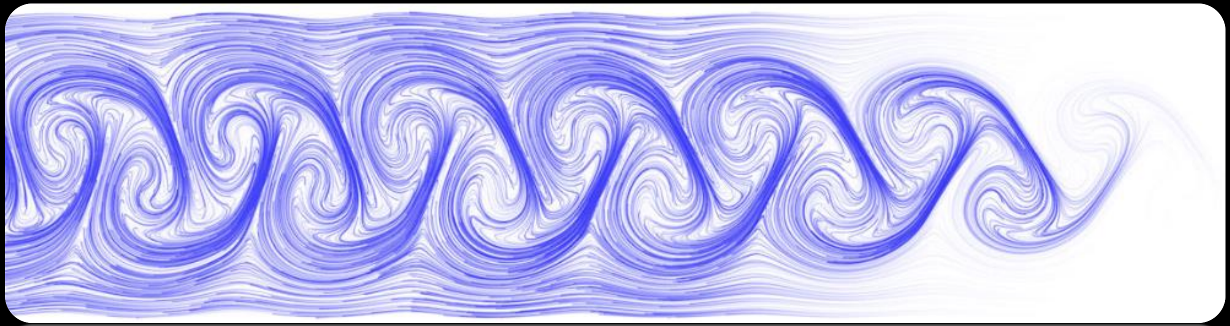
Image from "Streak Lines as Tangent Curves of a Derived Vector Field" (Weinkauf and Theisel; TVCG 2010)



Vector Field Processing

Lots of Material/Slides From…

Vector Field Processing on triangle meshes

Fernando de Goes
Pixar Animation Studios

Mathieu Desbrun
Caltech

Yiying Tong
Michigan State University

Render the Possibilities
SIGGRAPH2016

Check out course notes!

Additional Nice Reference

EUROGRAPHICS 2016
J. Madeira and G. Patow
(Guest Editors)

Volume 35 (2016), Number 2
STAR – State of The Art Report

Directional Field Synthesis, Design, and Processing

Amir Vaxman[1] Marcel Campen[2] Olga Diamanti[3] Daniele Panozzo[2,3] David Bommes[4] Klaus Hildebrandt[5] Mirela Ben-Chen[6]

[1]Utrecht University [2]New York University [3]ETH Zurich [4]RWTH Aachen University [5]Delft University of Technology [6]Technion

Abstract

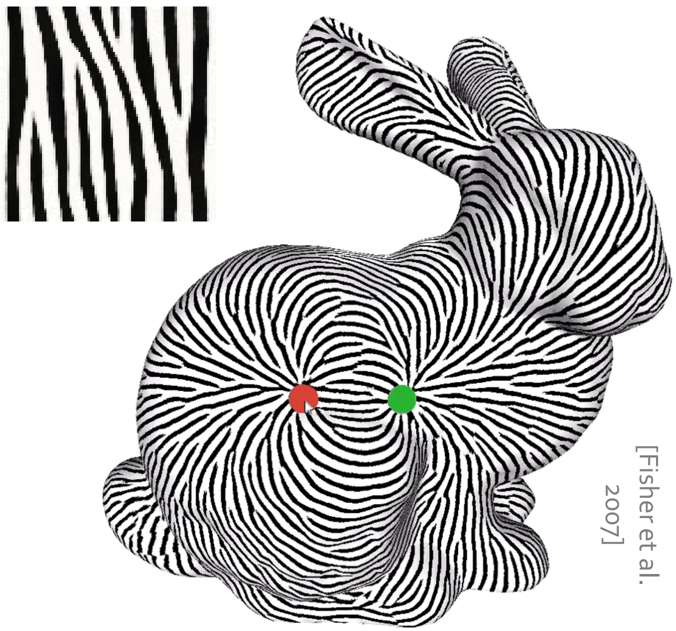
Direction fields and vector fields play an increasingly important role in computer graphics and geometry processing. The synthesis of directional fields on surfaces, or other spatial domains, is a fundamental step in numerous applications, such as mesh generation, deformation, texture mapping, and many more. The wide range of applications resulted in definitions for many types of directional fields: from vector and tensor fields, over line and cross fields, to frame and vector-set fields. Depending on the application at hand, researchers have used various notions of objectives and constraints to synthesize such fields. These notions are defined in terms of fairness, feature alignment, symmetry, or field topology, to mention just a few. To facilitate these objectives, various representations, discretizations, and optimization strategies have been developed. These choices come with varying strengths and weaknesses. This report provides a systematic overview of directional field synthesis for graphics applications, the challenges it poses, and the methods developed in recent years to address these challenges.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—

1. Introduction

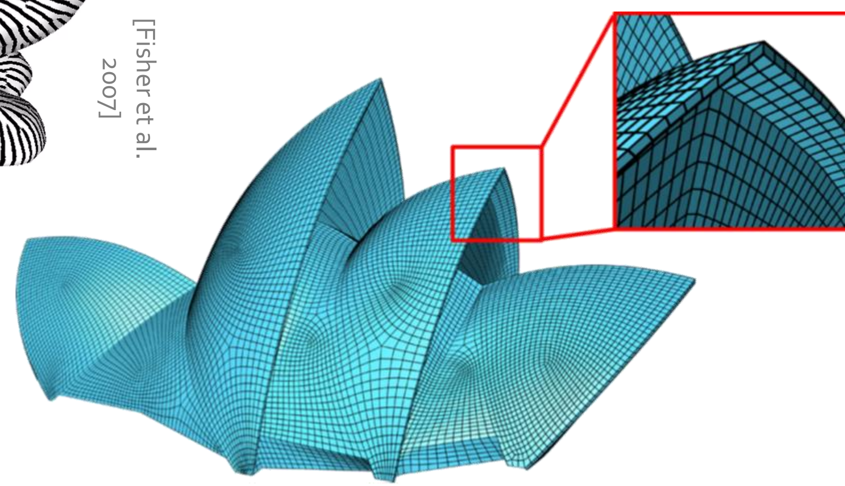
There have been significant developments in directional field synthesis over the past decade. These developments have been driven

Why Vector Fields?



[Fisher et al. 2007]

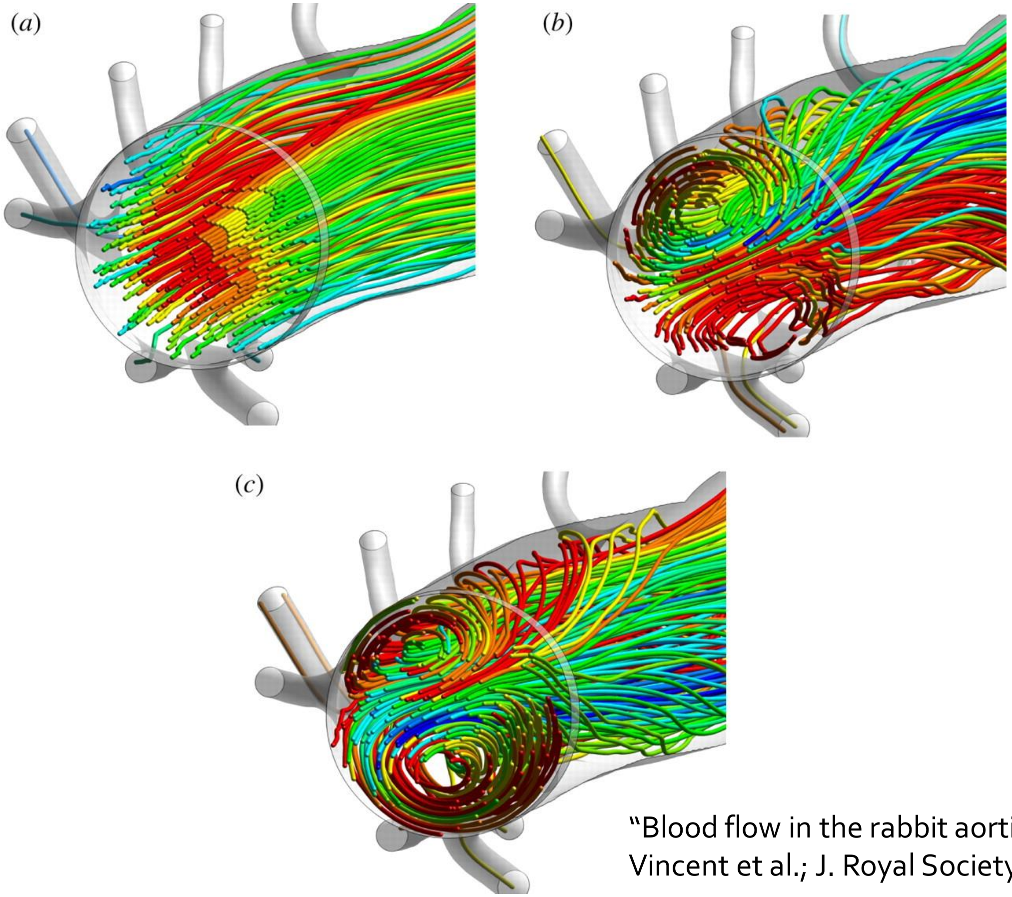
© Disney/Pixar



[Jiang et al. 2015]

Graphics

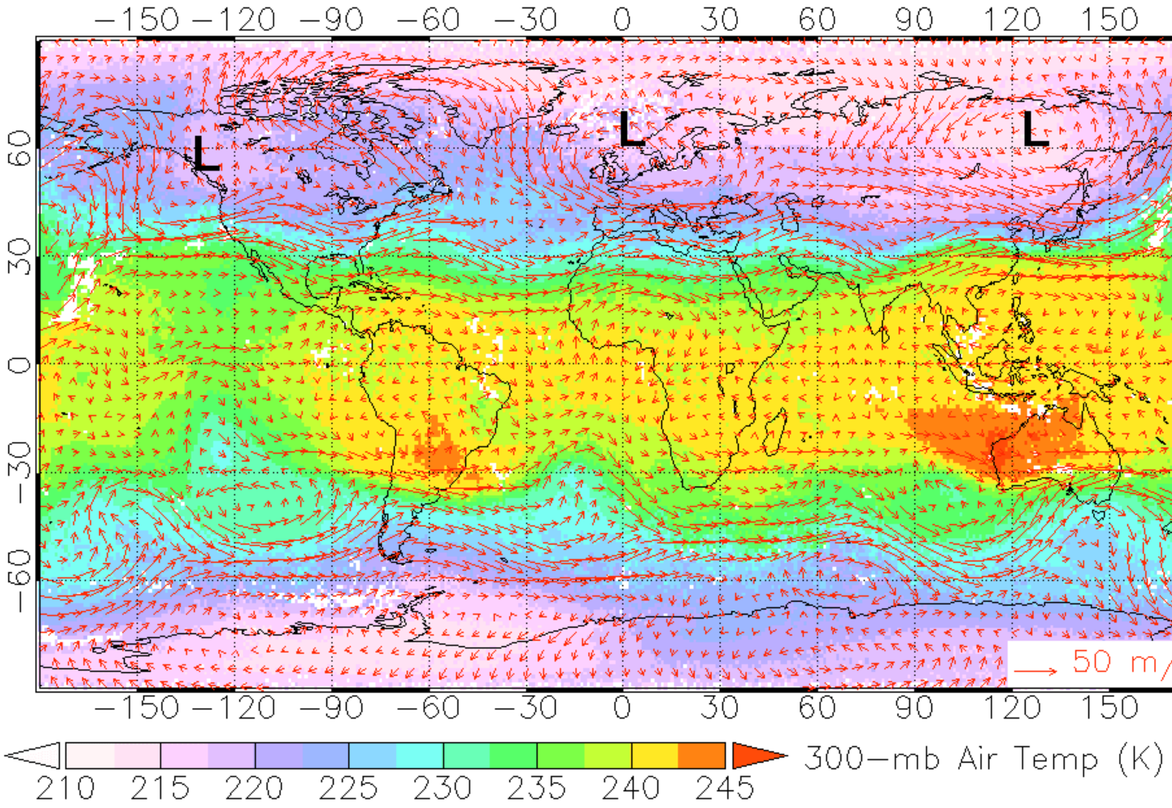
Why Vector Fields?



"Blood flow in the rabbit aortic arch and descending thoracic aorta"
Vincent et al.; J. Royal Society 2011

Biological science and imaging

Why Vector Fields?



https://disc.gsfc.nasa.gov/featured-items/airs-monitors-cold-weather

Weather modeling

Why Vector Fields?

https://forum.unity3d.com/threads/megaflow-vector-fields-fluid-flows-released.278000/

Simulation and engineering

Many Challenges

- Directional derivative?
- Purely intrinsic version?
- Singularities?
- Flow lines?
- …

Theoretical

- How to discretize?
- Discrete derivatives?
- Singularity detection?
- Flow line computation?
- …

Discrete

Plan

Crash course

in theory/discretization of vector fields.

