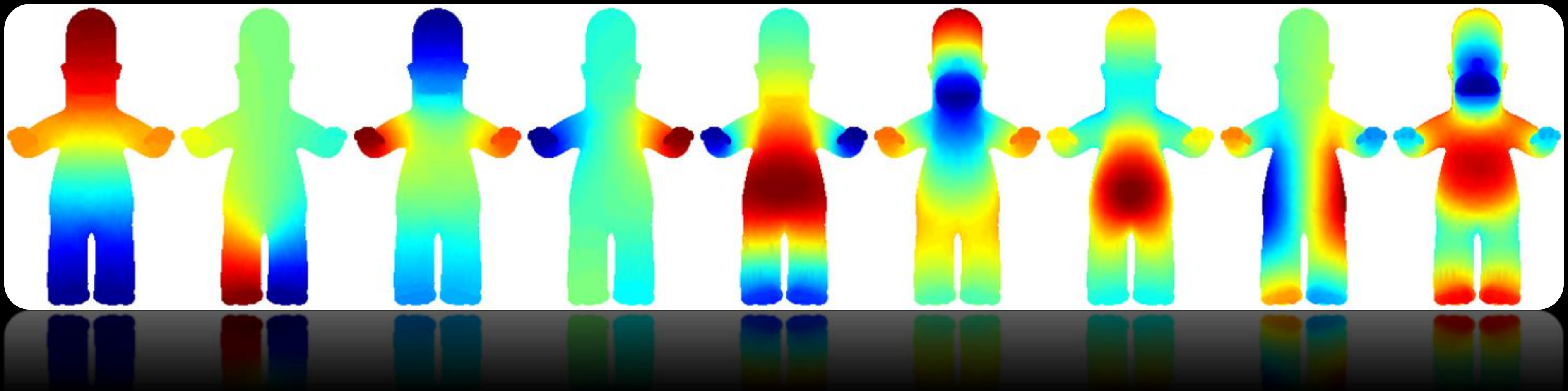


Applications of the Laplacian

Justin Solomon
MIT, Spring 2019

Review:

Rough Intuition: Spectral Geometry

http://pngimg.com/upload/hammer_PNG3886.png

You can learn a lot about a shape by **hitting it** (lightly) **with a hammer**!

Review:

Rough Definition

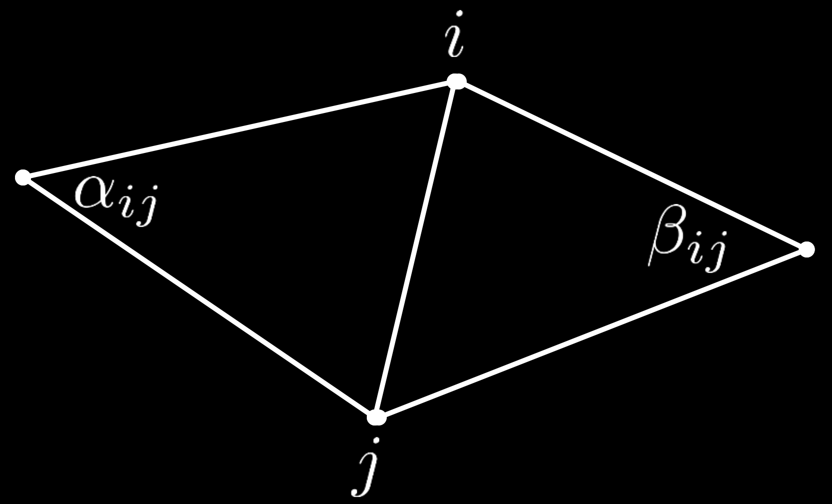
What can you learn about its shape from vibration frequencies and oscillation patterns?

$$\Delta f = \lambda f$$

Review:

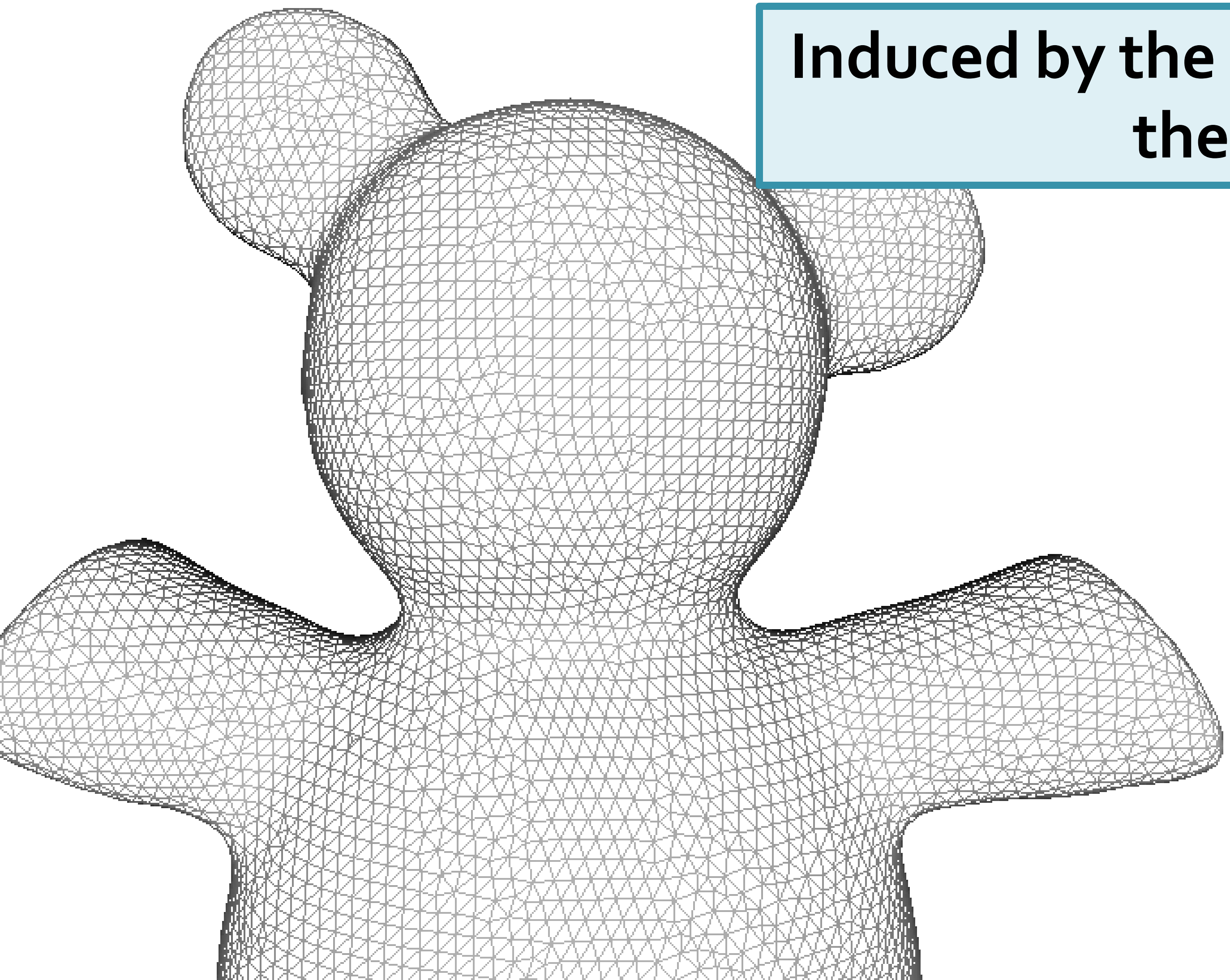
The Cotangent Laplacian

$$L_{ij} = \begin{cases} \frac{1}{2}\sum_{i\sim k}(\cot\alpha_{ik} + \cot\beta_{ik}) & \text{if } i = j \\ -\frac{1}{2}(\cot\alpha_{ij} + \cot\beta_{ij}) & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$



Key property:

Sparsity



Induced by the connectivity of the triangle mesh.

Our Next Topic

Discrete Laplacian operators:

What are they good for?

- Useful properties of the Laplacian
- Applications in graphics/shape analysis
- Applications in machine learning

A quick survey:
A popular field!

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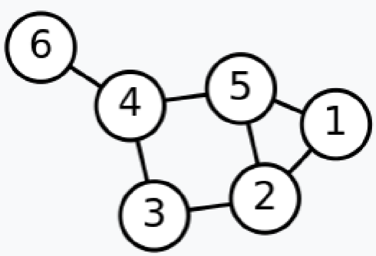
One Object, Many Interpretations

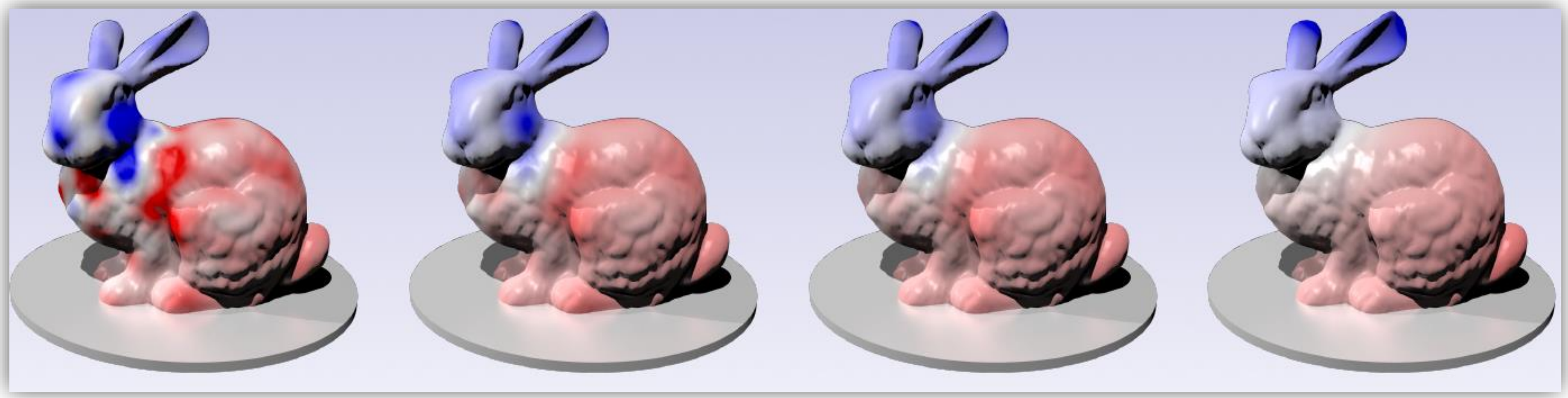
$$L_{vw} = A - D = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

Labeled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$

https://en.wikipedia.org/wiki/Laplacian_matrix

Deviation from neighbors

One Object, Many Interpretations



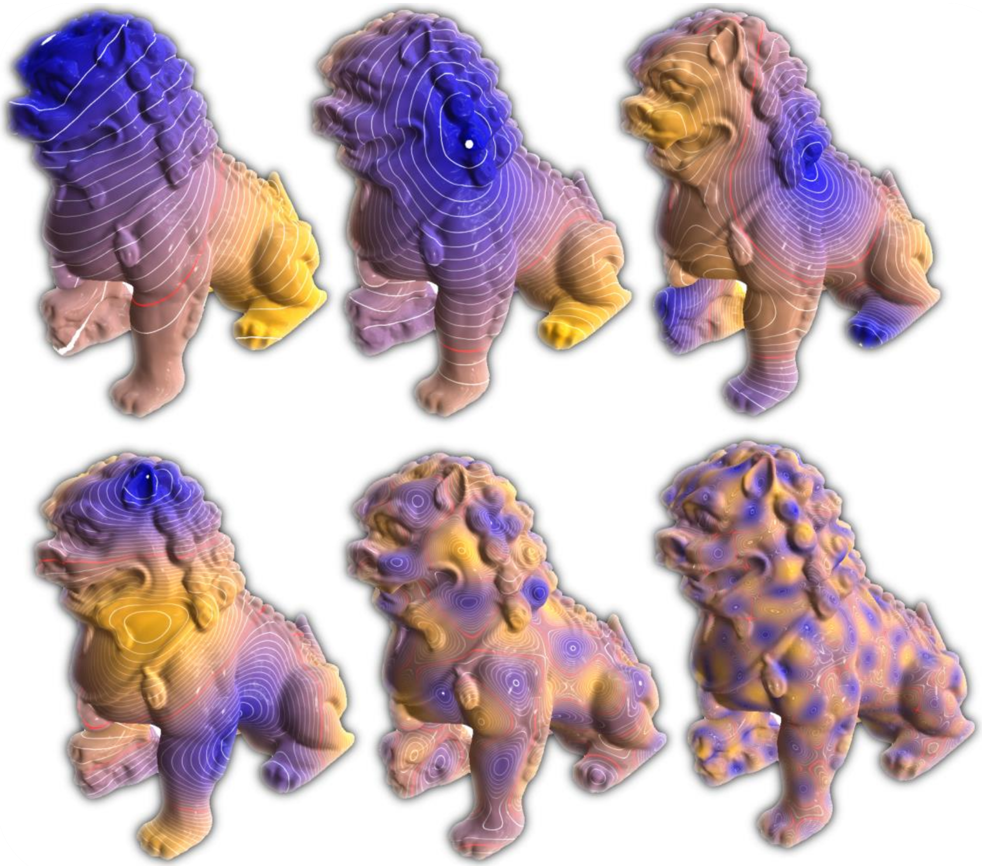
Decreasing *E*

$$E[f] := \int_S \|\nabla f\|_2^2 \, dA = \int_S f(x) \Delta f(x) \, dA(x)$$

Images made by E. Vouga

Dirichlet energy: Measures smoothness

One Object, Many Interpretations



$$\Delta \psi_i = \lambda_i \psi_i$$

Vibration modes of surface (not volume!)

http://alice.loria.fr/publications/papers/2008/ManifoldHarmonics//photo/dragon_mhb.png

