \rangle$$

$$\mathbf{v} = \sum v^i \mathbf{e}_i$$

$$\mathbf{v}^\flat = \sum \omega_i dx^i$$

$$\omega_i = \sum_j g_{ij} v^j$$

Bloch, Schelomo

Vector to covector (lowers index)

Musical Isomorphisms: Sharp

$$\xi(\mathbf{w}) := \langle \xi^\sharp, \mathbf{w} \rangle$$

$$\omega = \sum \omega_i dx^i$$

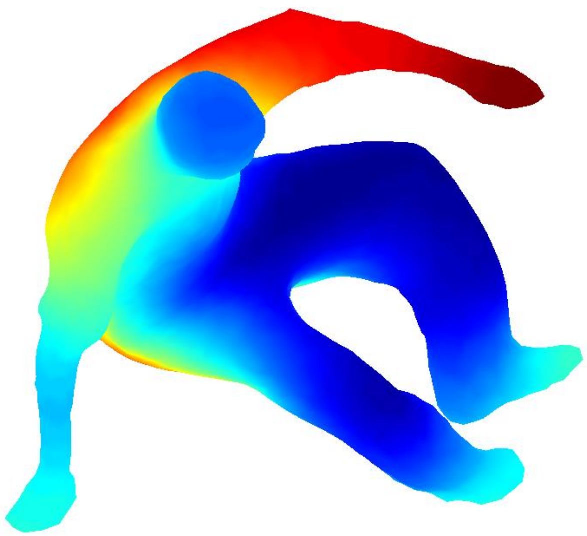
$$\omega^\sharp = \sum v^i \mathbf{e}_i$$

$$v^i = \sum_j g^{ij} \omega_j$$

Elgar, Cello Concerto

Covector to vector (raises index)

Forms on Surfaces



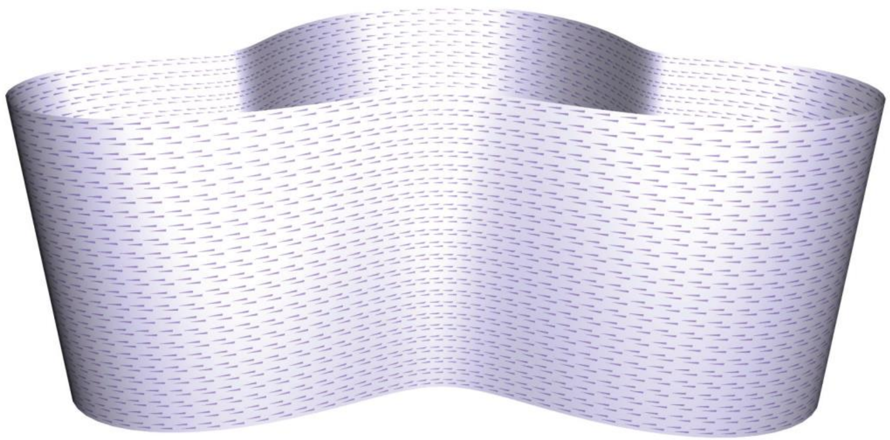
$$f : \Sigma \to \mathbb{R}$$

$\downarrow$

0-form

http://perception.inrialpes.fr/Publications/2011/ZBH11a/scale-9.png

Differential One-Forms



Vector field

$\vec{v} : \Sigma \to T\Sigma$

$\vec{v}^\flat \downarrow \uparrow \omega^\sharp$

1-form

$\omega(\vec{x}) = \vec{v} \cdot \vec{x}$

Evaluating One-Forms

$$\omega(\vec{v}) = \sum_i \omega^i v_i$$

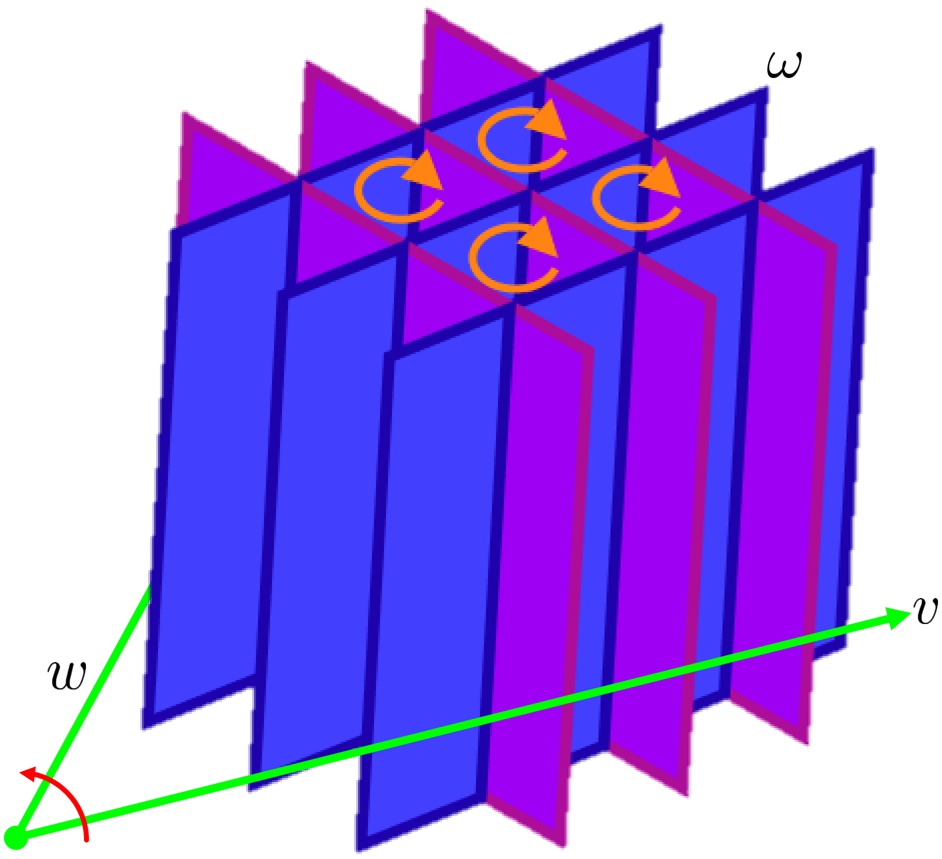
Motivate on the board!

No metric matrix *g*

Q:

What is a two-form?

Continuing Onion Analogy

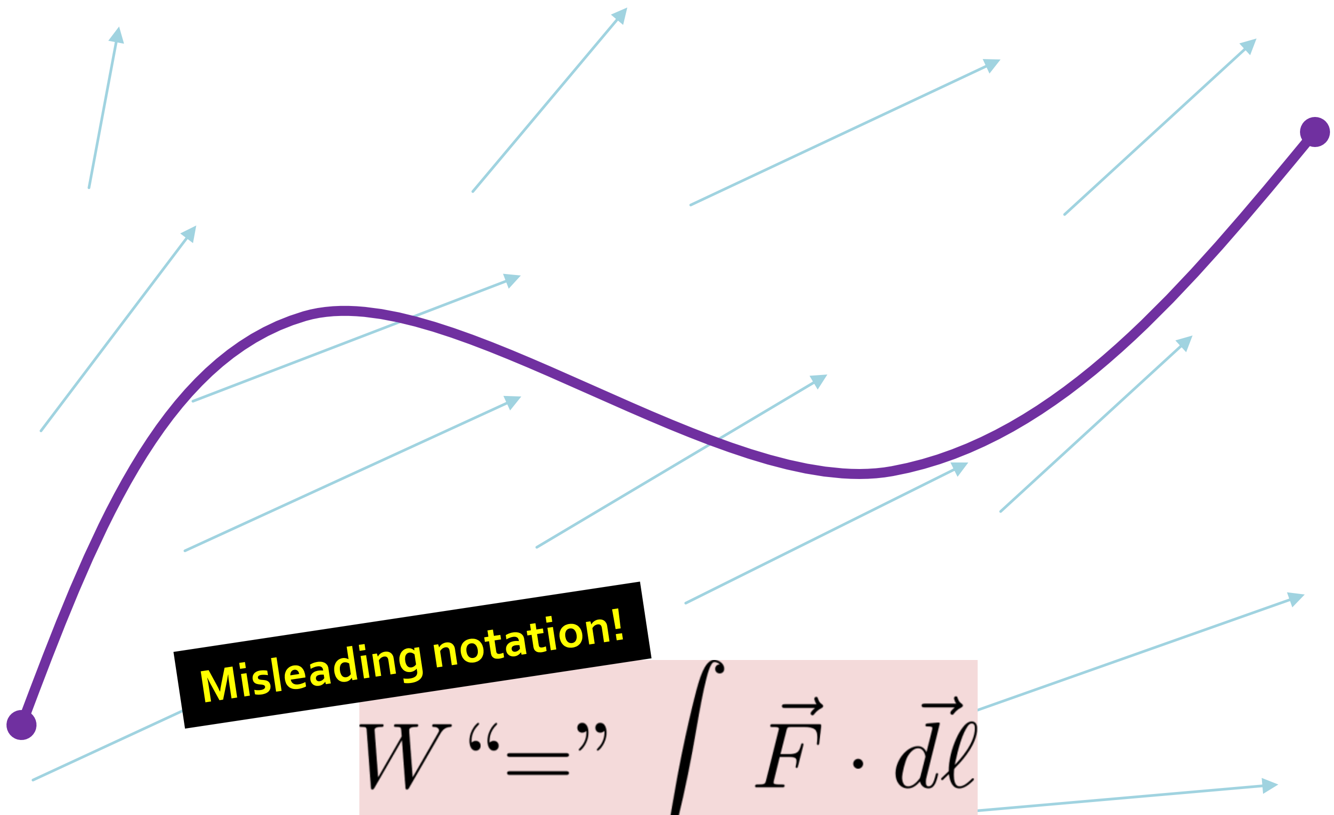


$\omega(v, w)$= number of oriented cells

Grid image from Wikipedia

Interlude: Line integral

Explanation from: http://www.math.ucla.edu/~tao/preprints/forms.pdf

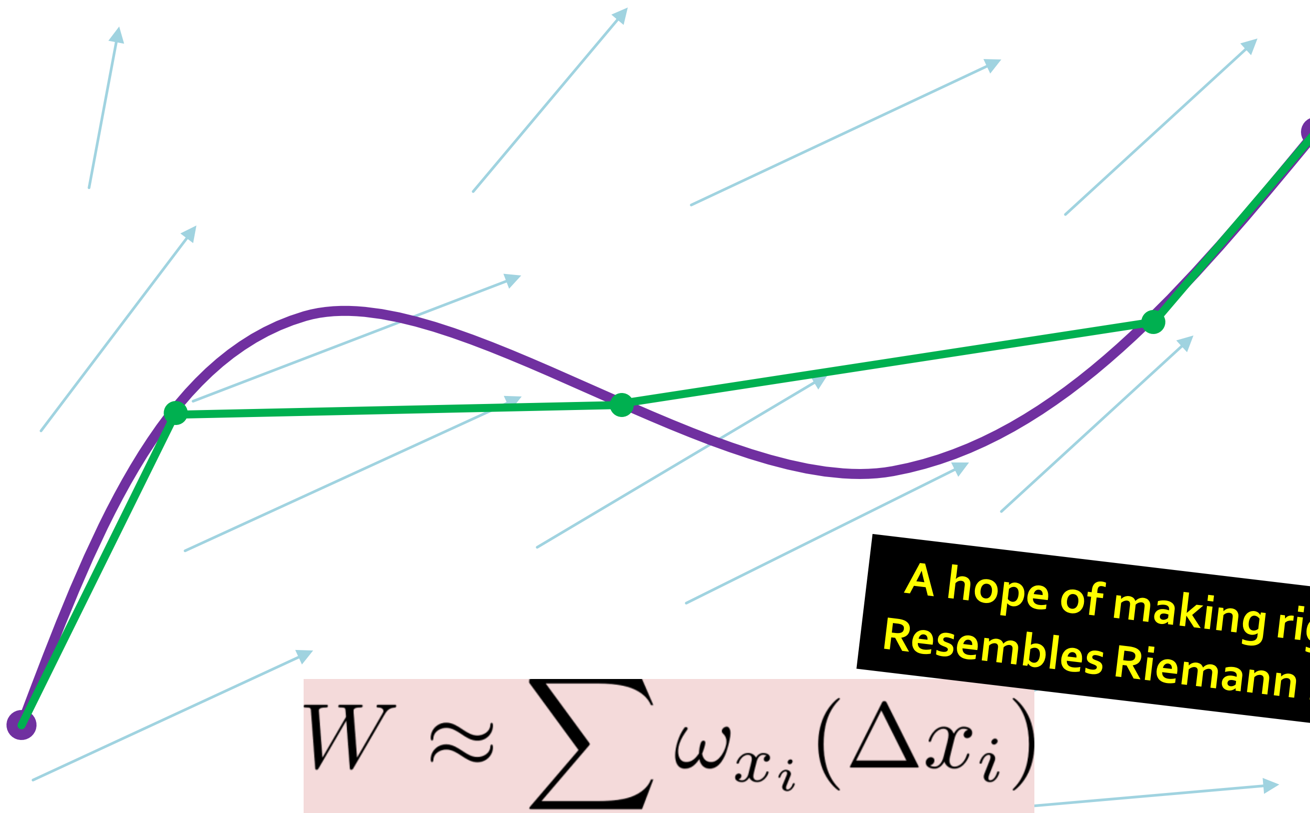


Misleading notation!

$$W \text{``}{=}\text{''} \int_\gamma \vec{F} \cdot \vec{d\ell}$$

Work = force * distance

Interlude: Line integral

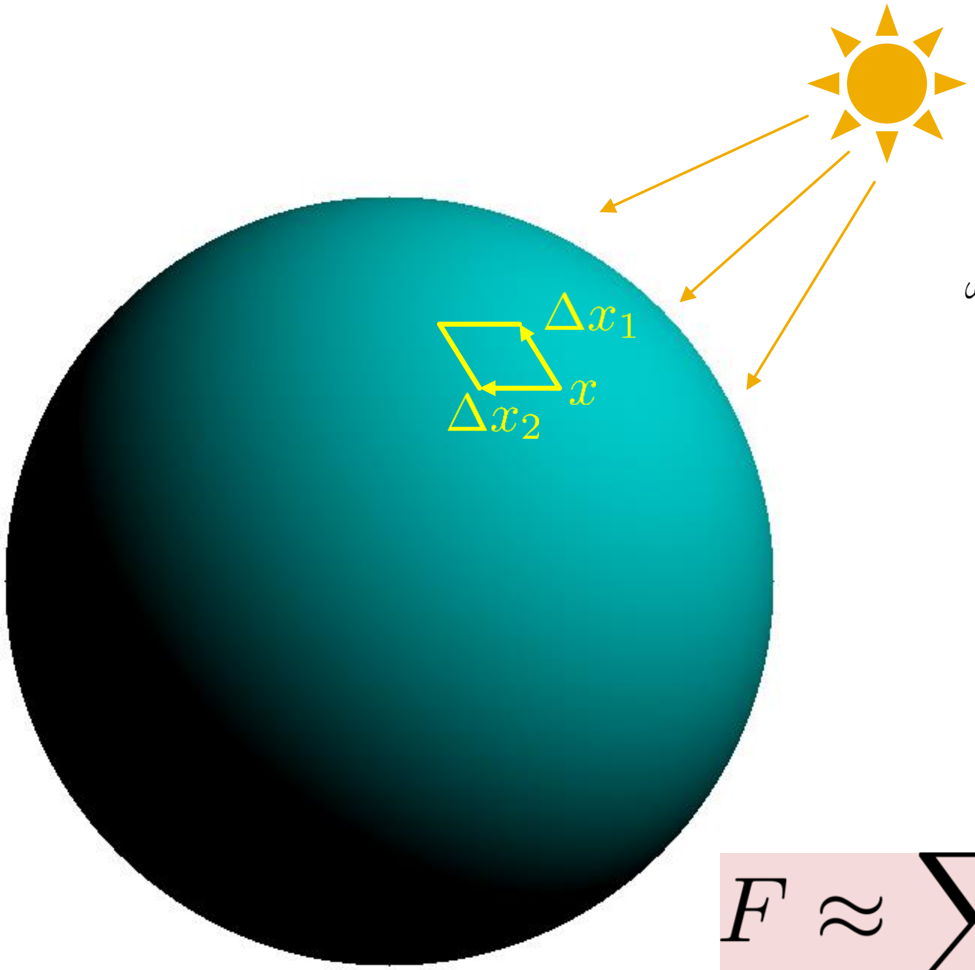


$$W \approx \sum_i \omega_{x_i}(\Delta x_i)$$

A hope of making rigorous:
Resembles Riemann integral

Work = force * distance

Incident Light Flux



Bilinear (same as 1D):

$$\omega_x(c\Delta x_1, \Delta x_2) = c\omega_x(\Delta x_1, \Delta x_2)$$
$$\omega_x((\Delta x_1 + \Delta x_1'), \Delta x_2) = \omega_x(\Delta x_1, \Delta x_2) + \omega_x(\Delta x_1', \Delta x_2)$$
$$\omega_x(\Delta x_1, c\Delta x_2) = c\omega_x(\Delta x_1, \Delta x_2)$$
$$\omega_x(\Delta x_1, \Delta x_2 + \Delta x_2') = \omega_x(\Delta x_1, \Delta x_2) + \omega_x(\Delta x_1, \Delta x_2')$$

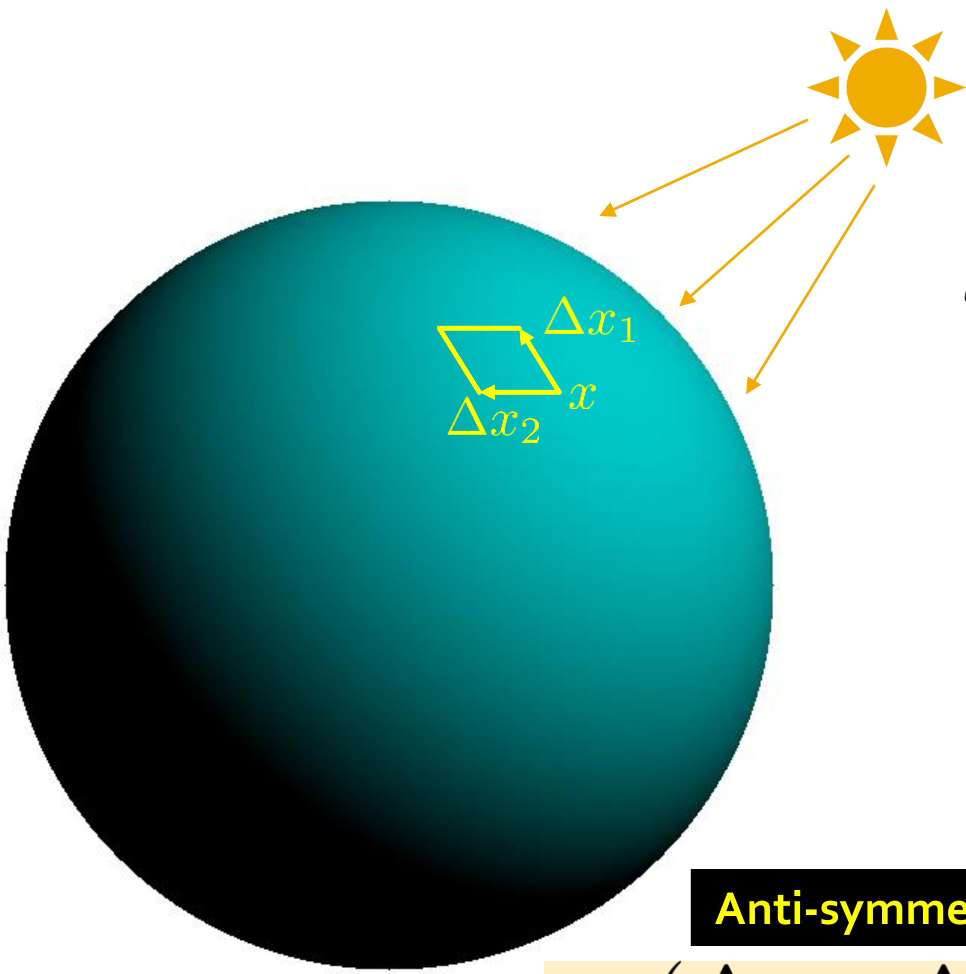
Flux through degenerate window:

$$\omega_x(\Delta x, \Delta x) = 0$$

Notice: Signed!