Many Challenges

- Directional derivative?
- Purely intrinsic version?
- Singularities?
- Flow lines?
- …

Theoretical

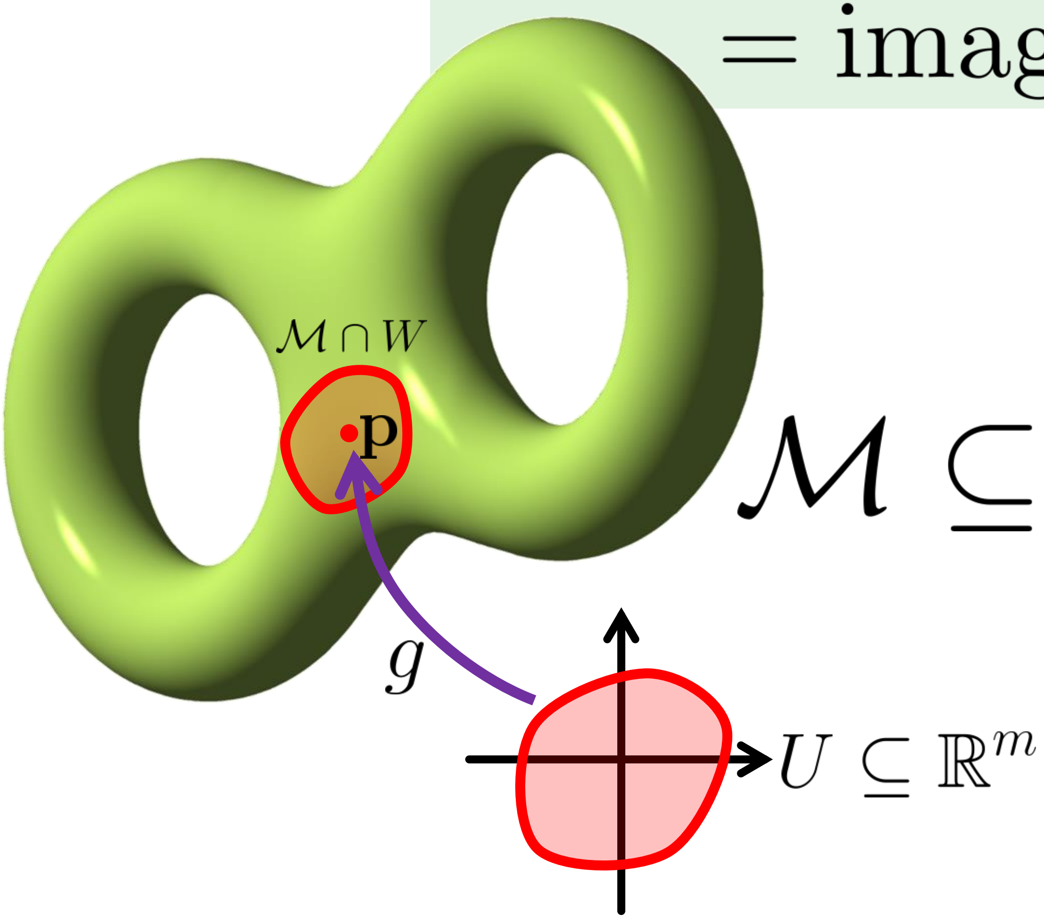
- How to discretize?
- Discrete derivatives?
- Singularity detection?
- Flow line computation?
- …

Discrete

Recall:

Tangent Space

$$T_{\mathbf{p}}\mathcal{M} = \gamma'(0), \text{ where } \gamma(0) = \mathbf{p}$$
$$= \text{image}(dg_{\mathbf{p}})$$



$\mathcal{M} \subseteq \mathbb{R}^n$

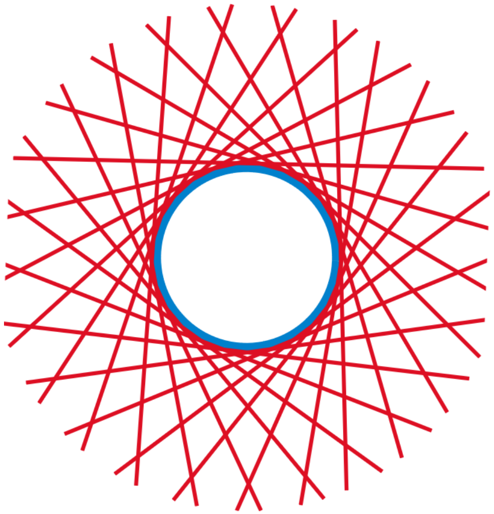
Some Definitions

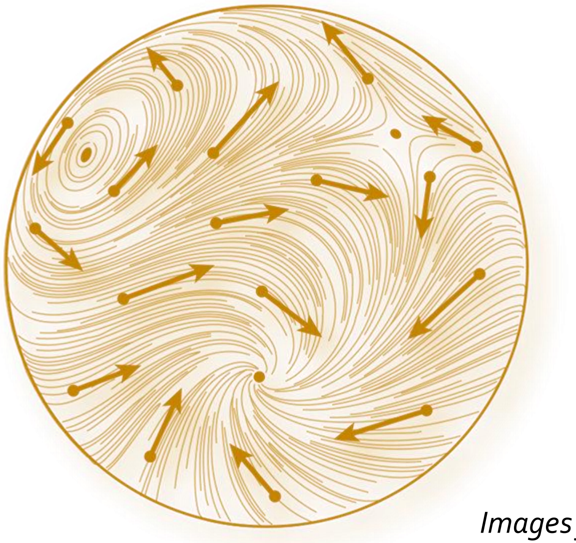
Tangent bundle:

$$TM := \{(p, v) : v \in T_pM\}$$

Vector field:

$$u : M \to TM \text{ with } u(p) = (p, v), v \in T_pM$$

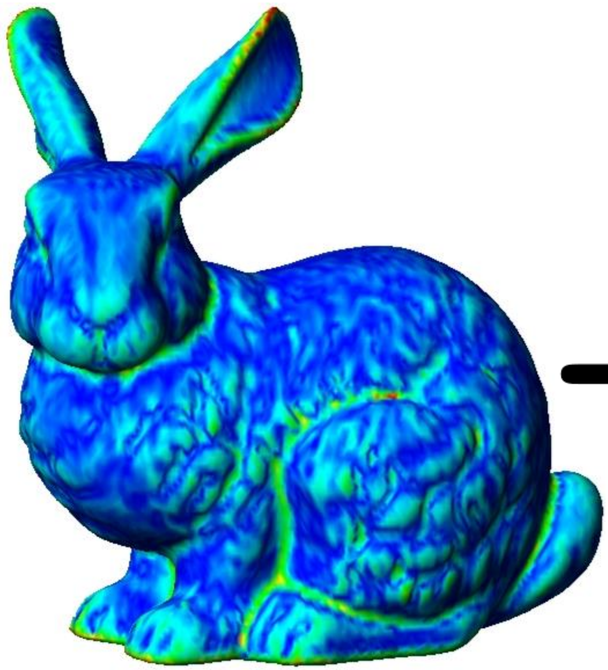




Images from Wikipedia, SIGGRAPH course

Recall:

Scalar Functions

$f :$  $\longrightarrow \mathbb{R}$

http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

Map points to real numbers

Recall:

Differential of a Map

Definition (Differential). *Suppose* $\varphi : \mathcal{M} \to \mathcal{N}$ *is a map from a submanifold* $\mathcal{M} \subseteq \mathbb{R}^k$ *into a submanifold* $\mathcal{N} \subseteq \mathbb{R}^\ell$. *Then, the* differential $d\varphi_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{M} \to T_{\varphi(\mathbf{p})}\mathcal{N}$ *of* φ *at a point* $\mathbf{p} \in \mathcal{M}$ *is given by*

$$d\varphi_{\mathbf{p}}(\mathbf{v}) := (\varphi \circ \gamma)'(0),$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ *is any curve with* $\gamma(0) = \mathbf{p}$ *and* $\gamma'(0) = \mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$.

Linear map of tangent spaces

$$d\varphi_{\mathbf{p}}(\gamma'(0)) := (\varphi \circ \gamma)'(0)$$

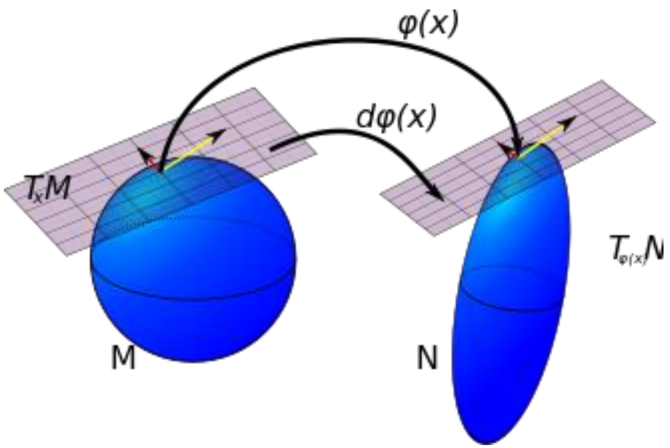
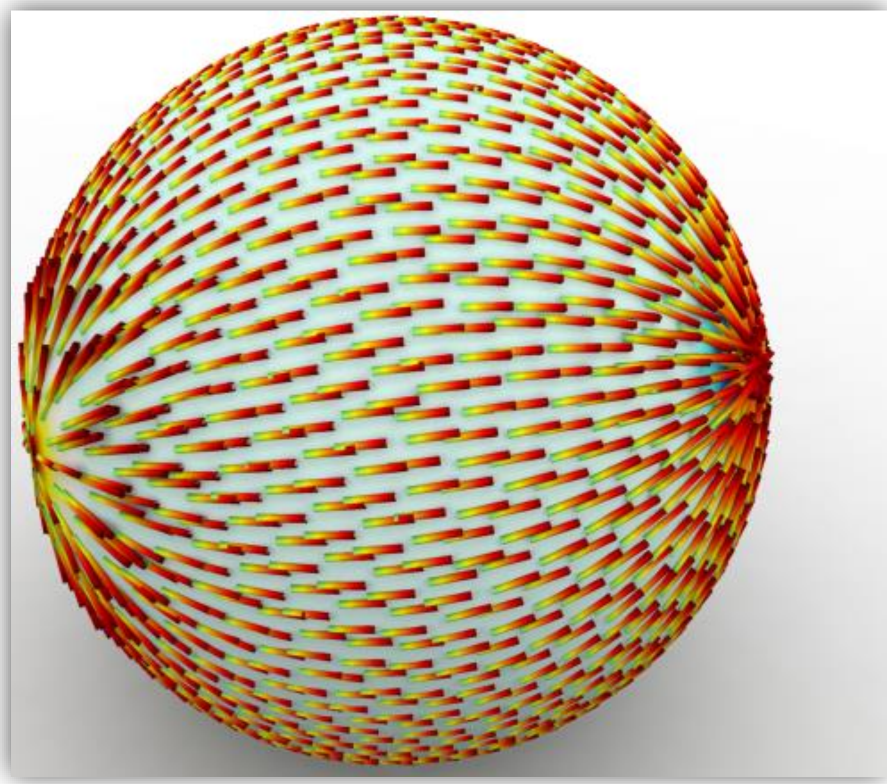


Image from Wikipedia

Recall:

Gradient Vector Field

Proposition 9.2. *For each* $\mathbf{p} \in \mathcal{M}$*, there exists a unique vector* $\nabla f(\mathbf{p}) \in T_{\mathbf{p}}\mathcal{M}$ *so that* $df_{\mathbf{p}}(\mathbf{v}) = \mathbf{v} \cdot \nabla f(\mathbf{p})$ *for all* $\mathbf{v} \in T_{\mathbf{p}}\mathcal{M}$*.*



Q:

How do you differentiate a vector field?

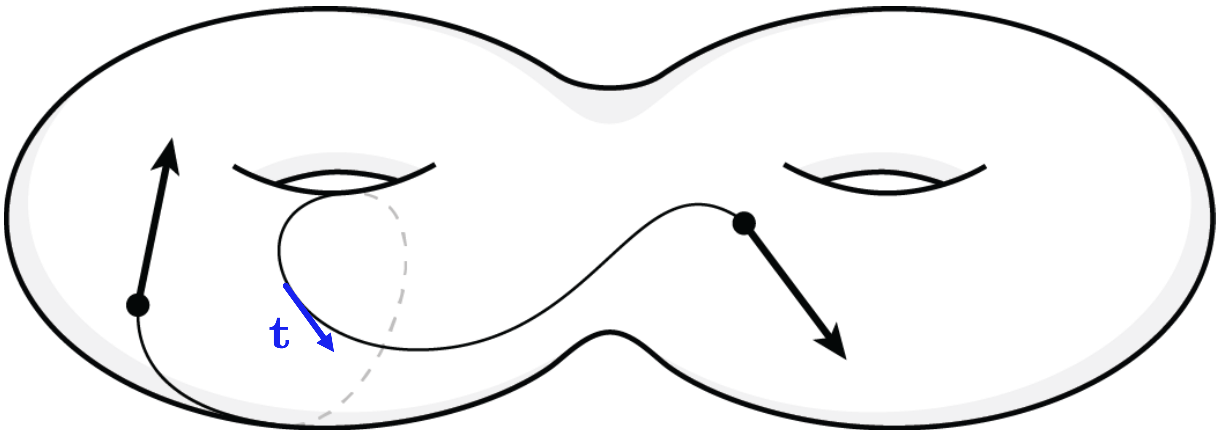
Common point of confusion.
(especially for your instructor)

http://www.wgnflag.com/xcart/images/P/G-50_StopSeatBeltsStockSign.jpg

Answer

http://www.relatably.com/m/img/complicated-memes/60260587.jpg

What's the issue?



How to identify different tangent spaces?

Many Notions of Derivative

- **Differential of covector**
  (defer for now)

- **Lie derivative**
  Weak structure, purely topological

- **Covariant derivative**
  Strong structure, involves geometry

Vector Field Flows: Diffeomorphism

$$\frac{d}{dt}\psi_t = V \circ \psi_t$$

Useful property: $\psi_{t+s}(x) = \psi_t(\psi_s(x))$

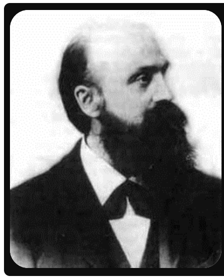
Diffeomorphism with inverse ψ_{-t}

Group structure!

Fun example:

Killing Vector Fields (KVFs)

Preserves distances infinitesimally



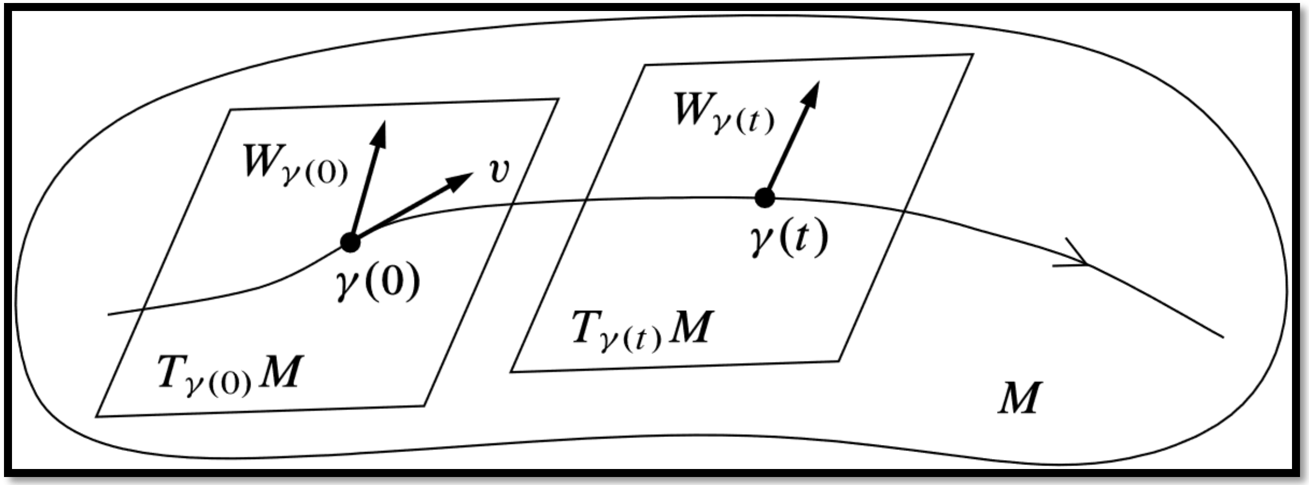
Wilhelm Killing

1847-1923

Germany

http://www.bradleycorp.com/image/985/9184b_highres.jpg

Differential of Vector Field Flow



$$d\psi_t(p) : T_pM \to T_{\psi_t(p)}M$$

Image from *Smooth Manifolds*, Lee

Lie Derivative

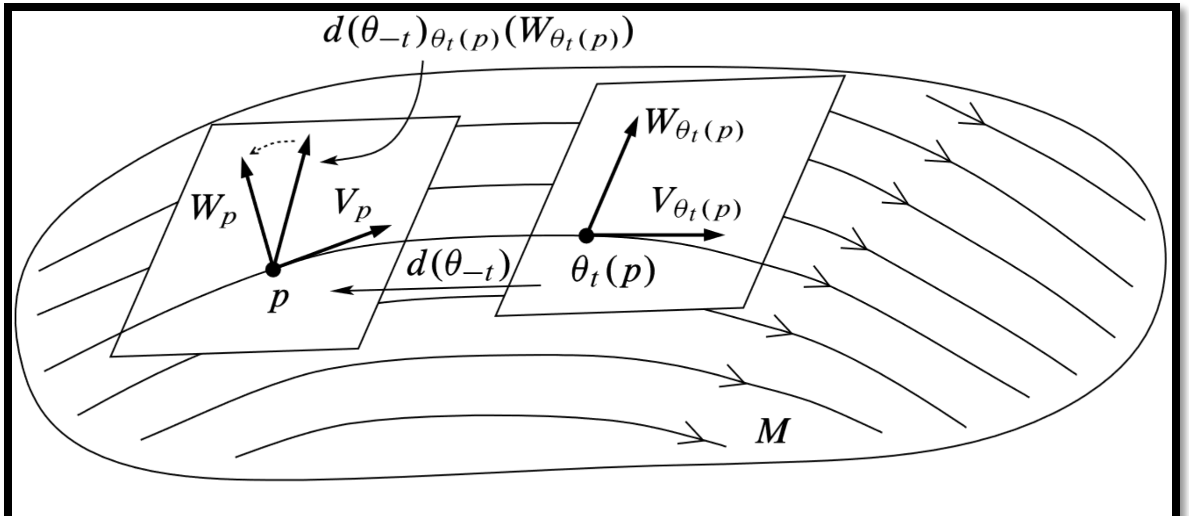


Fig. 9.13 The Lie derivative of a vector field

$$(\mathcal{L}_V W)_p := \lim_{t \to 0} \frac{1}{t} \left[(d\psi_{-t})_{\psi_t(p)}(W_{\psi_t(p)}) - W_p \right]$$