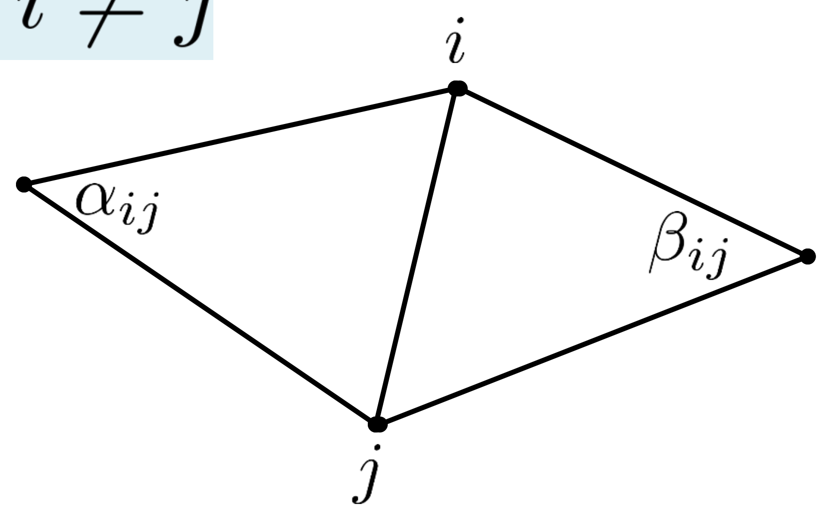
Vibration modes

Key Observation (in discrete case)

$$L_{ij} = \begin{cases} \frac{1}{2} \sum_{i \sim k} (\cot \alpha_{ik} + \cot \beta_{ik}) & \text{if } i = j \\ -\frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}) & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

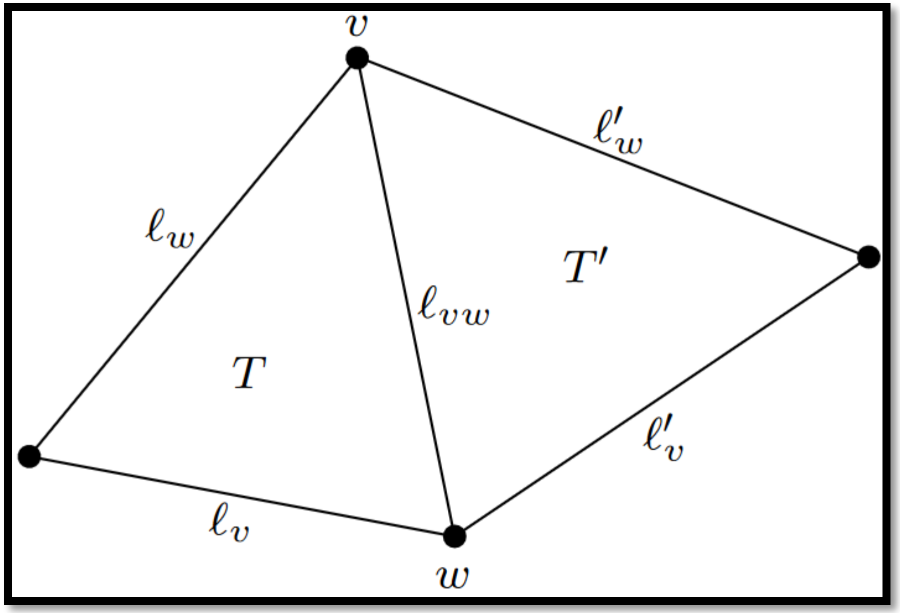
$$M_{ij} = \begin{cases} \frac{\text{one-ring area}}{6} & \text{if } i = j \\ \frac{\text{adjacent area}}{12} & \text{if } i \neq j \end{cases}$$

Can be written in terms of angles and areas!



After (More) Trigonometry

$$L_{vw} = \frac{1}{8}\begin{cases} -\sum_{u\sim v} L_{uv} & \text{when } v = w \\ \mu(T)^{-1}(\ell_{vw}^2 - \ell_v^2 - \ell_w^2) \\ \quad +\mu(T')^{-1}(\ell_{vw}^2 - \ell_v'^2 - \ell_w'^2) & \text{when } v \sim w \\ 0 & \text{otherwise} \end{cases}$$



Image/formula in "Functional Characterization of Instrinsic and Extrinsic Geometry," TOG 2017 (Corman et al.)

Laplacian only depends on edge lengths

Isometry

[ahy-**som**-i-tree]**:**

Bending without stretching.

Lots of Interpretations

Global isometry

$$d_1(x, y) = d_2(f(x), f(y))$$

Local isometry

$$\begin{aligned} g_1 &= f^* g_2 \\ g_1(v, w) &= g_2(f_* v, f_* w) \end{aligned}$$

Intrinsic Techniques

http://www.revedreams.com/crochet/yarncrochet/nonorientable-crochet/

Isometry invariant

Isometry Invariance: Hope

http://www.flickr.com/photos/melvinvoskuijl/galleries/72157624236168459

Isometry Invariance: Reality

"Rigidity"

http://www.4tnz.com/content/got-toilet-paper

Few shapes *can* deform isometrically

Isometry Invariance: Reality

Behavior for approximate isometry is important, too!

http://www.4tnz.com/content/got-toilet-paper

Few shapes *can* deform isometrically

Useful Fact

Graphical Models 74 (2012) 121–129

Contents lists available at SciVerse ScienceDirect

Graphical Models

journal homepage: www.elsevier.com/locate/gmod

Discrete heat kernel determines discrete Riemannian metric

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A R T I C L E   I N F O

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Discrete Riemannian metric
Laplace–Beltrami operator
Legendre duality principle

A B S T R A C T

The Laplace–Beltrami operator of a smooth Riemannian manifold is determined by the Riemannian metric. Conversely, the heat kernel constructed from the eigenvalues and eigenfunctions of the Laplace–Beltrami operator determines the Riemannian metric. This work proves the analogy on Euclidean polyhedral surfaces (triangle meshes), that the discrete heat kernel and the discrete Riemannian metric (unique up to a scaling) are mutually determined by each other. Given a Euclidean polyhedral surface, its Riemannian metric is represented as edge lengths, satisfying triangle inequalities on all faces. The Laplace–Beltrami operator is formulated using the cotangent formula, where the edge weight is defined as the sum of the cotangent of angles against the edge. We prove that the edge

Beware

Figure 1: Deformations of a glove (left) and a solid hand (right) are an illustration of the difference between boundary and volume isometries.

Image from: Raviv et al. "Volumetric Heat Kernel Signatures." 3DOR 2010.

But calculations on a volume are expensive!

(changing!)

Not the same.

Why Study the Laplacian?

- **Encodes intrinsic geometry**
  Edge lengths on triangle mesh, Riemannian metric on manifold
- **Multi-scale**
  Filter based on frequency
- **Geometry through linear algebra**
  Linear/eigenvalue problems, sparse positive definite matrices
- **Connection to physics**
  Heat equation, wave equation, vibration, …

Our Next Topic

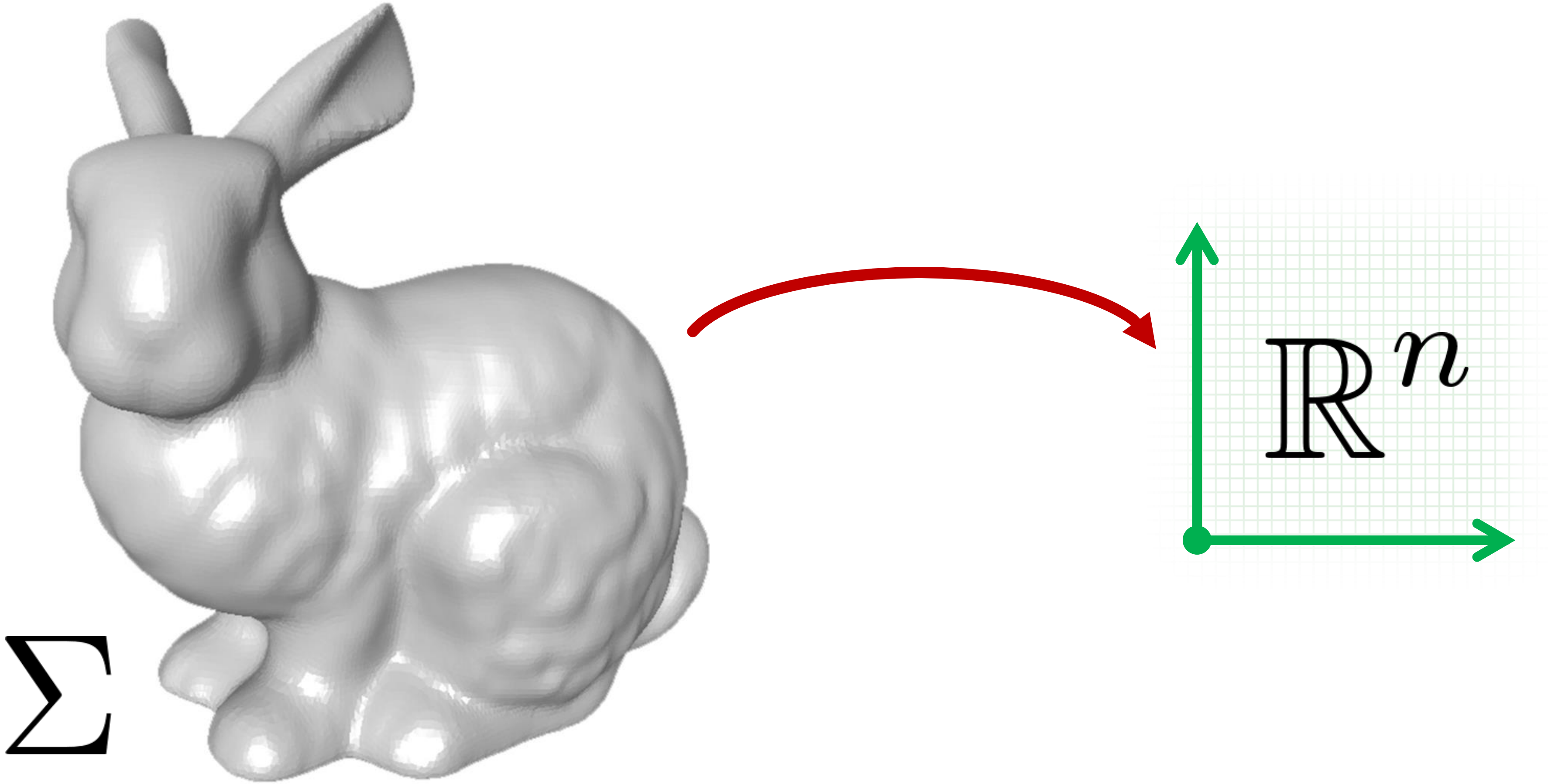
Discrete Laplacian operators:

What are they good for?

- Useful properties of the Laplacian
- Applications in graphics/shape analysis
- Applications in machine learning

A quick survey:
A popular field!

Example Task: Shape Descriptors



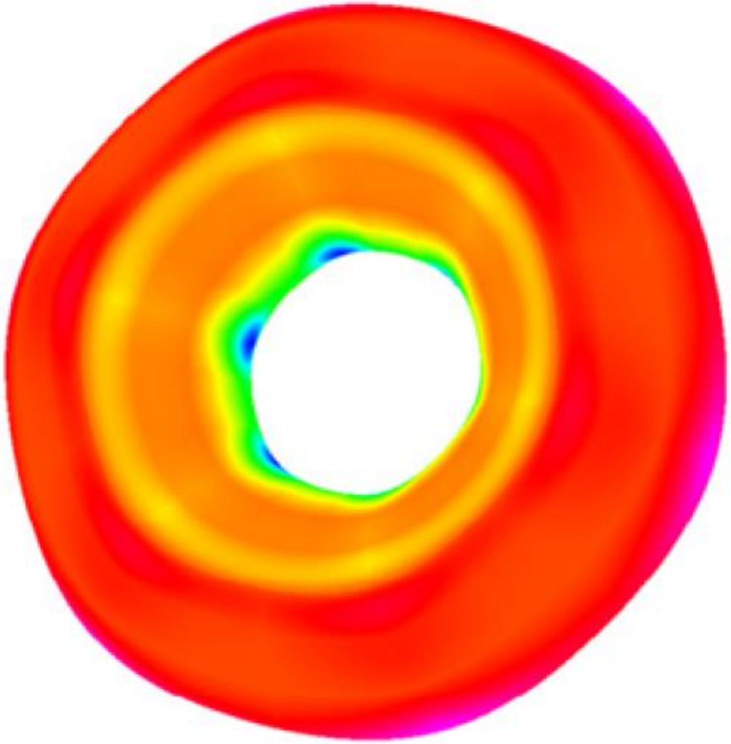
http://liris.cnrs.fr/meshbenchmark/images/fig_attacks.jpg

Pointwise quantity

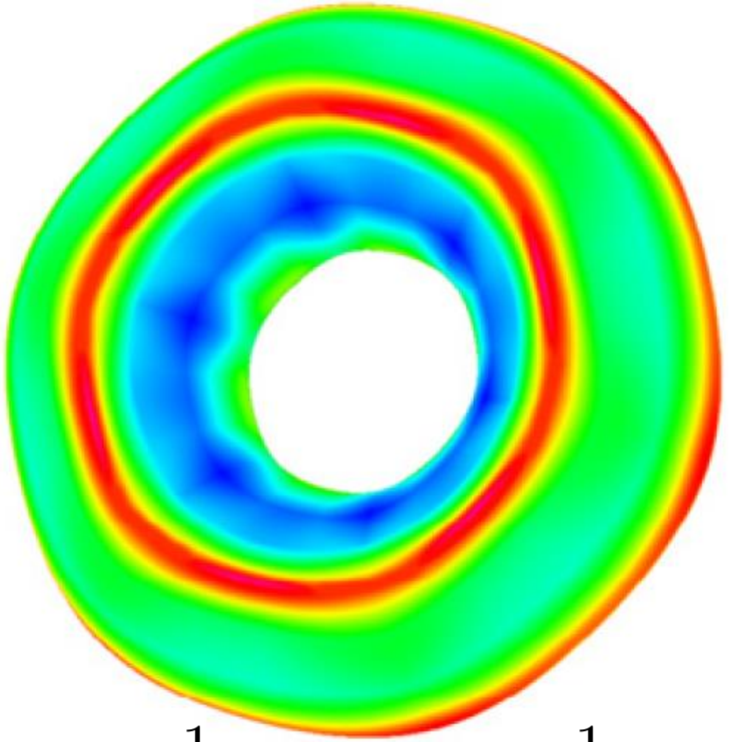
Descriptor Tasks

- **Characterize local geometry**
  Feature/anomaly detection
- **Describe point's role on surface**
  Symmetry detection, correspondence