$$F \approx \sum_i \omega_{x_i}(\Delta x_{i1}, \Delta x_{i2})$$

(Initially) Surprising Corollary



Bilinear (same as 1D):

$$\omega_x(c\Delta x_1, \Delta x_2) = c\omega_x(\Delta x_1, \Delta x_2)$$
$$\omega_x((\Delta x_1 + \Delta x_1'), \Delta x_2) = \omega_x(\Delta x_1, \Delta x_2) + \omega_x(\Delta x_1', \Delta x_2)$$
$$\omega_x(\Delta x_1, c\Delta x_2) = c\omega_x(\Delta x_1, \Delta x_2)$$
$$\omega_x(\Delta x_1, \Delta x_2 + \Delta x_2') = \omega_x(\Delta x_1, \Delta x_2) + \omega_x(\Delta x_1, \Delta x_2')$$

Flux through degenerate window:

$$\omega_x(\Delta x, \Delta x) = 0$$

Anti-symmetric (follows from properties above):

$$\omega_x(\Delta x_1, \Delta x_2) = -\omega_x(\Delta x_2, \Delta x_1)$$

Defining Two-Forms

Bilinear:

$$\omega_x(c\Delta x_1, \Delta x_2) = c\omega_x(\Delta x_1, \Delta x_2)$$
$$\omega_x((\Delta x_1 + \Delta x_1'), \Delta x_2) = \omega_x(\Delta x_1, \Delta x_2) + \omega_x(\Delta x_1', \Delta x_2)$$
$$\omega_x(\Delta x_1, c\Delta x_2) = c\omega_x(\Delta x_1, \Delta x_2)$$
$$\omega_x(\Delta x_1, \Delta x_2 + \Delta x_2') = \omega_x(\Delta x_1, \Delta x_2) + \omega_x(\Delta x_1, \Delta x_2')$$

Flux through degenerate window:

$$\omega_x(\Delta x, \Delta x) = 0$$

Alternative equivalent definition:
$\omega_x(\Delta x_1, \Delta x_2) = -\omega_x(\Delta x_2, \Delta x_1)$(alternating)

***k*-form: Same thing, *k* slots!**

More Concrete 2-Forms on $\mathbb{R}^n$

$$\omega \in \Lambda^2 \implies \omega(\mathbf{v},\mathbf{w}) = \mathbf{v}^\top M \mathbf{w}$$

where $M^\top = -M$ (“antisymmetric”)

In 2D:

$$\omega(\mathbf{v},\mathbf{w}) = \underset{\text{One DOF}}{c} \cdot \mathbf{v}^\top \underset{90^\circ \text{ rotation}}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \mathbf{w}$$

In 3D:

$$\omega(\mathbf{v},\mathbf{w}) = \mathbf{v}^\top \underset{\text{Three DOFs}}{\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}} \mathbf{w} = -\mathbf{c} \cdot (\mathbf{v} \times \mathbf{w})$$

DOFs agree with cross product!

Differential Forms

For each point *p* on a surface:

k vectors in the tangent space at *p* → ***Differential k-form*** → $\mathbb{R}$

k-linear

Alternating

Note

Two relevant details:

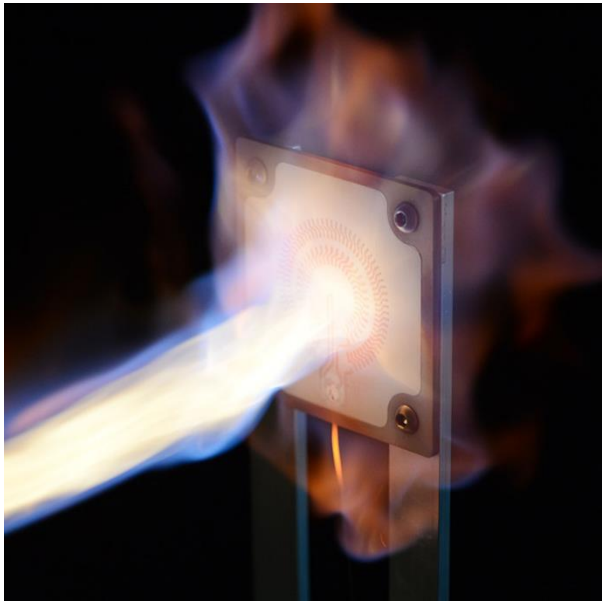
- ***k*** **= number of inputs**
- ***n*** **= dimension**

e.g. "a 2-form over $\mathbb{R}^3$"
(*k*=2, *n*=3)

Alternating *k*-Forms as Flux Sensors

One-form:

$\omega(\Delta x)$ = how much flux in direction Δx



Two-form:

$\omega(\Delta x_1, \Delta x_2)$ = how much flux in parallelogram $(\Delta x_1, \Delta x_2)$

http://www.waterworld.com/articles/print/products/2012/09/portable-velocity-flow-meter.html
http://www.mesoscribe.com/sensors/heat-flux-sensors/

Some Algebra

On the board:

Space of k-forms on $\mathbb{R}^k$ is one-dimensional.

On the board:

k-forms on $\mathbb{R}^n$ are zero when $k > n$.

$$\text{Area form: } dA$$

Products: Observations About ×

The units change:

$$\overrightarrow{\text{inches}} \times \overrightarrow{\text{inches}} = \overrightarrow{\text{inches}^2}$$

Not product-like behavior:

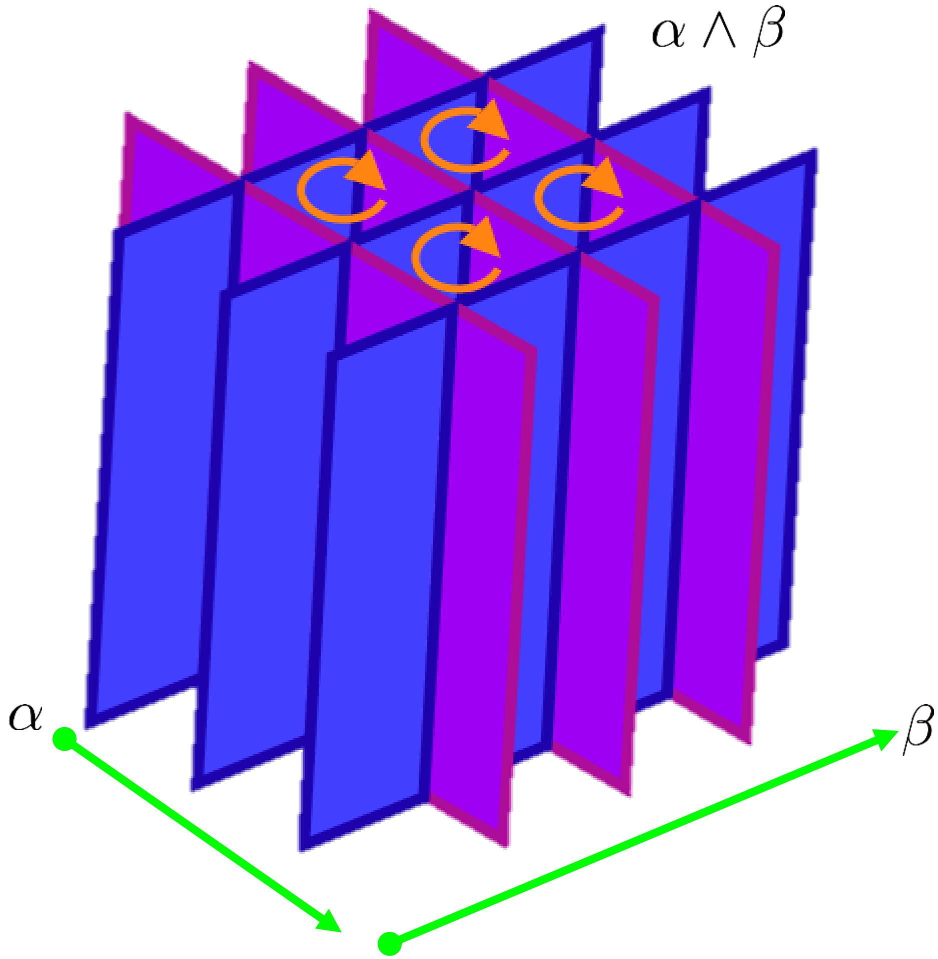
$$\vec{x} \times \vec{x} = \vec{0}$$

Dimensionality of cross product is variable:

$$\vec{\text{2D}} \times \vec{\text{2D}} = \text{scalar}$$
$$\vec{\text{3D}} \times \vec{\text{3D}} = \vec{\text{vector}}$$

Cross product of vectors is weird!

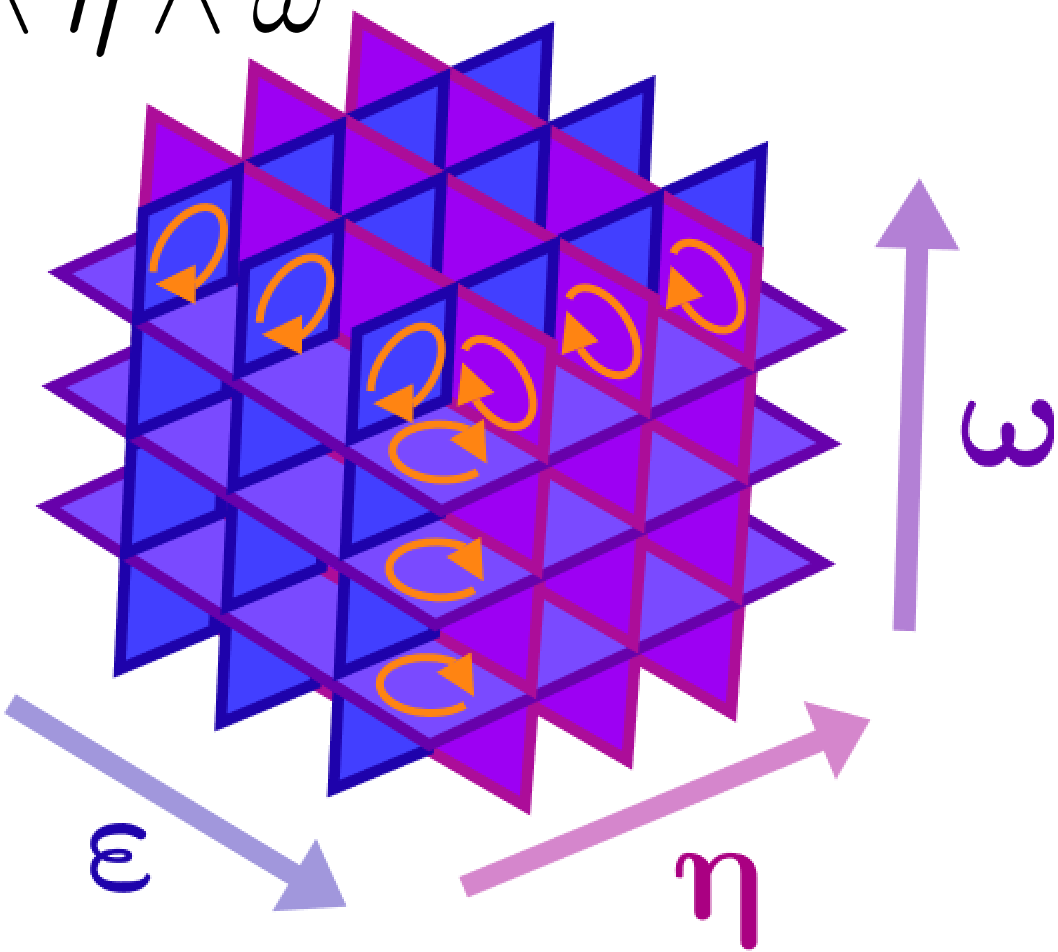
Wedge: Product of Onions



Grid image from Wikipedia

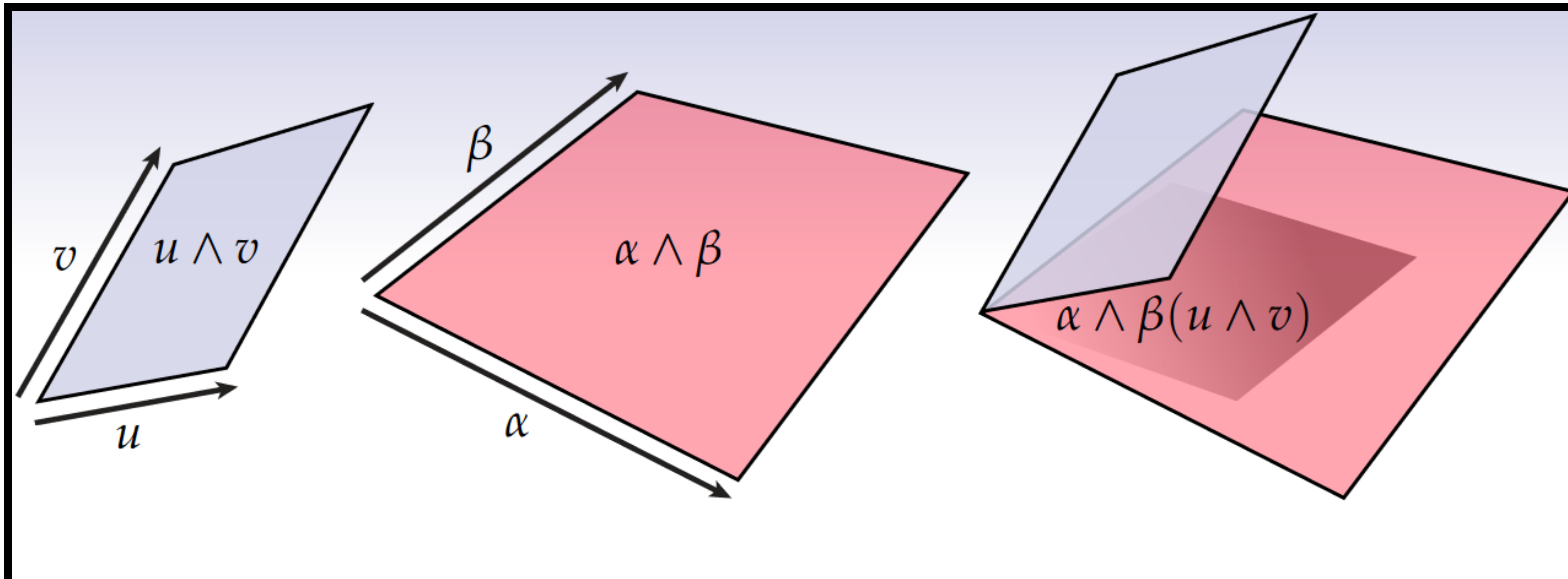
Wedge: Product of Onions

$\varepsilon \wedge \eta \wedge \omega$



Grid image from Wikipedia

Wedge Product of One-Forms

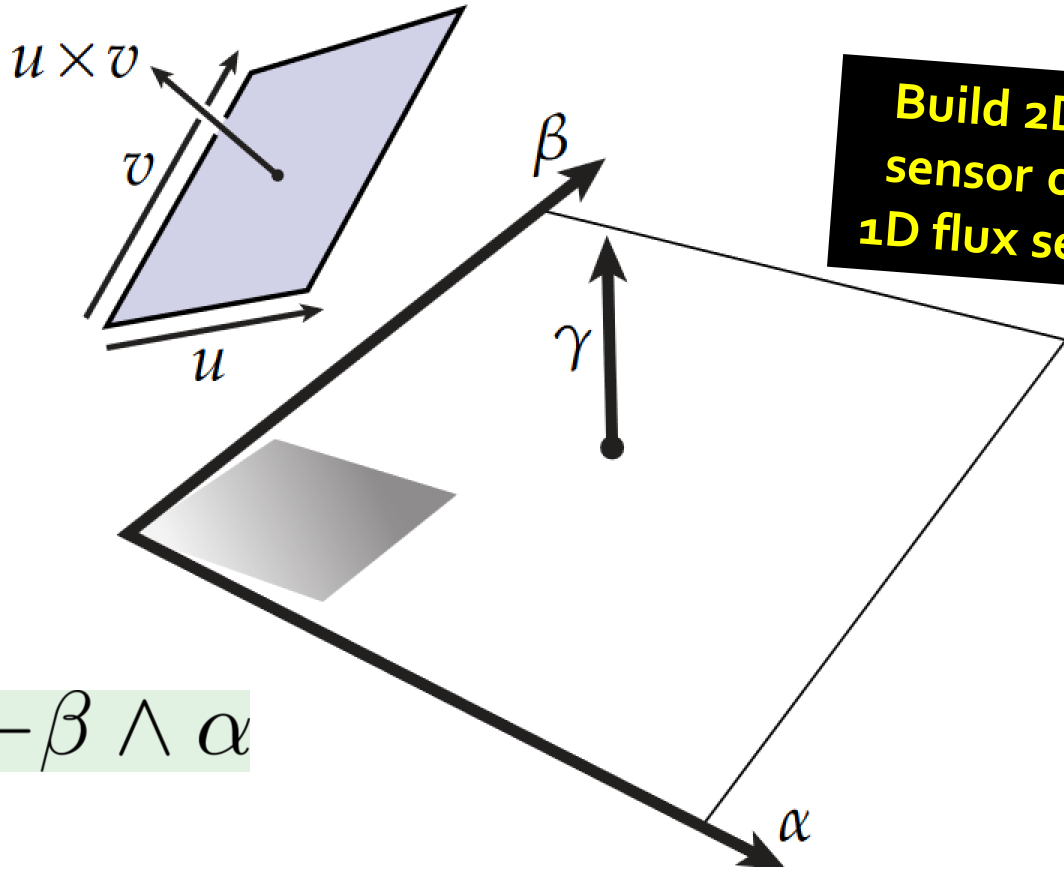


Key idea: 2-form *measures* size of parallelogram projected onto some (oriented) plane

Image courtesy K. Crane

Wedge of One-Forms

$$\alpha \wedge \beta(\mathbf{u}, \mathbf{v}) = \alpha(\mathbf{u})\beta(\mathbf{v}) - \alpha(\mathbf{v})\beta(\mathbf{u})$$



Build 2D flux sensor out of 1D flux sensors

$$\alpha \wedge \beta = -\beta \wedge \alpha$$

Image courtesy K. Crane

Relationship to Cross Product

$$\begin{aligned} \alpha(\mathbf{w}) &:= \mathbf{a} \cdot \mathbf{w} \\ \beta(\mathbf{w}) &:= \mathbf{b} \cdot \mathbf{w} \\ \implies (\alpha \wedge \beta)(\mathbf{u}, \mathbf{v}) &= \alpha(\mathbf{u})\beta(\mathbf{v}) - \alpha(\mathbf{v})\beta(\mathbf{u}) \\ &= (\mathbf{a} \cdot \mathbf{u})(\mathbf{b} \cdot \mathbf{v}) - (\mathbf{a} \cdot \mathbf{v})(\mathbf{b} \cdot \mathbf{u}) \\ &= (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{u} \times \mathbf{v}) \end{aligned}$$

For one-forms:

"How similar is parallelogram (a,b) to parallelogram (u,v)?"

Notice: All 2-forms are wedges of 1-forms.

Symbol of a Permutation

$$\varepsilon(P) = \begin{cases} +1 & \text{if } P \text{ has an even number of swaps} \\ -1 & \text{otherwise} \end{cases}$$

Examples:

(1234)

(1324)

(1342)

Wedge Product: Formal Definition

$$\alpha \in \Lambda^k, \beta \in \Lambda^\ell$$

$$\alpha \wedge \beta(\mathbf{v}_1, \ldots, \mathbf{v}_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in \mathrm{Perm}(k+\ell)} \varepsilon(\sigma) \alpha(\mathbf{v}_1, \ldots, \mathbf{v}_k) \beta(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+\ell})$$

$$\in \Lambda^{k+\ell}$$

$$\text{Antisymmetry: } \alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$$

$$\text{Associativity: } \alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$$

$$\text{Distributivity: } \alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$$

$$\implies \alpha \wedge \alpha \equiv 0$$

Basis for *k*-Forms

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

with no repeated indices.

