http://www.alvinomassage.com/images/knot.jpg

Curves: Continuous and Discrete

Justin Solomon

MIT, Spring 2017

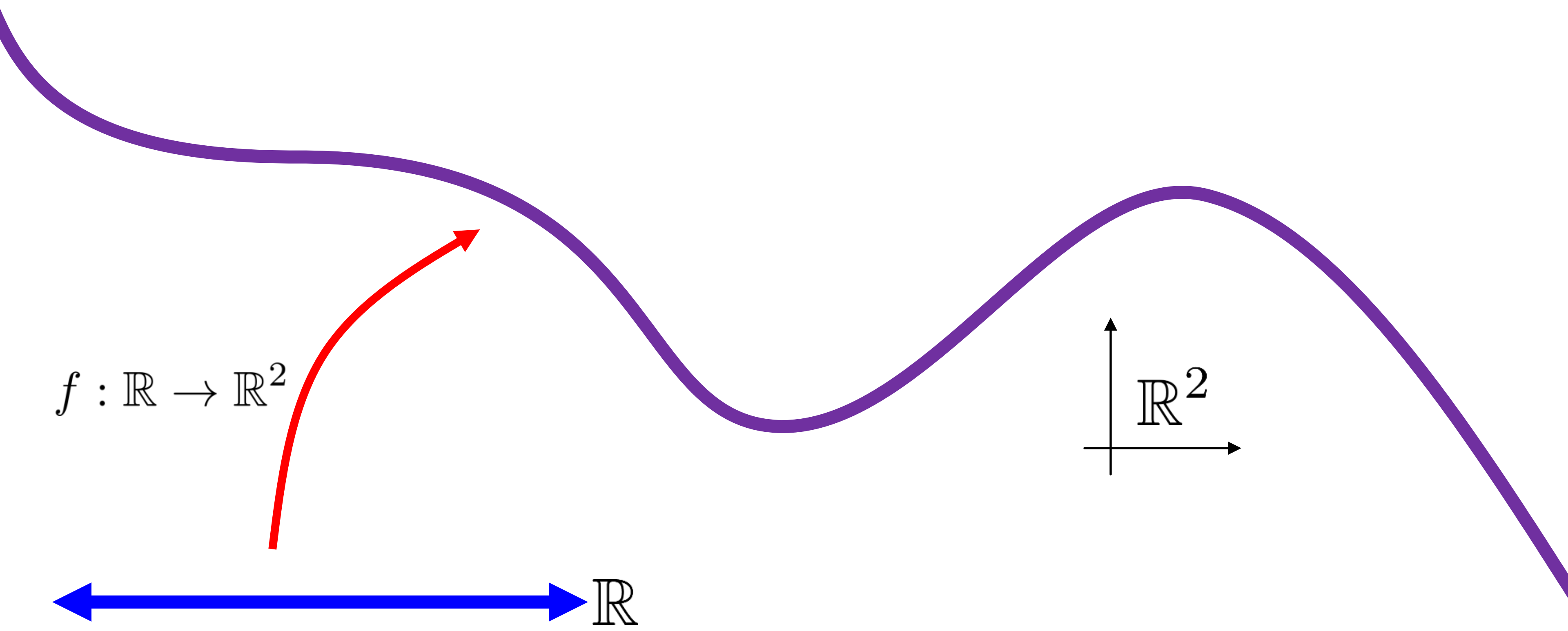
MIT EECS

Some materials from Stanford CS 468, spring 2013 (Butscher & Solomon)

Q:

What is a curve?

Defining "Curve"



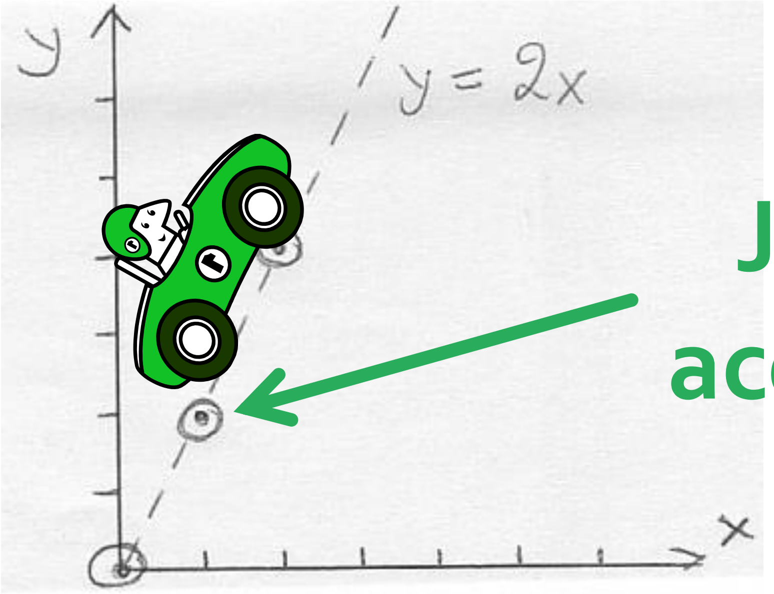
A function?