Image from *Smooth Manifolds*, Lee

It’s pronounced

“Lee”

Not “Lahy”

(BTW: It’s “oiler,” not “you-ler”)

What's Wrong with Lie Derivatives?

$$(\mathcal{L}_V W)_p := \lim_{t \to 0} \frac{1}{t} \left[(d\psi_{-t})_{\psi_t(p)} (W_{\psi_t(p)}) - W_p \right]$$

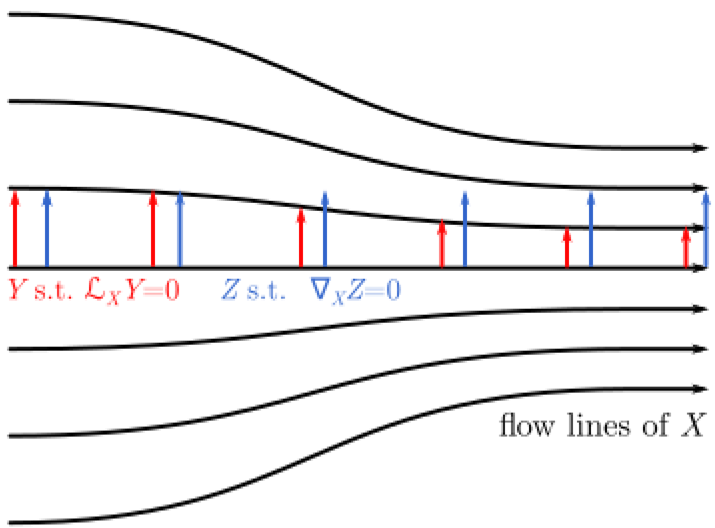
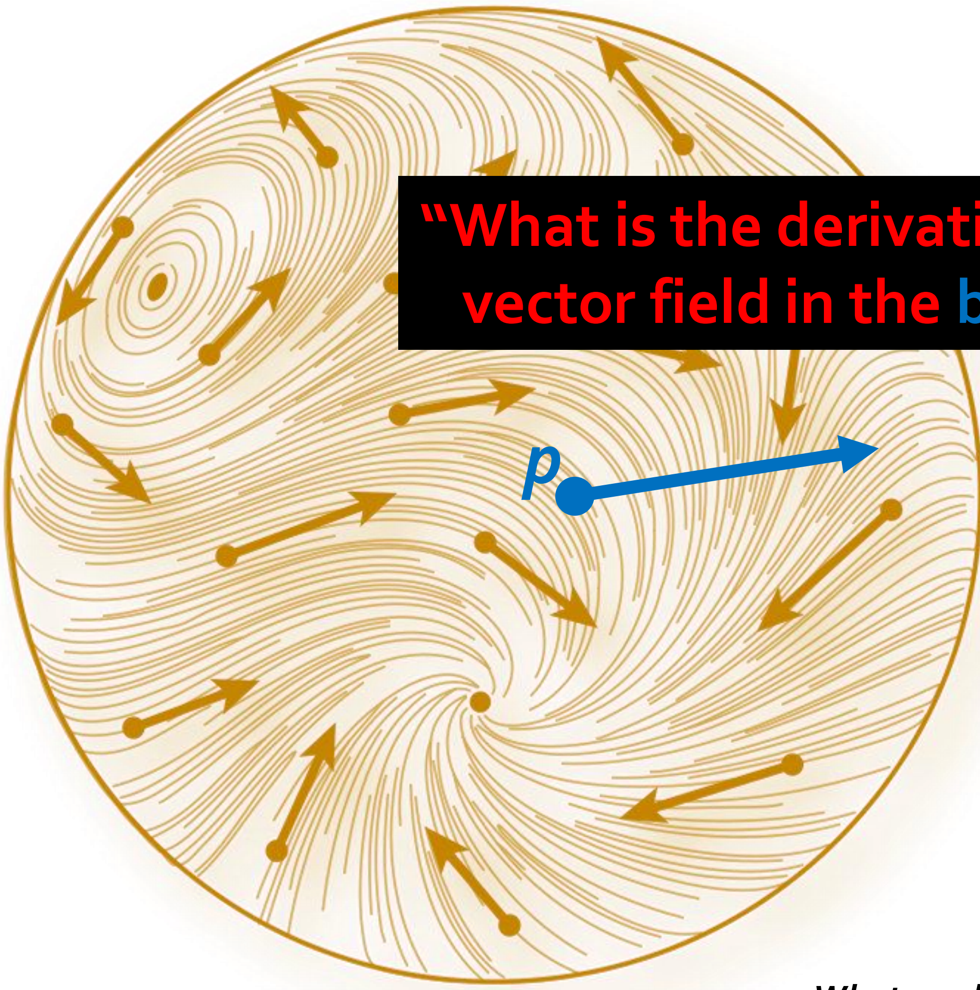


Image courtesy A. Carapetis

Depends on <u>structure</u> of *V*

What We Want

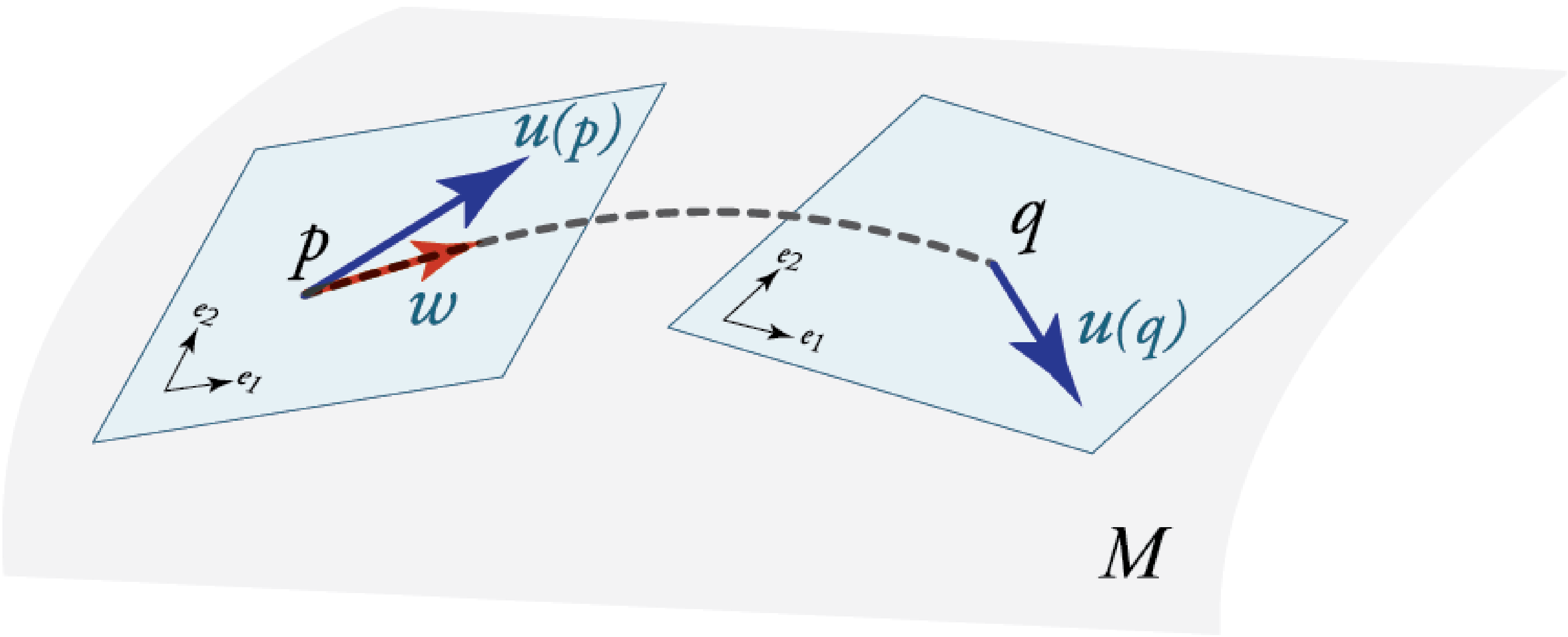


"What is the derivative of the orange vector field in the blue direction?"

What we don't want:

Specify blue direction anywhere but at *p*.

Parallel Transport



Canonical identification of tangent spaces

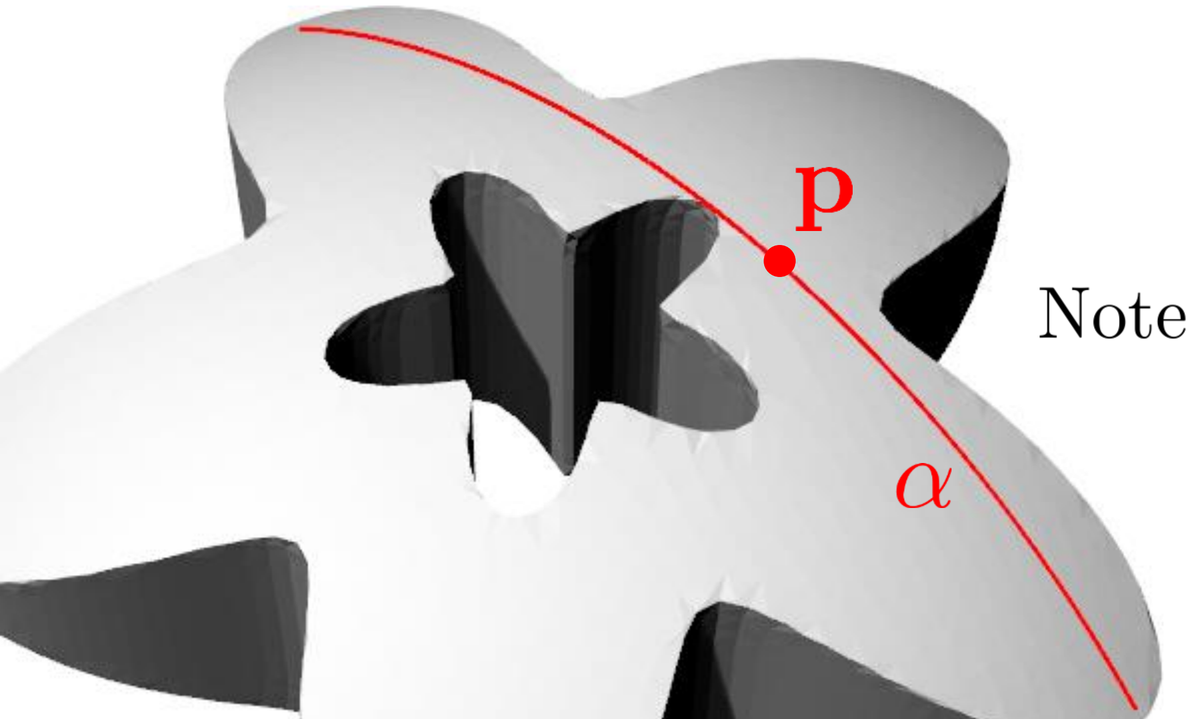
More later…

Covariant Derivative (Embedded)

$$\nabla_{\mathbf{v}}\mathbf{w} := [d\mathbf{w}(\mathbf{v})]^{\parallel} = \mathrm{proj}_{T_{\mathbf{p}}\mathcal{M}}(\mathbf{w} \circ \alpha)'(0)$$

Integral curve of v through p

Synonym: (Levi-Civita) Connection



$$\text{Note: } [d\mathbf{w}(\mathbf{v})]^{\perp} = \mathrm{I\!I}(\mathbf{v}, \mathbf{w})\mathbf{n}$$

Some Properties

Properties of the Covariant Derivative

As defined, $\nabla_V Y$ depends only on V_p and Y to first order along c.

Also, we have the Five Properties:

1. C^∞-linearity in the V-slot:
   $\nabla_{V_1 + fV_2} Y = \nabla_{V_1} Y + f \nabla_{V_2} Y$ where $f : S \to \mathbb{R}$
2. $\mathbb{R}$-linearity in the Y-slot:
   $\nabla_V (Y_1 + aY_2) = \nabla_V Y_1 + a \nabla_V Y_2$ where $a \in \mathbb{R}$
3. Product rule in the Y-slot:
   $\nabla_V (f\, Y) = f \cdot \nabla_V Y + (\nabla_V f) \cdot Y$ where $f : S \to \mathbb{R}$
4. The metric compatibility property:
   $\nabla_V \langle Y, Z \rangle = \langle \nabla_V Y, Z \rangle + \langle Y, \nabla_V Z \rangle$
5. The "torsion-free" property:
   $\nabla_{V_1} V_2 - \nabla_{V_2} V_1 = [V_1, V_2]$

The Lie bracket

$$[V_1, V_2](f) := D_{V_1} D_{V_2}(f) - D_{V_2} D_{V_1}(f)$$

Defines a vector field, which is tangent to S if V_1, V_2 are!

Slide by A. Butscher, Stanford CS 468

Recall:

Geodesic Equation

$$\text{proj}_{T_{\gamma(s)}\mathcal{M}}\left[\gamma''(s)\right] \equiv 0$$

- The only acceleration is out of the surface
  - No steering wheel!