Inner Product of 1-Forms

First clear appearance of geometry!

$$\langle \xi, \eta \rangle := \langle \xi^\sharp, \eta^\sharp \rangle$$

Borrow from vectors

Inner Product of *k*-Forms

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k \rangle := \det(\langle \alpha_i, \beta_j \rangle)$$

Example: **Inner product of 2-forms over $\mathbb{R}^3$**

$$\langle v^\flat \wedge w^\flat, a^\flat \wedge b^\flat \rangle = \det \begin{pmatrix} \mathbf{v} \cdot \mathbf{a} & \mathbf{v} \cdot \mathbf{b} \\ \mathbf{w} \cdot \mathbf{a} & \mathbf{w} \cdot \mathbf{b} \end{pmatrix}$$

$$= (\mathbf{v} \times \mathbf{w}) \cdot (\mathbf{a} \times \mathbf{b})$$

$$[= \mathbf{v}^\flat \wedge \mathbf{w}^\flat (\mathbf{a}, \mathbf{b})]$$

Again!

"How similar is parallelogram (v,w) to parallelogram (a,b)?"

Hodge Star

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle dA$$

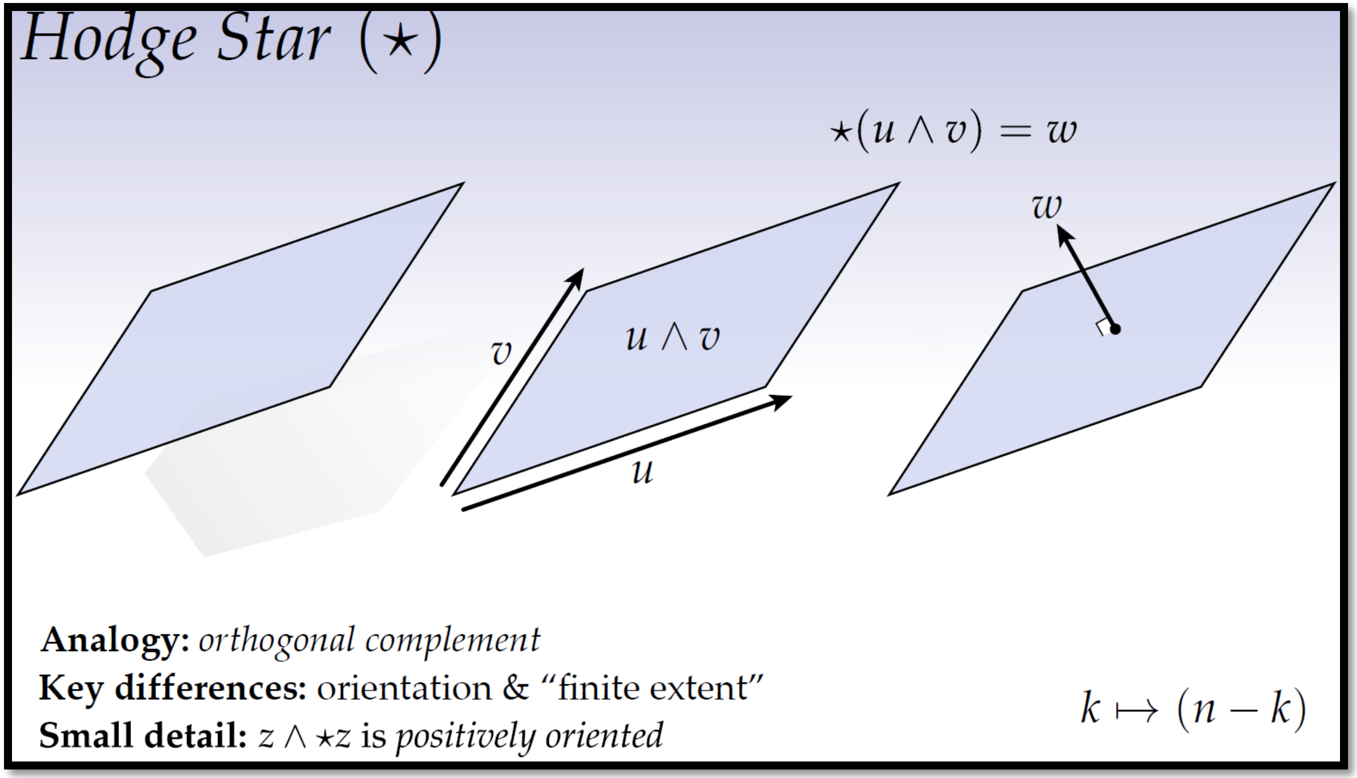
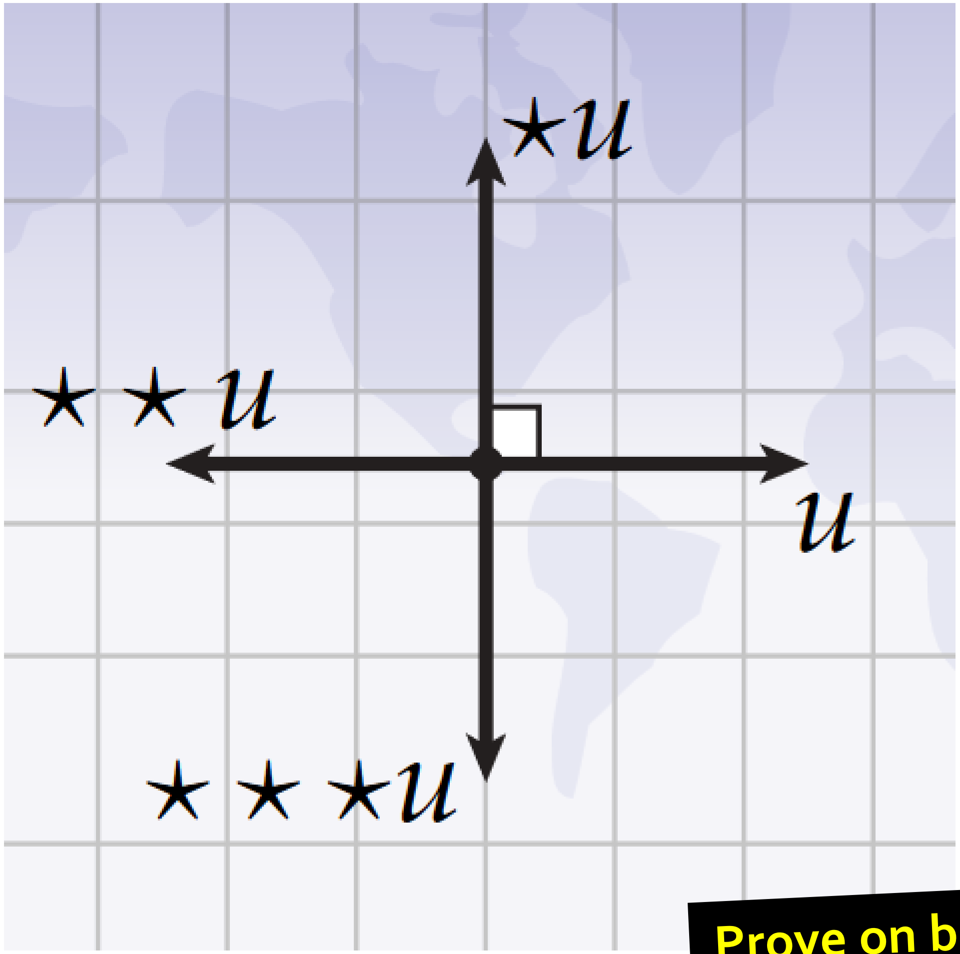


Image courtesy K. Crane

Hodge Star in 2D



Prove on board

Image courtesy K. Crane

Differential *k*-Forms on Manifolds

$$\Lambda^k := \{\text{alternating } k\text{-multilinear forms}\}$$

$$\Omega^k := \{\omega \text{ taking } \mathbf{p} \in \Sigma \mapsto \Lambda^k(T_{\mathbf{p}}\Sigma)\}$$

"One differential form per tangent plane"

Inner Product of k-Forms

$$\langle \alpha, \beta \rangle := \int_\Sigma \star\alpha \wedge \beta$$

Recall:

Differential of a Map

Definition (Differential). *Suppose $\varphi : \mathcal{M} \to \mathcal{N}$ is a map from a submanifold $\mathcal{M} \subseteq \mathbb{R}^k$ into a submanifold $\mathcal{N} \subseteq \mathbb{R}^\ell$. Then, the* differential $d\varphi_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \to T_{\varphi(\mathbf{p})}\mathcal{N}$ *of φ at a point $\mathbf{p} \in \mathcal{M}$ is given by*

$$d\varphi_{\mathbf{p}}(\mathbf{v}) := (\varphi \circ \gamma)'(0),$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ is any curve with $\gamma(0) = \mathbf{p}$ and $\gamma'(0) = \mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$.

Linear map of tangent spaces

$$d\varphi_{\mathbf{p}}(\gamma'(0)) := (\varphi \circ \gamma)'(0)$$

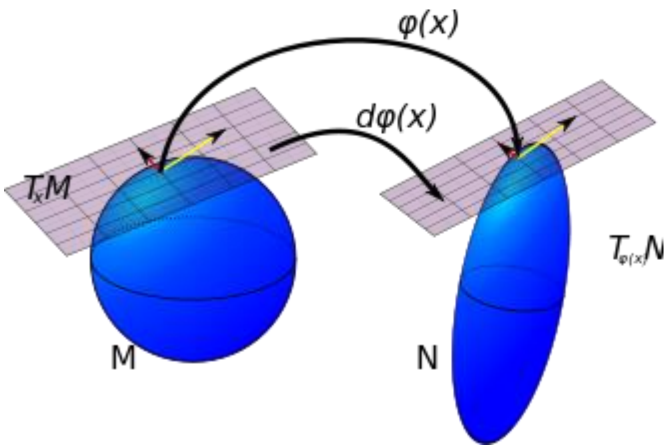


Image from Wikipedia

Recall:

Differential of a Map

Definition (Differential). *Suppose $\varphi : \mathcal{M} \to \mathcal{N}$ is a map from a submanifold $\mathcal{M} \subseteq \mathbb{R}^k$ into a submanifold $\mathcal{N} \subseteq \mathbb{R}^\ell$. Then, the* differential $d\varphi_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \to T_{\varphi(\mathbf{p})}\mathcal{N}$ *of φ at a point $\mathbf{p} \in \mathcal{M}$ is given by*

$$d\varphi_{\mathbf{p}}(\mathbf{v}) := (\varphi \circ \gamma)'(0),$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ is any curve with $\gamma(0) = \mathbf{p}$ and $\gamma'(0) = \mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$.

Linear map of tangent spaces

$$d\varphi_{\mathbf{p}}(\gamma'(0)) := (\varphi \circ \gamma)'(0)$$

"Exterior derivative of 0-form"

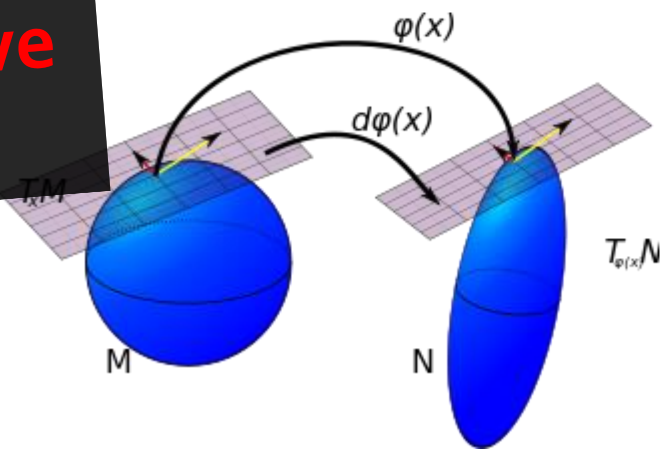


Image from Wikipedia

Fancy Notation

$$\nabla f = (df)^\sharp$$

$$(\nabla f)^\flat = df$$

Construction of Exterior Derivative

Given a 1-form α, when is there a function f with $\alpha = df$?

$$\alpha := \sum_i f_i dx^i$$

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} \implies \frac{\partial f_j}{\partial x^i} = \frac{\partial f_i}{\partial x^j}$$

$$\implies 0 = \sum_{ij} \frac{\partial f_j}{\partial x^i} dx^i \wedge dx^j$$

Alternating!

Transforms d on 0-forms to d on 1-forms…

Iterate!

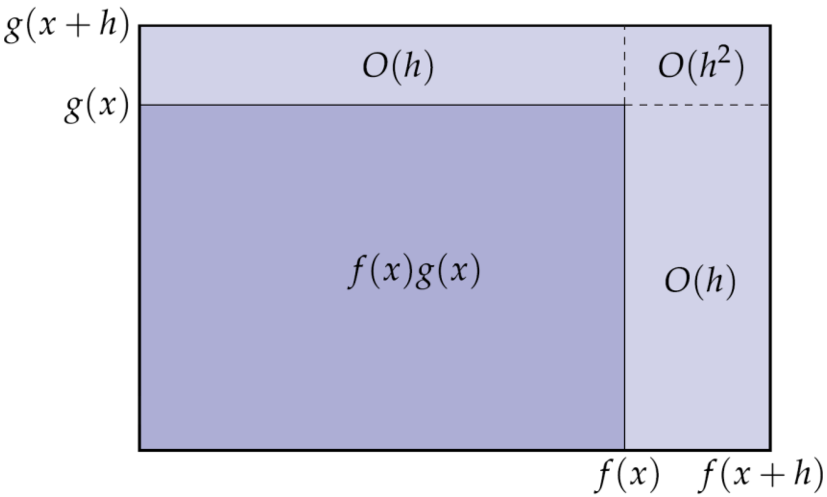
Exterior Derivative: Axiomatic

Differential: df agrees with directional derivative

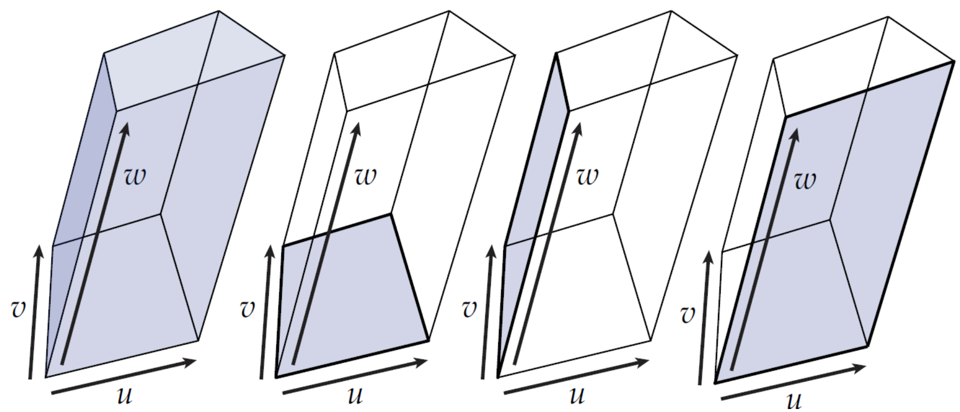
Product rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$

Exactness: $d^2 = 0$

Product Rule: Intuition

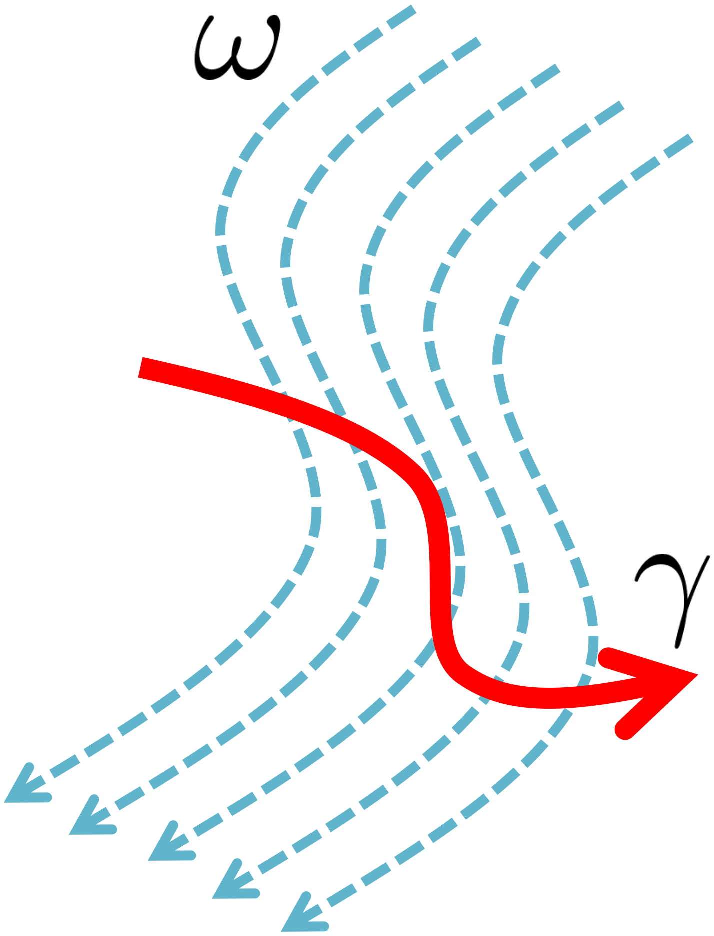


$$(fg)' = f'g + fg'$$



Images courtesy K. Crane

Integration of *k*-Forms



$$\int_\gamma \omega := \int_\gamma \omega(\mathbf{T})\, ds$$

Measures amount of ω parallel to γ

Integrate on *k*-dimensional objects

Stokes' Theorem

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

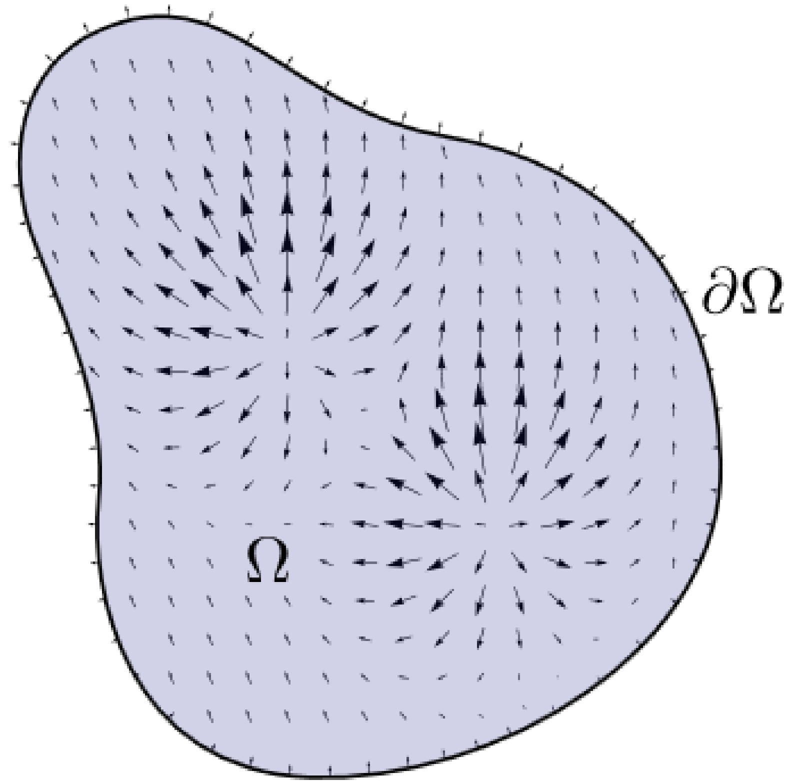
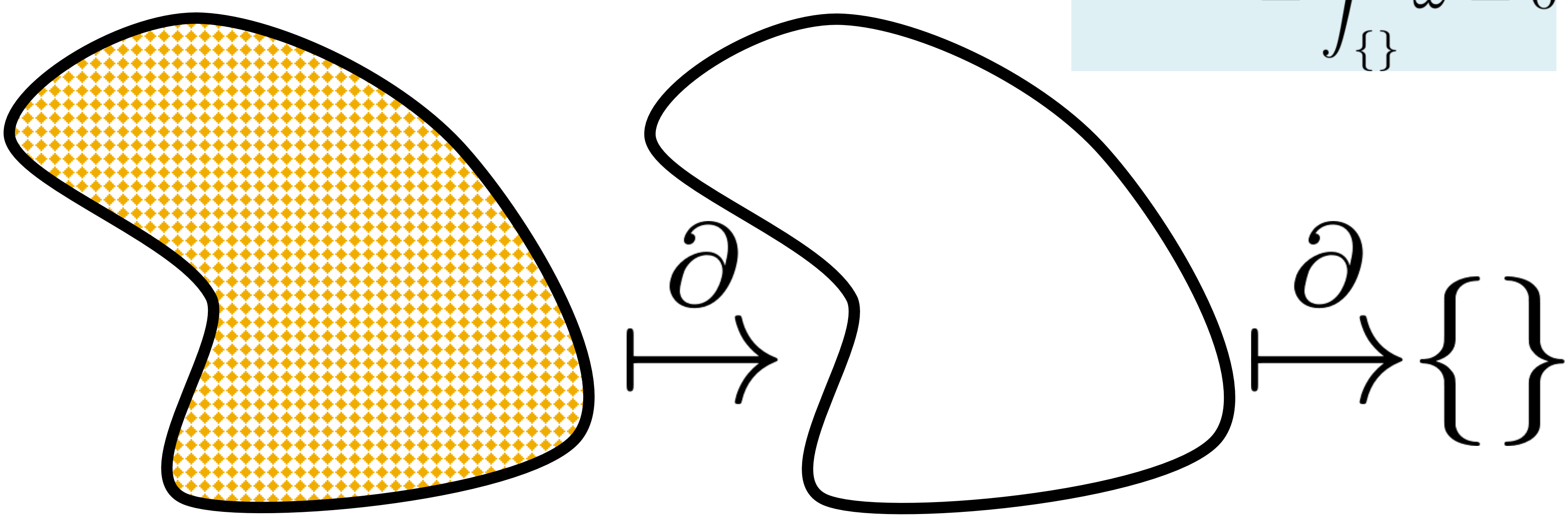


Image courtesy K. Crane

Intuition for Exactness

$$d^2 = 0$$

$$\begin{aligned}\int_\Omega d^2\omega &= \int_{\partial\Omega} d\omega \\ &= \int_{\partial\partial\Omega} \omega \\ &= \int_{\{\}} \omega = 0\end{aligned}$$



Translating Vector Calculus

$$\nabla f = (df)^\sharp$$

$$\nabla \cdot F = \star d \star (F^\flat)$$

$$\nabla \times F = (\star d(F^\flat))^\sharp$$

$$\Delta f = \star d \star df$$

Homework!