Descriptors We've Seen Before



$$K := \kappa_1 \kappa_2 = \det \mathrm{I\!I}$$



$$H := \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2} \operatorname{tr} \mathrm{I\!I}$$

http://www.sciencedirect.com/science/article/pii/S0010448510001983

Gaussian and mean curvature

Desirable Properties

- **Distinguishing**

  Provides useful information about a point

- **Stable**

  Numerically and geometrically

- **Intrinsic**

  No dependence on embedding

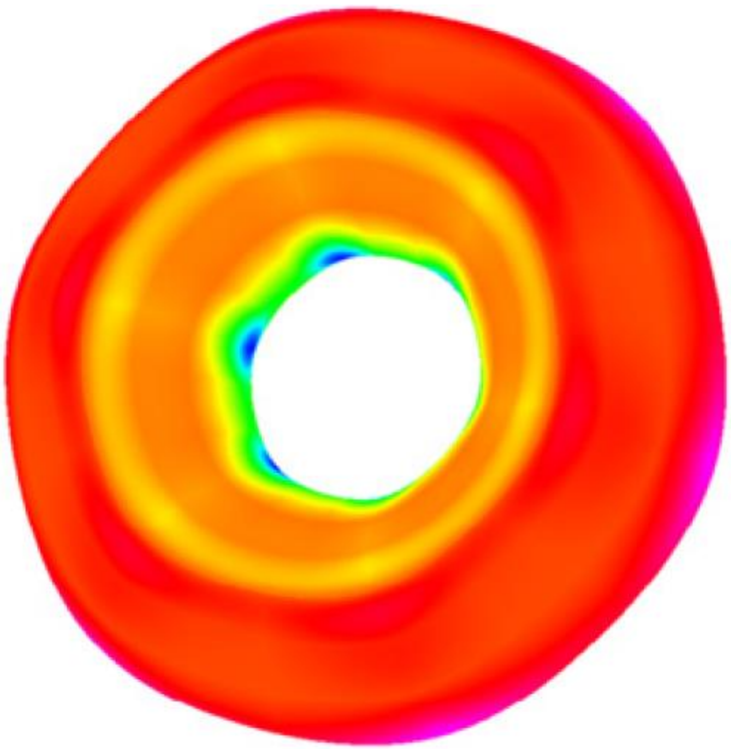
  Sometimes undesirable!

Intrinsic Descriptors

Invariant under

- **Rigid motion**
- **Bending without stretching**

Intrinsic Descriptor



$$K := \kappa_1 \kappa_2 = \det \mathrm{I\!I}$$

Theorema Egregium

("Totally Awesome Theorem"):

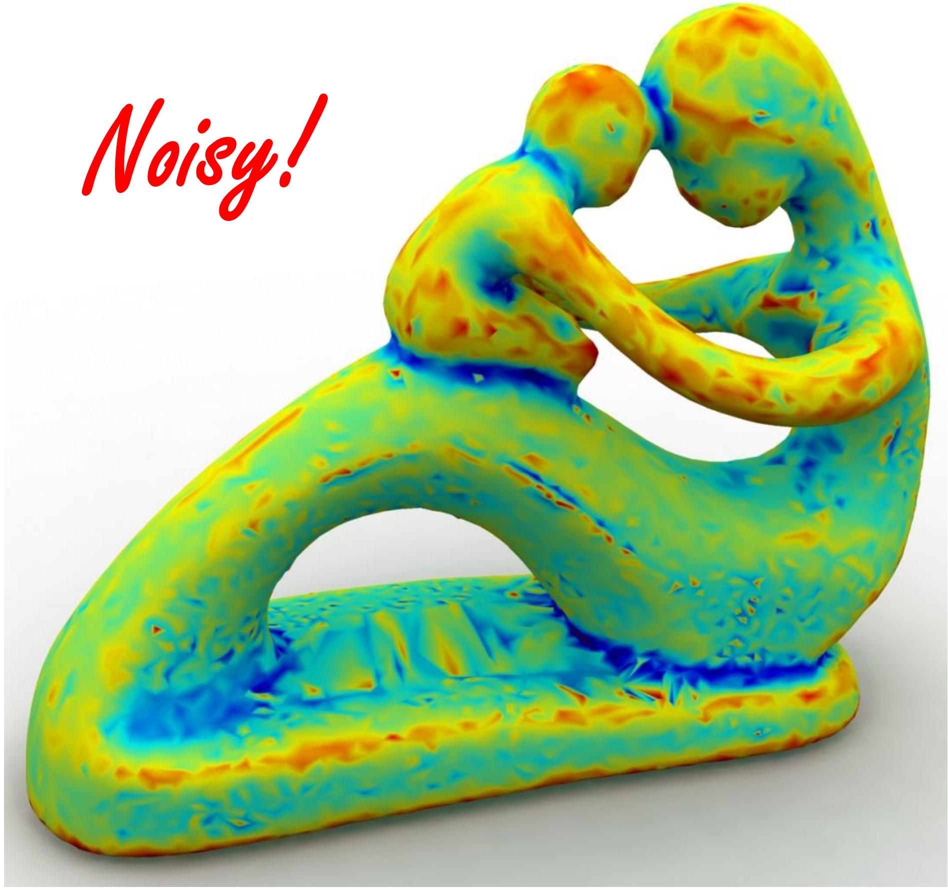
Gaussian curvature is intrinsic.

http://www.sciencedirect.com/science/article/pii/S0010448510001983

Gaussian curvature

End of the Story?

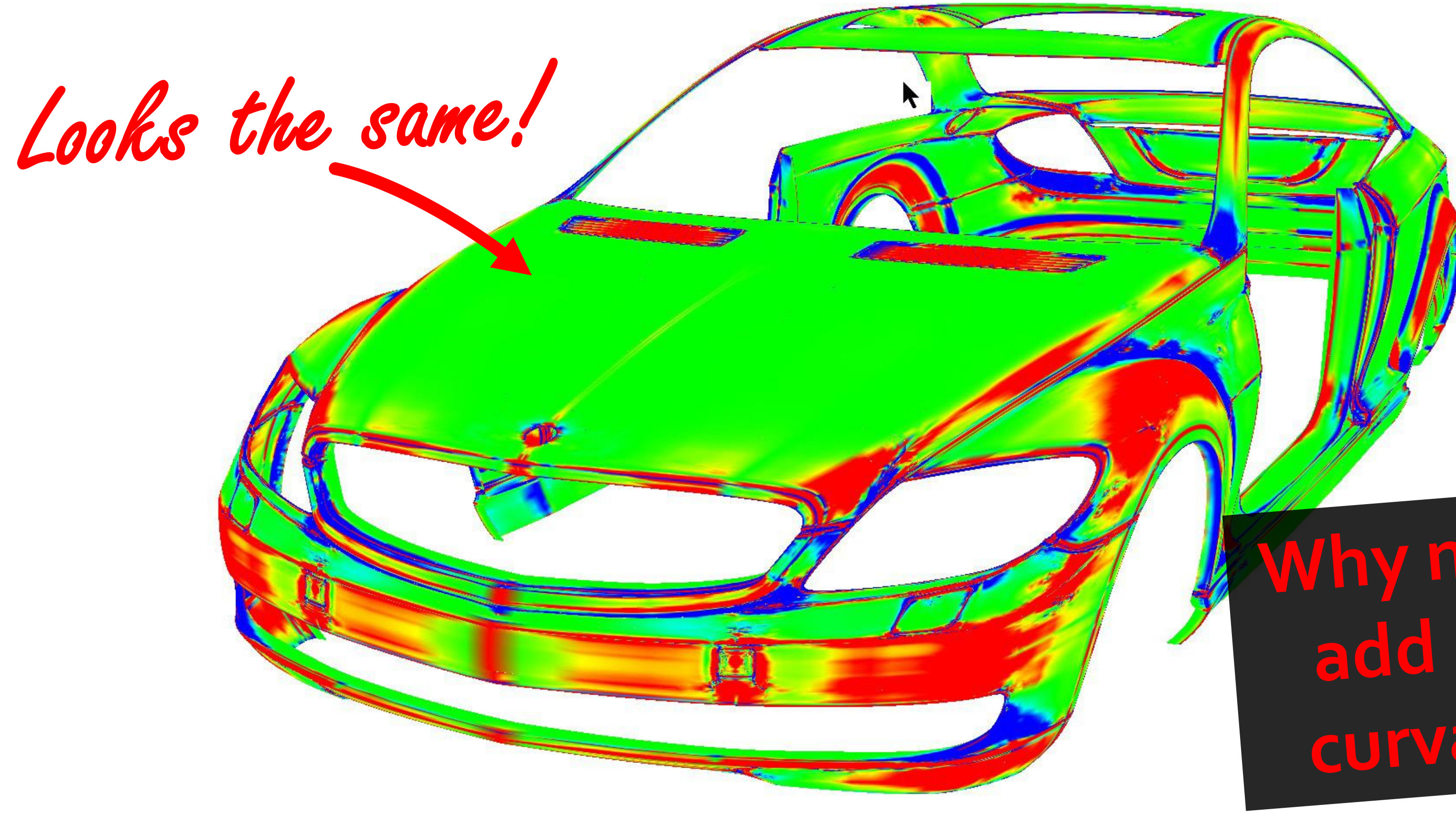
Noisy!



$$K = \kappa_1 \kappa_2$$

Second derivative quantity

End of the Story?



Why not just add mean curvature?

http://www.integrityware.com/images/MerceedesGaussianCurvature.jpg

Non-unique

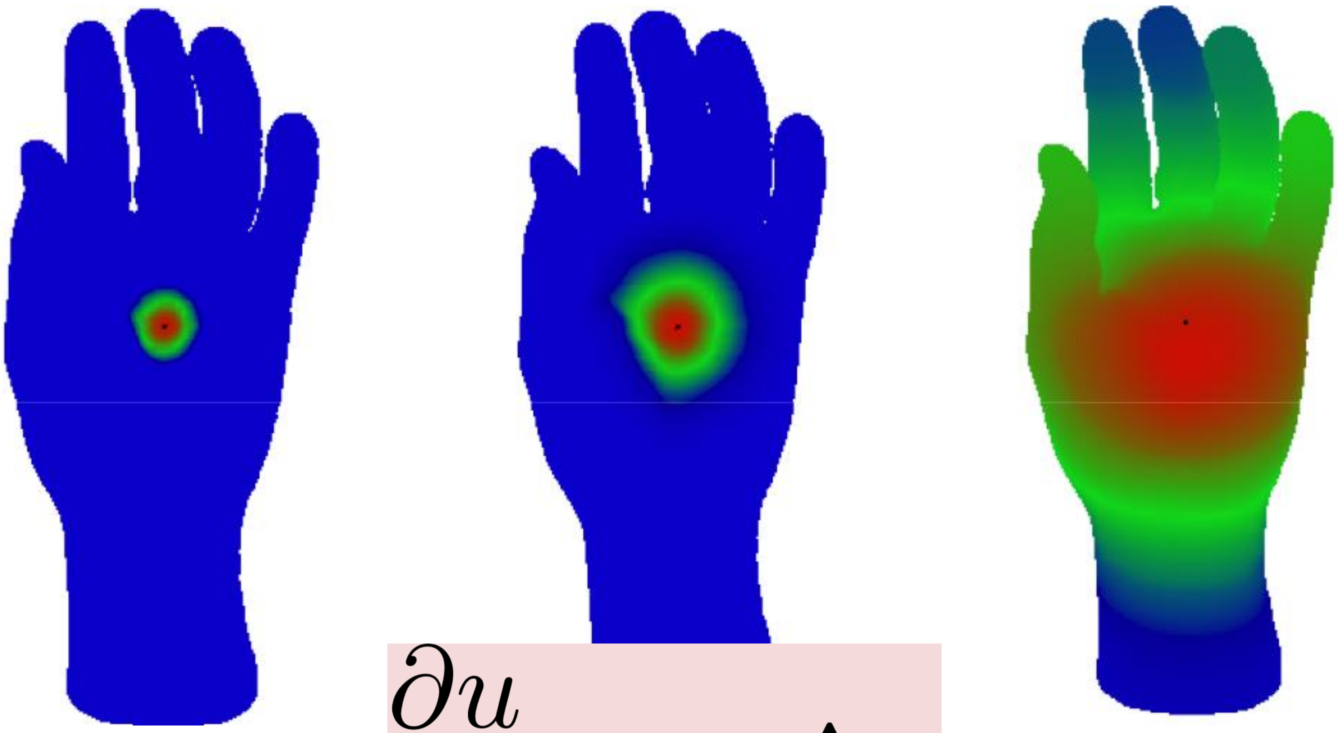
Desirable Properties

Incorporates neighborhood information in an intrinsic fashion

Stable under small deformation

Recall:

Connection to Physics



$$\frac{\partial u}{\partial t} = -\Delta u$$

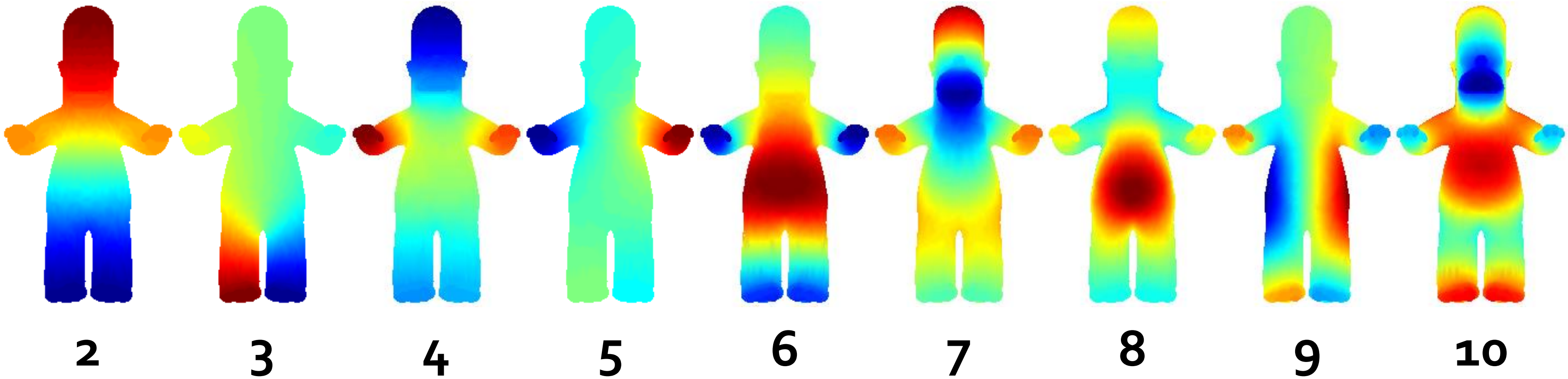
http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

Intrinsic Observation

Heat diffusion patterns are not affected if you bend a surface.