Intrinsic Geodesic Equation

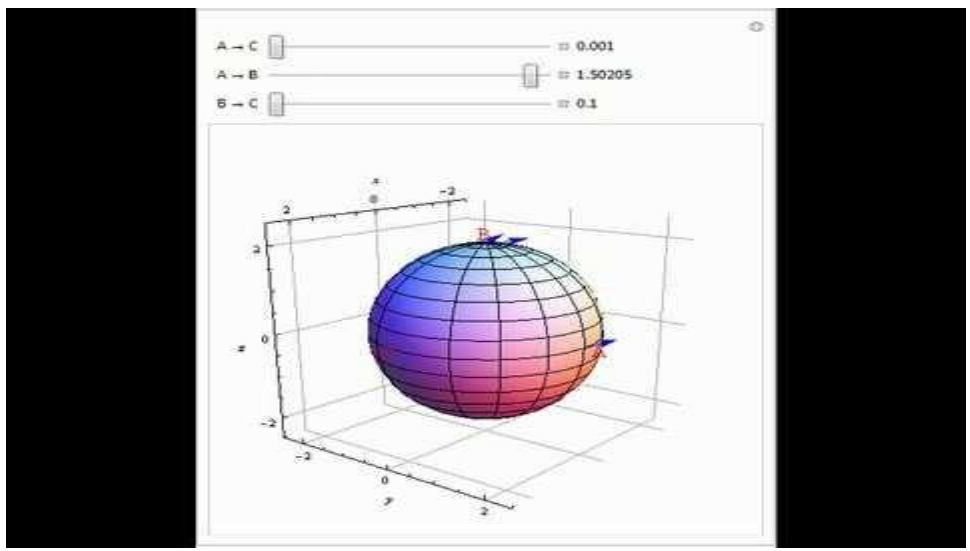
$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$$

- No stepping on the accelerator
- No steering wheel!

Parallel Transport

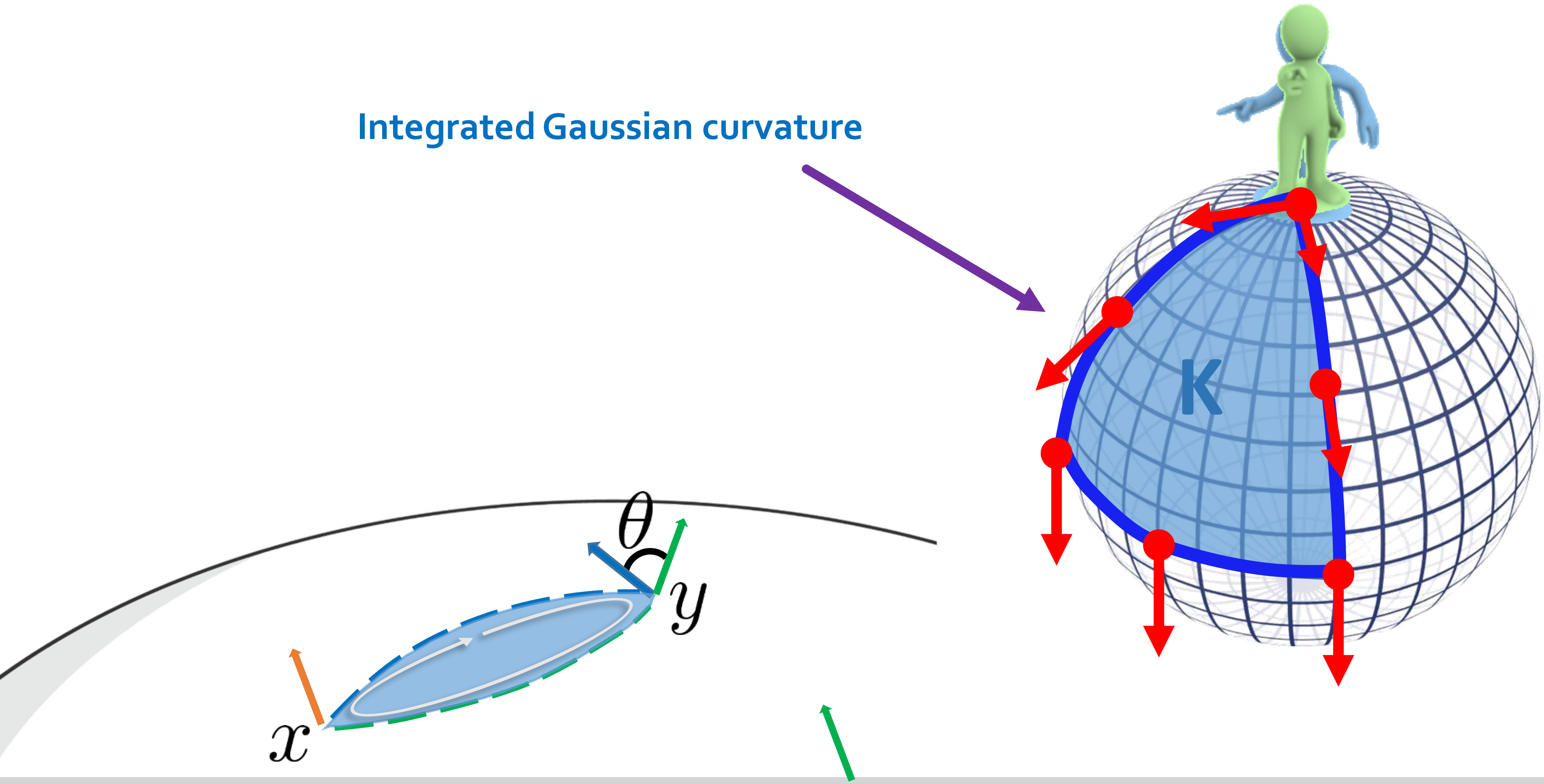
Only path-independent if domain is flat.

$$\mathbf{0} = \nabla_{\dot{\gamma}(t)} \mathbf{v}$$



Preserves length, inner product
(can be used to *define* covariant derivative)

Holonomy



Path dependence of parallel transport

Vector Field Topology

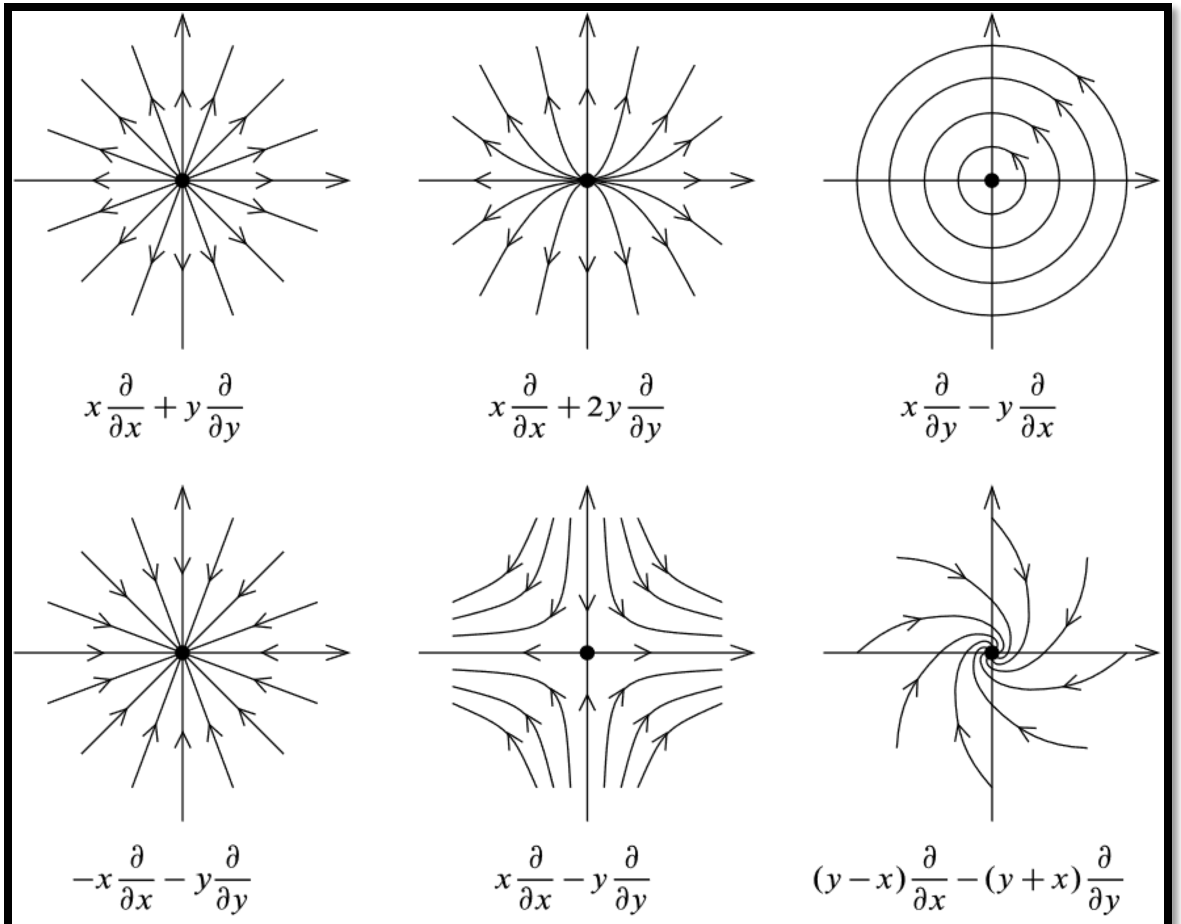


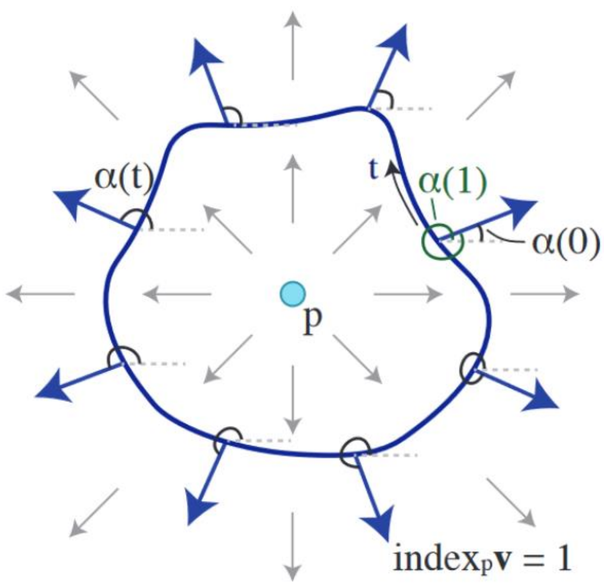
Fig. 9.8 Examples of flows near equilibrium points

Image from *Smooth Manifolds*, Lee

Poincaré-Hopf Theorem

$$\sum_i \operatorname{index}_{x_i}(v) = \chi(M)$$

where vector field v has isolated singularities $\{x_i\}$.



$$v(c(t)) = \|v(c(t))\| \begin{pmatrix} \cos\alpha(t) \\ \sin\alpha(t) \end{pmatrix}$$

Image from "Directional Field Synthesis, Design, and Processing" (Vaxman et al., EG STAR 2016)

Famous Corollary

© Keenan Crane

Hairy ball theorem

Many Challenges

- Directional derivative?
- Purely intrinsic version?
- Singularities?
- Flow lines?
- …

Theoretical

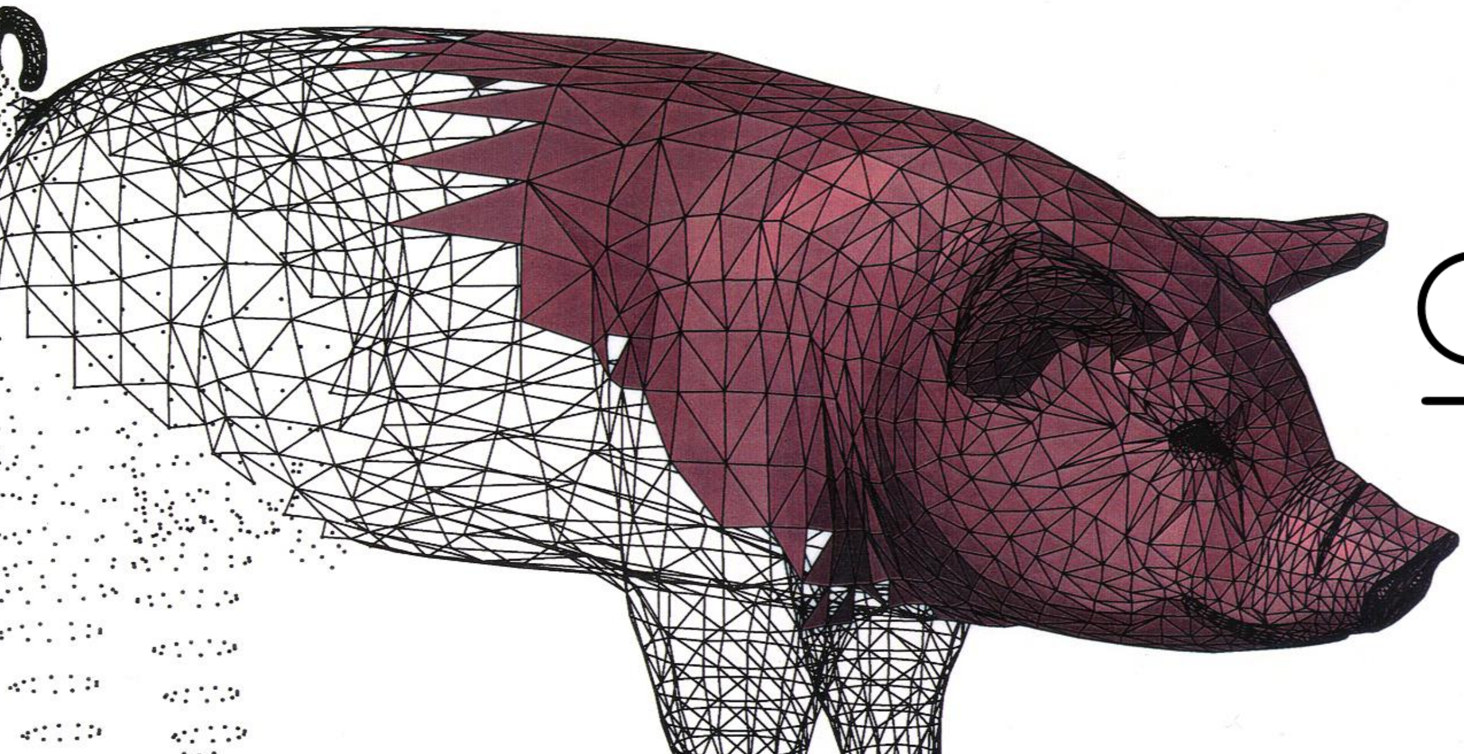
- How to discretize?
- Discrete derivatives?
- Singularity detection?
- Flow line computation?
- …

Discrete

Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based



$\subseteq \mathbb{R}^3$

Vector Fields on Triangle Meshes

No consensus:

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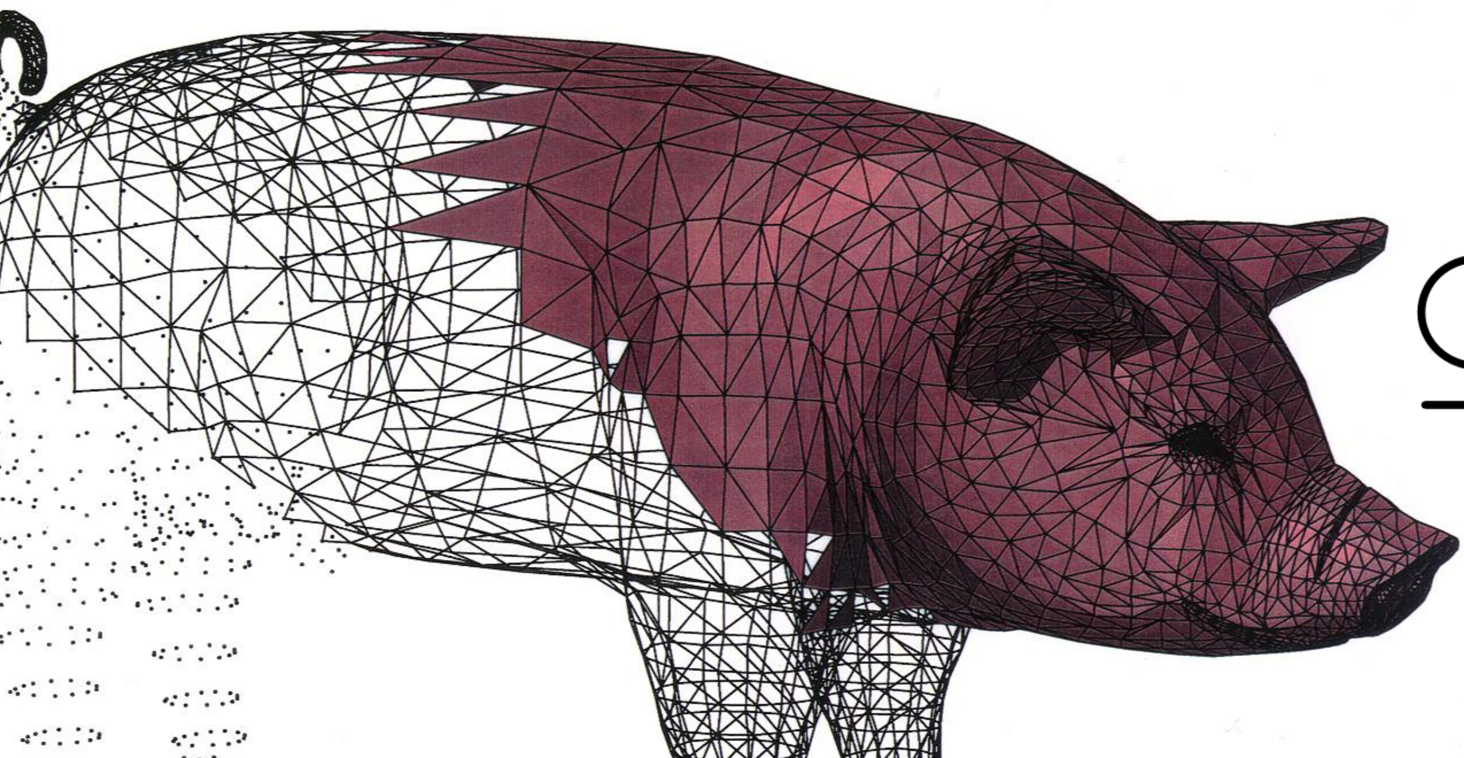
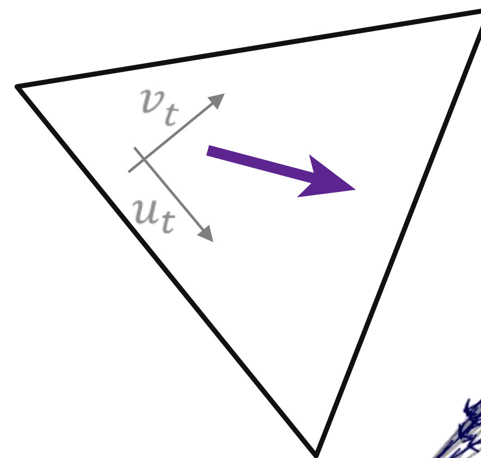
- Edge-based
- Vertex-based

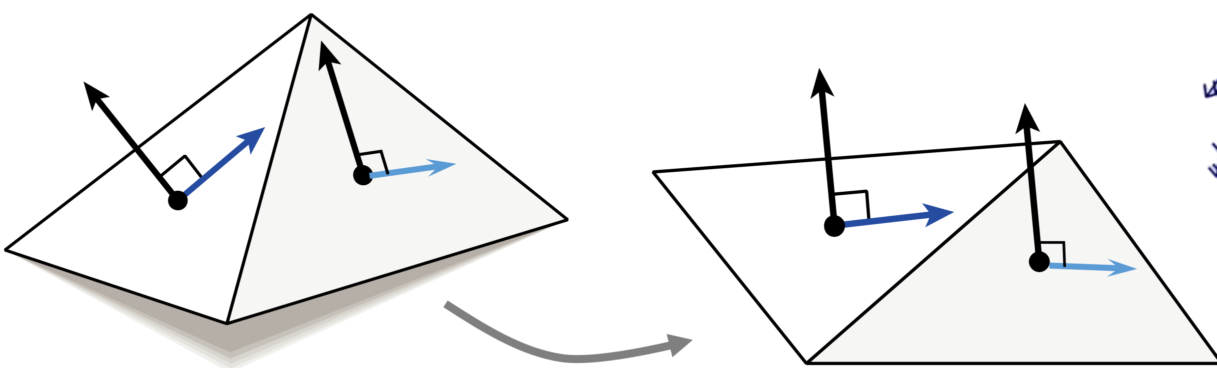


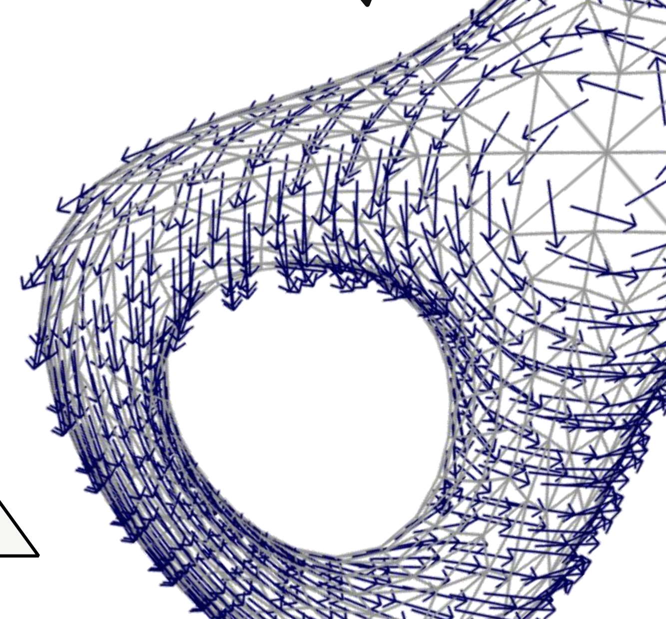
$\subseteq \mathbb{R}^3$

Triangle-Based

- Triangle as its **own tangent plane**
- One vector per triangle
  - "Piecewise constant"
  - Discontinuous at edges/vertices
- Easy to "unfold"/"hinge"

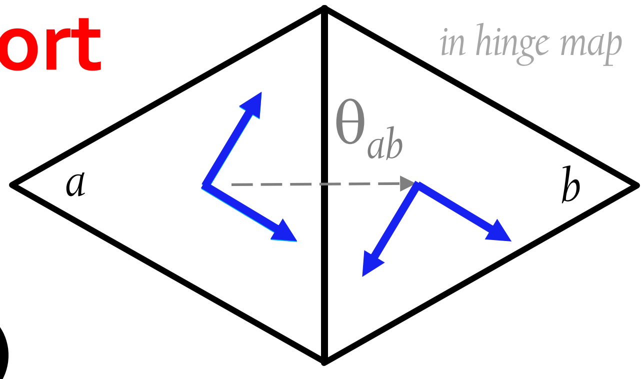


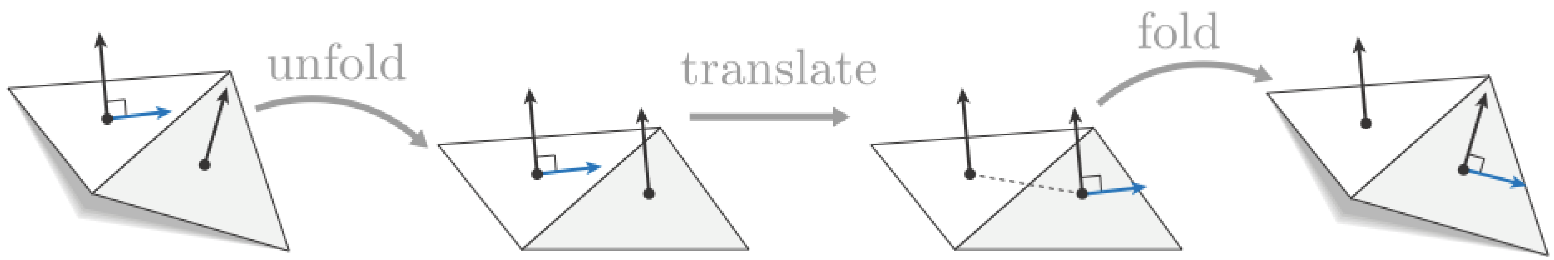


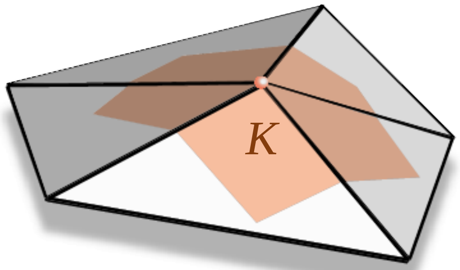


Discrete Levi-Civita Connection

- Simple notion of parallel transport
- Transport around vertex: Excess angle is (integrated) Gaussian curvature (holonomy!)

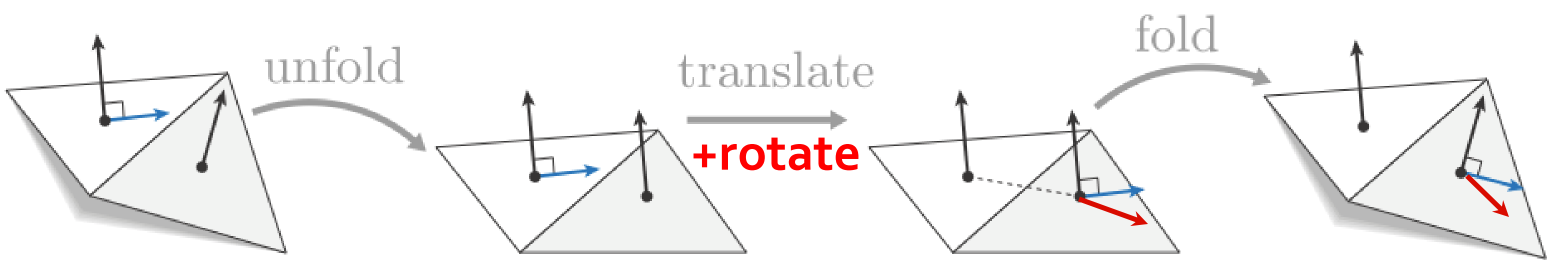






$$K = 2\pi - \sum_a \gamma_a$$

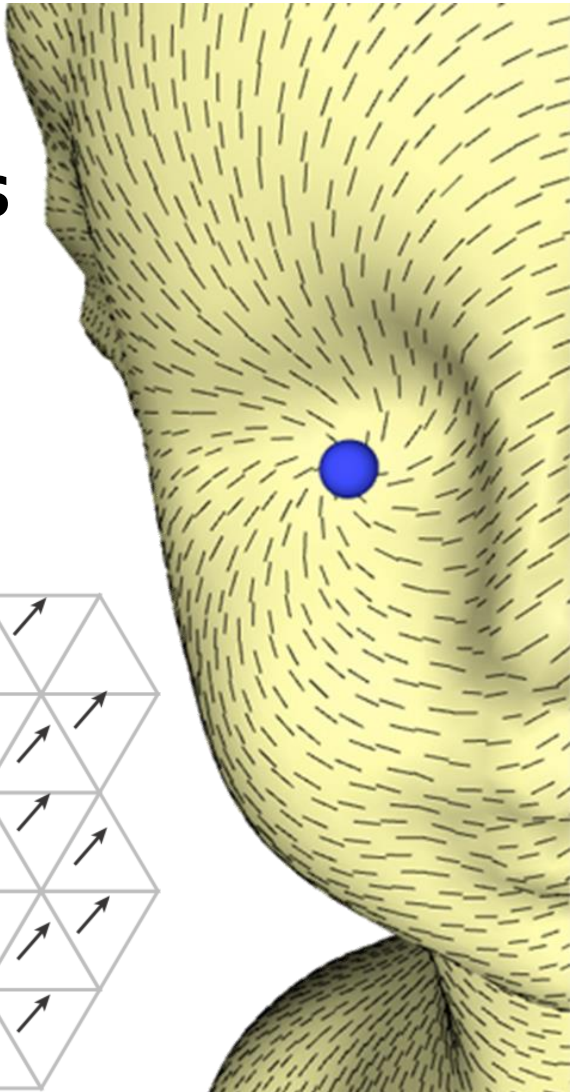
Arbitrary Connection

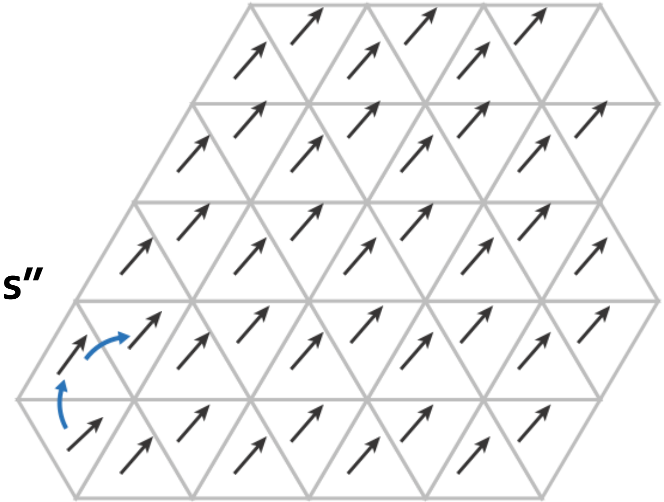


Represent using angle θ_{edge} of extra rotation.