Rough Outline

1. **Exterior calculus**
   Alternating *k*-forms, derivatives, and integration

2. **Discrete exterior calculus**
   All that, on a simplicial complex

Discrete Exterior Calculus (DEC)

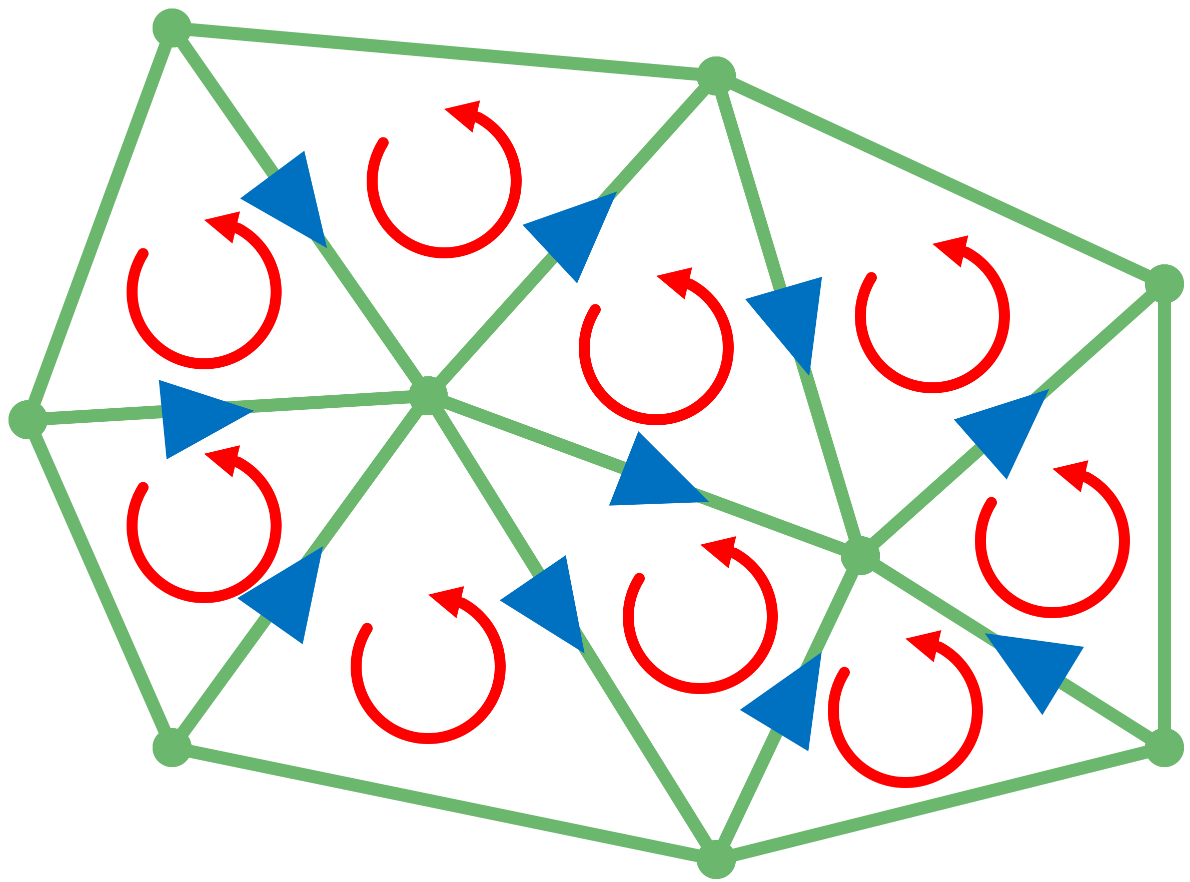
Discrete version of exterior calculus.

$$\omega^\sharp \quad v^\flat \quad \omega_1 \wedge \omega_2 \quad \star\omega \quad d\omega$$

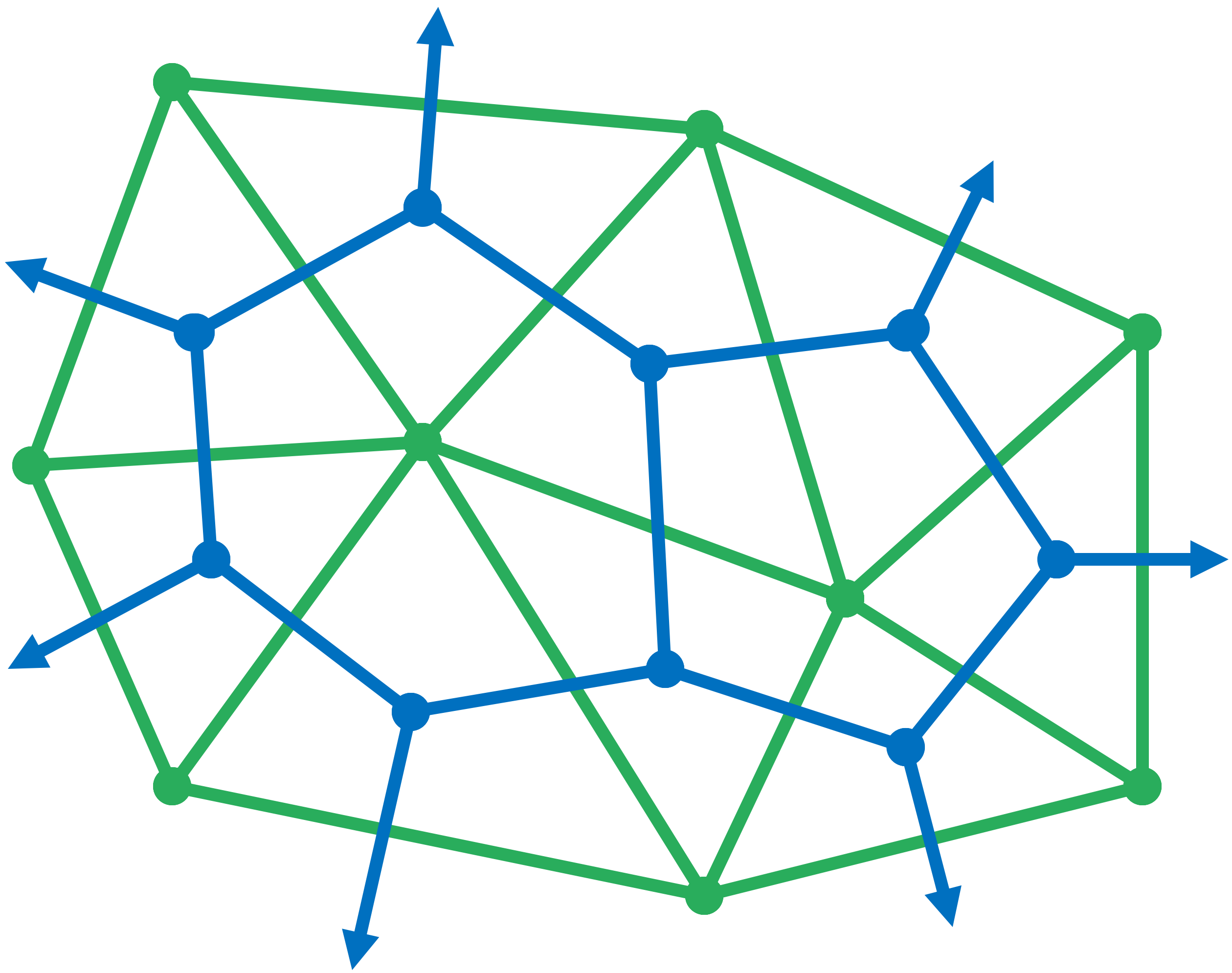
...

Recall:

Oriented Simplicial Complex



Recall: Dual Complex

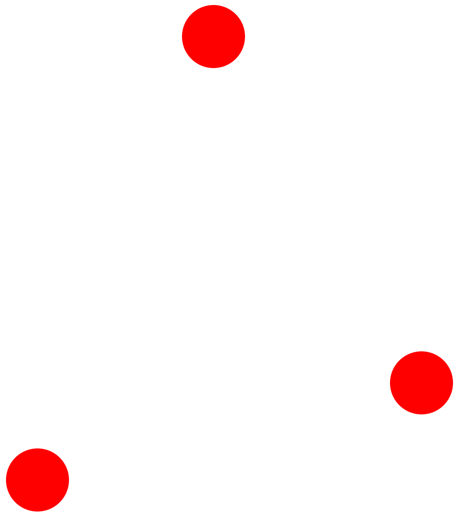


The Trick

Store *integrals* of forms!

Integrated *k*-forms

Discrete 0-form

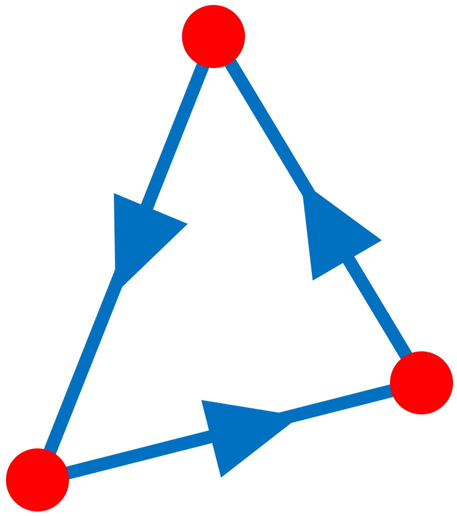


$$\int_v \omega = f(v) \in \mathbb{R}^{|V|}$$

Store *integrated* quantities!

Integrated *k*-forms

Discrete 1-form

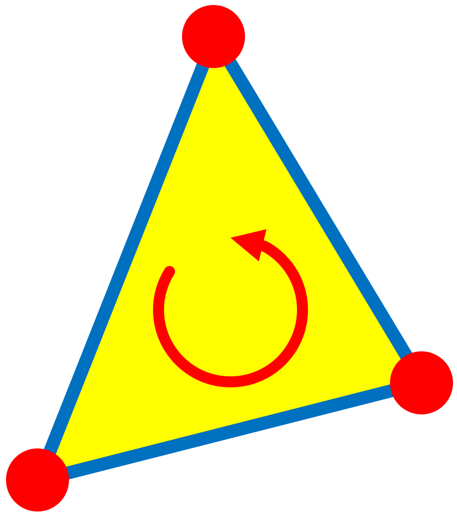


$$\int_e \omega \in \mathbb{R}^{|E|}$$

Store *integrated* quantities!

Integrated *k*-forms

Discrete 2-form



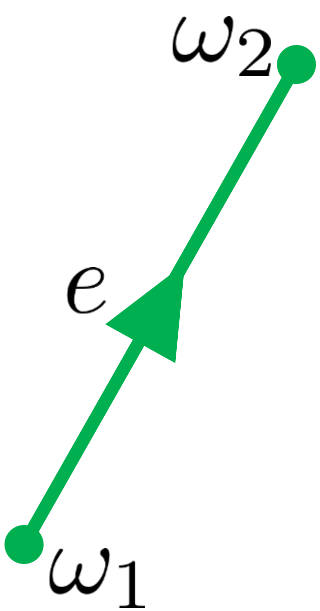
$$\int_t \omega \in \mathbb{R}^{|T|}$$

Store *integrated* quantities!

Exterior Derivative

$$\boxed{\int_{\Omega} d\omega = \int_{\partial\Omega} \omega}$$

Stokes' Theorem



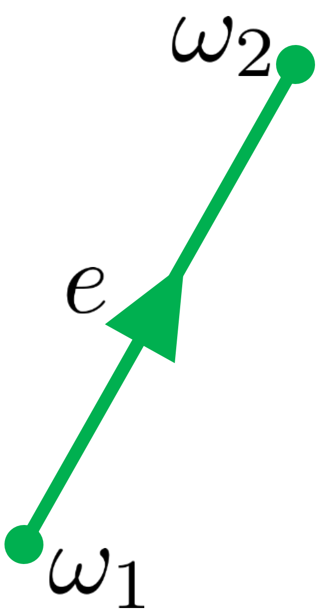
$$\int_{e} d\omega = \int_{\partial e} \omega = \omega_2 - \omega_1$$

$$d_{01} \in \{-1, 0, 1\}^{|E| \times |V|}$$

Exterior Derivative

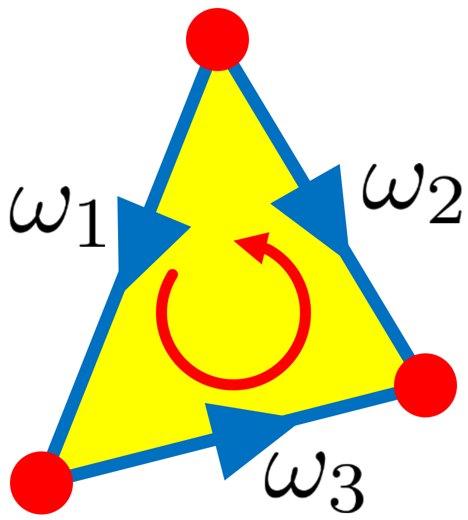
$$d \in \mathbb{R}^{|E| \times |V|}$$

consists of 1, 0, -1



$$\int_e d\omega = \int_{\partial e} \omega = \omega_2 - \omega_1$$

Exterior Derivative

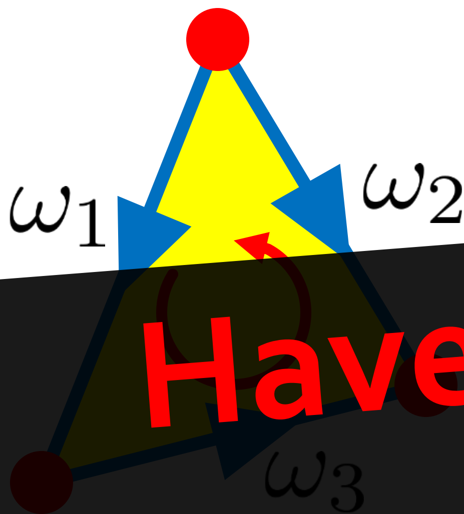


$$d \in \mathbb{R}^{|F| \times |E|}$$

consists of 1, 0, -1

$$\int_t d\omega = \int_{\partial t} \omega = \omega_1 - \omega_2 + \omega_3$$

Exterior Derivative



$$d \in \mathbb{R}^{|F| \times |E|}$$

consists of 1, 0, -1

Haven't made any approximations yet!

$$\int_t d\omega = \int_{\partial t} \omega = \omega_1 - \omega_2 + \omega_3$$

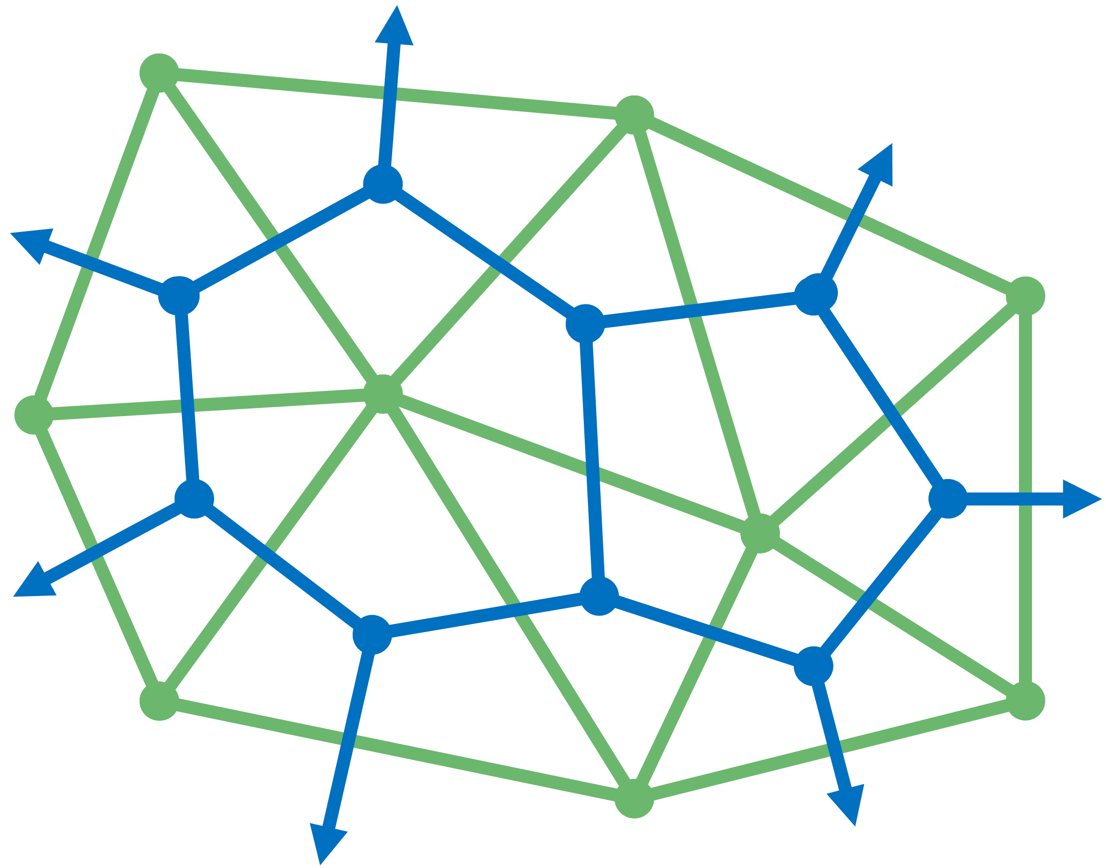
Observation

$$\text{“}d^2 = 0\text{”}$$

You proved this in your homework!