Subtlety

$$\gamma_3(t) := (0,0)$$

Not a curve

Different from Calculus



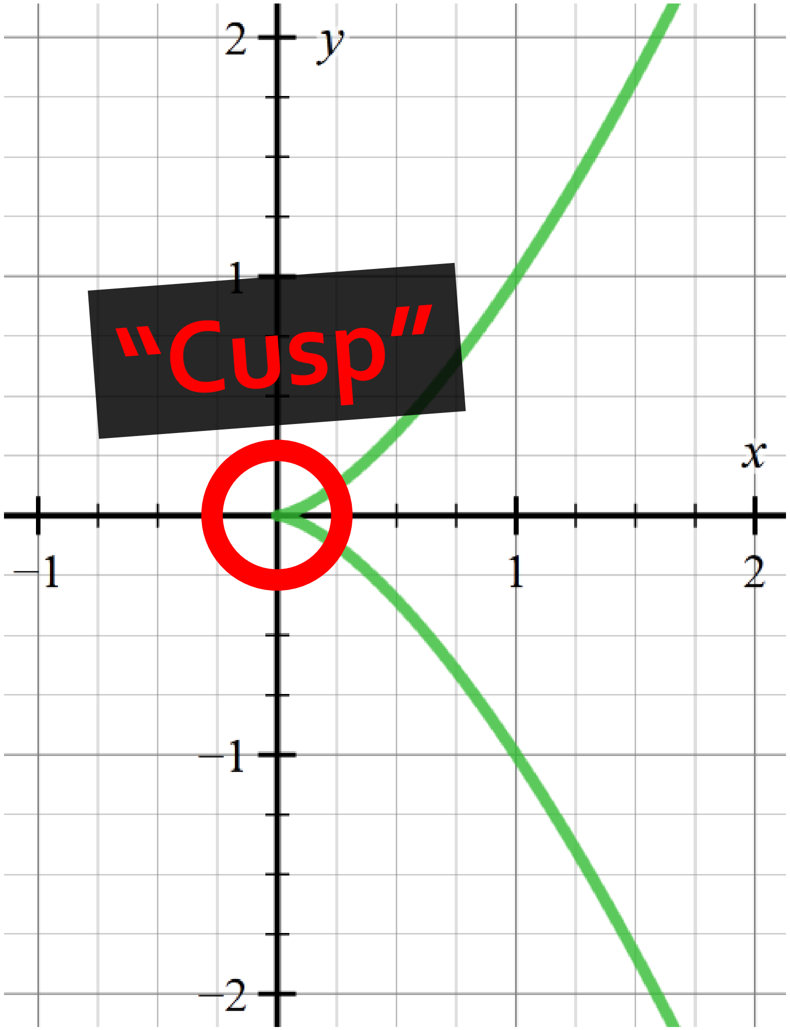
Jams on accelerator

$$f_1(t) = (t, 2t)$$

$$f_2(t) = \begin{cases} (t, 2t) & t \leq 1 \\ (2(t - \frac{1}{2}), 4(t - \frac{1}{2})) & t > 1 \end{cases}$$

http://sd271.k12.id.us/lchs/faculty/sjacobson/ibphysics/compendium/12_files/image003.jpg

Graphs of Smooth Functions



$$f(t) = (t^2, t^3)$$

http://en.wikipedia.org/wiki/Singular_point_of_a_curve

Geometry of a Curve

A curve is a

set of points

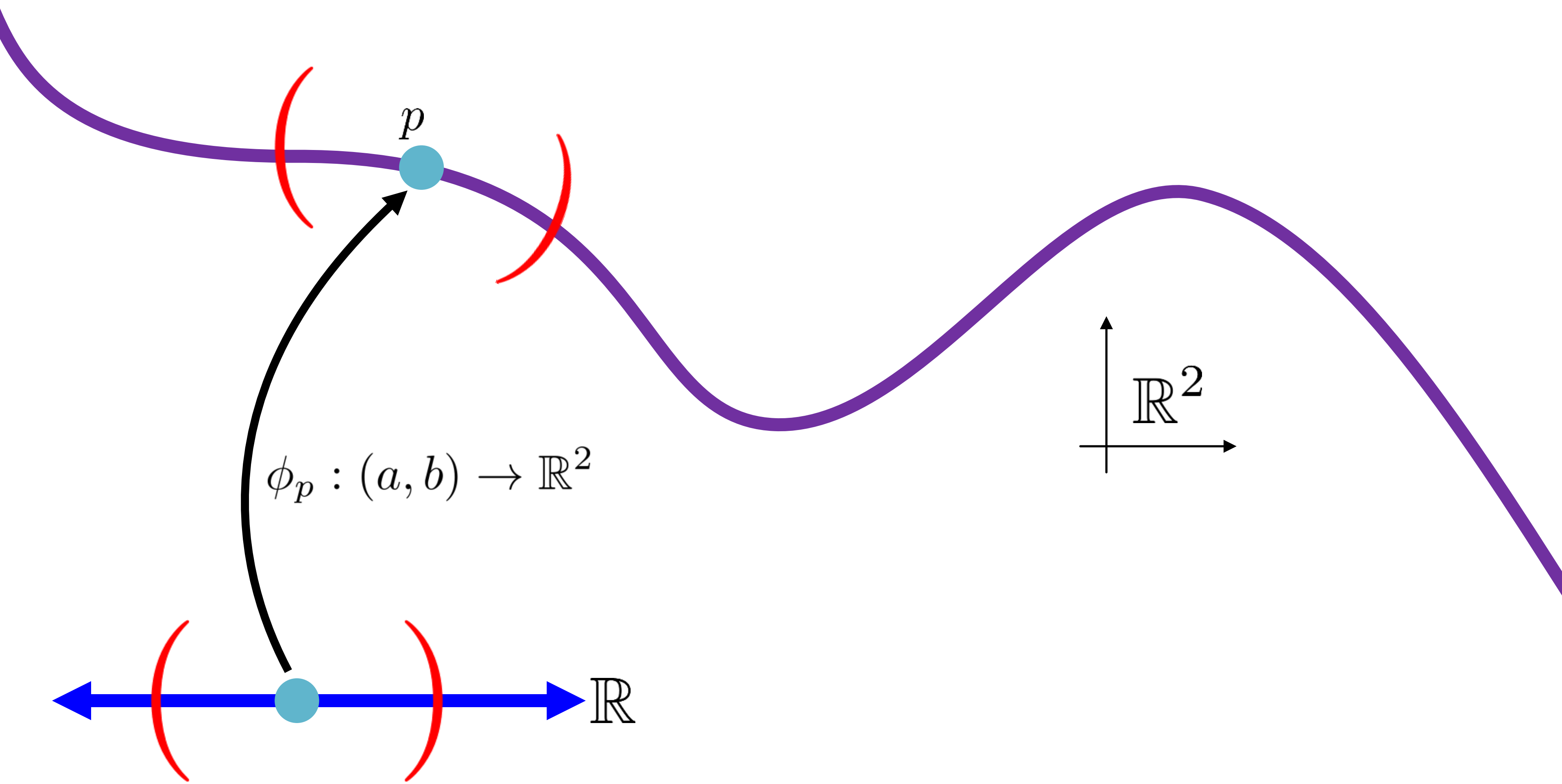
with certain properties.

It is not a function.