Trivial Connections

- Vector field design
- **Zero holonomy** on discrete cycles
  - Except for a few singularities
- Path-independent away from singularities





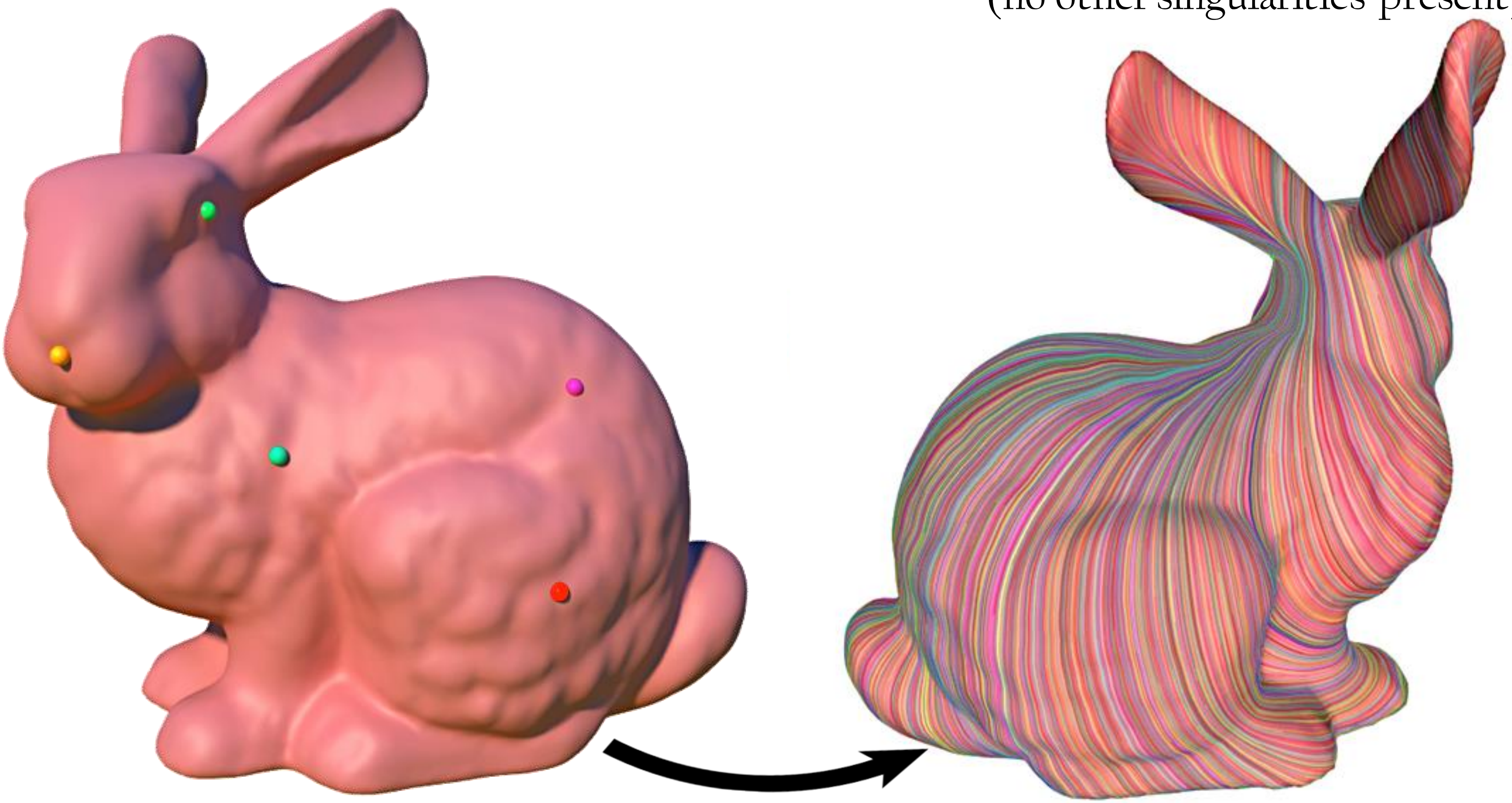
"Trivial Connections on Discrete Surfaces"
Crane et al., SGP 2010

Trivial Connections: Details

- Solve θ_{edge} of extra rotation per edge
- Linear constraint:
  Zero holonomy on basis cycles
  - V+2g constraints: Vertex cycles plus harmonic
  - Fix curvature at chosen singularities
- Underconstrained: Minimize $||\vec{\theta}||$
  - "Best approximation" of Levi-Civita

Result

Resulting trivial connection
(no other singularities present)



Linear system

Helmholtz-Hodge Decomposition

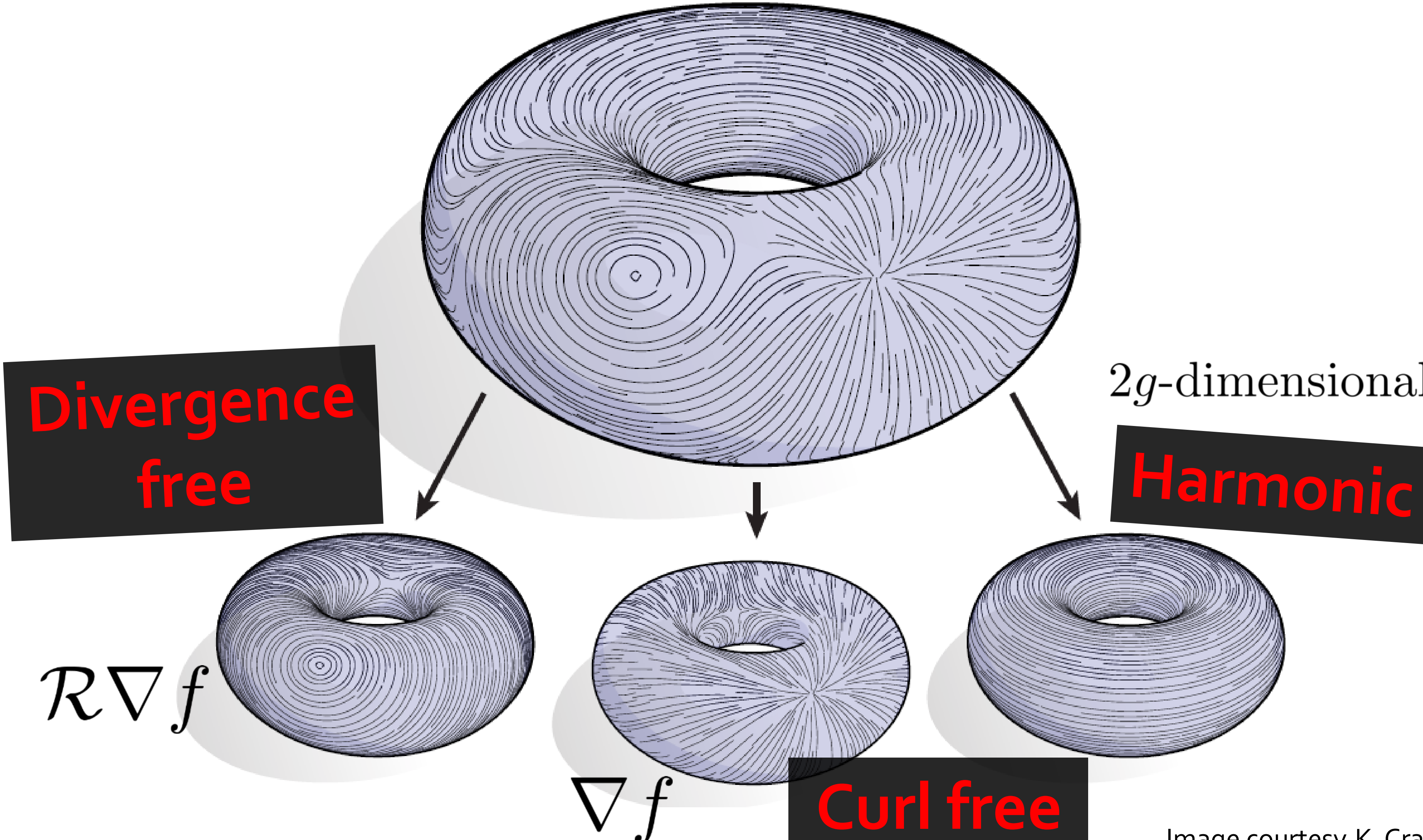
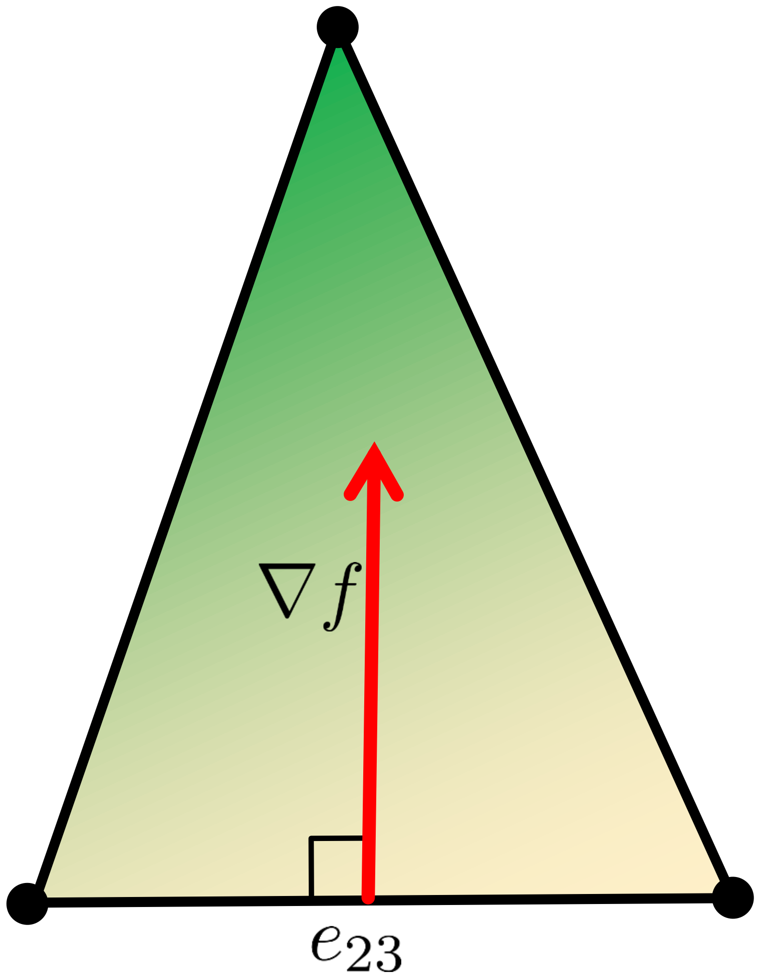


Image courtesy K. Crane

Recall:

Gradient of a Hat Function



$$\|\nabla f\| = \frac{1}{\ell_3 \sin\theta_3} = \frac{1}{h}$$

$$\nabla f = \frac{e_{23}^{\perp}}{2A}$$

Length of e_{23} cancels "base" in A

Recall:

Euler Characteristic

$$V - E + F := \chi$$

$$\chi = 2 - 2g$$

$g = 0$

$g = 1$

$g = 2$

Discrete Helmholtz-Hodge

$$2 - 2g = V - E + F$$
$$\implies 2F = (V - 1) + (E - 1) + 2g$$

Dimensionality works out perfectly!

Either

- Vertex-based gradients
- Edge-based rotated gradients

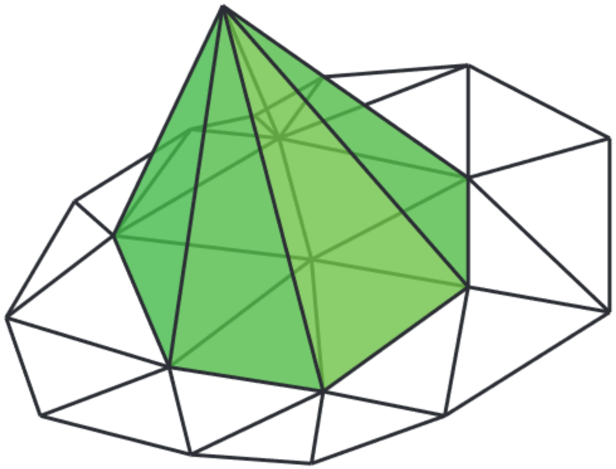
or

- Edge-based gradients
- Vertex-based rotated gradients

Can work out div/grad/curl

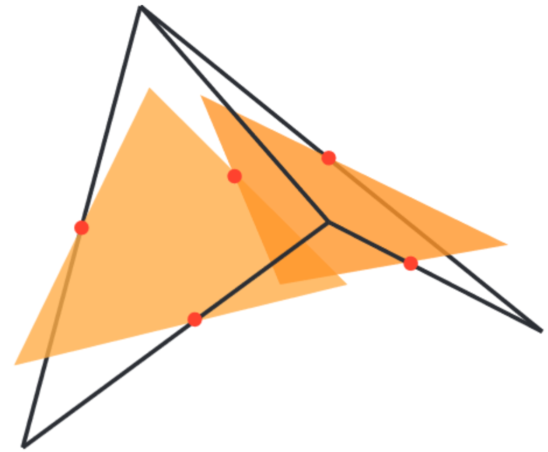
“Mixed” finite elements

Face-Based Calculus



Vertex-based

"Conforming"
Already did this in 6.838



Edge-based

"Nonconforming"
[Wardetzky 2006]

$$\text{Relationship: } \psi_{ij} = \phi_i + \phi_j - \phi_k$$

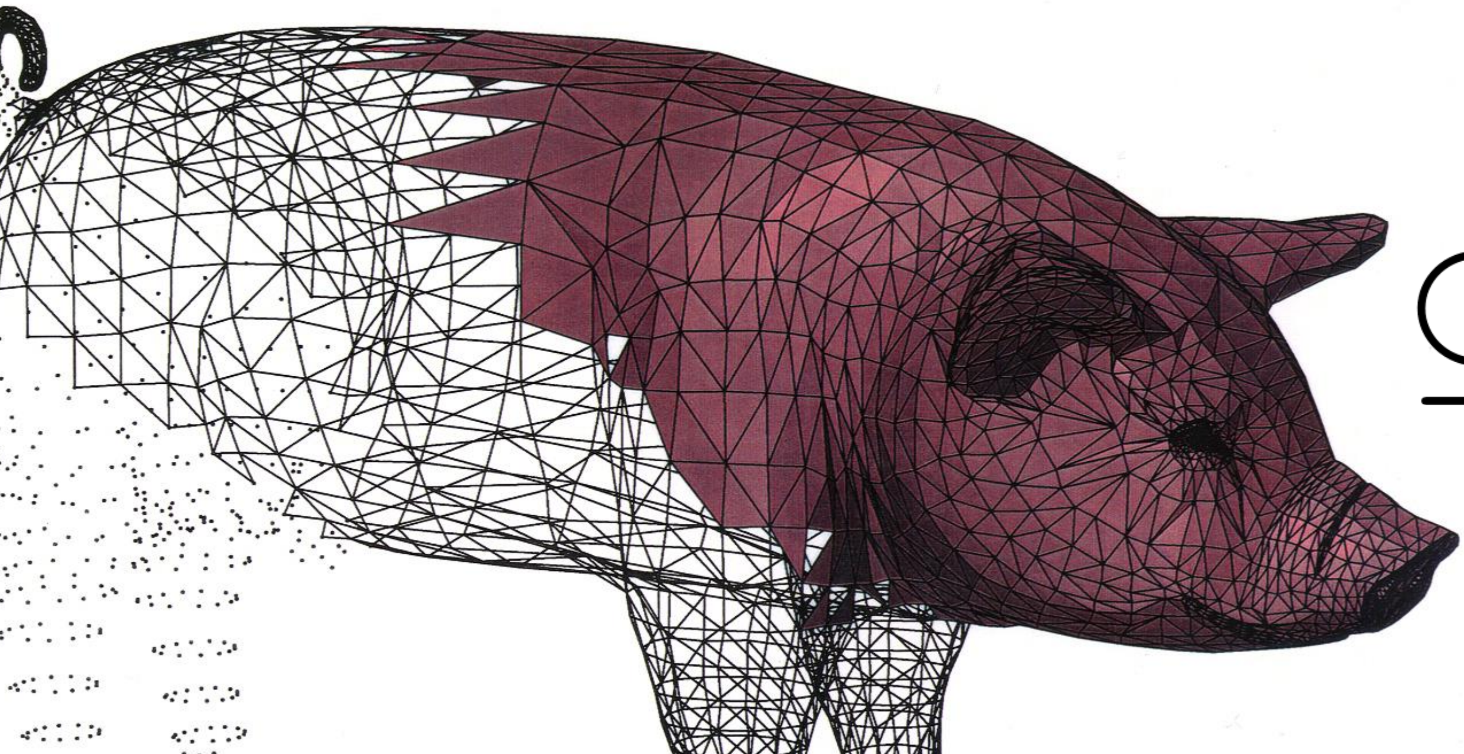
Gradient Vector Field

Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based

Defer to DEC!