Two different *d* matrices

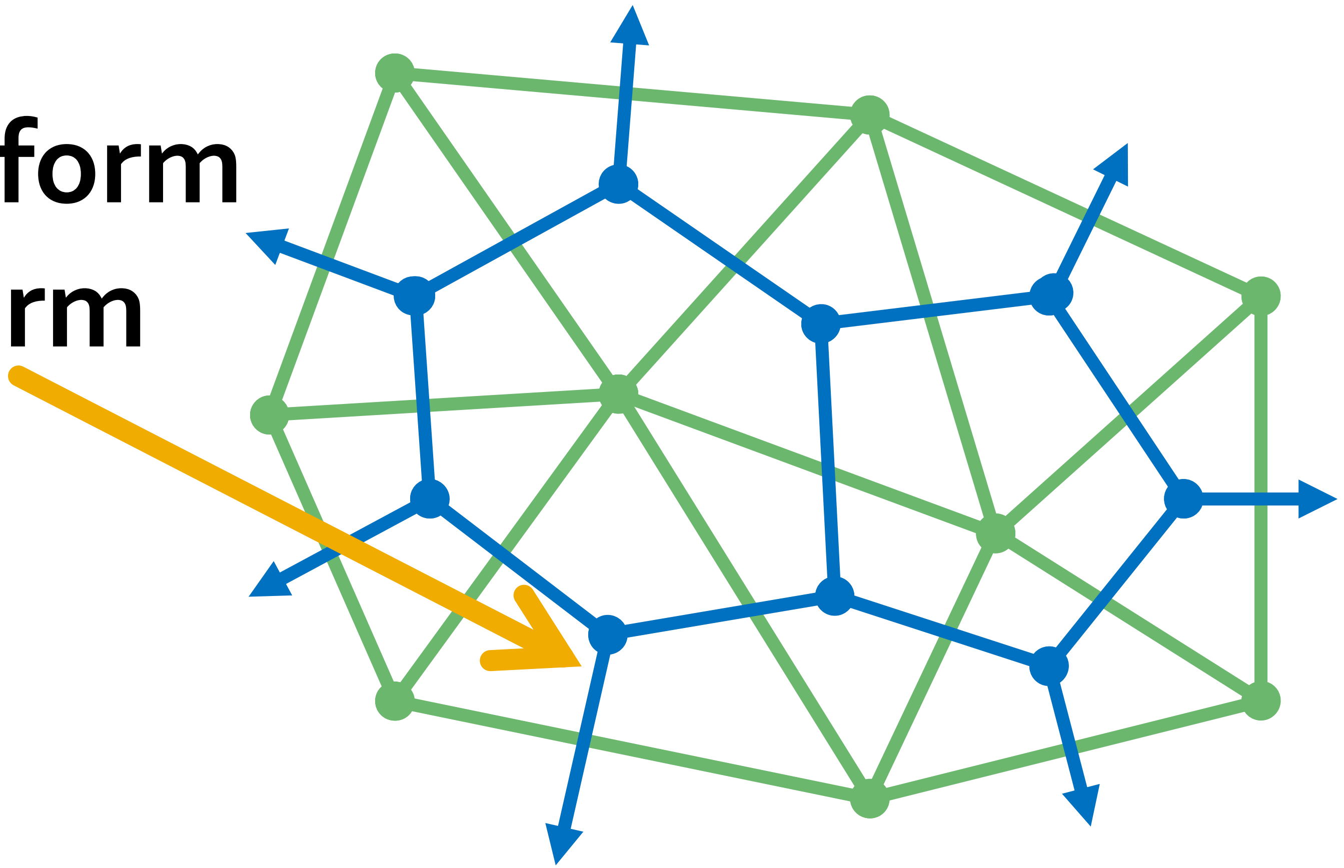
Hodge Star: Idea



Moves to dual mesh

Hodge Star

Primal 2-form
Dual 0-form

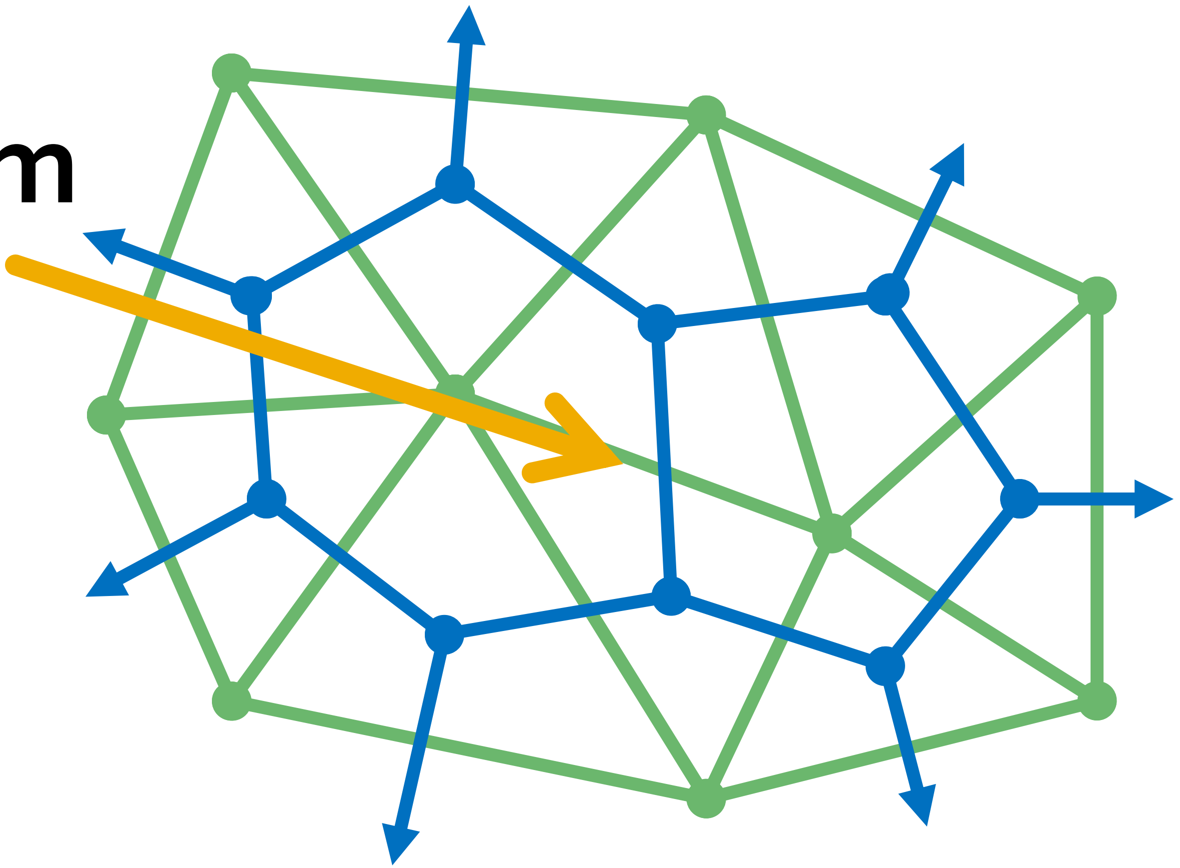


Moves to dual mesh

Hodge Star

Primal 1-form

Dual 1-form



Moves to dual mesh

Hodge Star Matrices

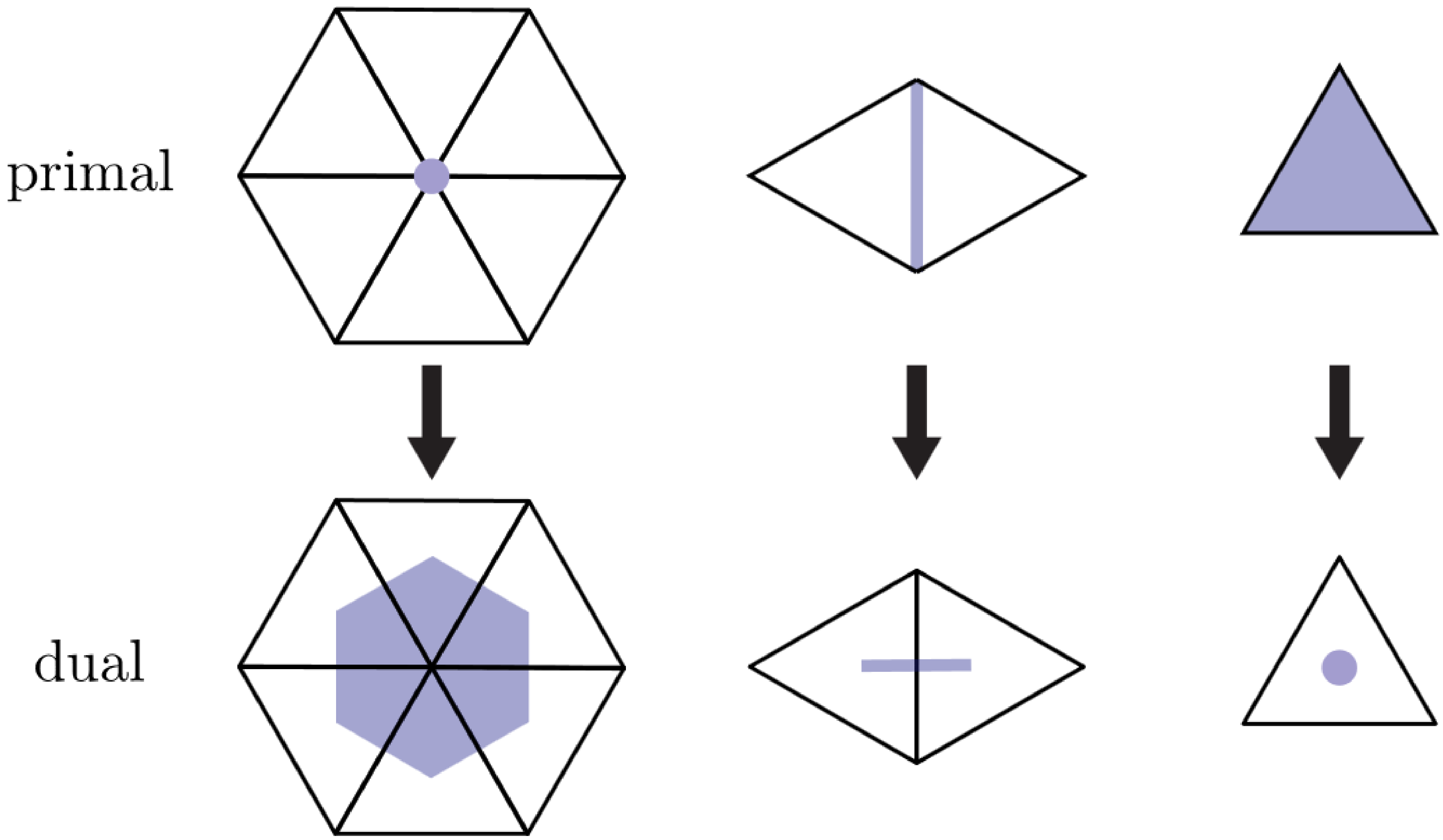
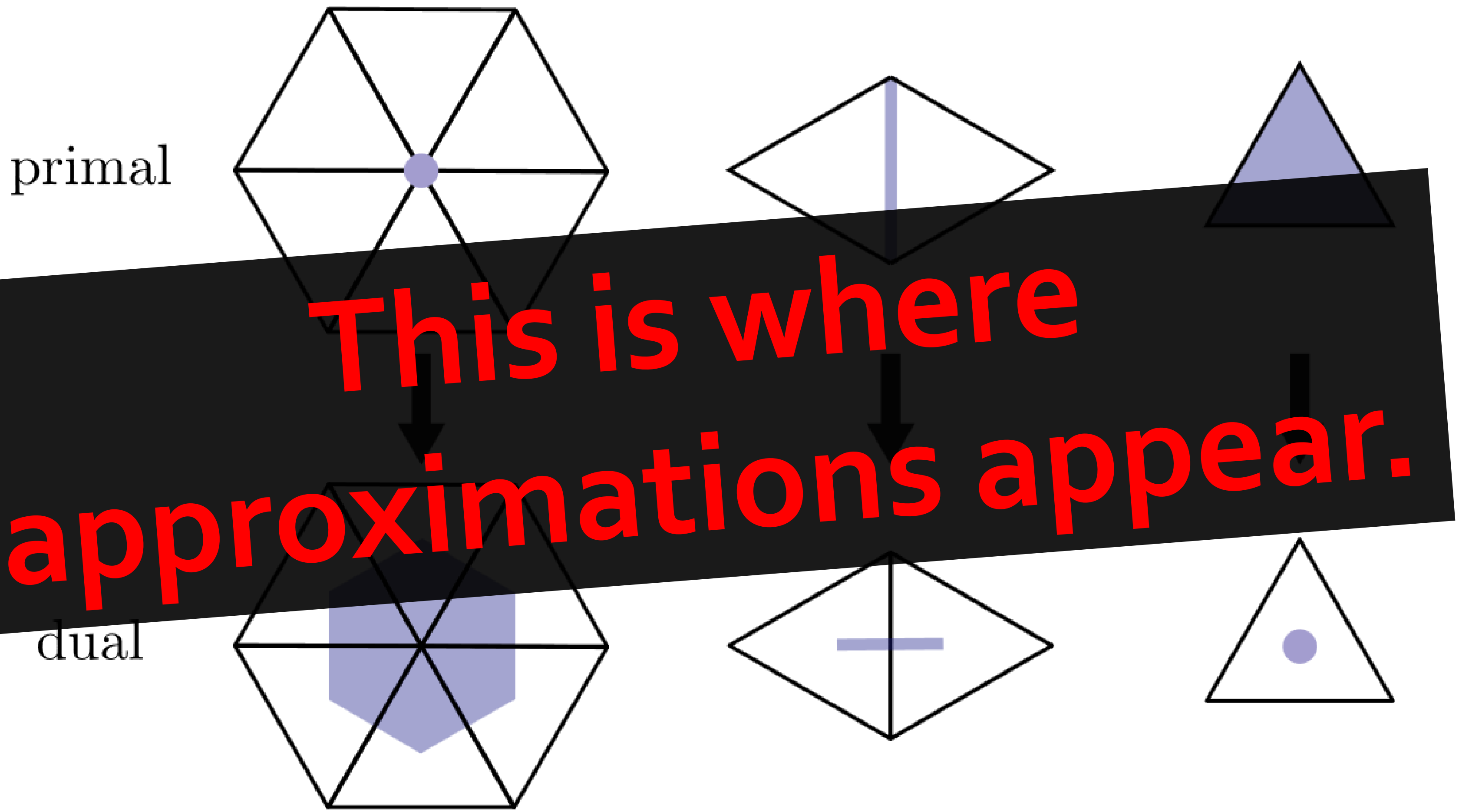


Image courtesy K. Crane

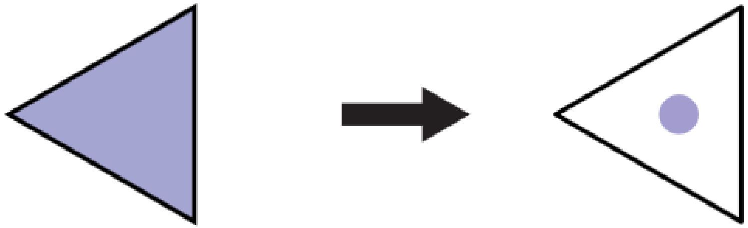
Hodge Star Matrices



This is where approximations appear.

Image courtesy K. Crane

Primal 2-Form / Dual 0-Form

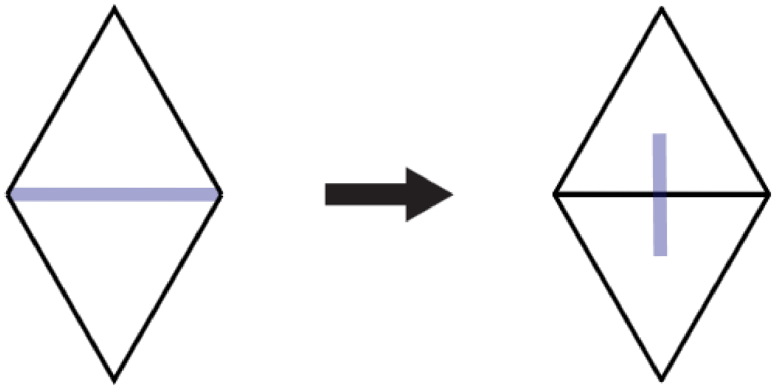


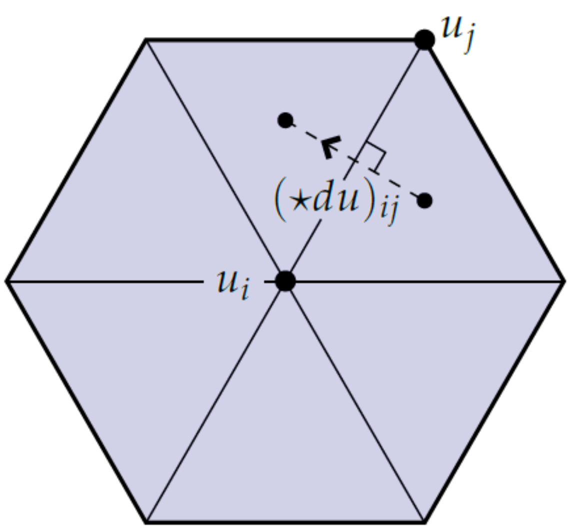
$$\star_{ii} = \mathrm{Area}(\mathrm{triangle}\ i)^{-1}$$

Just triangle areas

Primal 1-Form / Dual 1-Form

What do you think it is?





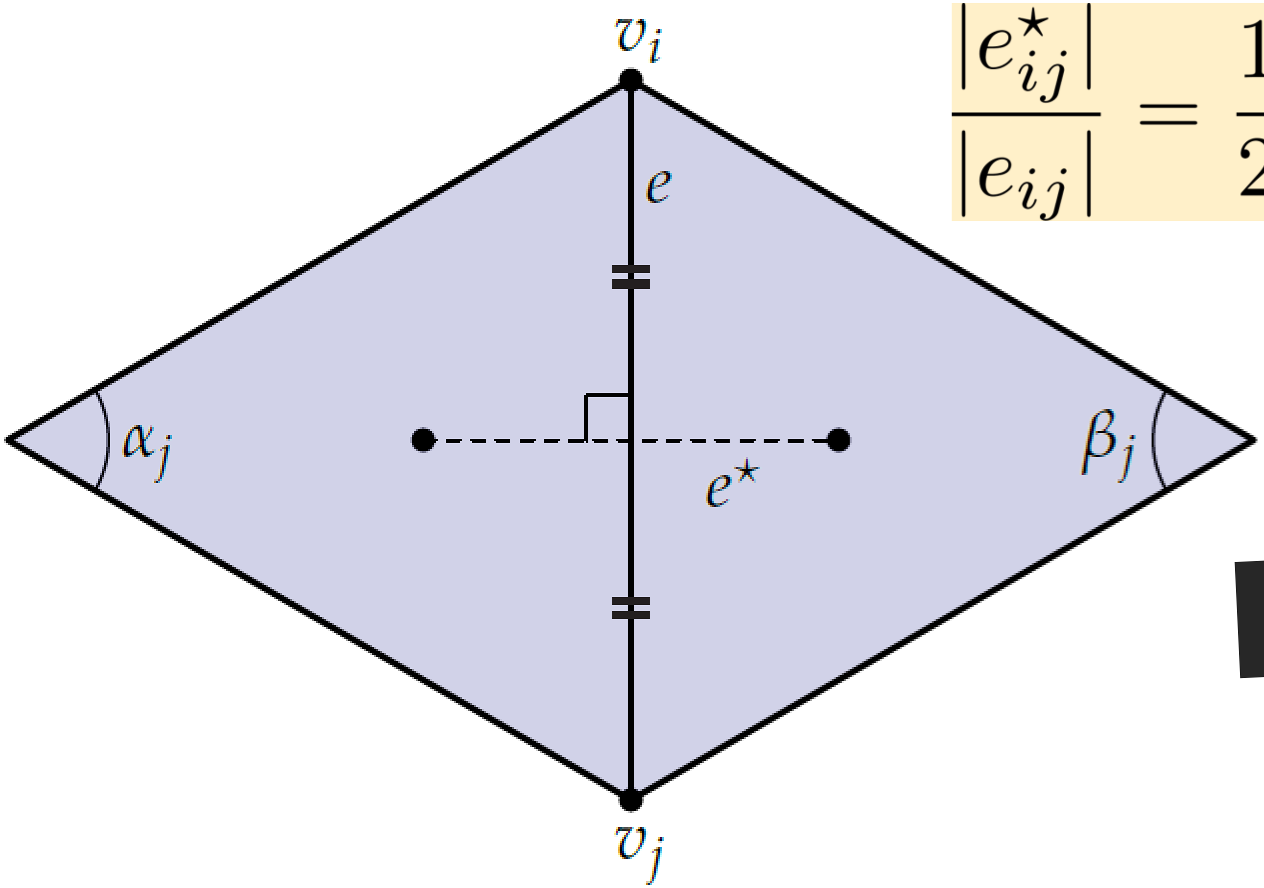
Careful with orientation/sign!

$$\star\omega = \frac{|e_\star|}{|e|}\omega$$

Image courtesy K. Crane

Ratio of edge lengths

Primal 1-Form / Dual 1-Form



$$\frac{|e^\star_{ij}|}{|e_{ij}|} = \frac{1}{2}(\cot \alpha_j + \cot \beta_j)$$

Homework

Image courtesy K. Crane

Choice of dual: Circumcenter

Nice Extension

Weighted Triangulations for Geometry Processing

Fernando de Goes
Caltech
and
Pooran Memari
CNRS-LTCI Telecom ParisTech
and
Patrick Mullen
Caltech
and
Mathieu Desbrun
Caltech

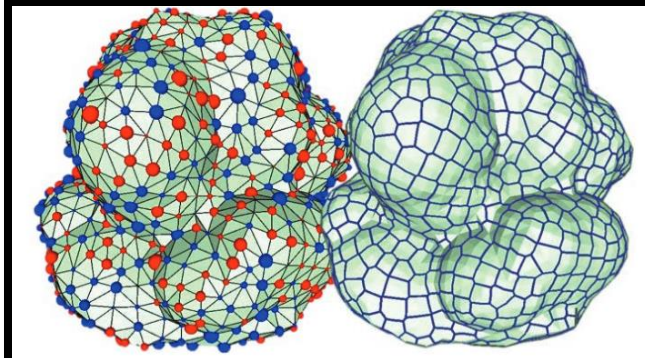


Fig. 1. **Weighted Triangulation.** Example of a triangle mesh equipped with vertex weights (left) displayed as spheres with squared radii corresponding to the weight magnitudes and colors according to their signs (red+, blue−). The set of weights endows the triangulation with an orthogonal, non-circumcentric dual structure (right).