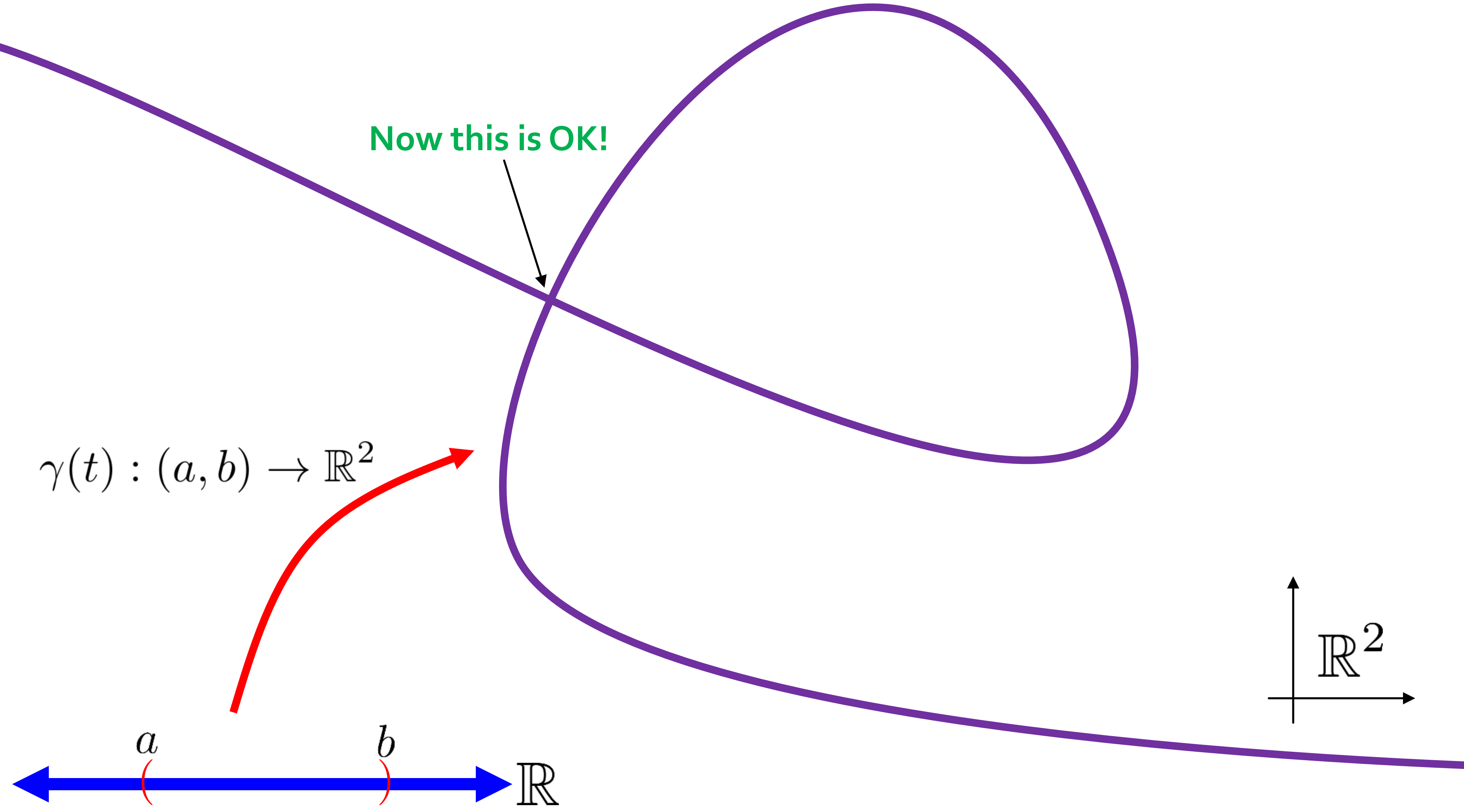
Geometric Definition

Set of points that locally looks like a line.

Differential Geometry Definition



Parameterized Curve



Some Vocabulary

- **Trace** of parameterized curve

$$\{\gamma(t) : t \in (a,b)\}$$

- **Component** functions

$$\gamma(t) = (x(t), y(t), z(t))$$

Change of Parameter

$$\bar{t} \mapsto \gamma(g(\bar{t})) = \gamma \circ g(\bar{t})$$

Geometric measurements should be

invariant

to changes of parameter.

Dependence of Velocity

$$\tilde{\gamma}(s) := \gamma(\phi(s))$$

On the board:

Effect on velocity and acceleration.

Arc Length

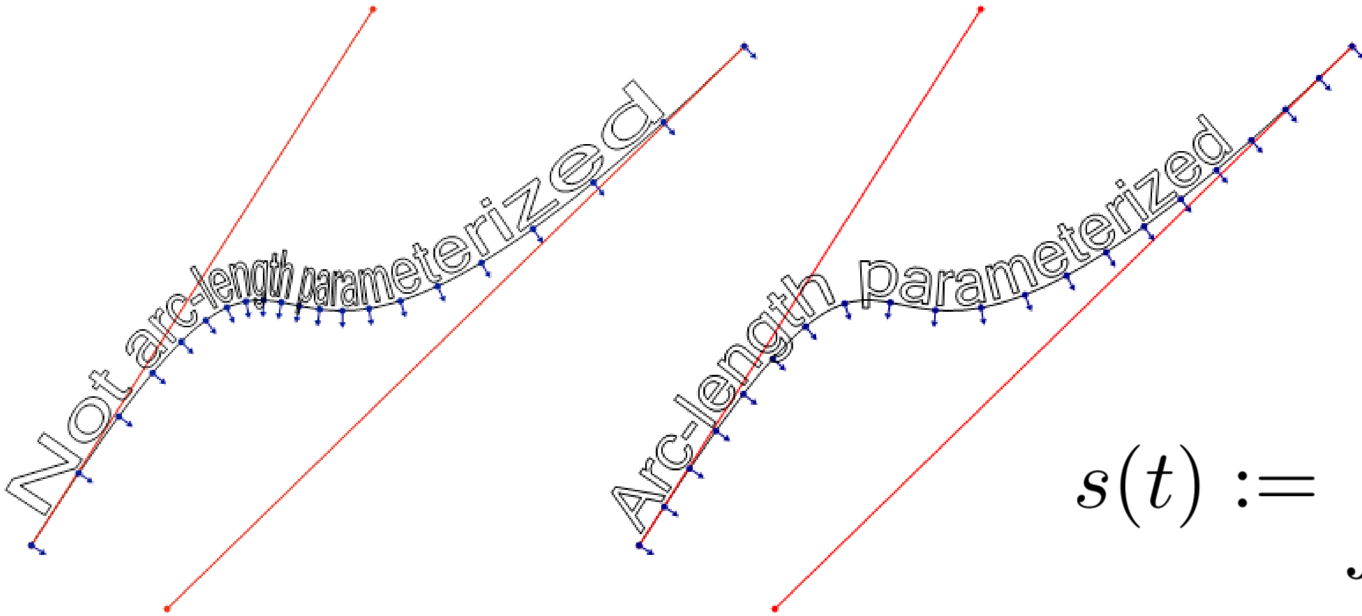
$$\int_a^b \|\gamma'(t)\|\,dt$$

On the board:

Independence of parameter

Parameterization by Arc Length

http://www.planetclegg.com/projects/WarpingTextToSplines.html



$$s(t) := \int_{t_0}^{t} \|\gamma'(t)\| \, dt$$

$$t(s) := \text{inverse of } s(t)$$

$$\gamma(s) := \gamma(s(t))$$

Constant-speed parameterization

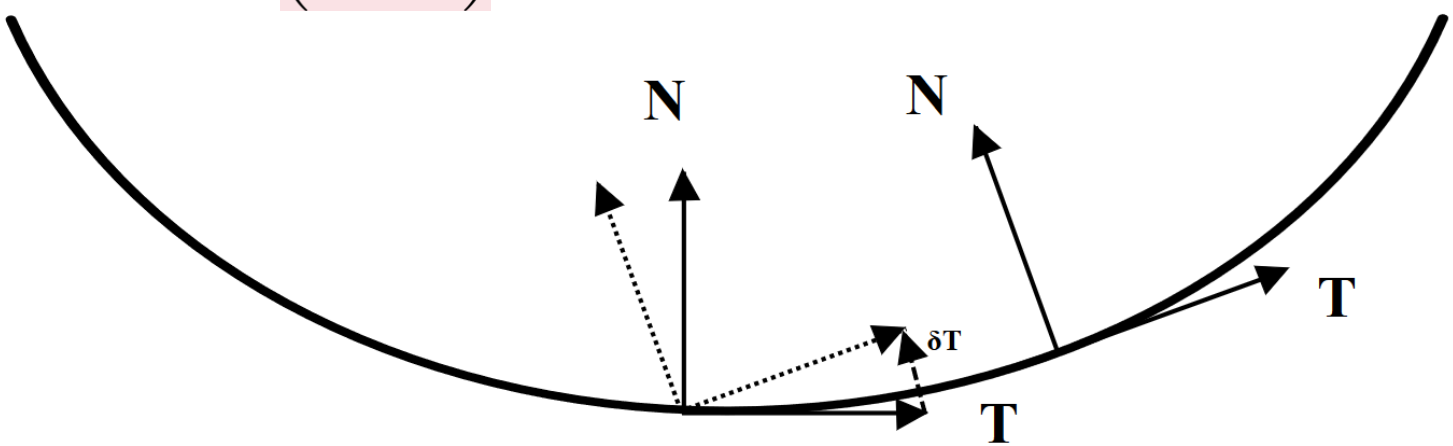
Moving Frame in 2D

$$T(s) := \gamma'(s)$$

$$\implies \quad \text{(on board)} \; \|T(s)\| \equiv 1$$

$$N(s) := JT(s)$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

Philosophical Point

Differential geometry "should" be

coordinate-invariant.

Referring to *x* and *y* is a hack!

(but sometimes convenient…)

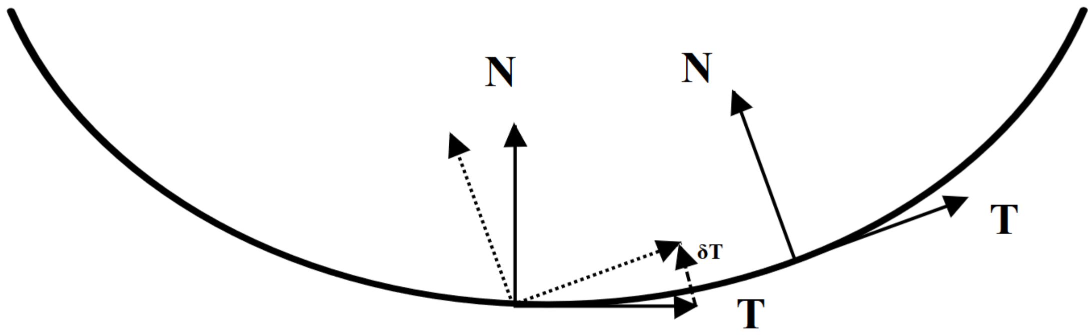
Q:

How do you characterize shape **without coordinates**?

Turtles All The Way Down

On the board:

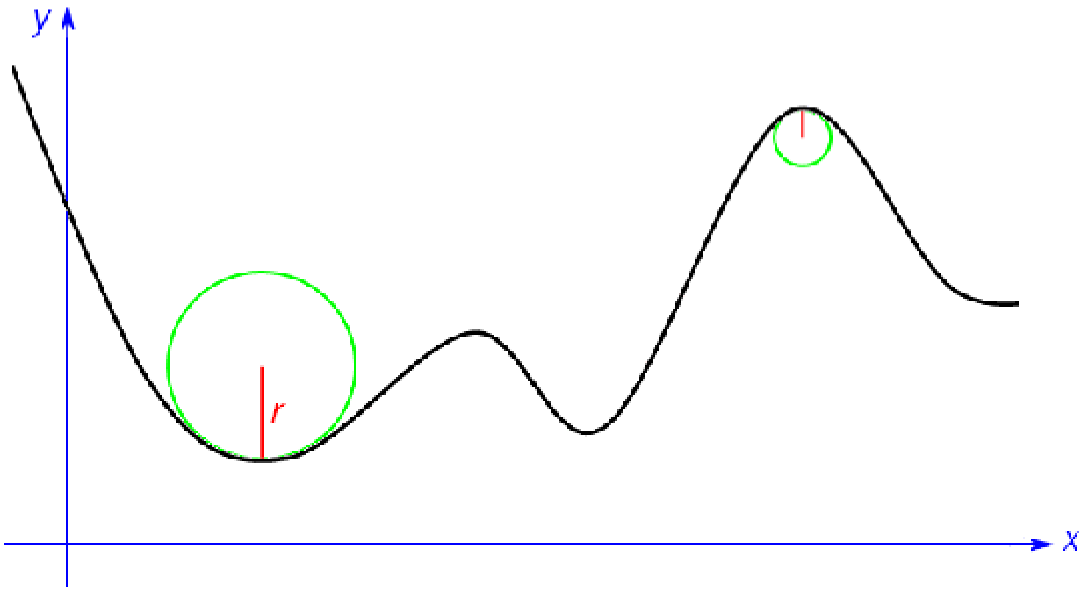
$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix} := \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix}$$



https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

Use coordinates *from* the curve to express its shape!

Radius of Curvature



$$r(s) := \frac{1}{k(s)}$$

https://www.quora.com/What-is-the-base-difference-between-radius-of-curvature-and-radius-of-gyration

Fundamental theorem of the local theory of plane curves:

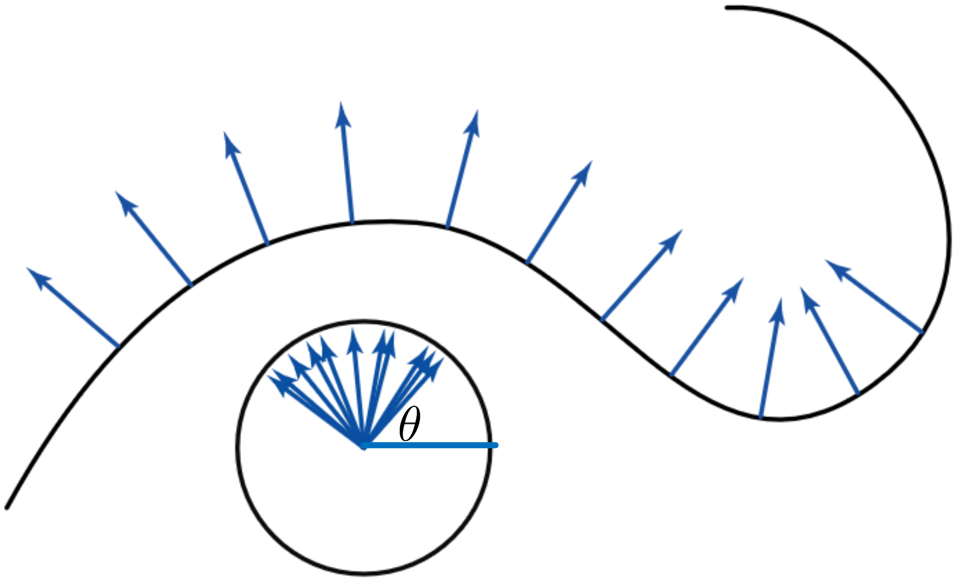
$k(s)$ characterizes a planar curve up to rigid motion.