$\subseteq \mathbb{R}^3$

Vector Fields on Triangle Meshes

No consensus:

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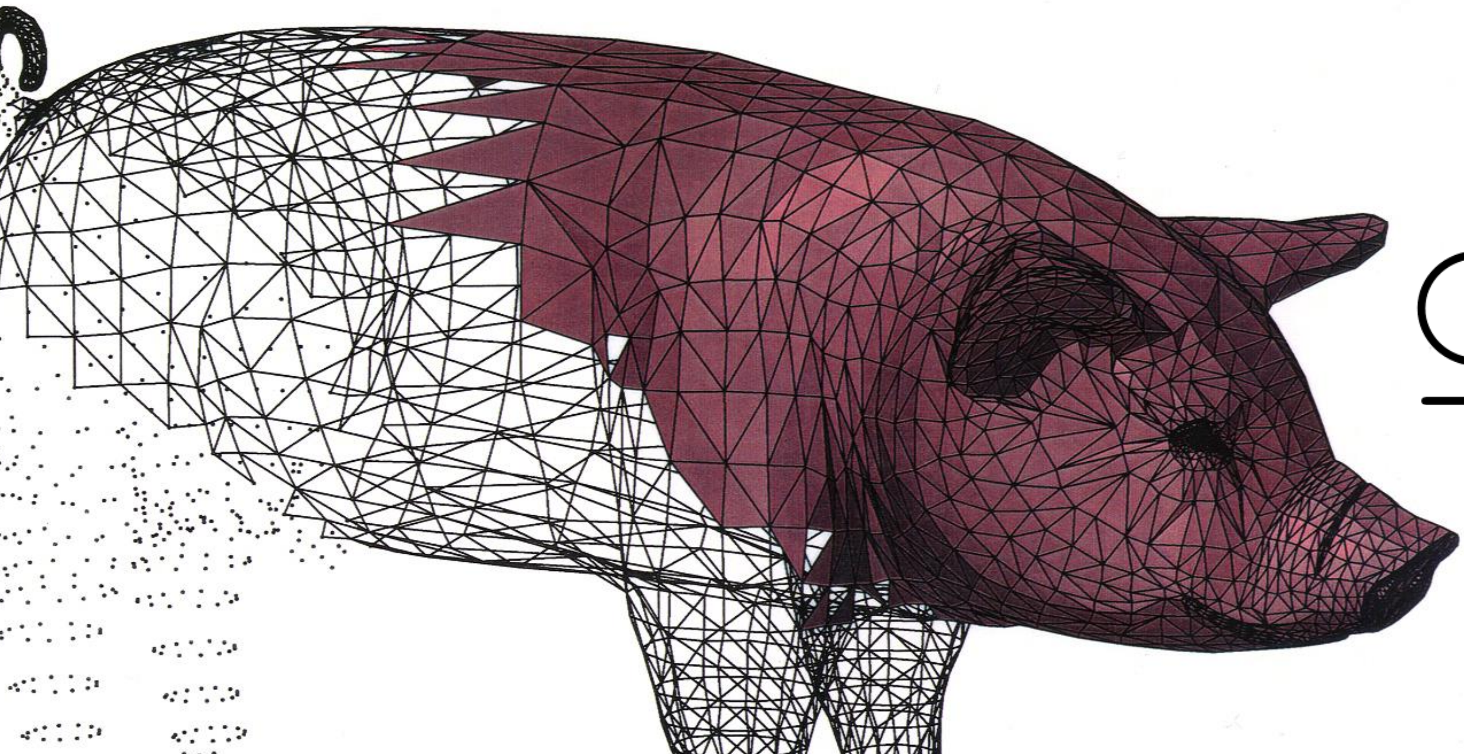
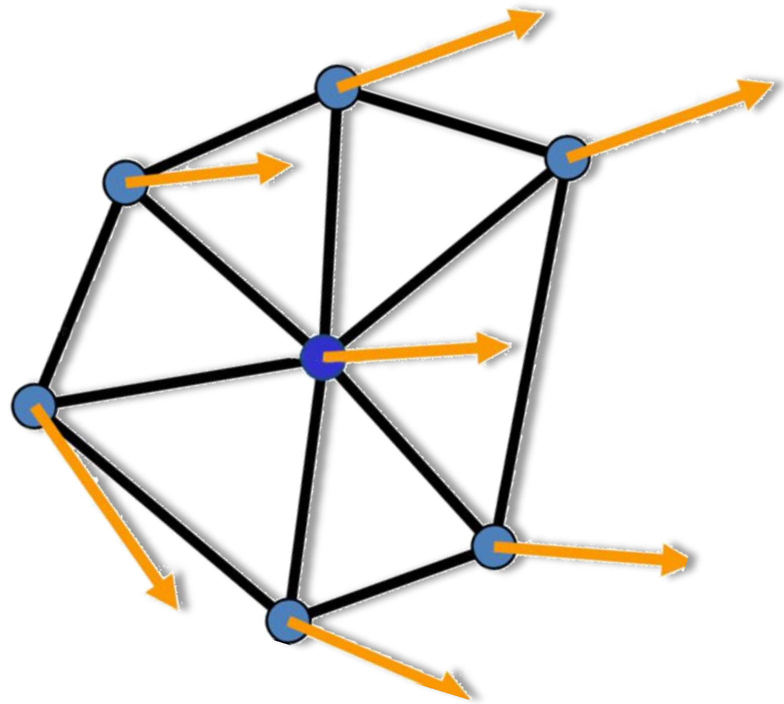


$\subseteq \mathbb{R}^3$

Vertex-Based Fields

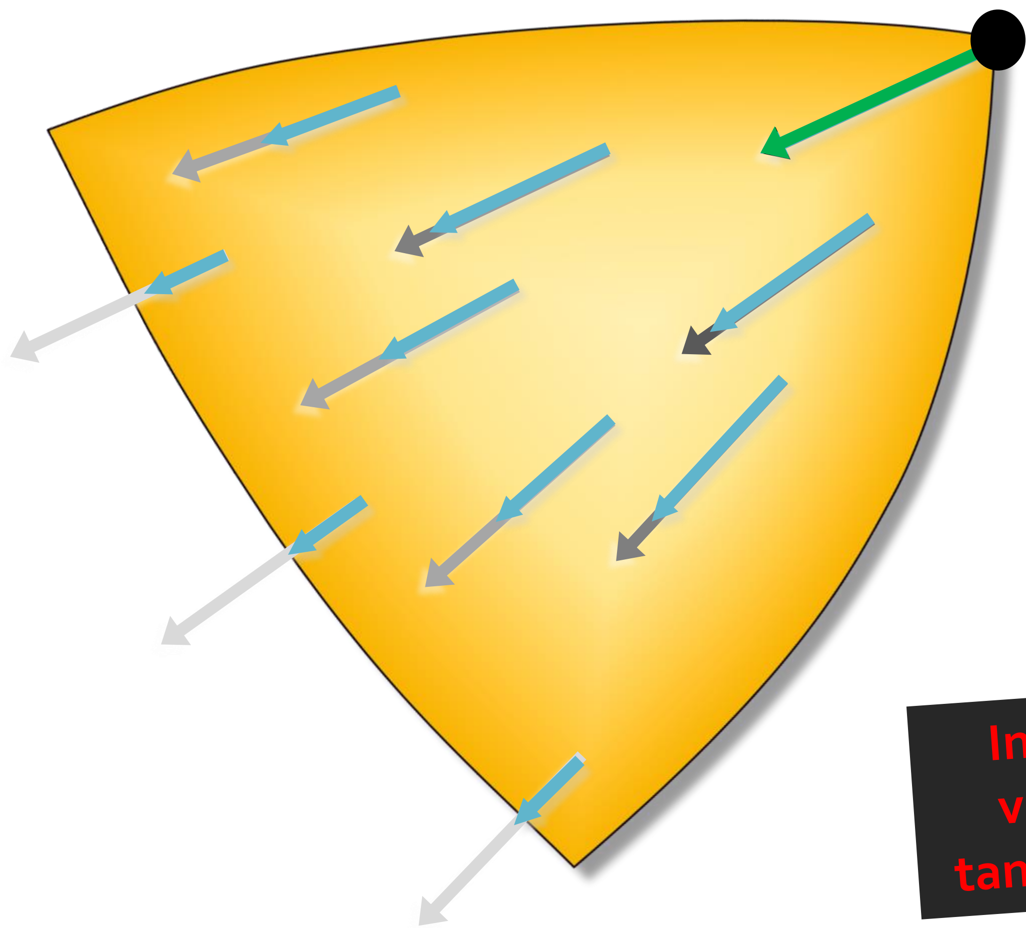
- Pros
  - Possibility of higher-order differentiation
- Cons
  - Vertices don’t have natural tangent spaces
  - Gaussian curvature concentrated

2D (Planar) Case: Easy



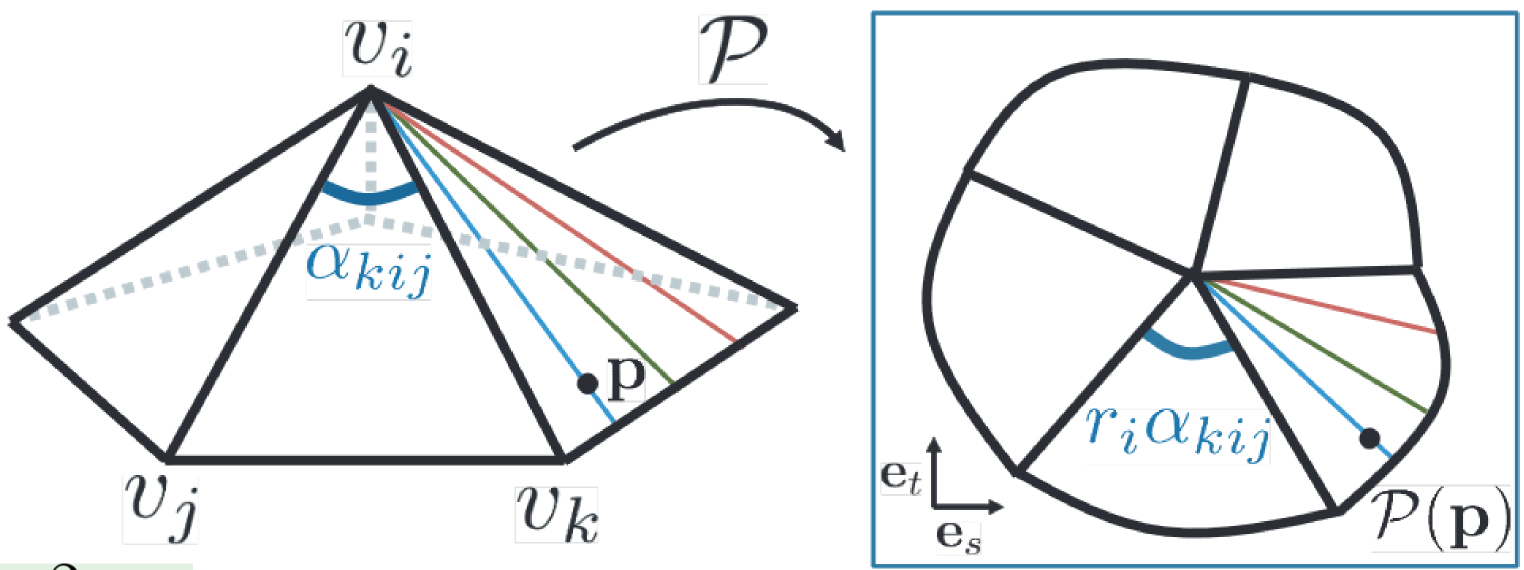
Piecewise-linear (x,y) components

3D Case: Ambiguous



Interpolate vector and tangent space!

Geodesic Polar Map



$$r_i := \frac{2\pi}{2\pi - \kappa_i}$$

Provides notion of tangency, but continuity issues

Preserve radial lines (change their spacing)

"Vector Field Design on Surfaces," Zhang et al., TOG 2006

Parallel transport radially from vertex

Recent Method for Continuous Fields

Discrete Connection and Covariant Derivative for Vector Field Analysis and Design

Beibei Liu and Yiying Tong
Michigan State University
and
Fernando de Goes and Mathieu Desbrun
California Institute of Technology

Includes basis, derivative operators

In this paper, we introduce a discrete definition of connection on simplicial manifolds, involving closed-form continuous expressions within simplices and finite rotations across simplices. The finite-dimensional parameters of this connection are optimally computed by minimizing a quadratic measure of the deviation to the (discontinuous) Levi-Civita connection induced by the embedding of the input triangle mesh, or to any metric connection with arbitrary cone singularities at vertices. From this discrete connection, a covariant derivative is constructed through exact differentiation, leading to explicit expressions for local integrals of first-order derivatives (such as divergence, curl and the Cauchy-Riemann operator), and for L_2-based energies (such as the Dirichlet energy). We finally demonstrate the utility, flexibility, and accuracy of our discrete formulations for the design and analysis of vector, n-vector, and n-direction fields.