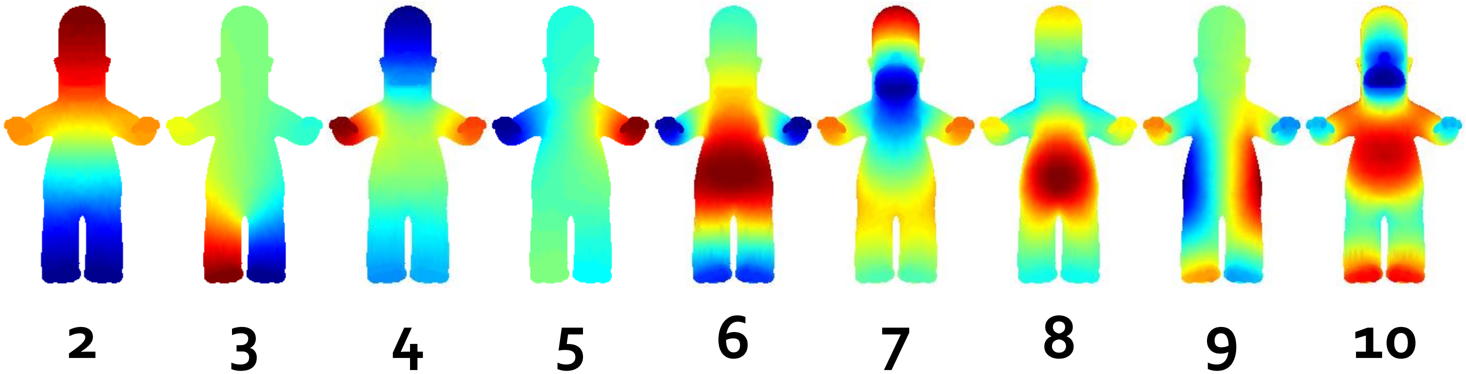
Global Point Signature



$$\mathrm{GPS}(p) := \left(-\frac{1}{\sqrt{\lambda_1}}\phi_1(p), -\frac{1}{\sqrt{\lambda_2}}\phi_2(p), -\frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

"Laplace-Beltrami Eigenfunctions for Deformation Invariant Shape Representation"
Rustamov, SGP 2007

Global Point Signature

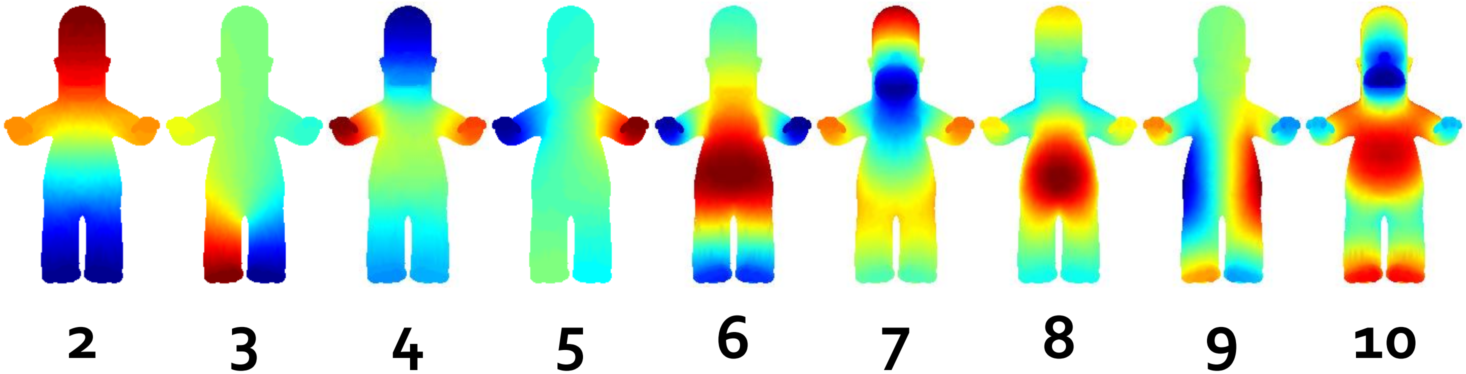


$$\mathrm{GPS}(p) := \left(-\frac{1}{\sqrt{\lambda_1}}\phi_1(p), -\frac{1}{\sqrt{\lambda_2}}\phi_2(p), -\frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

If surface does not self-intersect, neither does the GPS embedding.

Proof: Laplacian eigenfunctions span $L^2(\Sigma)$; if GPS(p)=GPS(q), then all functions on Σ would be equal at p and q.

Global Point Signature



$$\mathrm{GPS}(p) := \left(-\frac{1}{\sqrt{\lambda_1}}\phi_1(p), -\frac{1}{\sqrt{\lambda_2}}\phi_2(p), -\frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

GPS is isometry-invariant.

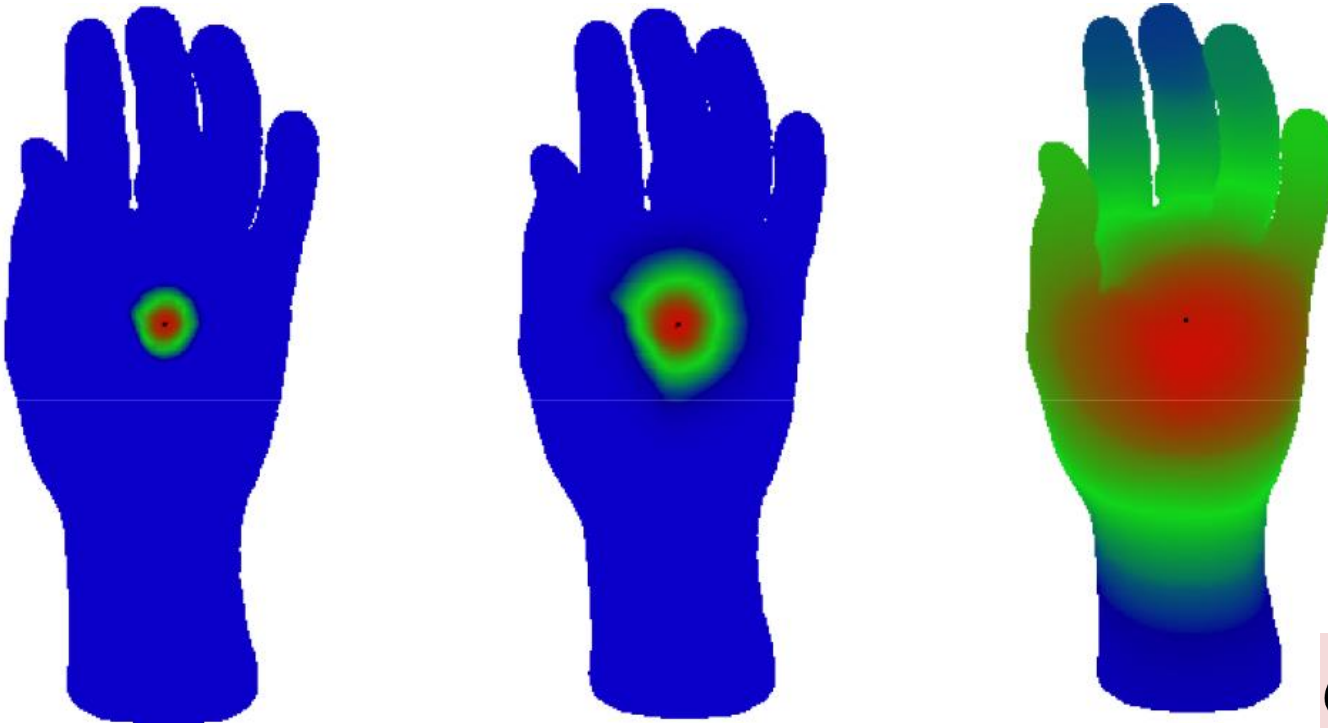
Proof: Comes from the Laplacian.

Drawbacks of GPS

- Assumes unique λ's
- Potential for eigenfunction "switching"
- Nonlocal feature

New idea:

PDE Applications of the Laplacian

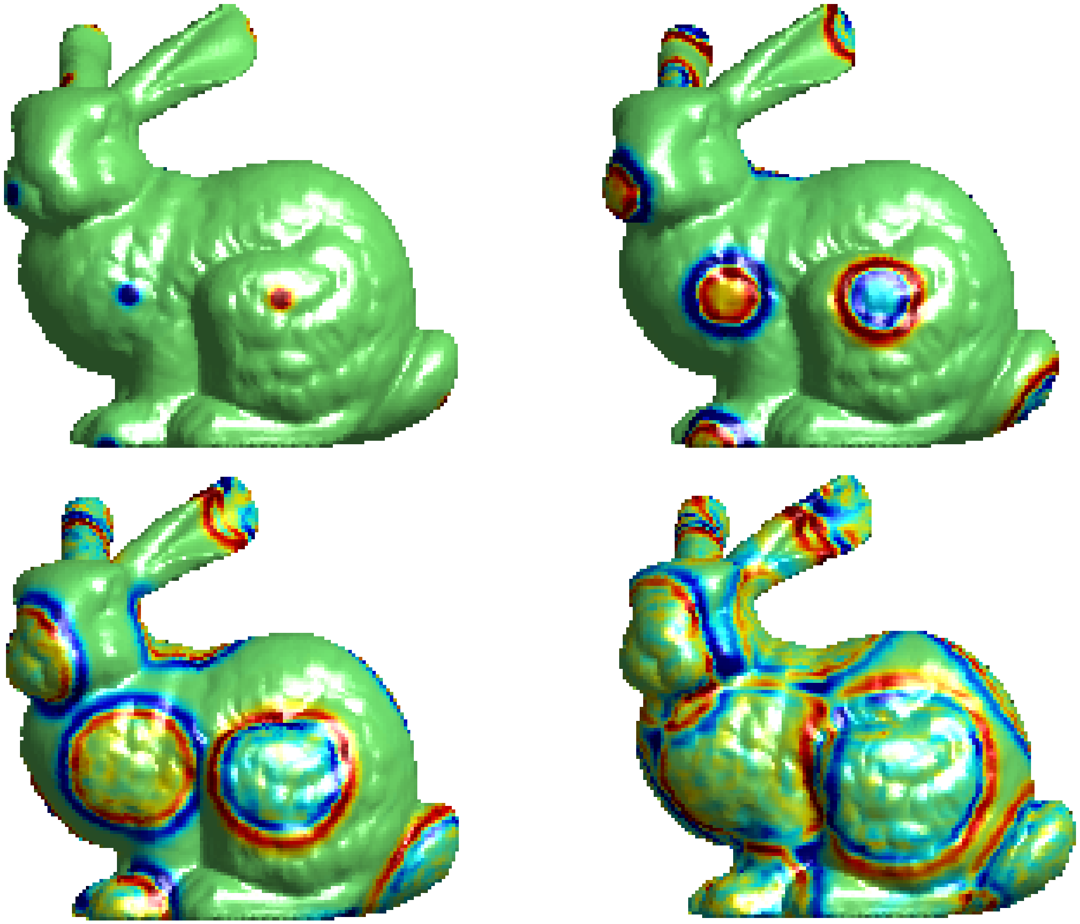


$$\frac{\partial u}{\partial t} = -\Delta u$$

http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

PDE Applications of the Laplacian



$$\frac{\partial^2 u}{\partial t^2} = -i\Delta u$$

Image courtesy G. Peyré

Wave equation

PDE Applications of the Laplacian

Use this behavior to characterize shape.

$$\frac{\partial^2 u}{\partial t^2} = -i\Delta u$$

Image courtesy G. Peyré

Wave equation

Solutions in the LB Basis

$$\frac{\partial u}{\partial t} = -\Delta u$$

Heat equation

$$u = \sum_{n=0}^{\infty} a_n e^{-\lambda_n t} \phi_n(x)$$

$$a_n = \int_{\Sigma} u_0(x) \cdot \phi_n(x) \, dA$$

Heat Kernel Signature (HKS)

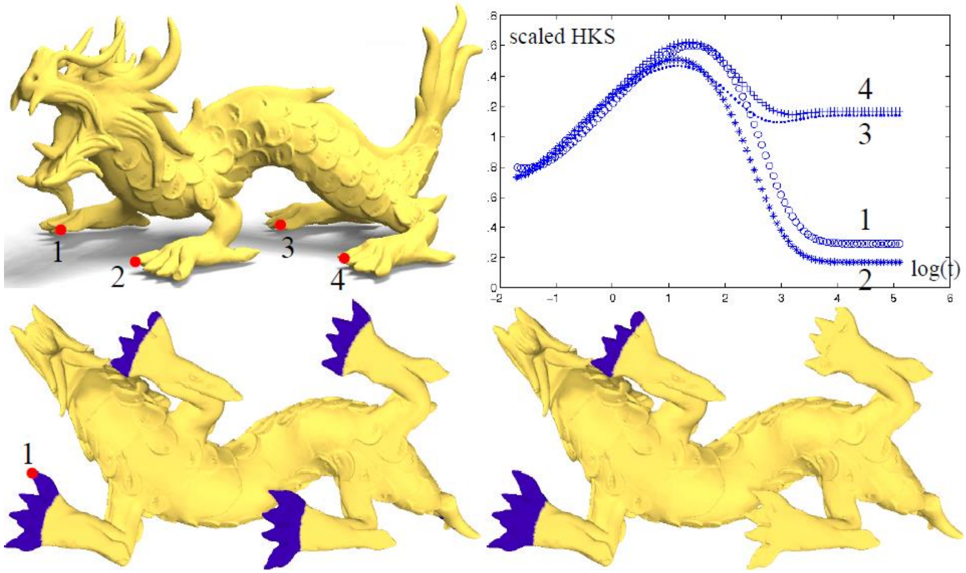
$$k_t(x,x) = \sum_{n=0}^{\infty} e^{-\lambda_i t} \phi_n(x)^2$$

Continuous function of $t \in [0, \infty)$

How much heat diffuses from *x* to itself in time *t*?