Fundamental theorem of the local theory of plane curves:

$k(s)$ characterizes a planar curve up to rigid motion.

Statement shorter than the name!

Idea of Proof



$$T(s) := (\cos\theta(s), \sin\theta(s))$$

$$\implies k(s) := \theta'(s)$$

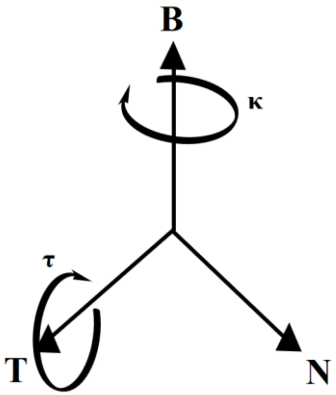
Image from DDG course notes by E. Grinspun

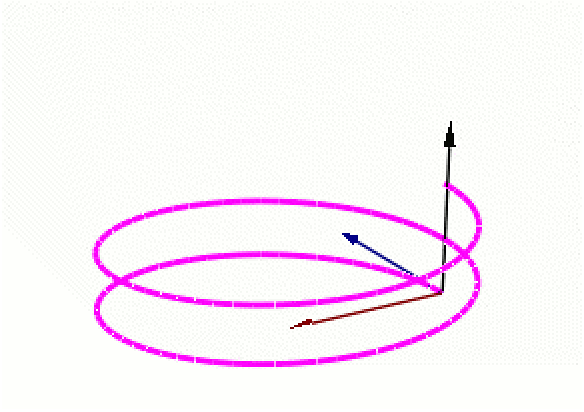
Provides intuition for curvature

Frenet Frame: Curves in $\mathbb{R}^3$

$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

- **Binormal:** $\boldsymbol{T} \times \boldsymbol{N}$
- **Curvature:** In-plane motion
- **Torsion:** Out-of-plane motion





Fundamental theorem of the local theory of space curves:

Curvature and torsion characterize a 3D curve up to rigid motion.

Aside: Generalized Frenet Frame

$$\gamma(s) : \mathbb{R} \to \mathbb{R}^n$$

$$\frac{d}{ds}\begin{pmatrix} e_1(s) \\ e_2(s) \\ \vdots \\ e_n(s) \end{pmatrix} = \begin{pmatrix} 0 & \chi_1(s) & & 0 \\ -\chi_1(s) & \ddots & \ddots & \\ & \ddots & 0 & \chi_{n-1}(s) \\ 0 & & -\chi_{n-1}(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ \vdots \\ e_n(s) \end{pmatrix}$$

Suspicion: Application to time series analysis? ML?

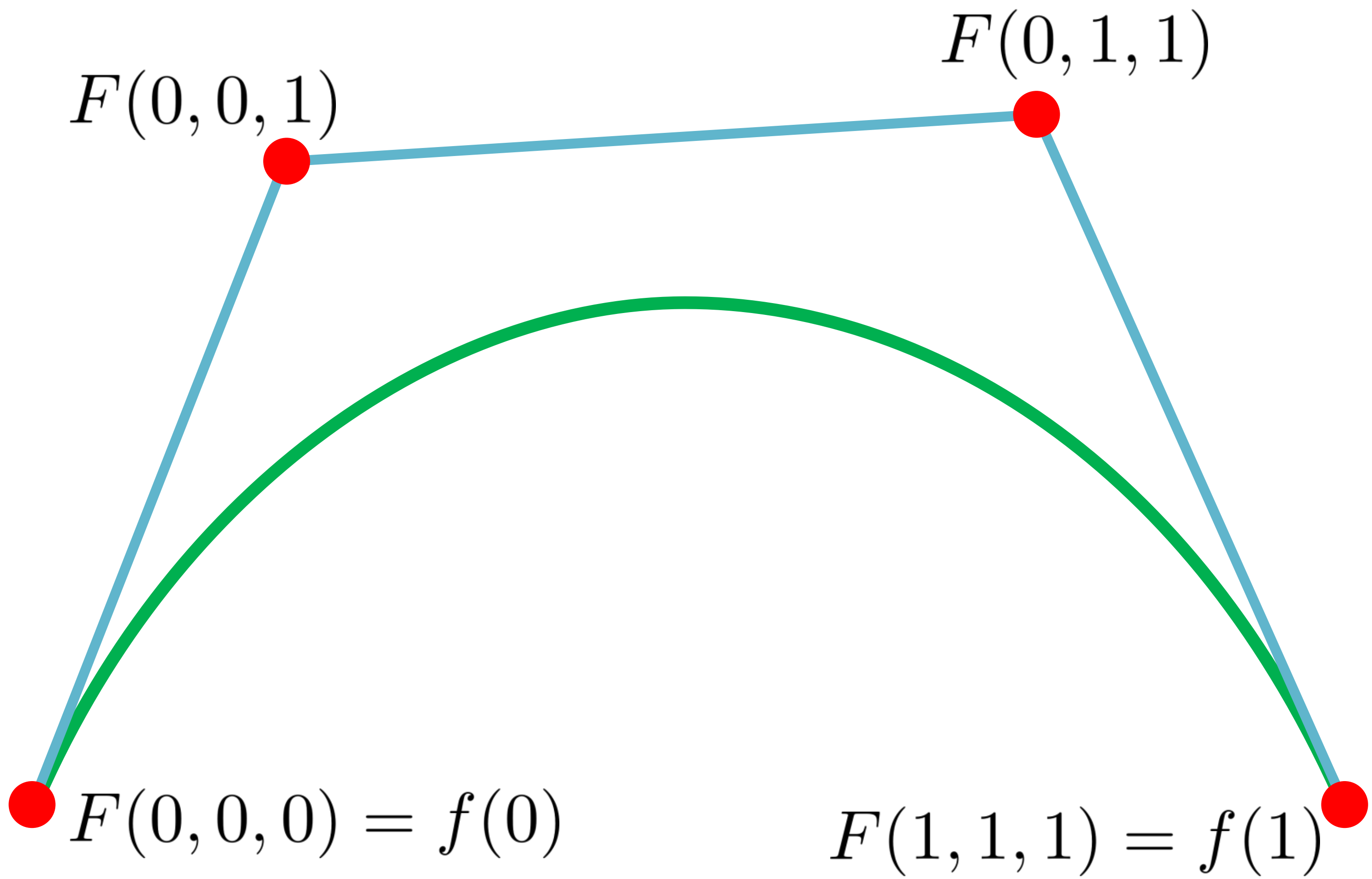
C. Jordan, 1874

Gram-Schmidt on first *n* derivatives

Q:

What do these calculations look like in software?