In this paper, we investigate the use of weighted triangulations as discrete, augmented approximations of surfaces for digital geometry processing. By incorporating a scalar weight per mesh vertex, we introduce a new notion of discrete metric that defines an orthogonal dual structure for arbitrary triangle meshes and thus extends weighted Delaunay triangulations to surface meshes. We also present alternative characterizations of this primal-dual structure (through combinations of angles, areas, and lengths) and, in the process, uncover closed-form expressions of mesh energies that were previously known in implicit form only. Finally, we demonstrate how weighted triangulations provide a faster and more robust approach to a series of geometry processing applications, including the generation of well-centered meshes, self-supporting surfaces, and sphere packing.

Categories and Subject Descriptors: I.3.5 [**Computer Graphics**]: Computational Geometry and Object Modeling—*Curve, surface, solid, and object representations*

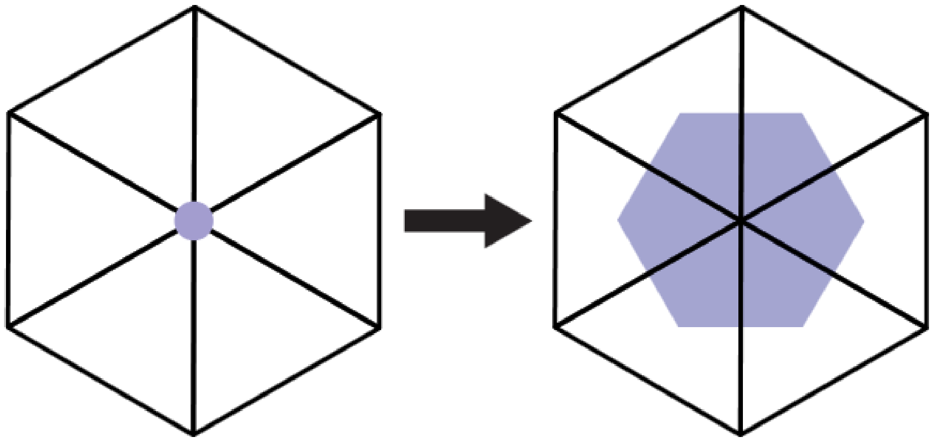
Additional Key Words and Phrases: discrete differential geometry, discrete metric, weighted triangulations, orthogonal dual diagram.

1. INTRODUCTION

Triangle meshes are arguably the predominant discretization of surfaces in graphics, and by now there is a large body of literature on the theory and practice of simplicial meshes for computations. However, many geometry processing applications rely, overtly or covertly, on an orthogonal dual structure to the primal mesh. The use of such a dual structure is very application-dependent, with circumcentric and power duals being found, for instance, in physical simulation [Elcott et al. 2007; Batty et al. 2010], architecture modeling [Liu et al. 2013; de Goes et al. 2013] and parameterization [Mercat 2001; Jin et al. 2008]. While most of these results are limited to planar triangle meshes, little attention has been paid to exploring orthogonal duals for triangulated surface meshes.

In this paper, we advocate the use of orthogonal dual structures to enrich simplicial approximations of arbitrary surfaces. We introduce an extended definition of metric for these discrete surfaces with which one can not only measure length and area of simplices,

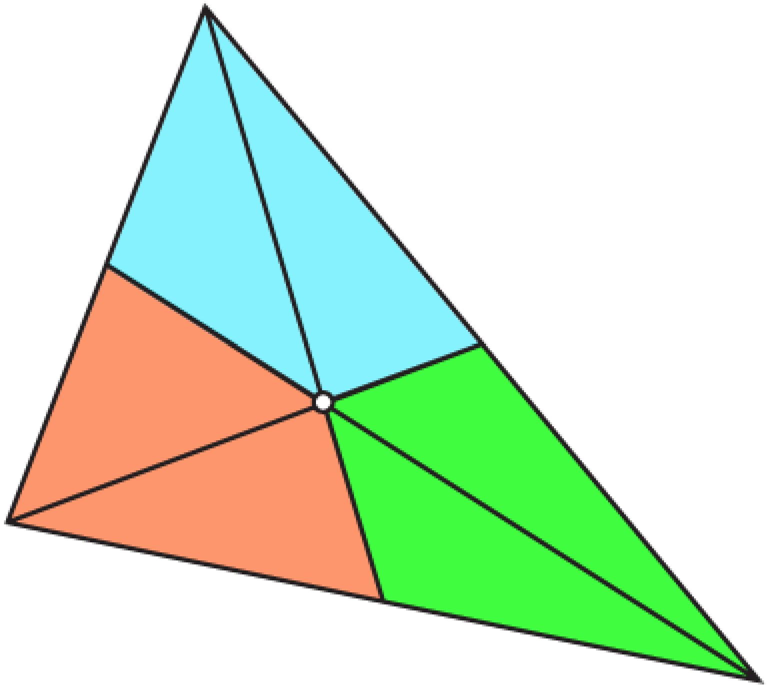
Primal 0-Form / Dual 2-Form

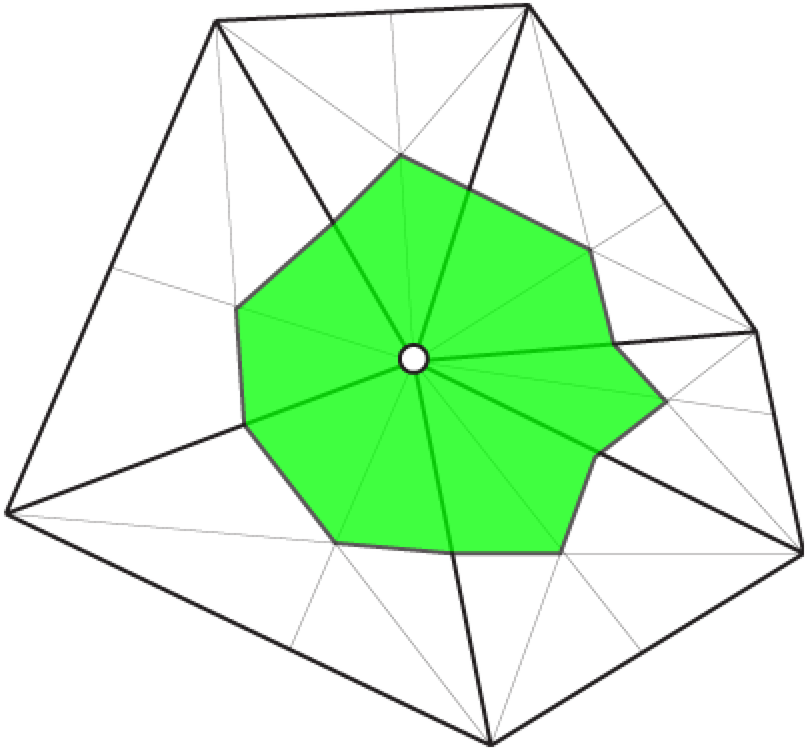


$$\star_{ii} = \text{Area}(\text{triangle } i)$$

Area of dual cell

Recall: Barycentric Lumped Mass

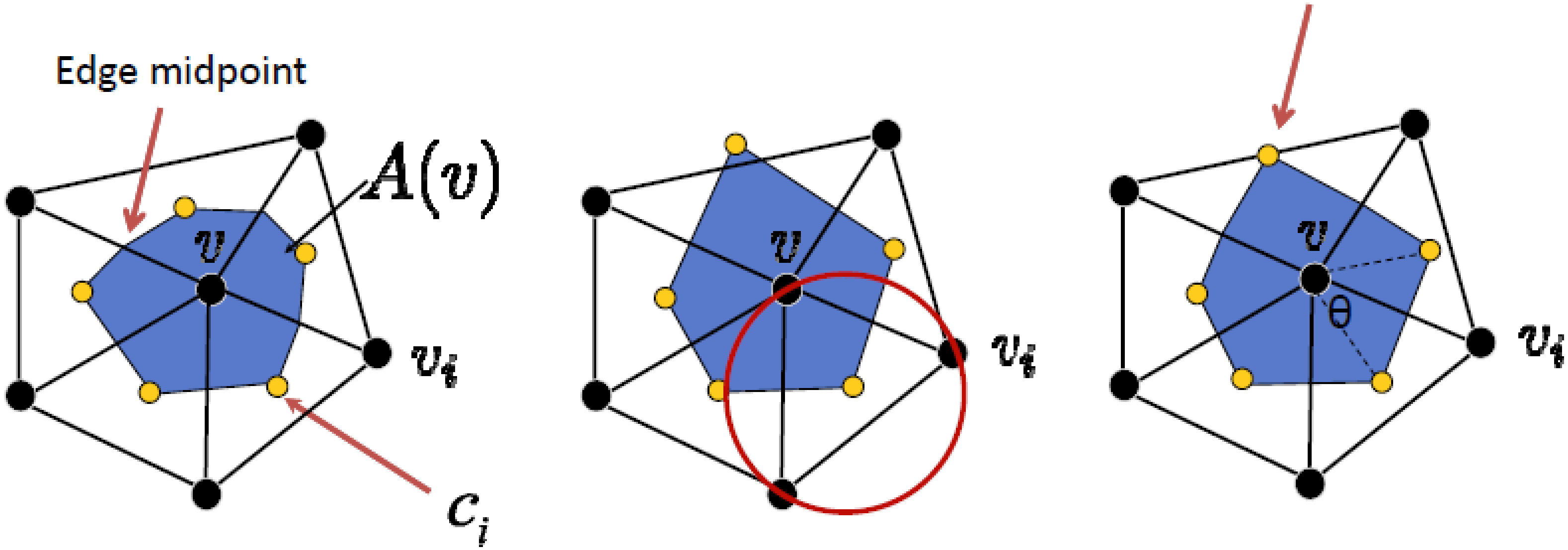




http://www.alecjacobson.com/weblog/?p=1146

Area/3 to each vertex

Additional Options



Barycentric cell

c_i = barycenter of triangle

Voronoi cell

c_i = circumcenter of triangle

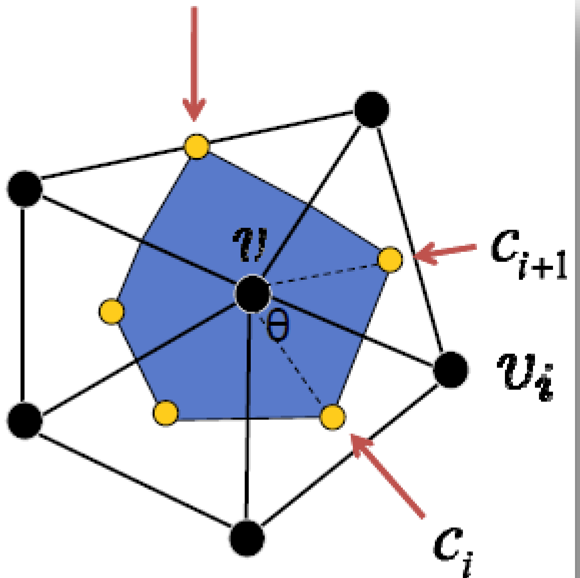
Mixed cell

Slide courtesy M. Ben-Chen

Mixed Voronoi Cell

If $\theta < \pi/2$, c_i is the circumcenter of the triangle $(v_i\ , v, v_{i+1}\ )$

If $\theta \geq \pi/2$, c_i is the midpoint of the edge $(v_i\ , v_{i+1}\ )$



$$A(v) = \sum_{v_i \in \mathcal{N}(v)} \left(Area(c_i, v, (v+v_i)/2) + Area(c_{i+1}, v, (v+v_i)/2) \right)$$

Slide courtesy M. Ben-Chen

Interesting Reading

HOT: Hodge-Optimized Triangulations

Patrick Mullen Pooran Memari Fernando de Goes Mathieu Desbrun
Caltech

Abstract

We introduce Hodge-optimized triangulations (HOT), a family of well-shaped primal-dual pairs of complexes designed for fast and accurate computations in computer graphics. Previous work most commonly employs barycentric or circumcentric duals; while barycentric duals guarantee that the dual of each simplex lies within the simplex, circumcentric duals are often preferred due to the induced orthogonality between primal and dual complexes. We instead promote the use of weighted duals ("power diagrams"). They allow greater flexibility in the location of dual vertices while keeping primal-dual orthogonality, thus providing a valuable extension to the usual choices of dual by only adding one additional scalar per primal vertex. Furthermore, we introduce a family of functionals on pairs of complexes that we derive from bounds on the errors induced by diagonal Hodge stars, commonly used in discrete computations. The minimizers of these functionals, called HOT meshes, are shown to be generalizations of Centroidal Voronoi Tesselations and Optimal Delaunay Triangulations, and to provide increased accuracy and flexibility for a variety of computational purposes.

Keywords: Optimal triangulations, Discrete Exterior Calculus, Discrete Hodge Star, Optimal Transport.

Links: DL PDF WEB

1 Introduction

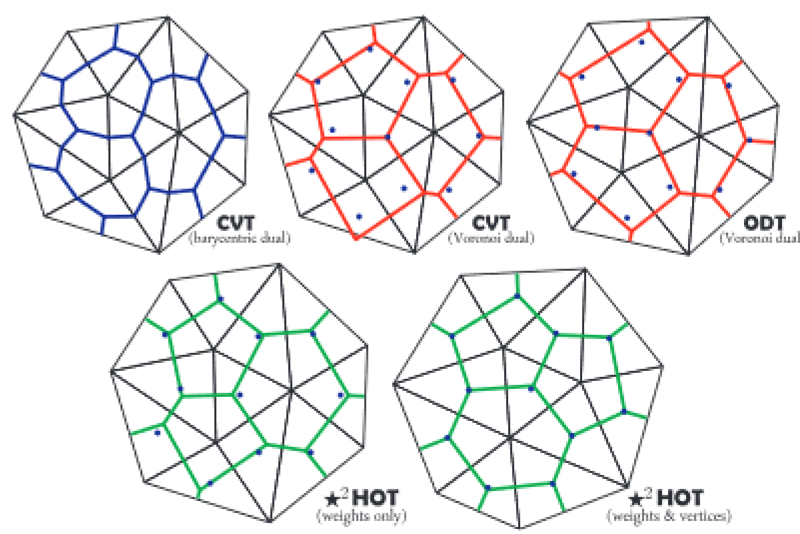
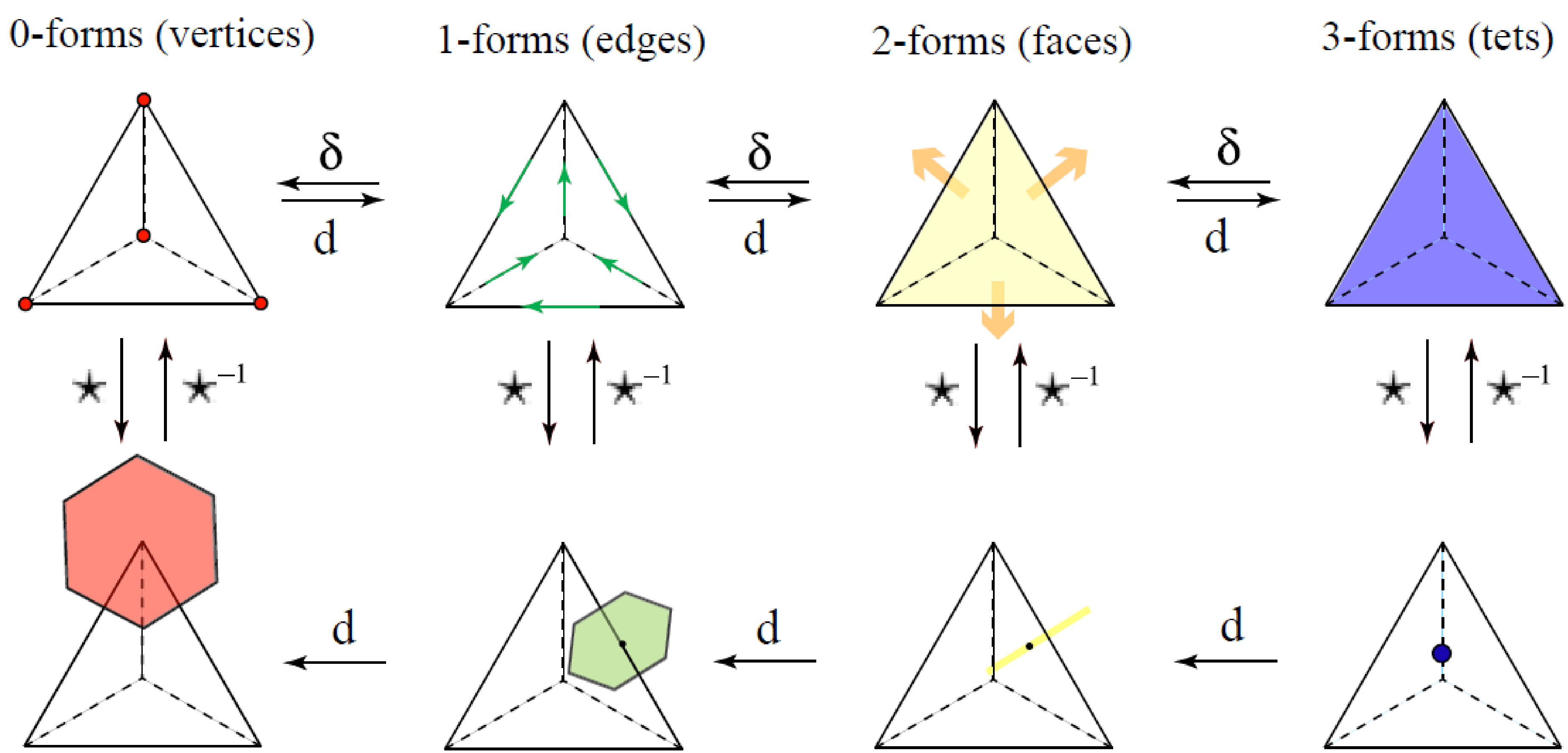


Figure 1: ***Primal/Dual Triangulations**: Using the barycentric dual (top-left) does not generally give dual meshes orthogonal to the primal mesh. Circumcentric duals, both in Centroidal Voronoi Tesselations (CVT, top-middle) and Optimal Delaunay Triangulations (ODT, top-right), can lead to dual points far from the barycenters of the triangles (blue points). Leveraging the freedom provided by weighted circumcenters, our Hodge-optimized triangulations (HOT) can optimize the dual mesh alone (bottom-left) or both the primal and dual meshes (bottom-right), e.g., to make them more*

Discrete deRham Complex



http://ddg.cs.columbia.edu/SIGGRAPH06/DDGCourse2006.pdf

In Practice

- **Build up tons of matrices**
- **Multiply them together for complicated operators**

$$d_{01}, d_{12}, \star_{02}, \ldots$$

Inner Product of Forms

Dot product:

One primal, one dual.