Heat Kernel Signature (HKS)

$$k_t(x,x) = \sum_{n=0}^{\infty} e^{-\lambda_i t} \phi_n(x)^2$$



"A concise and provably informative multi-scale signature based on heat diffusion"
Sun, Ovsjanikov, and Guibas; SGP 2009

Heat Kernel Signature (HKS)

$$k_t(x,x) = \sum_{n=0}^{\infty} e^{-\lambda_i t} \phi_n(x)^2$$

Good properties:

- Isometry-invariant
- Multiscale
- Not subject to switching
- Easy to compute
- Related to curvature at small scales

Heat Kernel Signature (HKS)

$$k_t(x,x) = \sum_{n=0}^{\infty} e^{-\lambda_i t} \phi_n(x)^2$$

Bad properties:

- **Issues remain with repeated eigenvalues**
- **Theoretical guarantees require (near-)isometry**

Wave Kernel Signature (WKS)

$$\mathrm{WKS}(E,x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T |\psi_E(x,t)|^2\,dt = \sum_{n=0}^{\infty} \phi_n(x)^2 f_E(\lambda_n)^2$$

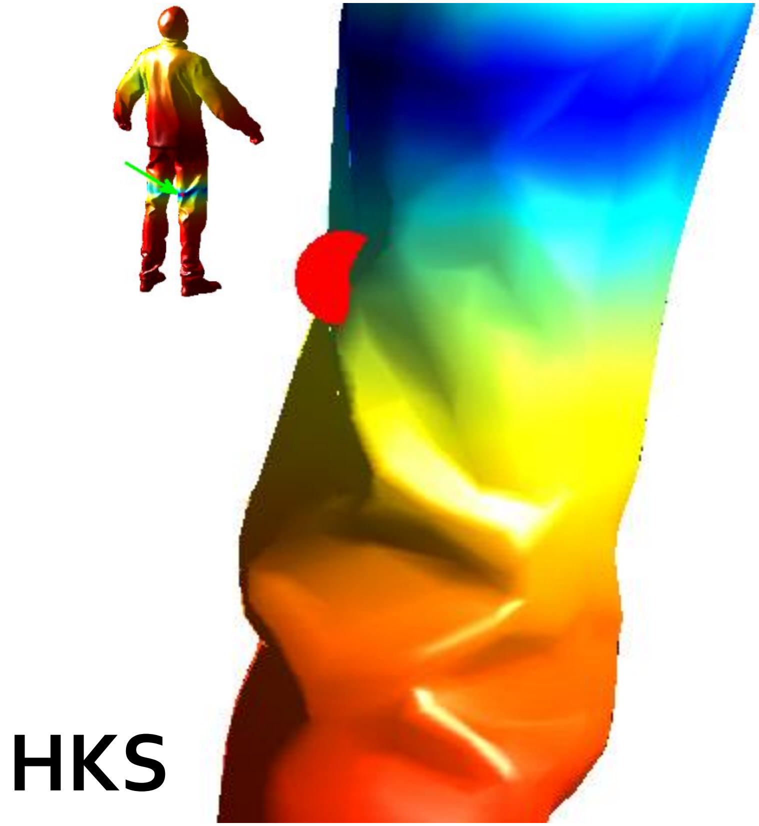
Initial energy distribution

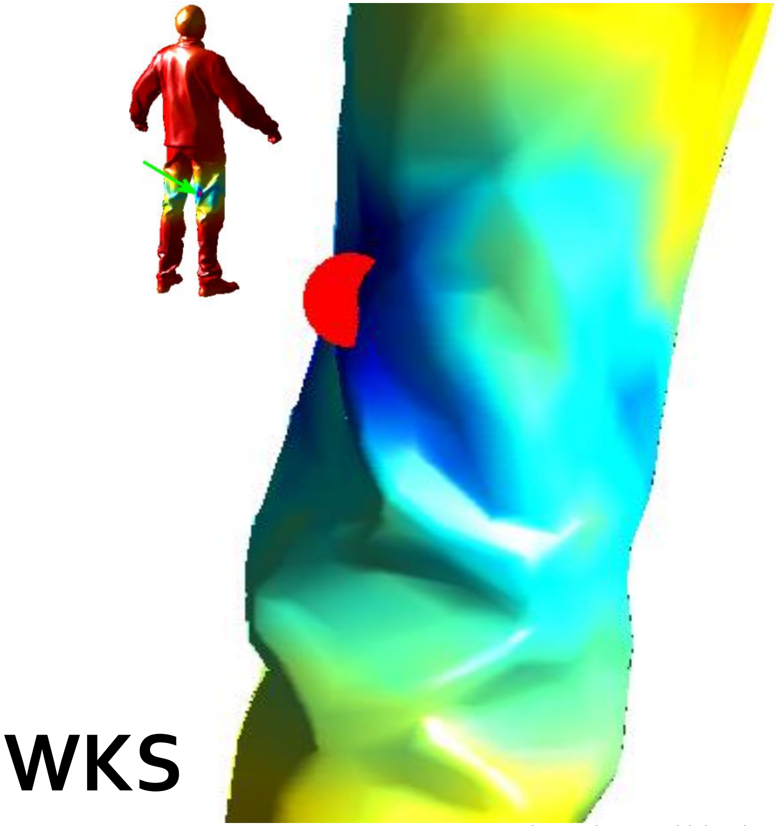
Average probability over time that particle is at *x*.

"The Wave Kernel Signature: A Quantum Mechanical Approach to Shape Analysis"
Aubry, Schlickewei, and Cremers; ICCV Workshops 2012

Wave Kernel Signature (WKS)

$$\mathrm{WKS}(E,x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T |\psi_E(x,t)|^2\, dt = \sum_{n=0}^{\infty} \phi_n(x)^2 f_E(\lambda_n)^2$$





vision.in.tum.de/_media/spezial/bib/aubry-et-al-4dmod11.pdf

Wave Kernel Signature (WKS)

$$\mathrm{WKS}(E,x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T |\psi_E(x,t)|^2\,dt = \sum_{n=0}^{\infty} \phi_n(x)^2 f_E(\lambda_n)^2$$

Good properties:

- [Similar to HKS]
- Localized in frequency
- Stable under some non-isometric deformation
- Some multi-scale properties

Wave Kernel Signature (WKS)

$$\mathrm{WKS}(E,x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T |\psi_E(x,t)|^2\, dt = \sum_{n=0}^{\infty} \phi_n(x)^2 f_E(\lambda_n)^2$$

Bad properties:

- [Similar to HKS]
- Can filter out *large*-scale features

Many Others

Lots of spectral descriptors in terms of Laplacian eigenstructure.

Combination with Machine Learning

$$p(x) = \sum_k f(\lambda_k)\phi_k^2(x)$$

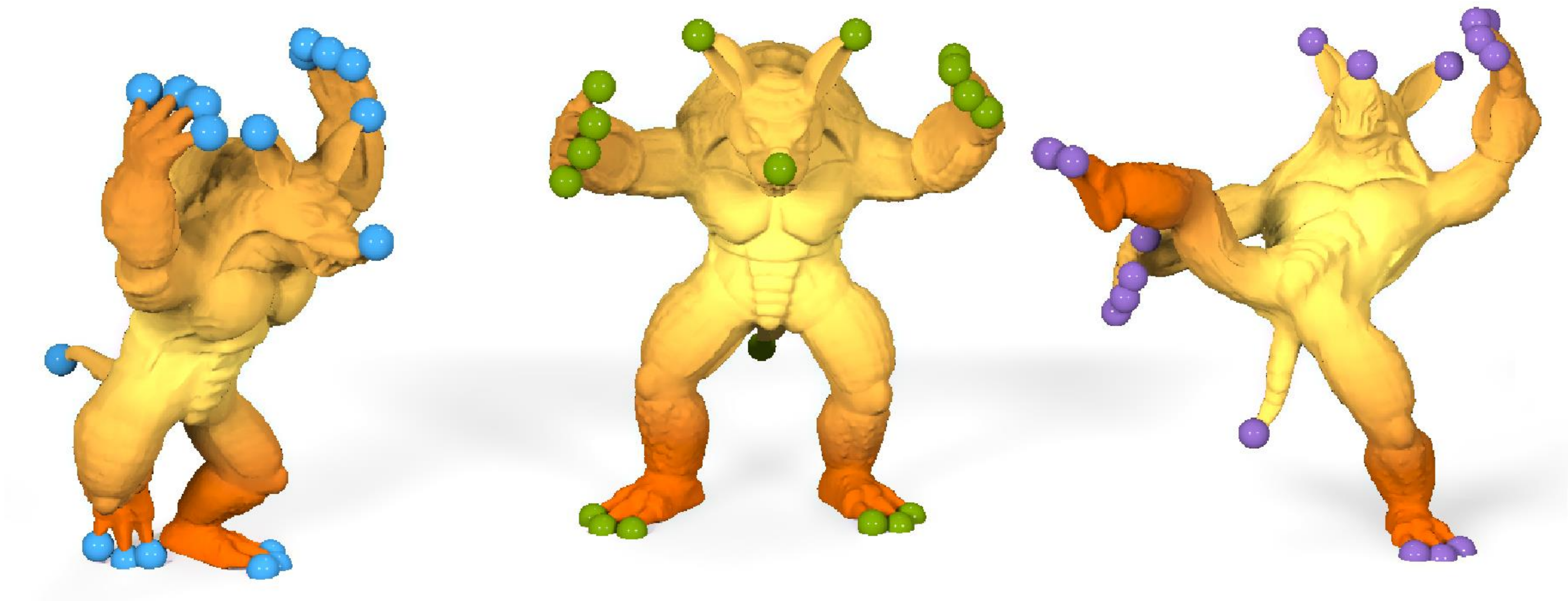
Learn *f* rather than defining it

Fig. 3. Correspondences computed on TOSCA shapes using the spectral matching algorithm [30]. Shown are the matches with geodesic distance distortion below 10 percent of the shape diameter, from left to right: HKS (34 matches), WKS (30 matches), and trained descriptor (54 matches).

Learning Spectral Descriptors for Deformable Shape Correspondence

Litman and Bronstein; PAMI 2014

Application: Feature Extraction

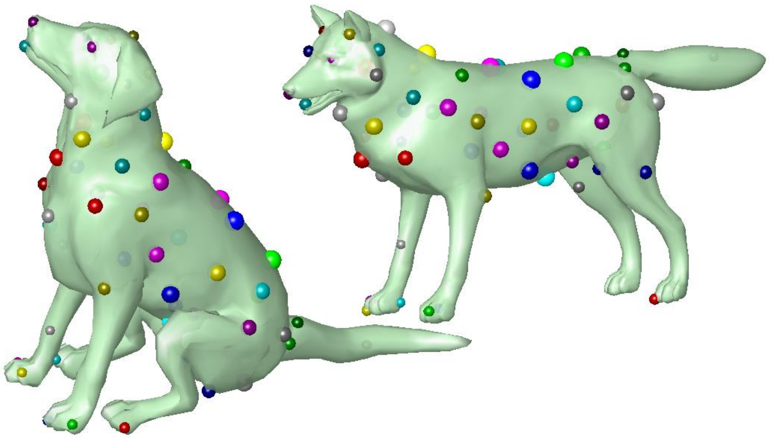


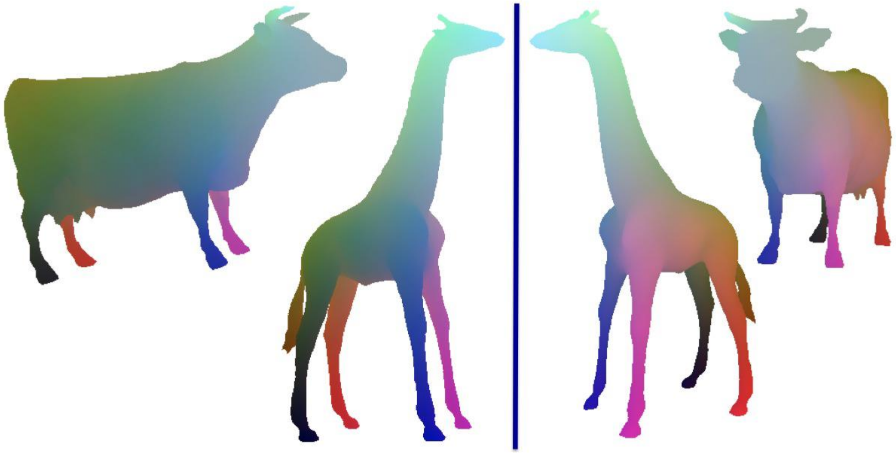
Maxima of $k_t(x,x)$ over x for large t.