Old-School Approach



Piecewise smooth approximations

Question

What is the arc length of a cubic Bézier curve?

$$\int_a^b \|\gamma'(t)\|\,dt$$

Question

What is the arc length of a cubic Bézier curve?

$$\int_a^b \|\gamma'(t)\|\,dt$$

Not known in closed form.

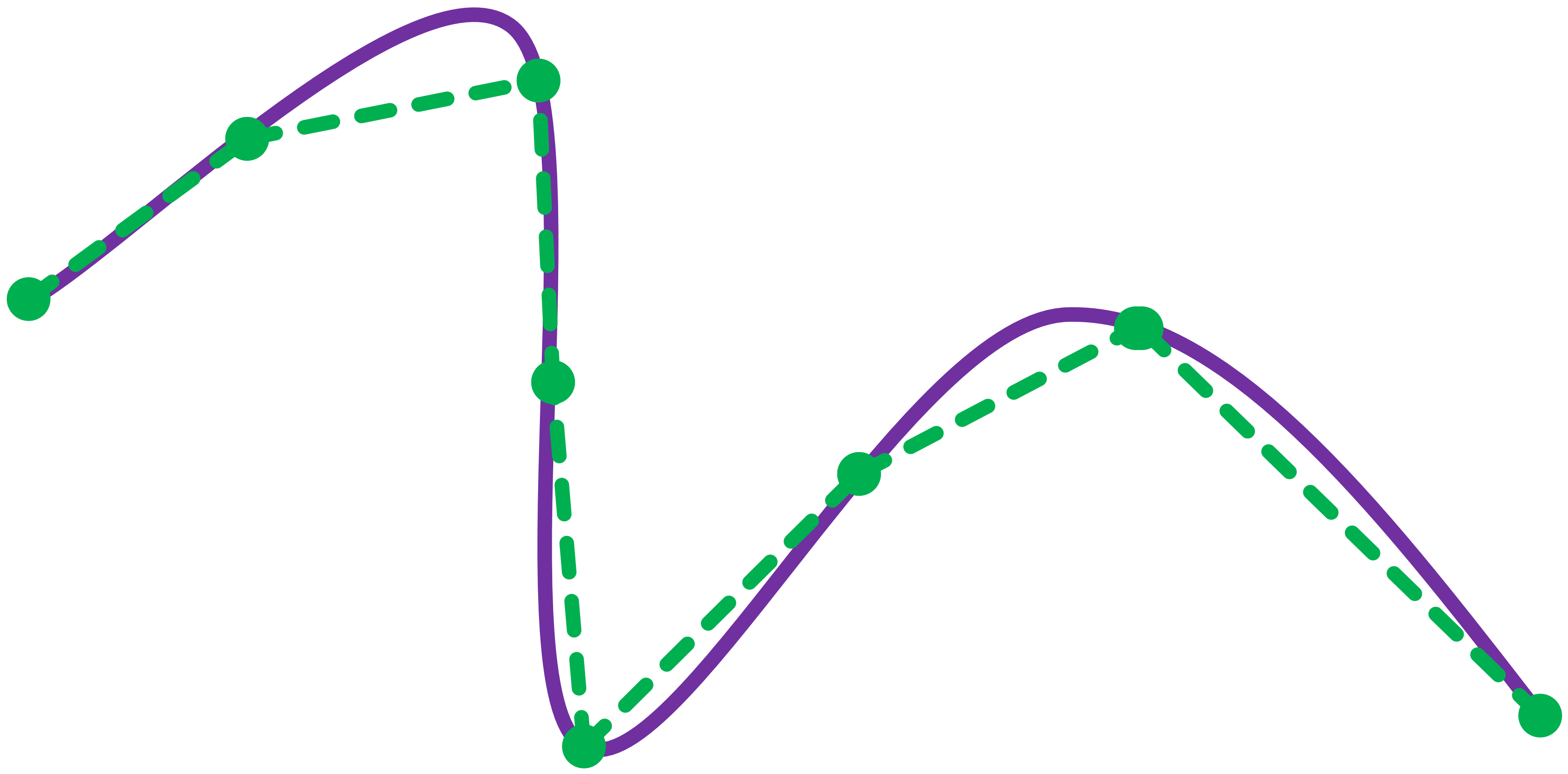
Sad fact:

Closed-form expressions rarely exist. When they do exist, they usually are messy.