(Already integrated!)

Co-Differential

$$\langle d\beta, \alpha \rangle = -\langle \beta, \star d \star \alpha \rangle$$

$$\Big\Downarrow \text{ grad} \rightarrow \text{div}$$

$$\delta := -\star d \star$$

http://www.uib.no/filearchive/noteshodge27sep.pdf

Yet Another Cotan Laplacian

$$L = d_{12} \star_{11} d_{01}$$

$$M = \star_{02}$$

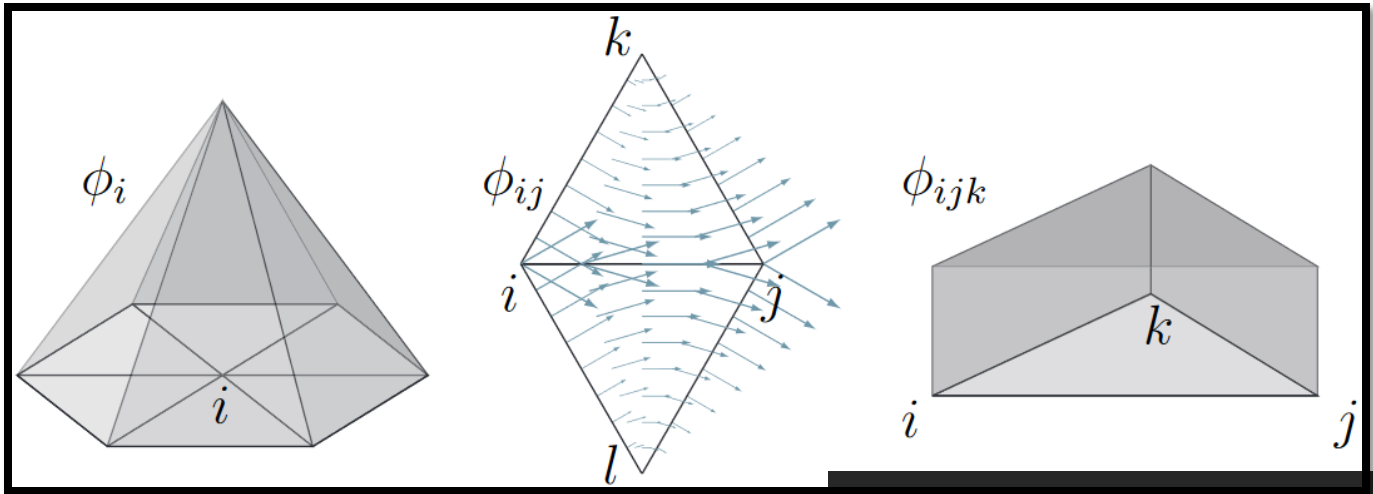
Hodge Laplacian

$$\Delta := d \star d \star + \star d \star d$$

What happens for 0-forms?
2-forms on a surface?

Whitney Elements

$$\phi_{ij}(p) = \phi_i(p)d\phi_j - \phi_j(p)d\phi_i, (d\phi_i)^\sharp = \nabla\phi_i$$

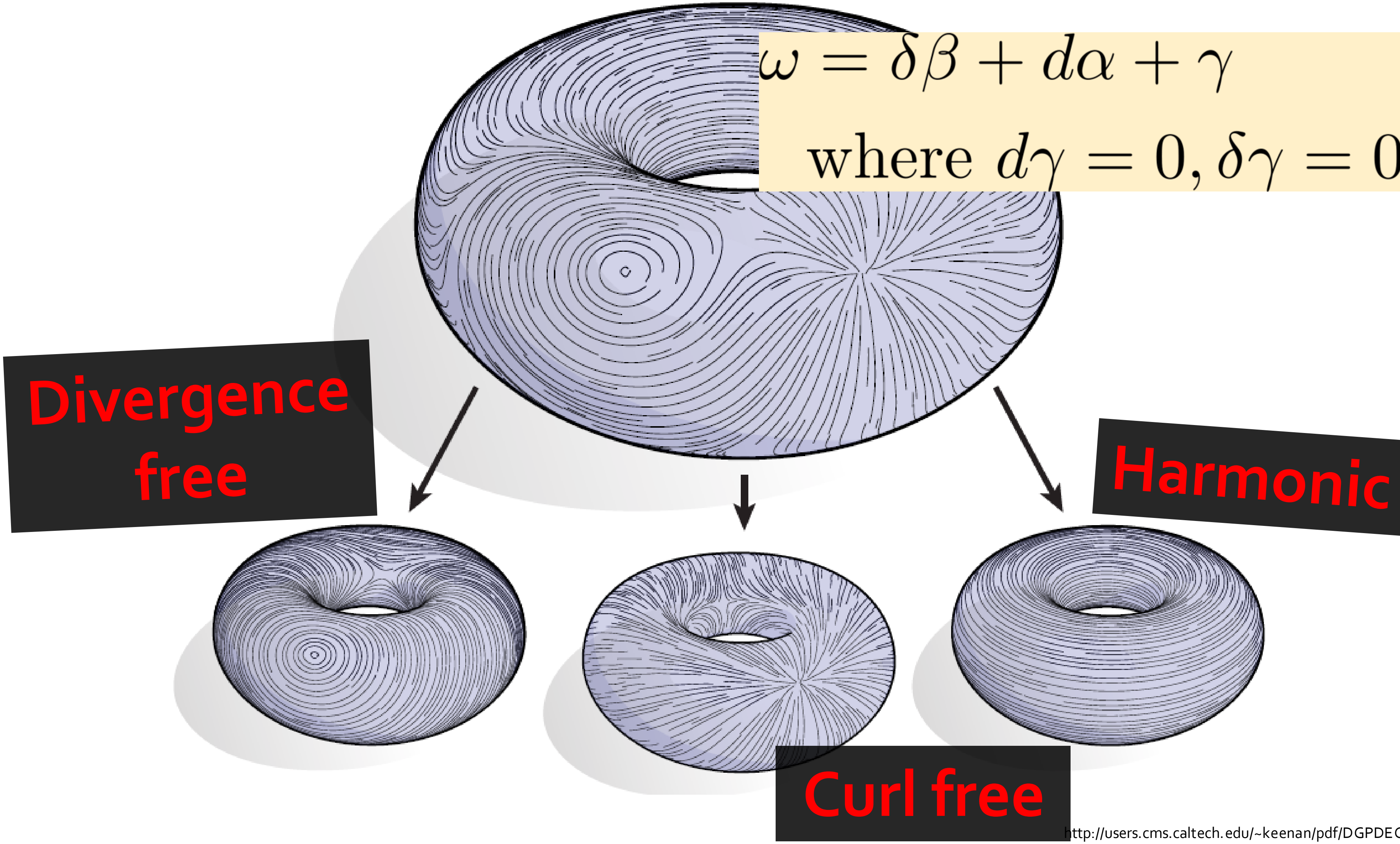


Discontinuous along edges!

Image courtesy F. de Goes

Interpolate one-form over triangle

Helmholtz-Hodge Decomposition



$$\omega = \delta\beta + d\alpha + \gamma$$
$$\text{where } d\gamma = 0, \delta\gamma = 0$$

http://users.cms.caltech.edu/~keenan/pdf/DGPDEC.pdf

Computing the Decomposition

$$\omega = \delta\beta + d\alpha + \gamma$$

$$\text{where } d\gamma = 0, \delta\gamma = 0$$

Also exists a simple topological algorithm

$$\delta d\alpha = \delta\omega$$

$$d\delta\beta = d\omega$$

$$\gamma = \omega - \delta\beta - d\alpha$$

One-Form Laplacian Eigenforms

$$\omega = \delta\beta + d\alpha + \gamma$$

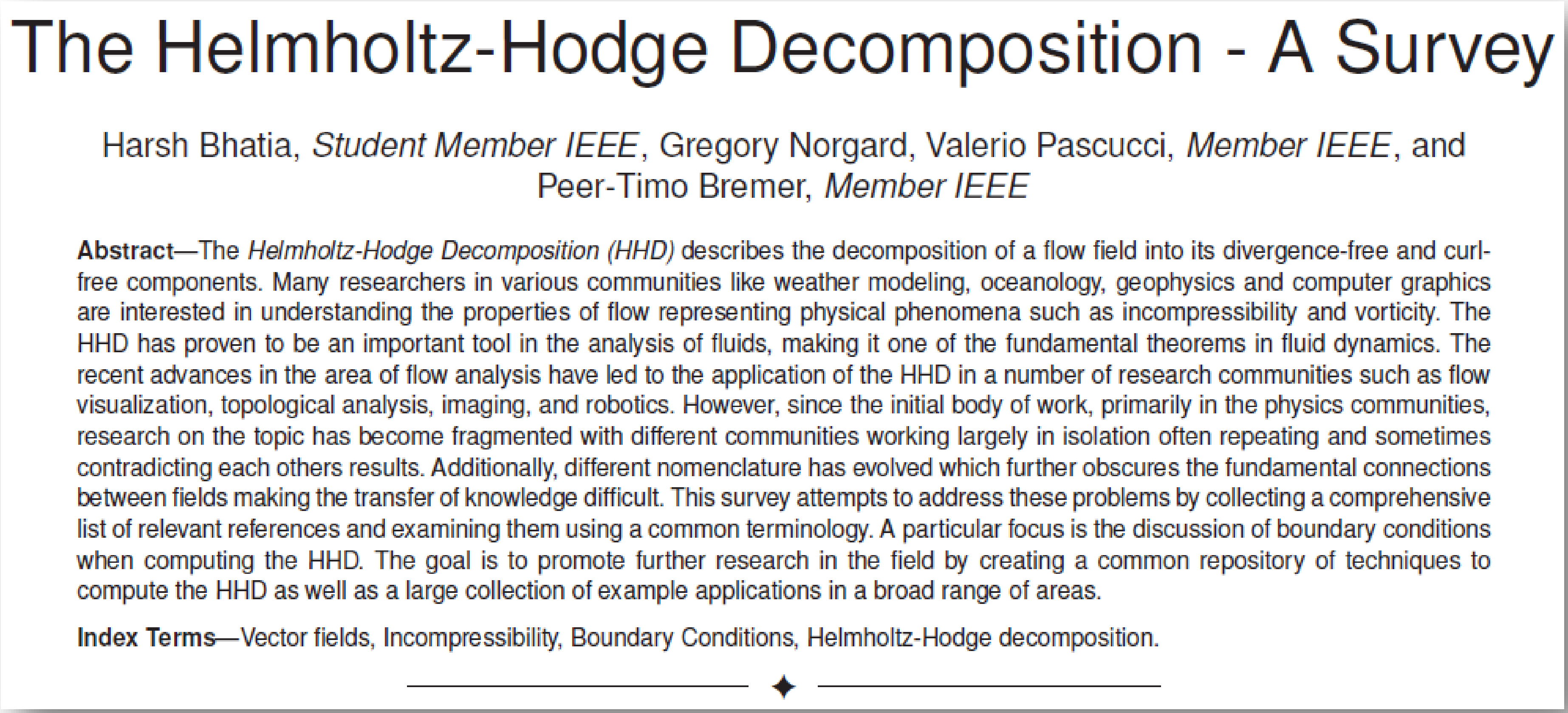
$$\text{where } d\gamma = 0, \delta\gamma = 0$$

$$\bar{\beta} := \star\beta$$

$$\begin{aligned}
\lambda(-\star d\bar{\beta} + d\alpha + \gamma) = \lambda\omega &= \Delta\omega \\
&= (d\star d\star + \star d\star d)(\delta\beta + d\alpha + \gamma) \\
&= (d\star d\star + \star d\star d)(-\star d\star\beta + d\alpha) \\
&= -\star d\star d\star d\star\beta + d\star d\star d\alpha \\
&= -\star d\Delta\bar{\beta} + d\Delta\alpha
\end{aligned}$$

Conclusion: For $\lambda \neq 0$, they're obtained by d and $\star d$ of Laplacian eigenfunctions.

Recommended Reading



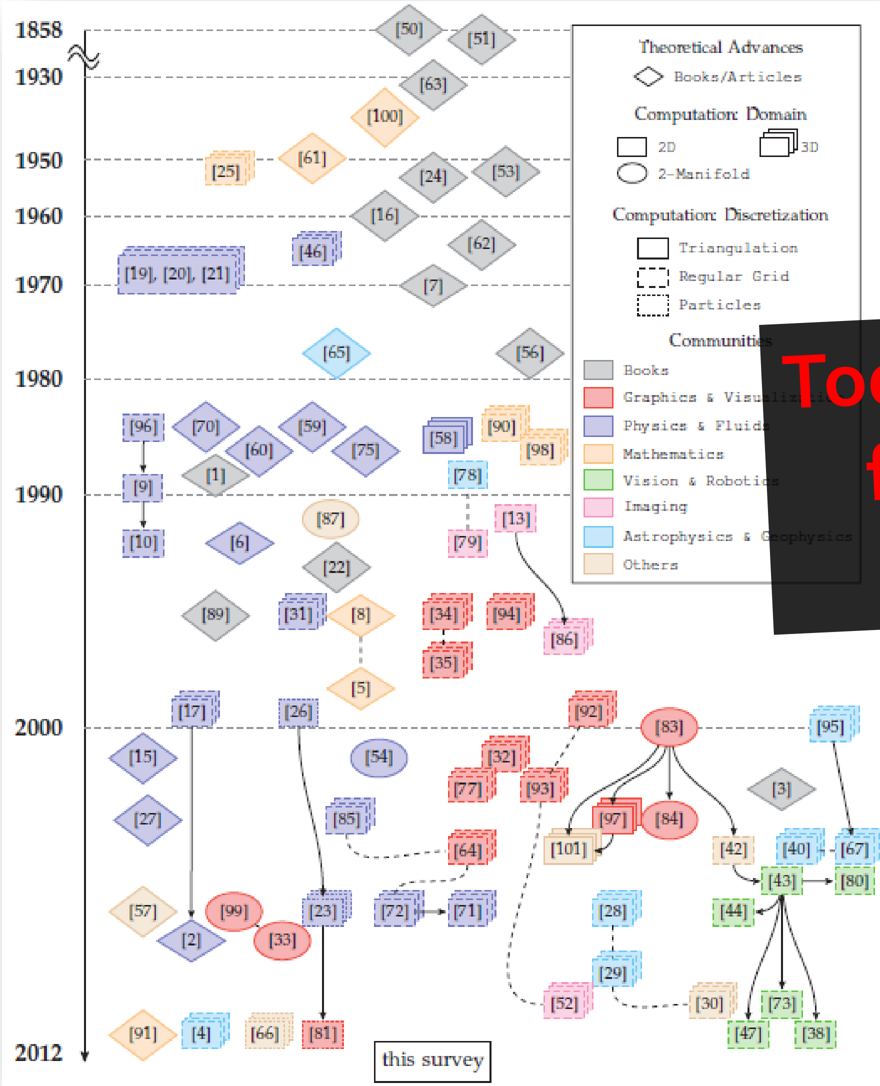
The Helmholtz-Hodge Decomposition - A Survey

Harsh Bhatia, *Student Member IEEE*, Gregory Norgard, Valerio Pascucci, *Member IEEE*, and Peer-Timo Bremer, *Member IEEE*

Abstract—The *Helmholtz-Hodge Decomposition (HHD)* describes the decomposition of a flow field into its divergence-free and curl-free components. Many researchers in various communities like weather modeling, oceanology, geophysics and computer graphics are interested in understanding the properties of flow representing physical phenomena such as incompressibility and vorticity. The HHD has proven to be an important tool in the analysis of fluids, making it one of the fundamental theorems in fluid dynamics. The recent advances in the area of flow analysis have led to the application of the HHD in a number of research communities such as flow visualization, topological analysis, imaging, and robotics. However, since the initial body of work, primarily in the physics communities, research on the topic has become fragmented with different communities working largely in isolation often repeating and sometimes contradicting each others results. Additionally, different nomenclature has evolved which further obscures the fundamental connections between fields making the transfer of knowledge difficult. This survey attempts to address these problems by collecting a comprehensive list of relevant references and examining them using a common terminology. A particular focus is the discussion of boundary conditions when computing the HHD. The goal is to promote further research in the field by creating a common repository of techniques to compute the HHD as well as a large collection of example applications in a broad range of areas.

Index Terms—Vector fields, Incompressibility, Boundary Conditions, Helmholtz-Hodge decomposition.

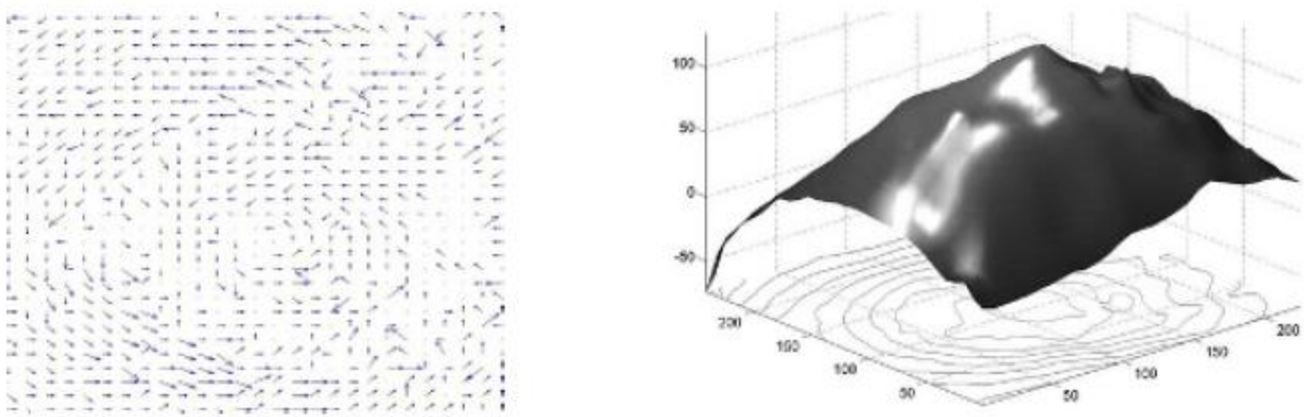
Recommended Reading



Today will take a few random samples

Simple Application

Fig. 2. Sequence of images from the Hurricane Luis sequence, with eye segmented



(a) (b)

Fig. 1. (a) Motion field in a anticlockwise rotating hurricane sequence extracted using the BMA. (b) The divergence free potential function with a distinct maximum and corresponding contours.

Palit, Basu, Mandal. "Applications of the Discrete Hodge Helmholtz Decomposition to Image and Video Processing." LNCS.