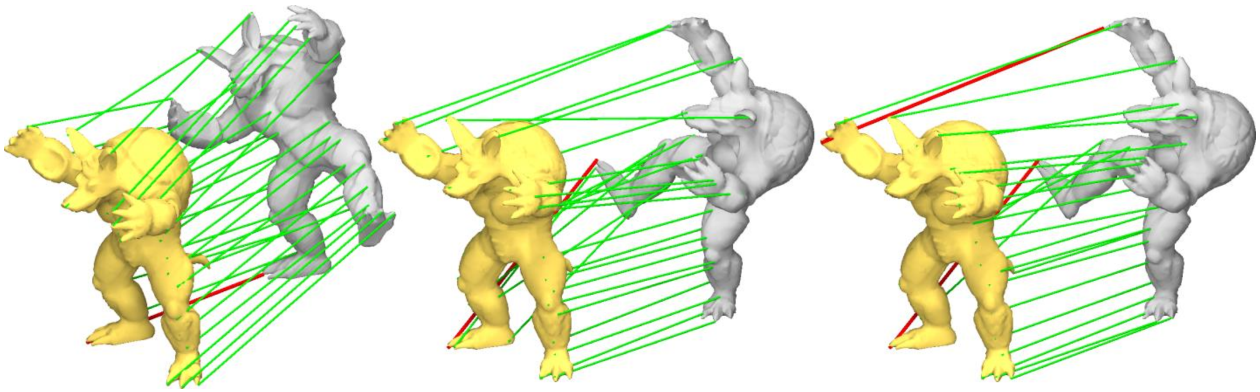
A Concise and Provably Informative Multi-Scale Signature Based on Heat Diffusion
Sun, Ovsjanikov, and Guibas; SGP 2009

Feature points

Preview: Correspondence





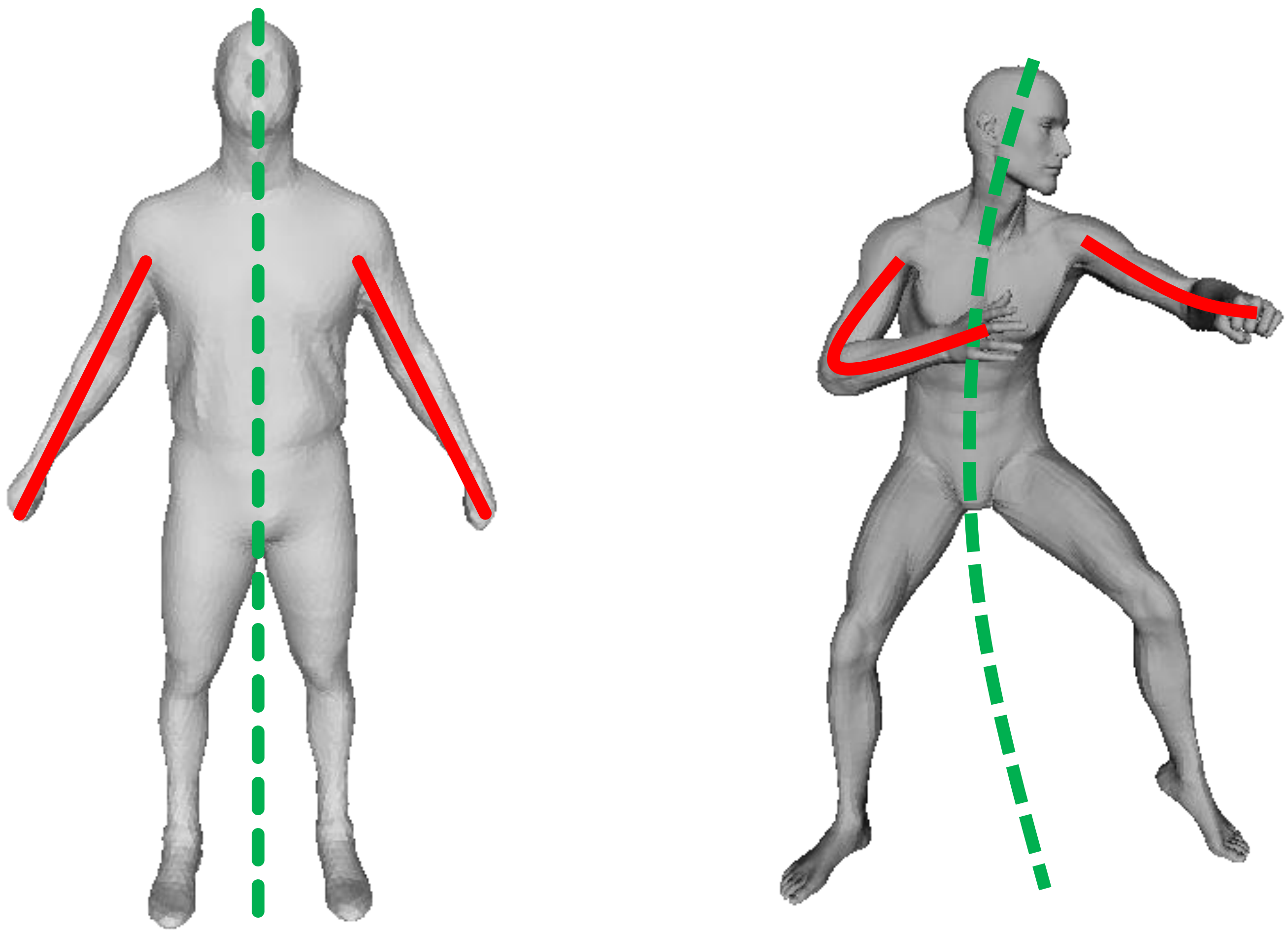


http://graphics.stanford.edu/projects/lgl/papers/ommg-opimhk-10/ommg-opimhk-10.pdf
http://www.cs.princeton.edu/~funk/sig11.pdf
http://gfx.cs.princeton.edu/pubs/Lipman_2009_MVF/mobius.pdf

Descriptor Matching

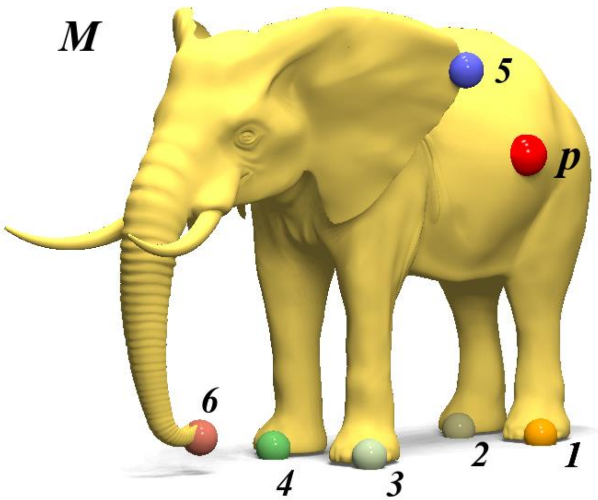
Simply match closest points in descriptor space.

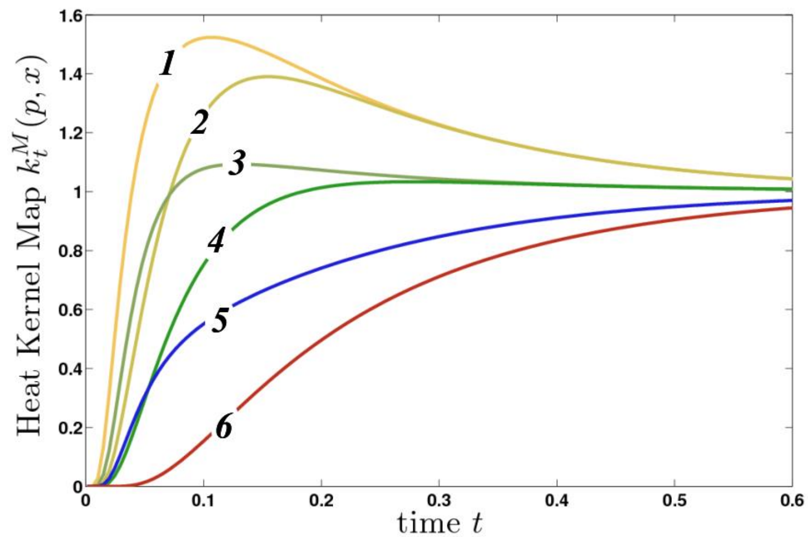
Descriptor Matching Problem



Symmetry

Heat Kernel Map





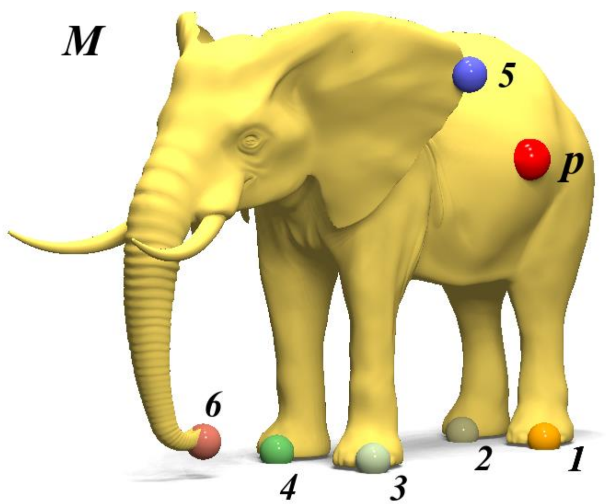
$$\mathrm{HKM}_p(x,t) := k_t(p,x)$$

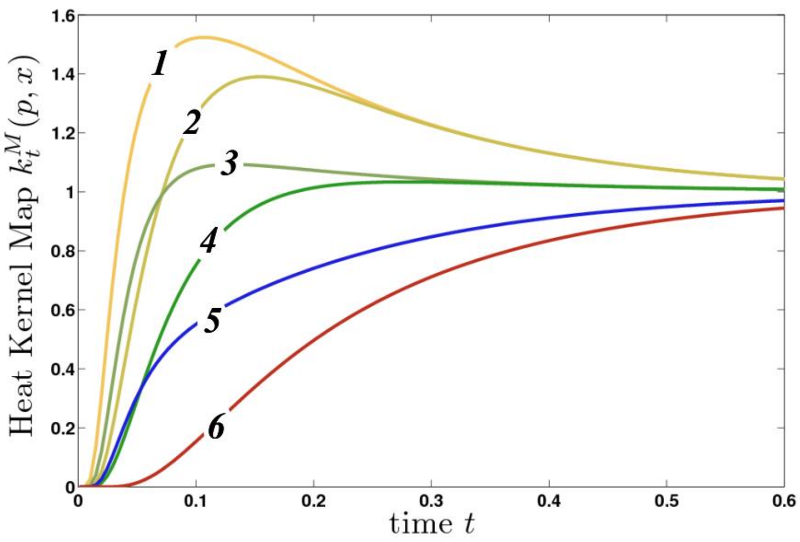
How much heat diffuses from *p* to *x* in time *t*?

One Point Isometric Matching with the Heat Kernel

Ovsjanikov et al. 2010

Heat Kernel Map





$$\mathrm{HKM}_p(x,t) := k_t(p,x)$$

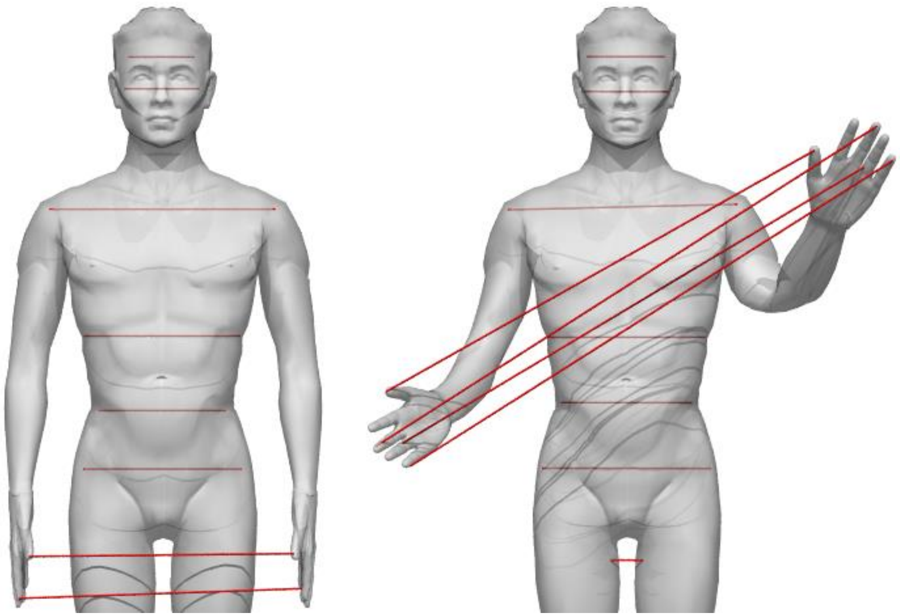
Theorem: Only have to match one point!

KNN

One Point Isometric Matching with the Heat Kernel

Ovsjanikov et al. 2010

Self-Map: Symmetry



Intrinsic symmetries become extrinsic in GPS space!