Only Approximations Anyway

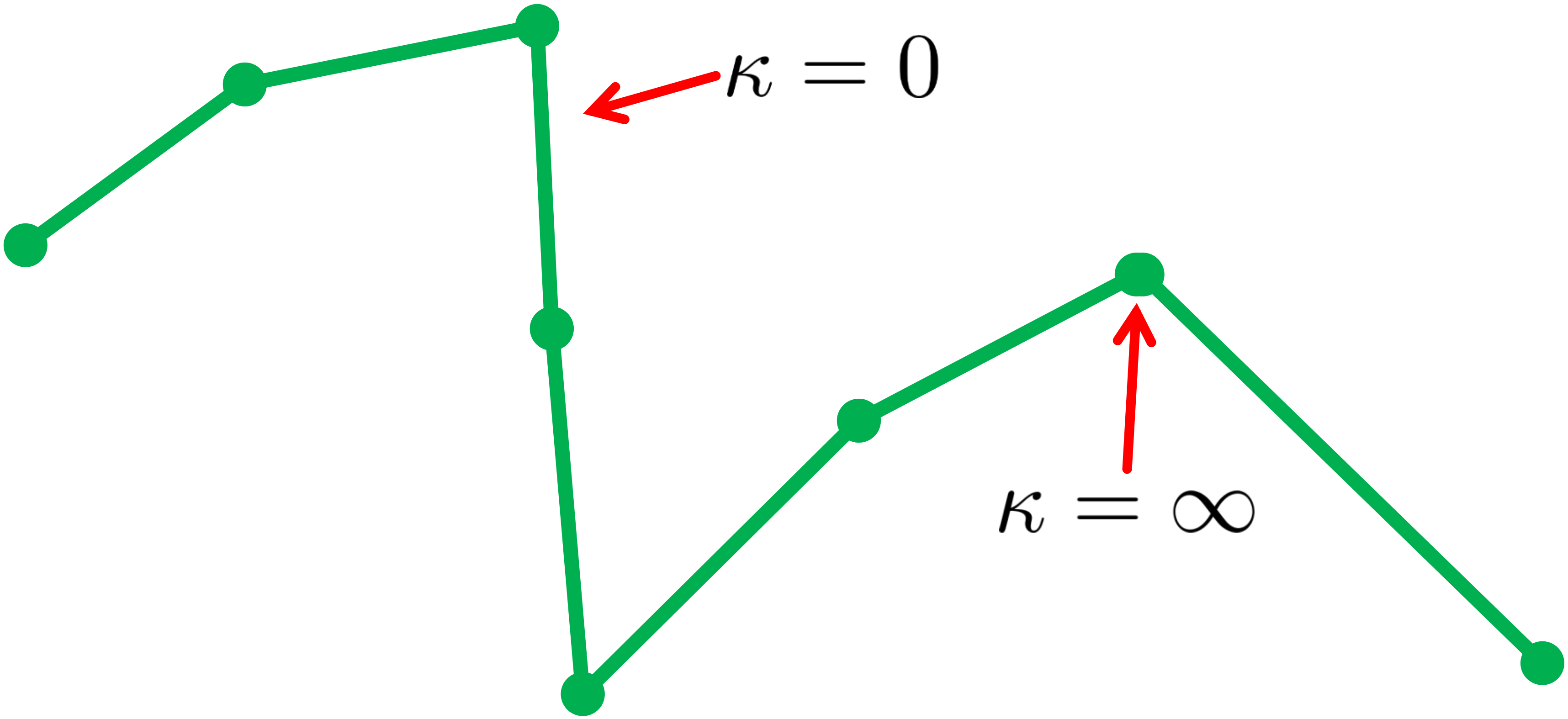
$$\{\text{Bézier curves}\} \subsetneq \{\gamma : \mathbb{R} \to \mathbb{R}^3\}$$

Equally Reasonable Approximation



Piecewise linear

Big Problem



Boring differential structure

Finite Difference Approach

$$f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$$

THEOREM: As $\Delta h \to 0$, [insert statement].

Reality Check

$$f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$$

THEOREM $h > 0$ tatement].

Two Key Considerations

- **Convergence** to continuous theory
- Discrete **behavior**

Goal

Examine discrete theories of differentiable curves.

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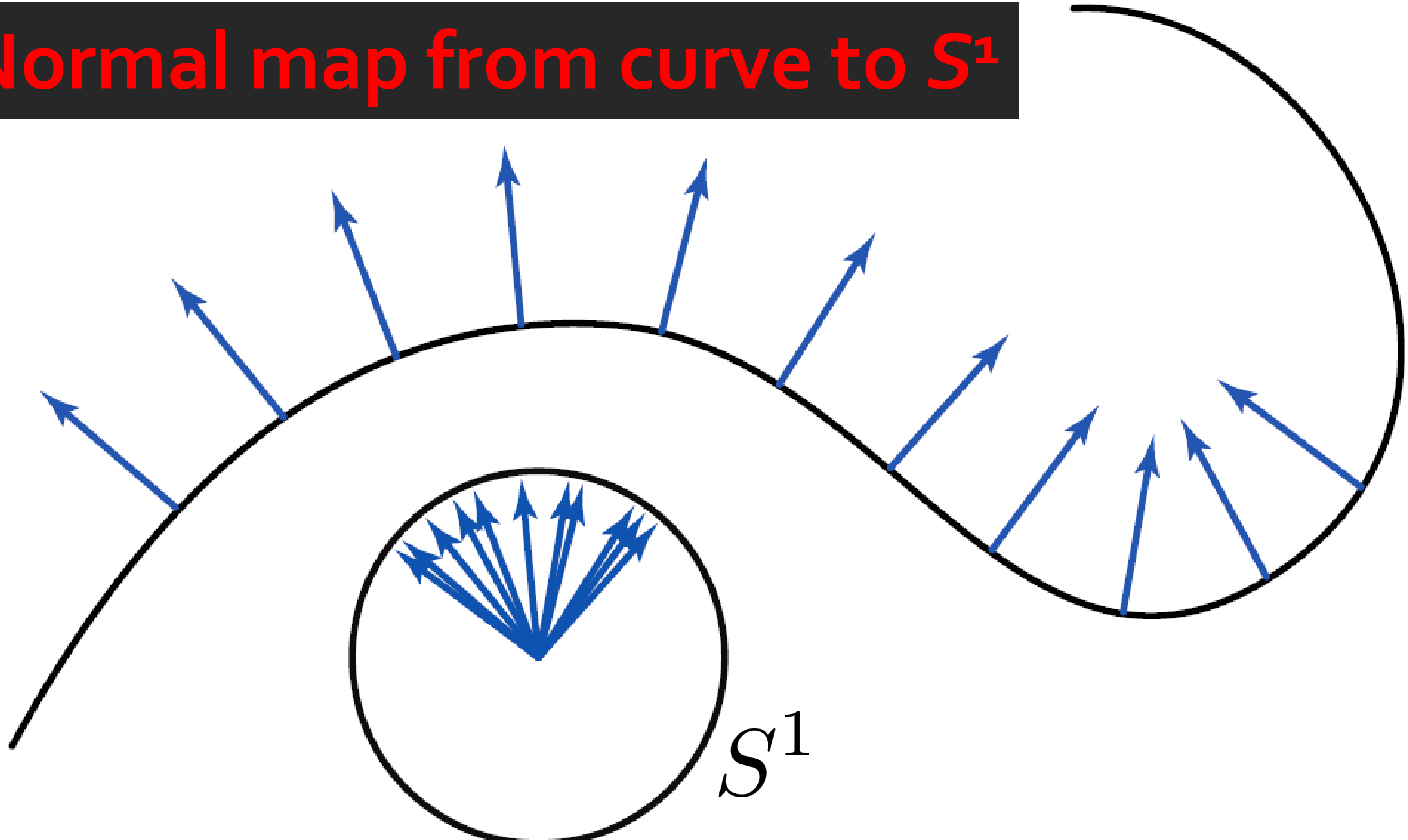
Examine discrete theor<u>ies</u> of differentiable curves.