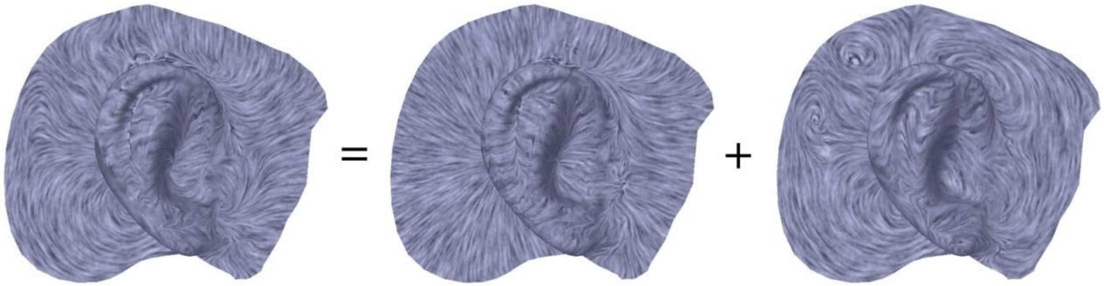
Fluid Simulation

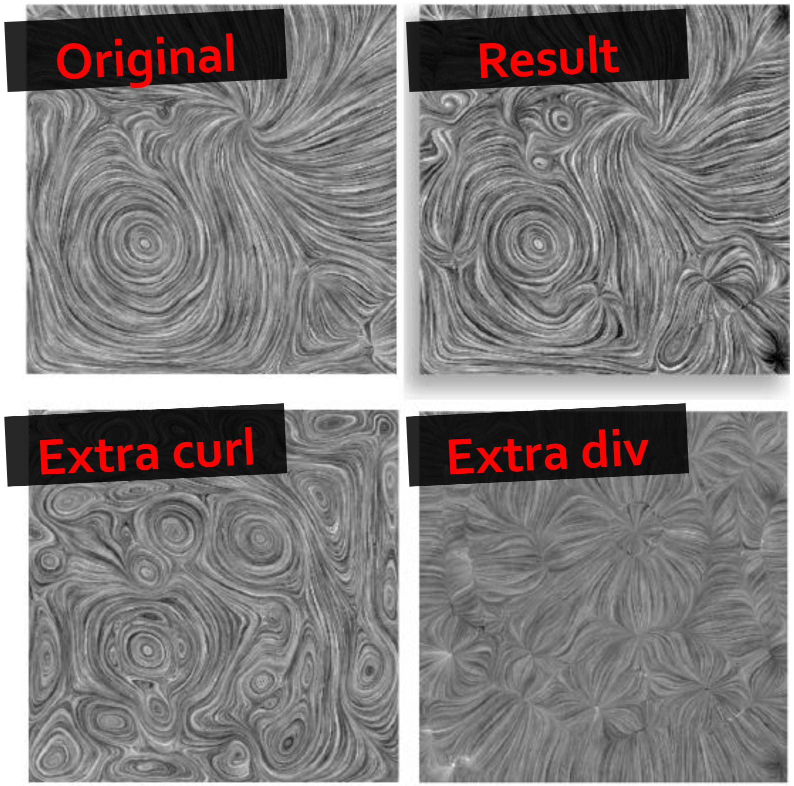
Divergence-free projection

Stam. "Stable Fluids." SIGGRAPH 1999. (and many others)

Incompressible: No divergence

Vector Field Editing

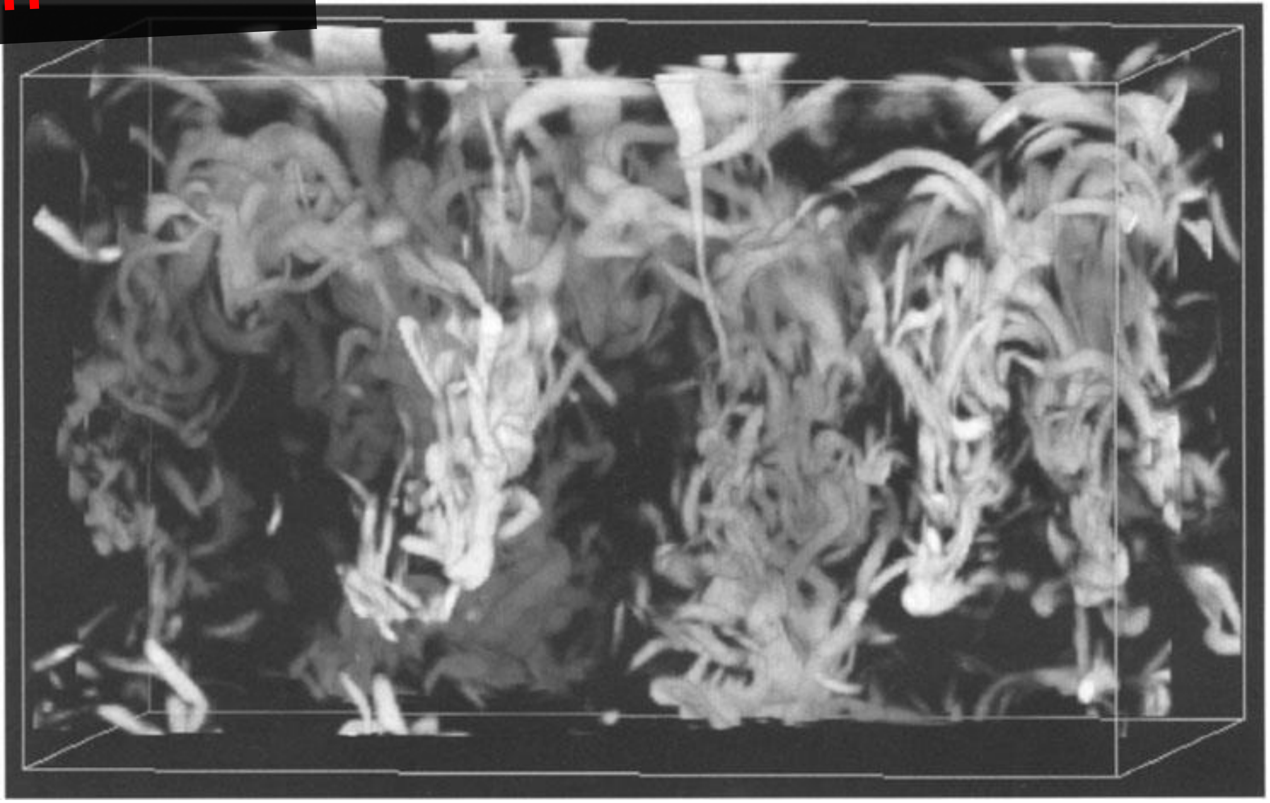




Tong et al. "Discrete Multiscale Vector Field Decomposition." TOG 2003.

Computational Physics

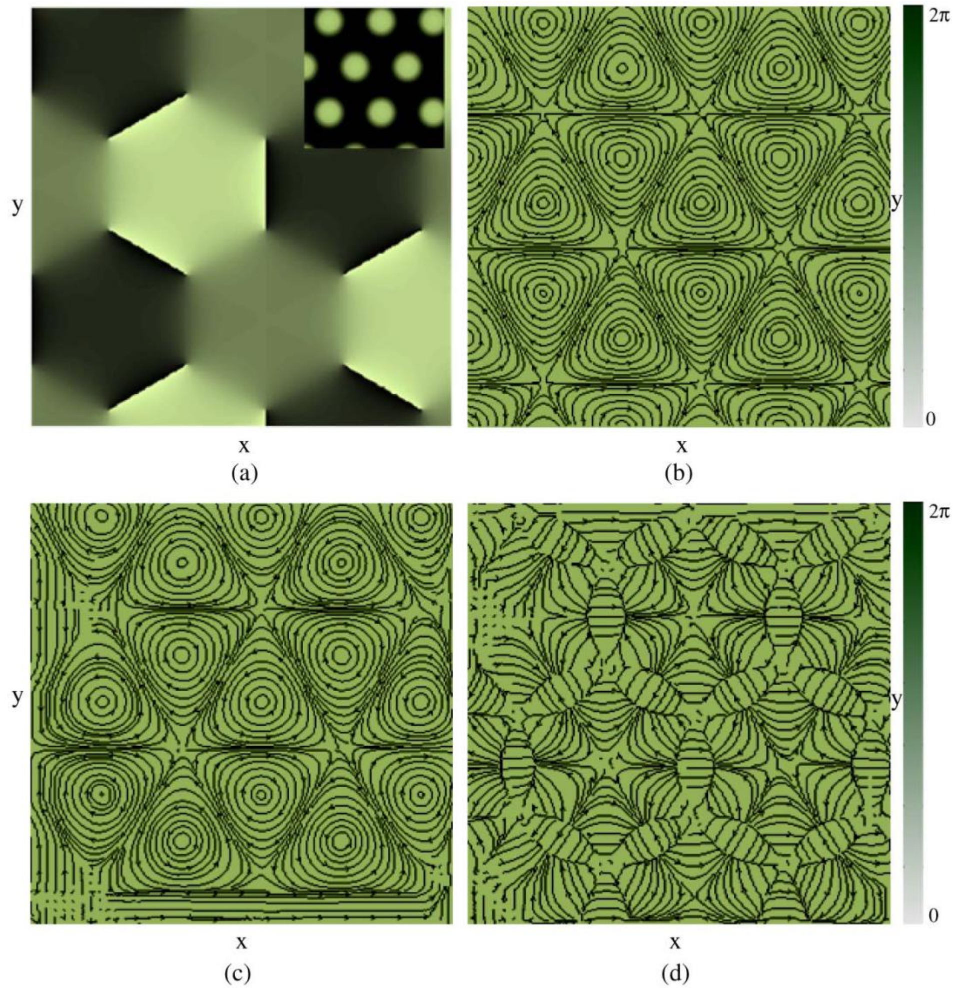
Separate turbulence from acoustics in solar simulation



Stein and Nordlund. "Realistic Solar Convection Simulations."
***Solar Physics* 2000.**

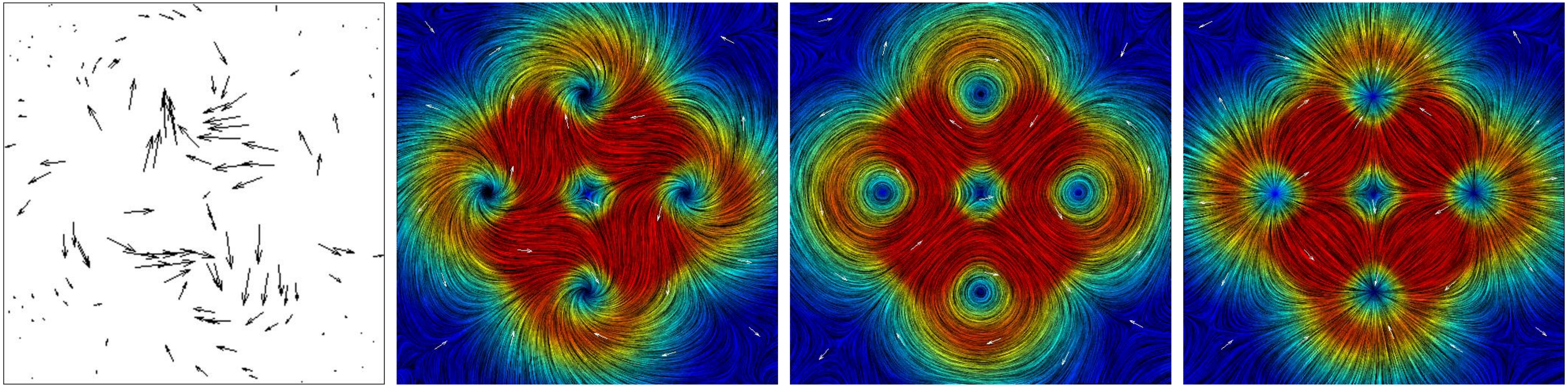
Computational Physics

Analyze interference and diffraction optics



Bahl and Senthilkumaran. "Helmholtz Hodge Decomposition of Scalar Optical Fields." J. Opt. Soc. Am. A 2012.

Reconstruct VF from Noisy Samples



$$\Phi_{df}(x) = H\phi(x) - tr\left\{H\phi(x)\right\} I$$

$$\Phi_{cf}(x) = -H\phi(x)$$

Macedo and Castro.
"Learning Divergence-Free and Curl-Free Vector Fields with Matrix-Valued Kernels."

Extension to Smooth Surfaces

Subdivision Exterior Calculus for Geometry Processing

Fernando de Goes, Pixar Animation Studios
Mathieu Desbrun, Caltech
Mark Meyer, Pixar Animation Studios
Tony DeRose, Pixar Animation Studios

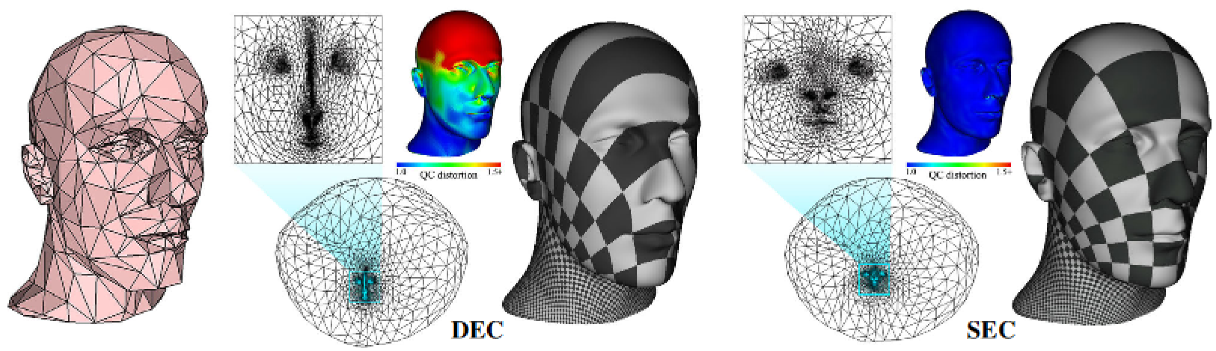


Figure 1: ***Subdivision Exterior Calculus (SEC).*** *We introduce a new technique to perform geometry processing applications on subdivision surfaces by extending Discrete Exterior Calculus (DEC) from the polygonal to the subdivision setting. With the preassemble of a few operators on the control mesh, SEC outperforms DEC in terms of numerics with only minor computational overhead. For instance, while the spectral conformal parameterization [Mullen et al. 2008] of the control mesh of the mannequin head (left) results in large quasi-conformal distortion* (mean = 1.784, max = 9.4) *after subdivision (middle), simply substituting our SEC operators for the original DEC operators significantly reduces distortion* (mean = 1.005, max = 3.0) *(right). Parameterizations, shown at level 1 for clarity, exhibit substantial differences.*

Abstract

This paper introduces a new computational method to solve differential equations on subdivision surfaces. Our approach adapts the numerical framework of Discrete Exterior Calculus (DEC) from the polygonal to the subdivision setting by exploiting the refinability of subdivision basis functions. The resulting *Subdivision Exterior Calculus* (SEC) provides significant improvements in accuracy compared to existing polygonal techniques, while offering exact finite-dimensional analogs of continuum structural identities such as Stokes' theorem and Helmholtz-Hodge decomposition. We demonstrate the versatility and efficiency of SEC on common geometry processing tasks including parameterization, geodesic distance computation, and vector field design.

Keywords: Subdivision surfaces, discrete exterior calculus, dis-

and Schröder 2000; Warren and Weimer 2001]. In spite of this prominence, little attention has been paid to numerically solving differential equations on subdivision surfaces. This is in sharp contrast to a large body of work in geometry processing that developed discrete differential operators for polygonal meshes [Botsch et al. 2010] serving as the foundations for several applications ranging from parameterization to fluid simulation [Crane et al. 2013a].

Among the various polygonal mesh techniques, Discrete Exterior Calculus (DEC) [Desbrun et al. 2008] is a coordinate-free formalism for solving scalar and vector valued differential equations. In particular, it reproduces, rather than merely approximates, essential properties of the differential setting such as Stokes' theorem. Given that the control mesh of a subdivision surface is a polygonal mesh, applying existing DEC methods directly to the control mesh

What About Symmetric Tensors?

Discrete 2-Tensor Fields on Triangulations

Fernando de Goes[1] Beibei Liu[2] Max Budninskiy[1] Yiying Tong[2] Mathieu Desbrun[1,3]

[1]Caltech [2]MSU [3]INRIA Sophia-Antipolis Méditerranée

Abstract

Geometry processing has made ample use of discrete representations of tangent vector fields and antisymmetric tensors (i.e., forms) on triangulations. Symmetric 2-tensors, while crucial in the definition of inner products and elliptic operators, have received only limited attention. They are often discretized by first defining a coordinate system per vertex, edge or face, then storing their components in this frame field. In this paper, we introduce a representation of arbitrary 2-tensor fields on triangle meshes. We leverage a coordinate-free decomposition of continuous 2-tensors in the plane to construct a finite-dimensional encoding of tensor fields through scalar values on oriented simplices of a manifold triangulation. We also provide closed-form expressions of pairing, inner product, and trace for this discrete representation of tensor fields, and formulate a discrete covariant derivative and a discrete Lie bracket. Our approach extends discrete/finite-element exterior calculus, recovers familiar operators such as the weighted Laplacian operator, and defines discrete notions of divergence-free, curl-free, and traceless tensors—thus offering a numerical framework for discrete tensor calculus on triangulations. We finally demonstrate the robustness and accuracy of our operators on analytical examples, before applying them to the computation of anisotropic geodesic distances on discrete surfaces.

Categories and Subject Descriptors (according to ACM CCS): Computer Graphics [I.3.5]: Computational Geometry and Object Modeling—Curve and surface representations.

More to do here!

(rank-1 tensor) fields have been staples of geometry processing, the use of rank-2 tensor fields has steadily grown over the last decade in applications ranging from non-photorealistic rendering to anisotropic meshing. Unlike their lower rank counterparts

frames defined either on vertices or on faces. A continuous vector field over a mesh is evaluated from this finite set of vectors based on piecewise constant interpolation [PP00] or, to increase smoothness, using non-linear basis functions derived from the geodesic polar map [ZMT06, KCPS13]. In an effort to remove the need for coordinate systems, scalar

Summary

Pros

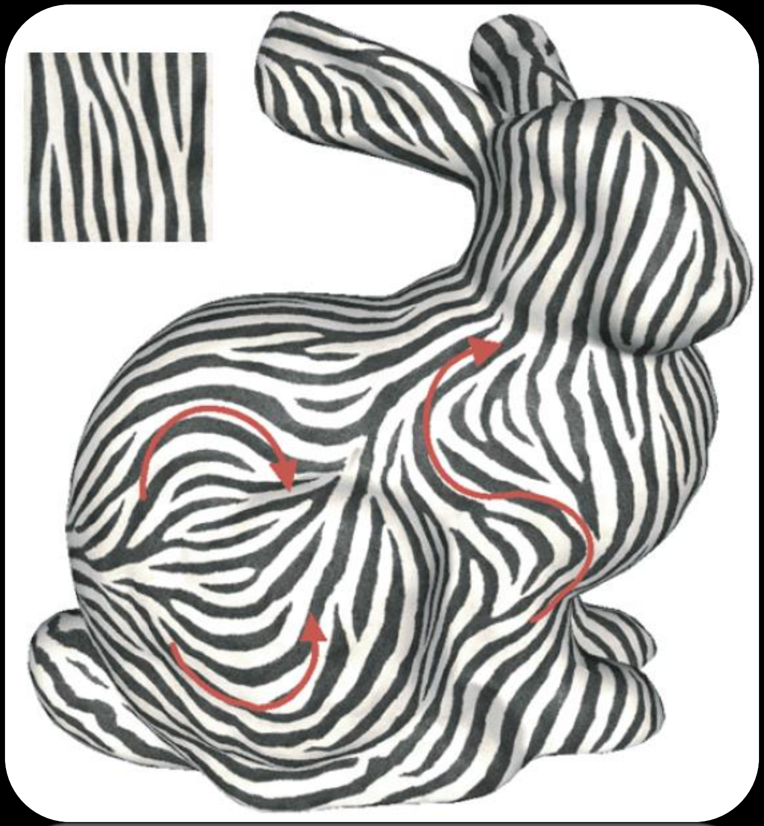
- Coordinate-free representation using only one scalar value per edge.
- Simple interpolation of edge values.
- Simple differential operators leveraging the DEC literature.

Cons

- Discontinuous reconstruction for low-order Whitney basis functions.
- No clear vector at vertices, so incompatible with vertex-based deformation of meshes.
- Generalization to n-vector fields has not been studied.

From *Vector Field Processing on Triangle Meshes*
de Goes, Desbrun, and Tong (SIGAsia 2015)

Fisher et al. "Design of Tangent Vector Fields" (SIGGRAPH 2007)



Discrete Exterior Calculus

Justin Solomon

MIT, Spring 2019

Original version from Stanford CS 468, spring 2013 (Butscher & Solomon)