Categories and Subject Descriptors: I.3.5 [**Computer Graphics**]: Computational Geometry & Object Modeling—*Curve & surface representations.*

CCS Concepts: •**Computing methodologies** → **Mesh models;**

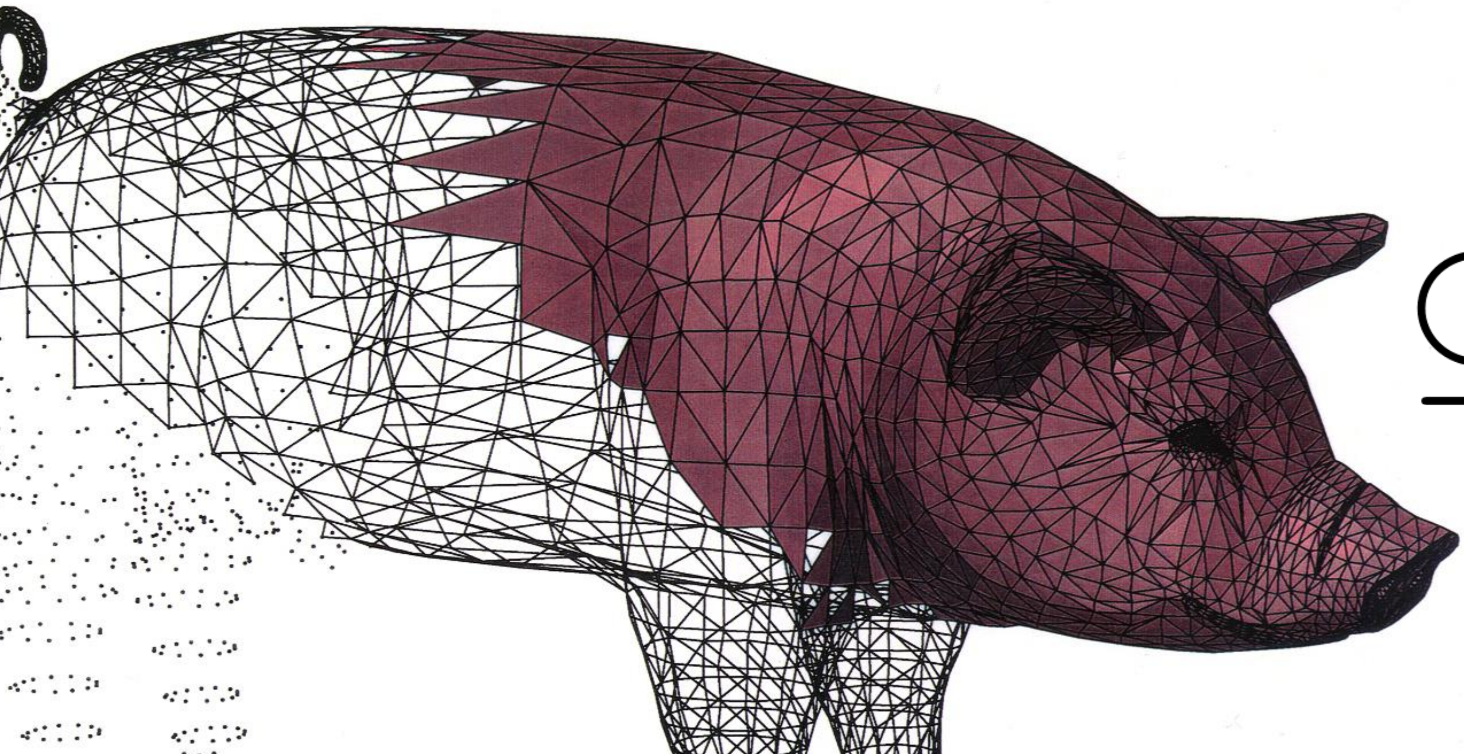
digital geometry processing, with applications ranging from texture synthesis to shape analysis, meshing, and simulation. However, existing discrete counterparts of such a differential operator acting on simplicial manifolds can either approximate local derivatives (such as divergence and curl) or estimate global integrals (such as the Dirichlet energy), but not both simultaneously.

In this paper, we present a unified discretization of the covariant derivative that offers closed-form expressions for both local and global first-order derivatives of vertex-based tangent vector fields on triangulations. Our approach is based on a new construction of discrete connections that provides consistent interpolation of tangent vectors within and across mesh simplices, while minimizing the deviation to the Levi-Civita connection induced by the 3D embedding of the input mesh—or more generally, to any metric connection with arbitrary cone singularities at vertices. We demonstrate the relevance of our contributions by providing new computational tools to design and edit vector and n-direction fields.

Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based

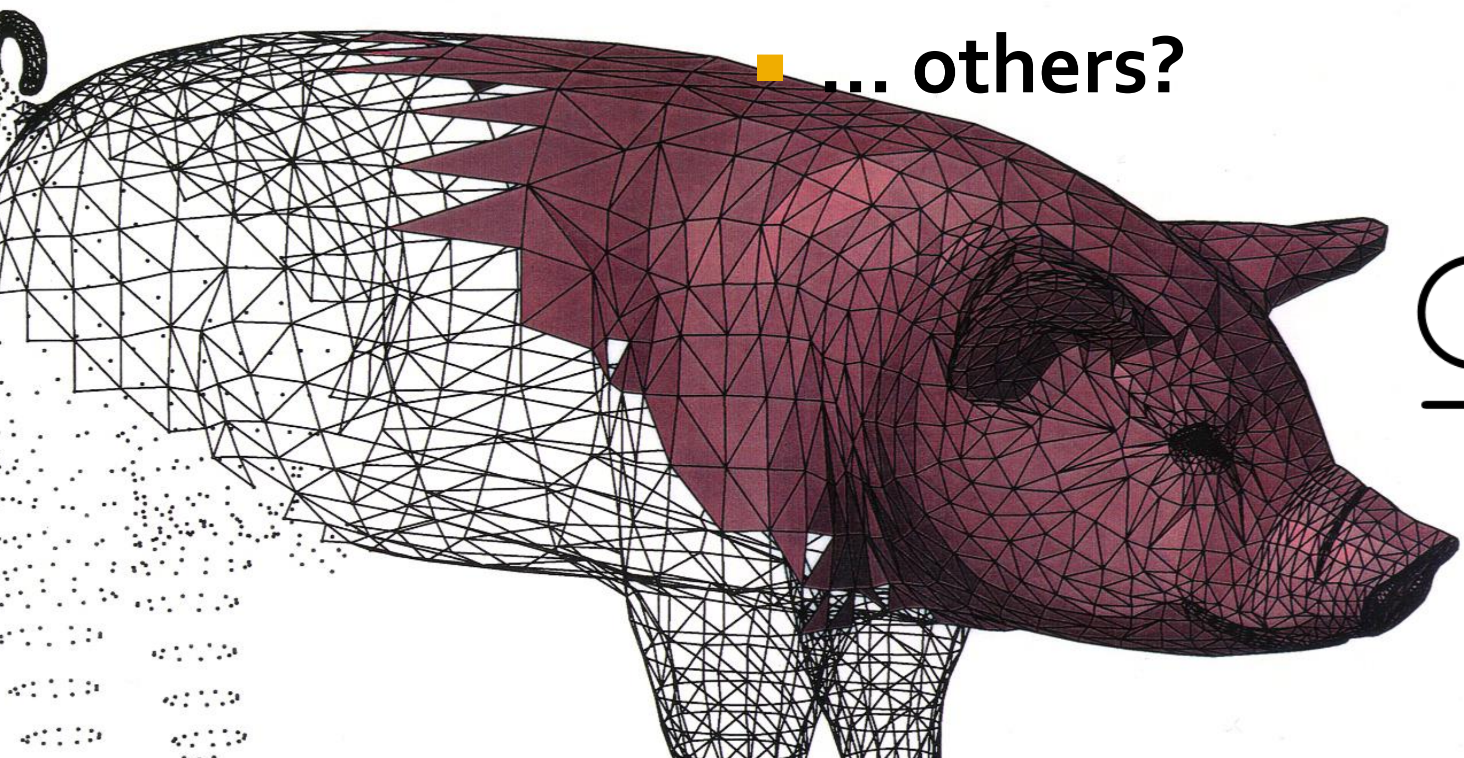


$\subseteq \mathbb{R}^3$

Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based
- … others?



$\subseteq \mathbb{R}^3$

More Exotic Choice

Eurographics Symposium on Geometry Processing 2013
Yaron Lipman and Richard Hao Zhang
(Guest Editors)

Volume 32 (2013), Number 5

An Operator Approach to Tangent Vector Field Processing

Omri Azencot[1] and Mirela Ben-Chen[1] and Frédéric Chazal[2] and Maks Ovsjanikov[3]

[1]Technion - Israel Institute of Technology
[2] Geometrica, INRIA
[3]LIX, École Polytechnique

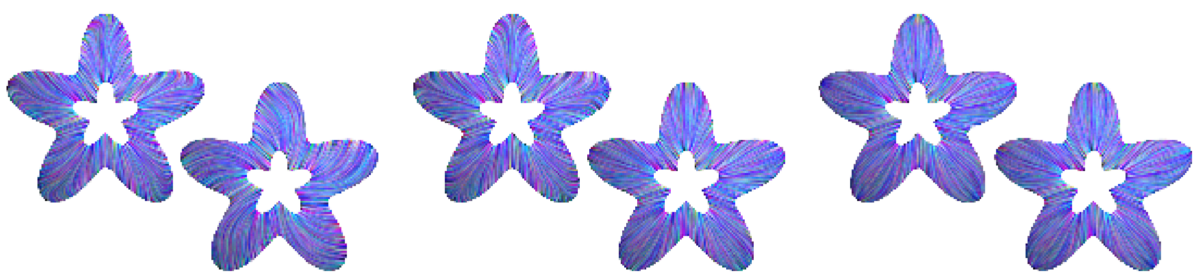


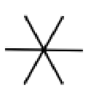
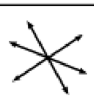
Figure 1: *Using our framework various vector field design goals can be easily posed as linear constraints. Here, given three symmetry maps: rotational (S1), bilateral (S2) and front/back (S3), we can generate a symmetric vec... (left), S1 + S2 (center) and S1 + S2 + S3 (right). The top row shows the front of the 3D model, and the bottom row its back.*

Vector fields as derivative operators

Abstract

In this paper, we introduce a novel coordinate-free method for manipulating and analyzing vector fields on d... surfaces. Unlike the commonly used representations of a vector field as an assignment of vectors to the faces of the mesh, or as real values on edges, we argue that vector fields can also be naturally viewed as operators whose

Extension: Direction Fields

	1-vector field	One vector, classical "vector field"
	2-direction field	Two directions with π symmetry, "line field", "2-RoSy field"
	1^3-vector field	Three independent vectors, "3-polyvector field"
	4-vector field	Four vectors with $\pi/2$ symmetry, "non-unit cross field"
	4-direction field	Four directions with $\pi/2$ symmetry, "unit cross field", "4-RoSy field"
	2^2-vector field	Two pairs of vectors with π symmetry each, "frame field"
	2^2-direction field	Two pairs of directions with π symmetry each, "non-ortho. cross field"
	6-direction field	Six directions with $\pi/3$ symmetry, "6-RoSy"
	2^3-vector field	Three pairs of vectors with π symmetry each

"Directional Field Synthesis, Design, and Processing" (Vaxman et al., EG STAR 2016)

Polyvector Fields

Eurographics Symposium on Geometry Processing 2014
Thomas Funkhouser and Shi-Min Hu
(Guest Editors)

Volume 33 (2014), Number 5

Designing *N*-PolyVector Fields with Complex Polynomials

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[1]ETH Zurich, Switzerland
[2]Vienna Institute of Technology, Austria

$$\{u_0, u_1, \ldots, u_k\}$$
$$\mapsto f(z) := (z - u_0) \cdot \cdots \cdot (z - u_k)$$
$$\mapsto f(z) = z^{k+1} + a_k z^k + \cdots + a_1 z + a_0$$
$$\mapsto \{a_0, \ldots, a_k\}$$

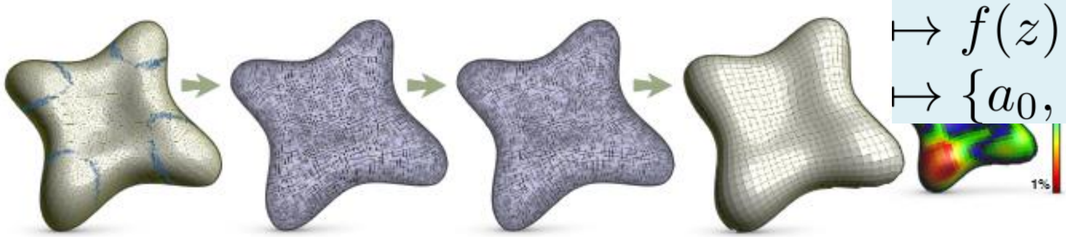
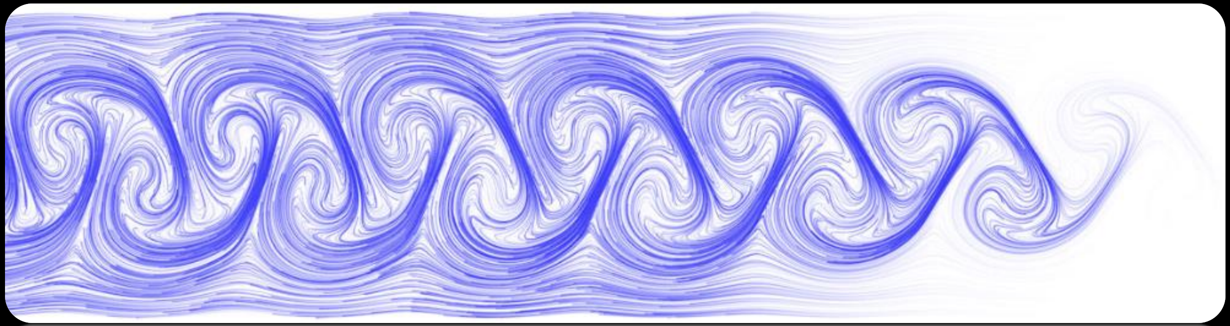


Figure 1: *A smooth 4-PolyVector field is generated from a sparse set of principal direction constraints (faces in light blue). We optimize the field for conjugacy and use it to guide the generation of a planar-quad mesh. Pseudocolor represents planarity.*

Abstract
We introduce N-PolyVector fields, a generalization of N-RoSy fields for which the vectors are neither necessarily orthogonal nor rotationally symmetric. We formally define a novel representation for N-PolyVectors as the root sets of complex polynomials and analyze their topological and geometric properties. A smooth N-PolyVector field can be efficiently generated by solving a sparse linear system without integer variables. We exploit the flexibility of N-PolyVector fields to design conjugate vector fields, offering an intuitive tool to generate planar quadrilateral meshes.

One encoding of direction fields

Image from “Streak Lines as Tangent Curves of a Derived Vector Field” (Weinkauf and Theisel; TVCG 2010)



Vector Field Processing