Global Intrinsic Symmetries of Shapes
Ovsjanikov, Sun, and Guibas 2008

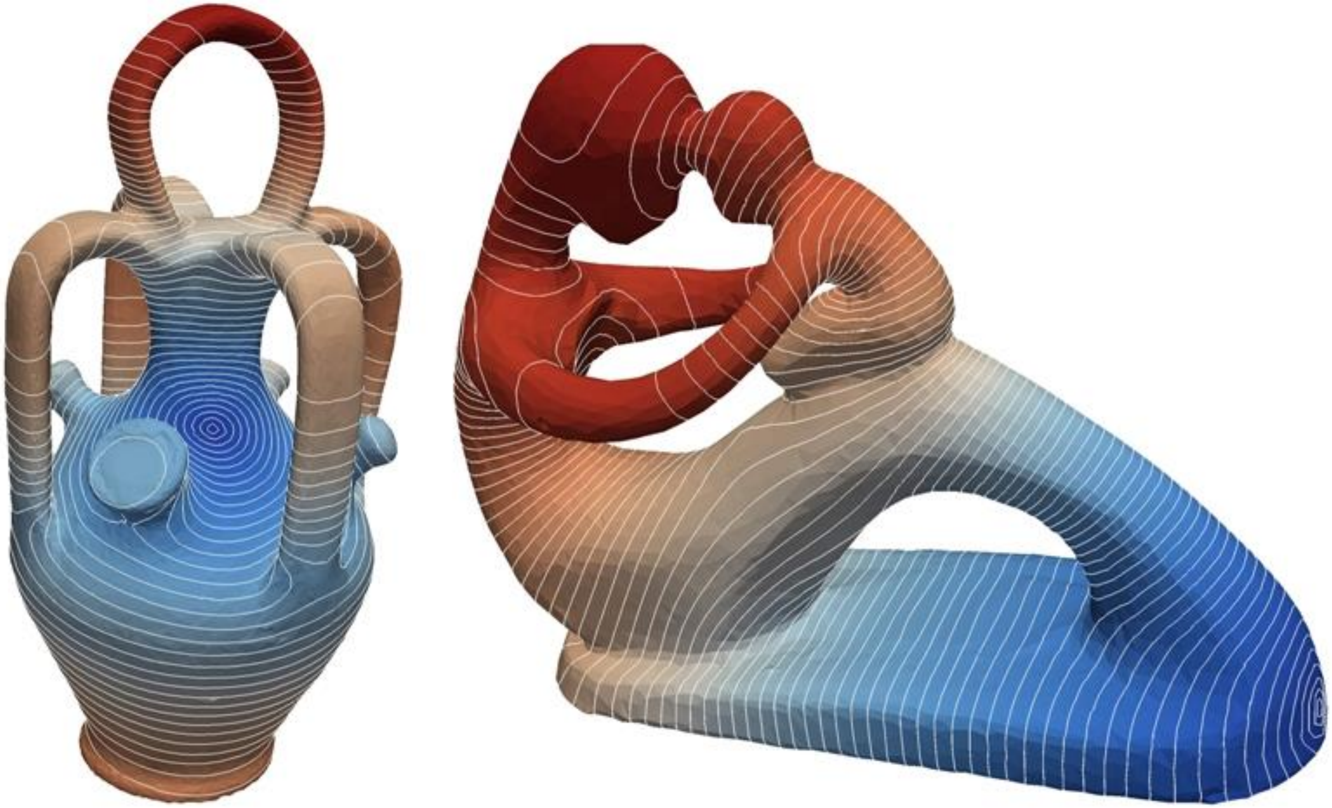
"Discrete intrinsic" symmetries

All Over the Place

Laplacians appear everywhere in shape analysis and geometry processing.

Biharmonic Distances

$$d_b(p,q) := \|g_p - g_q\|_2, \ \text{where } \Delta g_p = \delta_p$$

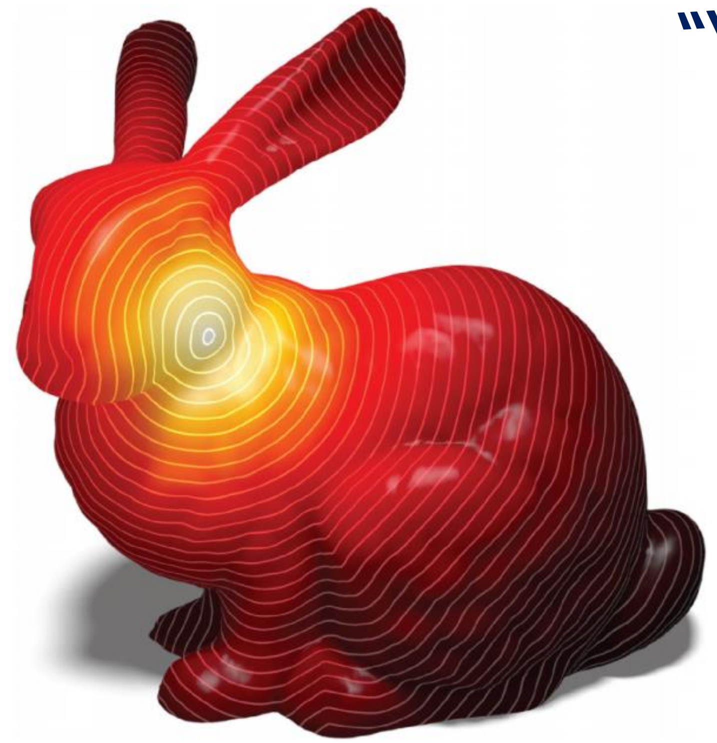


"Biharmonic distance"
Lipman, Rustamov & Funkhouser, 2010

Geodesic Distances

$$d_g(p,q) = \lim_{t\to 0} \sqrt{-4t \log k_{t,p}(q)}$$

"Varadhan's Theorem"



"Geodesics in heat"
Crane, Weischedel, and Wardetzky; TOG 2013

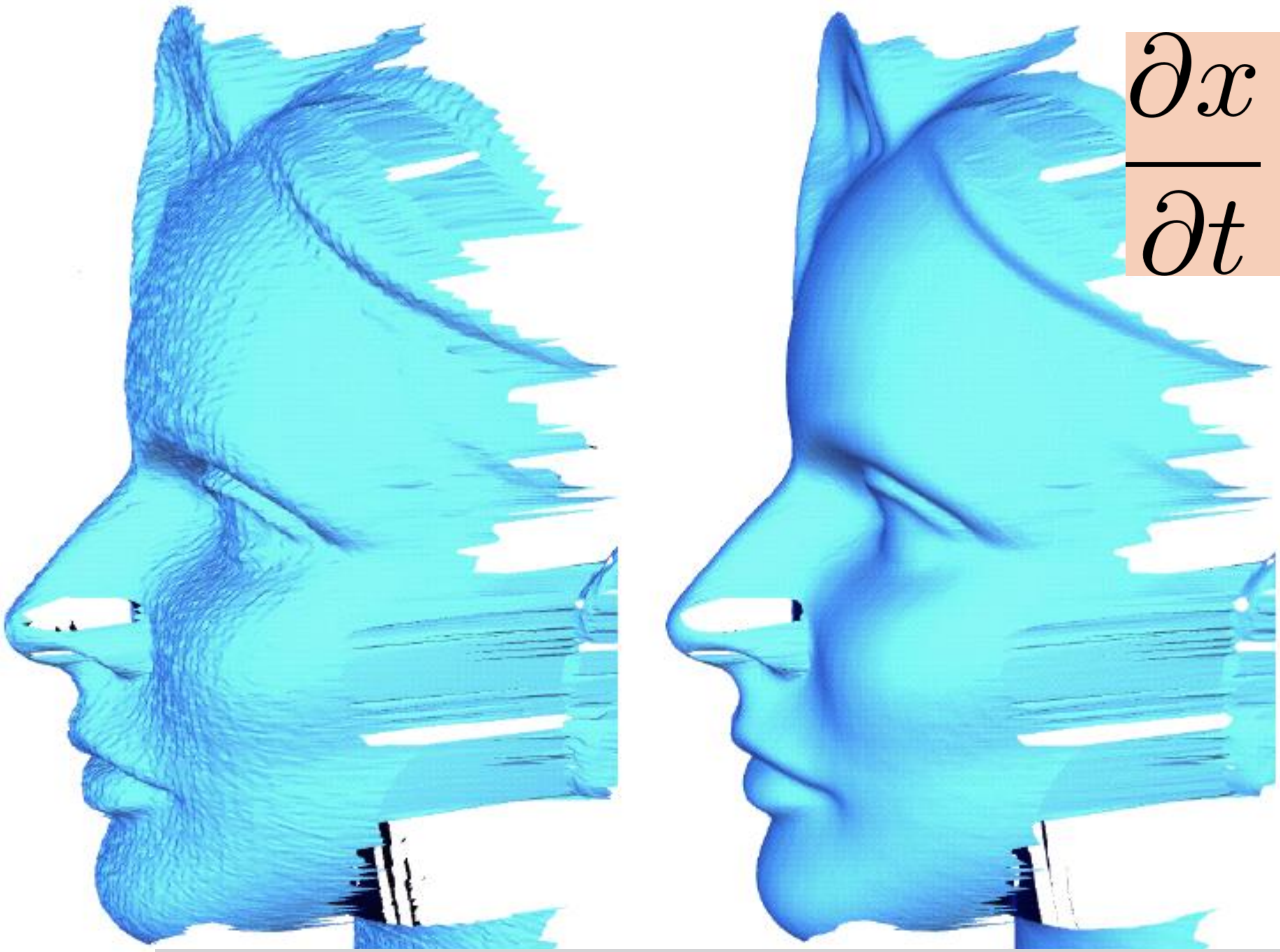
Alternative to Eikonal Equation

Algorithm 1 The Heat Method

I. Integrate the heat flow $\dot{u} = \Delta u$ for time t.
II. Evaluate the vector field $X = -\nabla u / |\nabla u|$.
III. Solve the Poisson equation $\Delta \phi = \nabla \cdot X$.

Crane, Weischedel, and Wardetzky. "Geodesics in Heat." TOG, 2013.

Implicit Fairing: Mean Curvature Flow



$$\frac{\partial x}{\partial t} = \Delta(x) \cdot x$$

"Implicit fairing of irregular meshes using diffusion and curvature flow"
Desbrun et al., 1999

Useful Technique

$$\frac{\partial f}{\partial t} = -\Delta f \text{ (heat equation)}$$

$$\to M\frac{\partial f}{\partial t} = Lf \text{ after discretization in space}$$

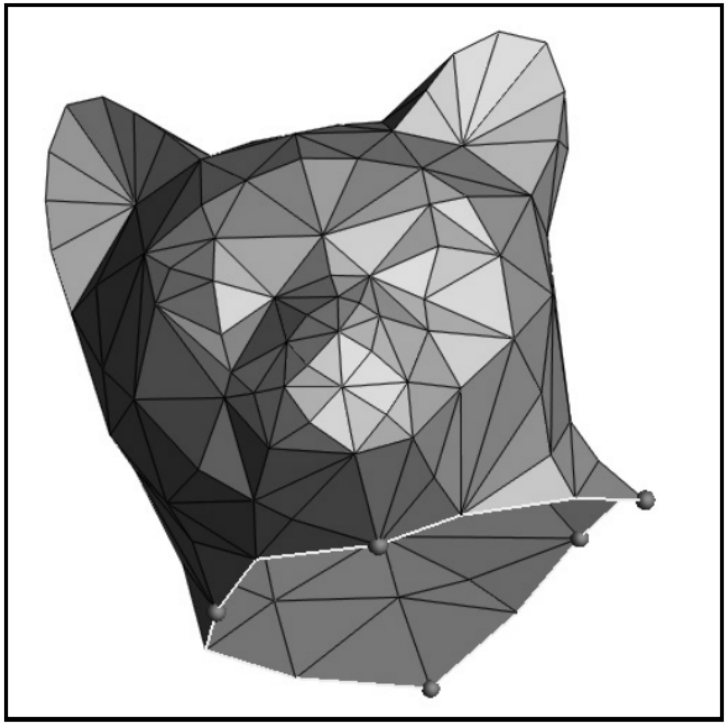
$$\to M\frac{f_T - f_0}{T} = Lf_T \text{ after time discretization}$$

Choice: Evaluate at time T

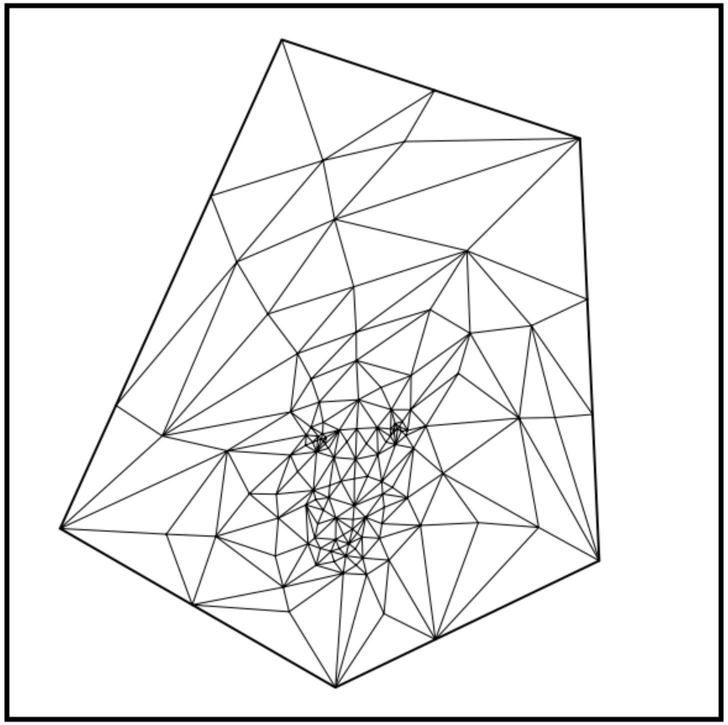
Unconditionally stable, but not necessarily accurate for large T!

Implicit time stepping

Parameterization: Harmonic Map



(a) Original mesh tile



(b) Harmonic embedding

Recall:
Mean value principle

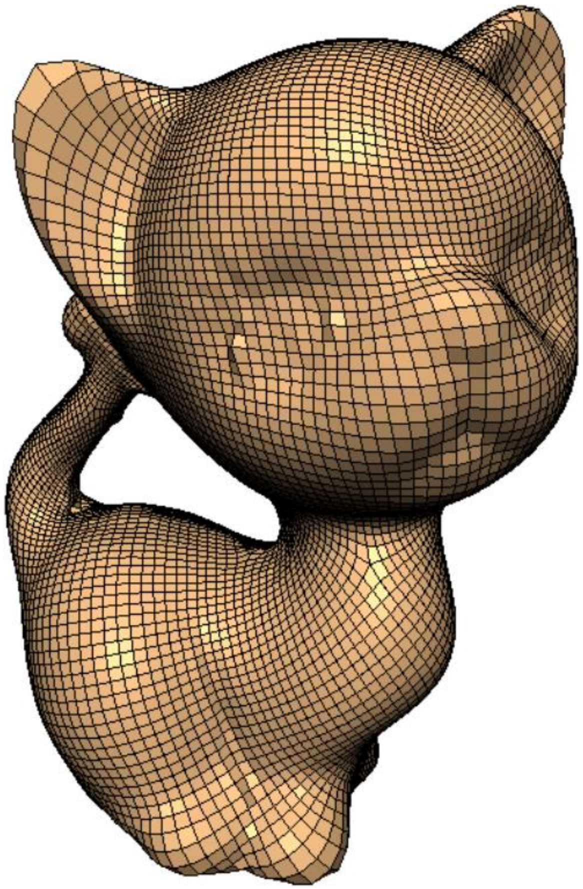
"Multiresolution analysis of arbitrary meshes"
Eck et al., 1995 (and many others!)

Others

- **Shape retrieval from Laplacian eigenvalues**
  "Shape DNA" [Reuter et al., 2006]

- **Quadrangulation**
  Nodal domains [Dong et al., 2006]

- **Surface deformation**
  "As-rigid-as-possible" [Sorkine & Alexa, 2007]



Our Next Topic

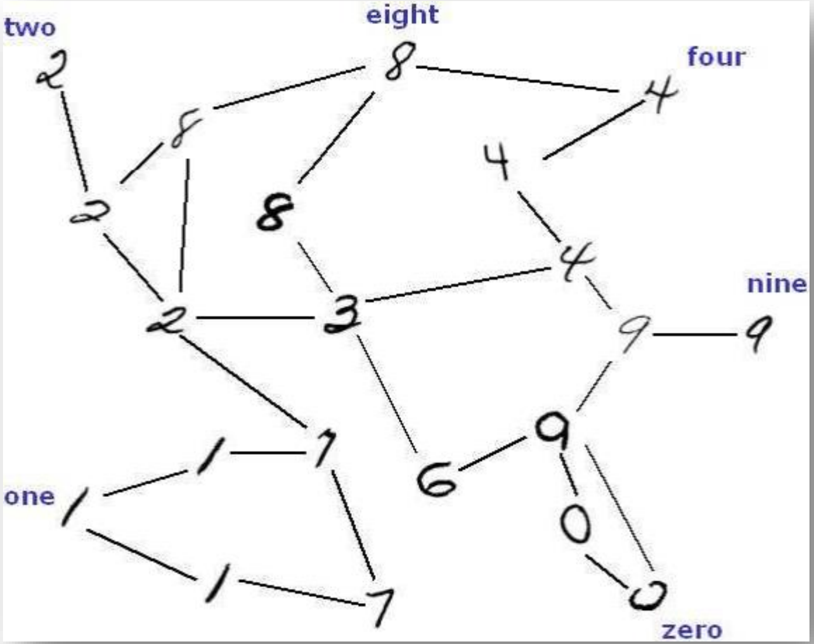
Discrete Laplacian operators:

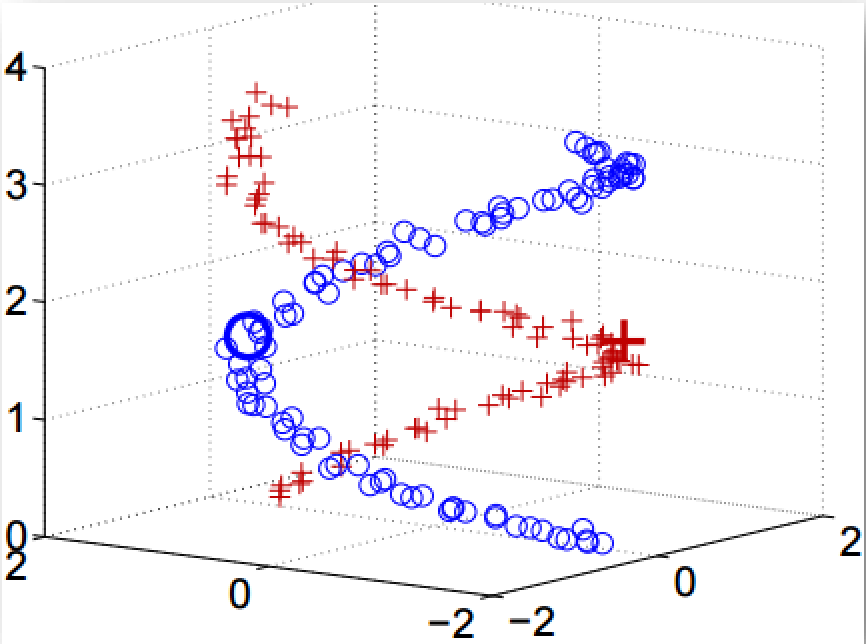
What are they good for?

- Useful properties of the Laplacian
- Applications in graphics/shape analysis
- Applications in machine learning

A quick survey:
A popular field!

Semi-Supervised Learning





"Semi-supervised learning using Gaussian fields and harmonic functions"
Zhu, Ghahramani, & Lafferty 2003

Semi-Supervised Technique

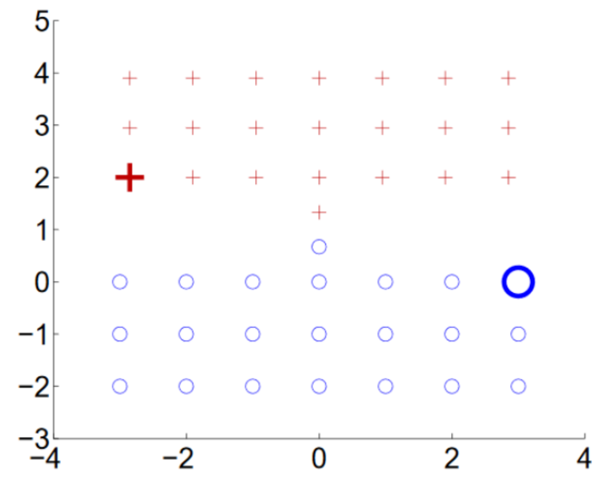
Given: ℓ labeled points $(x_1, y_1), \ldots, (x_\ell, y_\ell); y_i \in \{0, 1\}$

u unlabeled points $x_{\ell+1}, \ldots, x_{\ell+u}; \ell \ll u$

$$\min \frac{1}{2} \sum_{ij} w_{ij} (f(i) - f(j))^2$$

$$\text{s.t. } f(k) \text{ fixed } \forall k \leq \ell$$

Dirichlet energy → Linear system of equations (Poisson)



Related Method

- **Step 1:**
  Build *k*-NN graph

- **Step 2:**
  Compute *p* smallest Laplacian eigenvectors

- **Step 3:**
  Solve semi-supervised problem in subspace

"Using Manifold Structure for Partially Labelled Classification"
Belkin and Niyogi; NIPS 2002

Buyer Beware: Ill-Posed in Limit?

Semi-Supervised Learning with the Graph Laplacian: The Limit of Infinite Unlabelled Data

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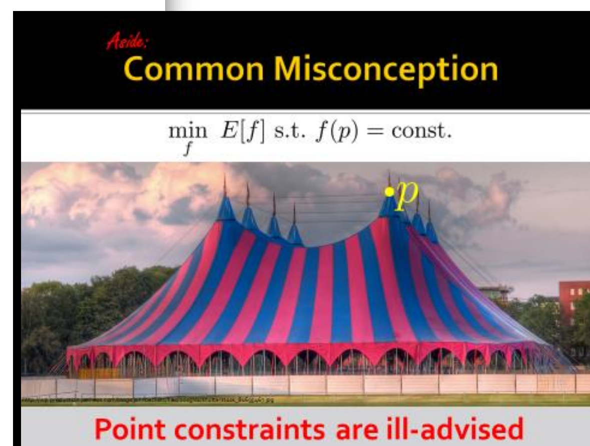
Potential fix:
Higher-order operators



Abstract

We study the behavior of the popular Laplacian Regularization method for Semi-Supervised Learning at the regime of a fixed number of labeled points but a large

Manifold Regularization

Regularized learning: $\displaystyle \arg\min_{f \in \mathcal{H}} \frac{1}{\ell} \sum_{i=1}^{\ell} V(f(x_i), y_i) + \gamma \|f\|^2$

Loss function

Regularizer

Dirichlet energy

$$\|f\|_I^2 := \int \|\nabla f(x)\|^2 \, dx \approx f^\top L f$$

"Manifold Regularization:
A Geometric Framework for Learning from Labeled and Unlabeled Examples"
Belkin, Niyogi, and Sindhwani; JMLR 2006

Examples of Manifold Regularization

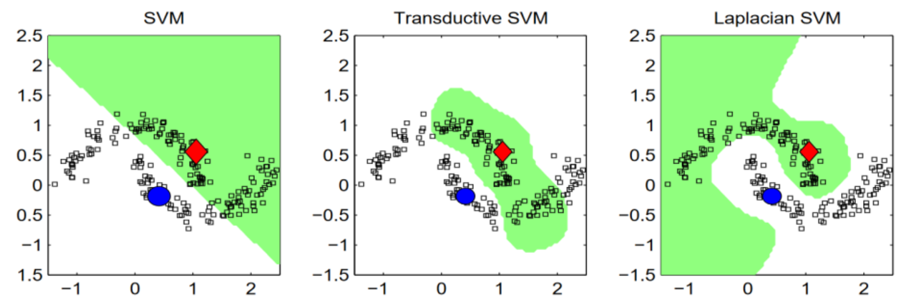
- Laplacian-regularized least squares (LapRLS)

$$\arg\min_{f\in\mathcal{H}} \frac{1}{\ell}\sum_{i=1}^{\ell}(f(x_i)-y_i)^2+\gamma\|f\|_I^2+\text{Other}[f]$$

- Laplacian support vector machine (LapSVM)

$$\arg\min_{f\in\mathcal{H}} \frac{1}{\ell}\sum_{i=1}^{\ell}\max(0,1-y_i f(x_i))+\gamma\|f\|_I^2+\text{Other}[f]$$

"On Manifold Regularization"
Belkin, Niyogi, Sindhwani; AISTATS 2005



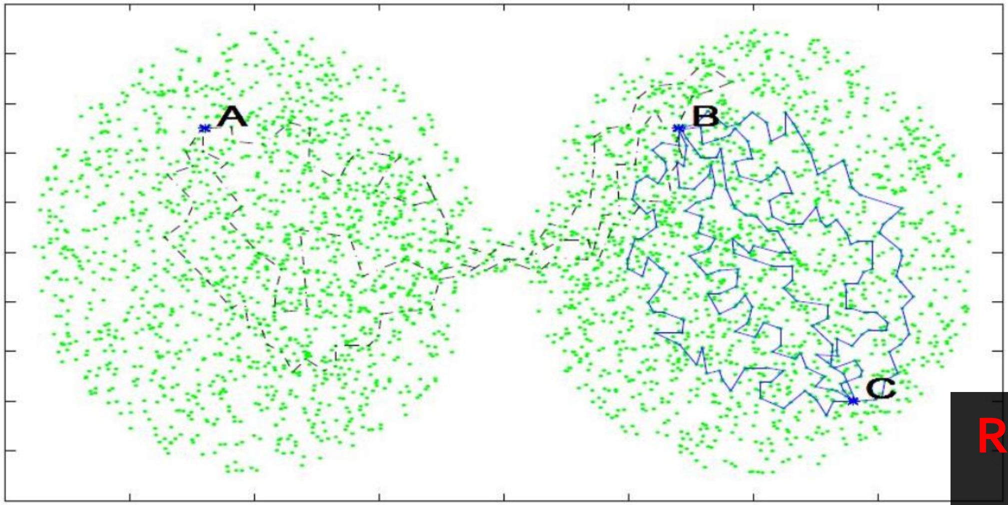
Diffusion Maps

Embedding from first *k* eigenvalues/vectors:

$$\Psi_t(x) := \left(\lambda_1^t \psi_1(x), \lambda_2^t \psi_2(x), \ldots, \lambda_k^t \psi_k(x)\right)$$

Roughly:

$|\boldsymbol{\Psi}_{\mathbf{t}}(\mathbf{x}) - \boldsymbol{\Psi}_t(y)|$ **is probability that *x*, *y* diffuse to the same point in time *t*.**

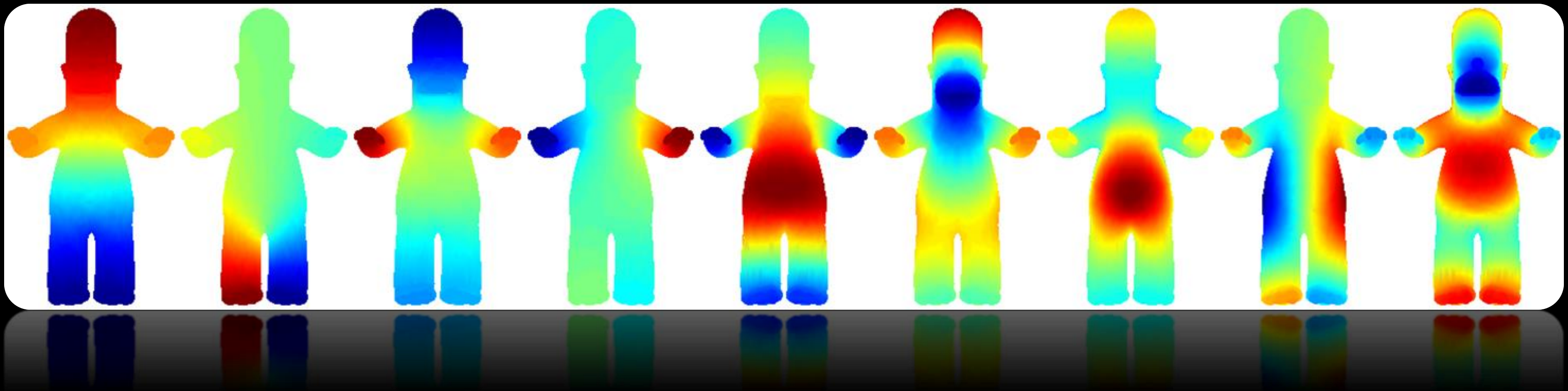


Robust to sampling and noise

"Diffusion Maps"

Coifman and Lafon; Applied and Computational Harmonic Analysis, 2006

Image from http://users.math.yale.edu/users/gw289/CpSc-445-545/Slides/CPSC445%20-%20Topic%2010%20-%20Diffusion%20Maps.pdf (nice slides!)



Applications of the Laplacian

Justin Solomon

MIT, Spring 2019