Gauss Map

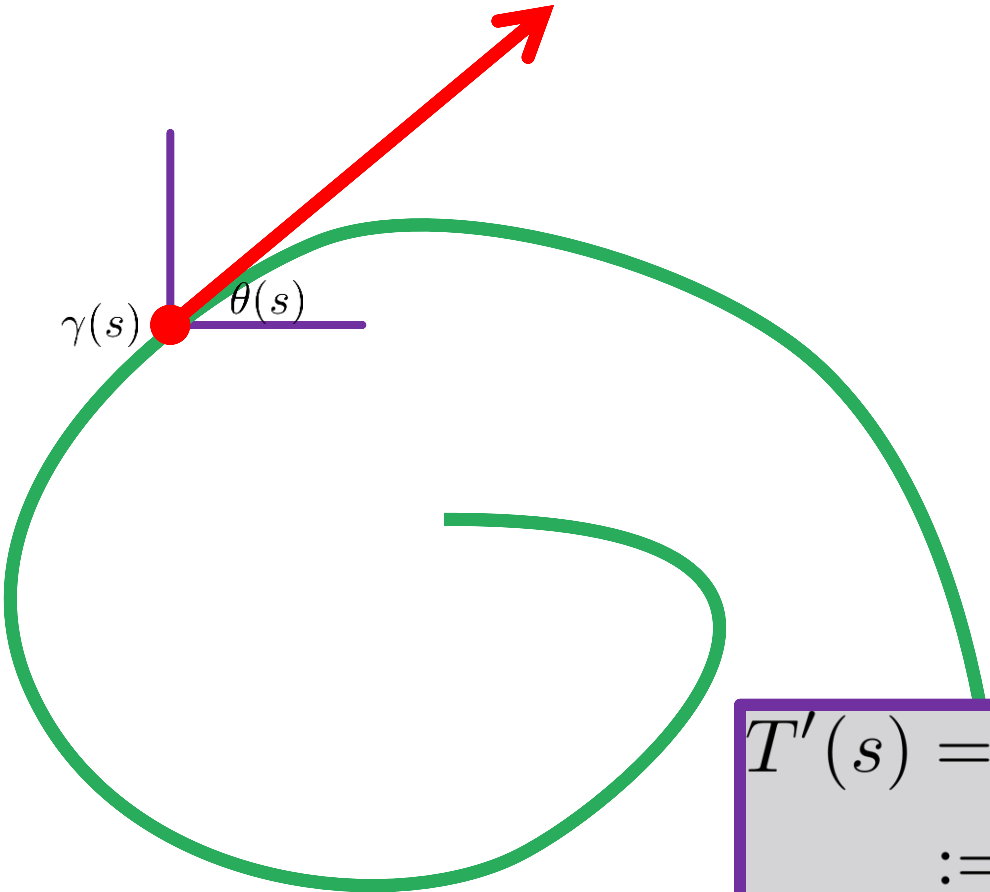
Normal map from curve to S^1



http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

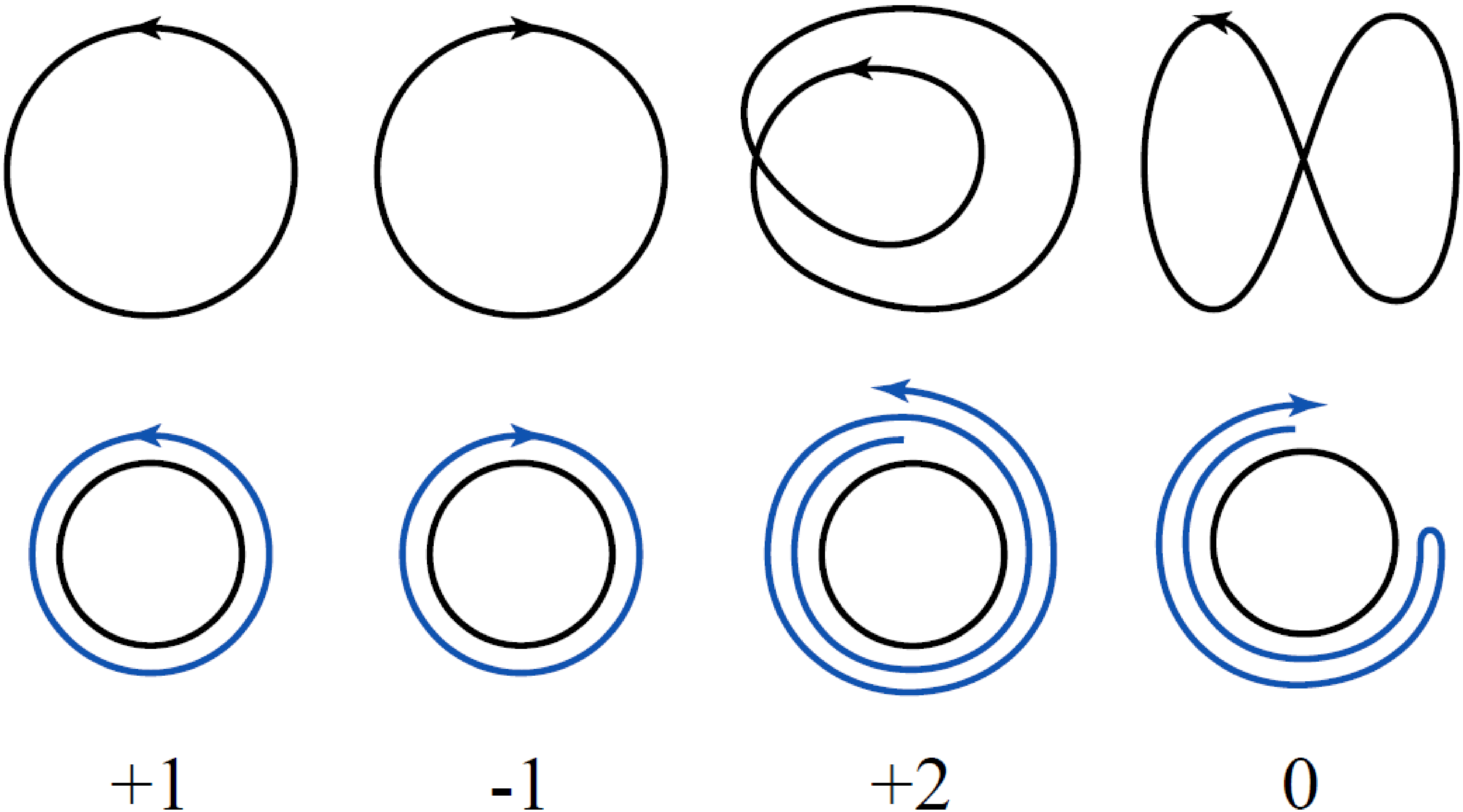
Signed Curvature on Plane Curves

$$T(s) = (\cos\theta(s), \sin\theta(s))$$



$$T'(s) = \theta'(s)(-\sin\theta(s), \cos\theta(s))$$
$$:= \kappa(s)N(s)$$

Turning Numbers



http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

Recovering Theta

$$\theta'(s) = \kappa(s)$$

$$\Downarrow$$

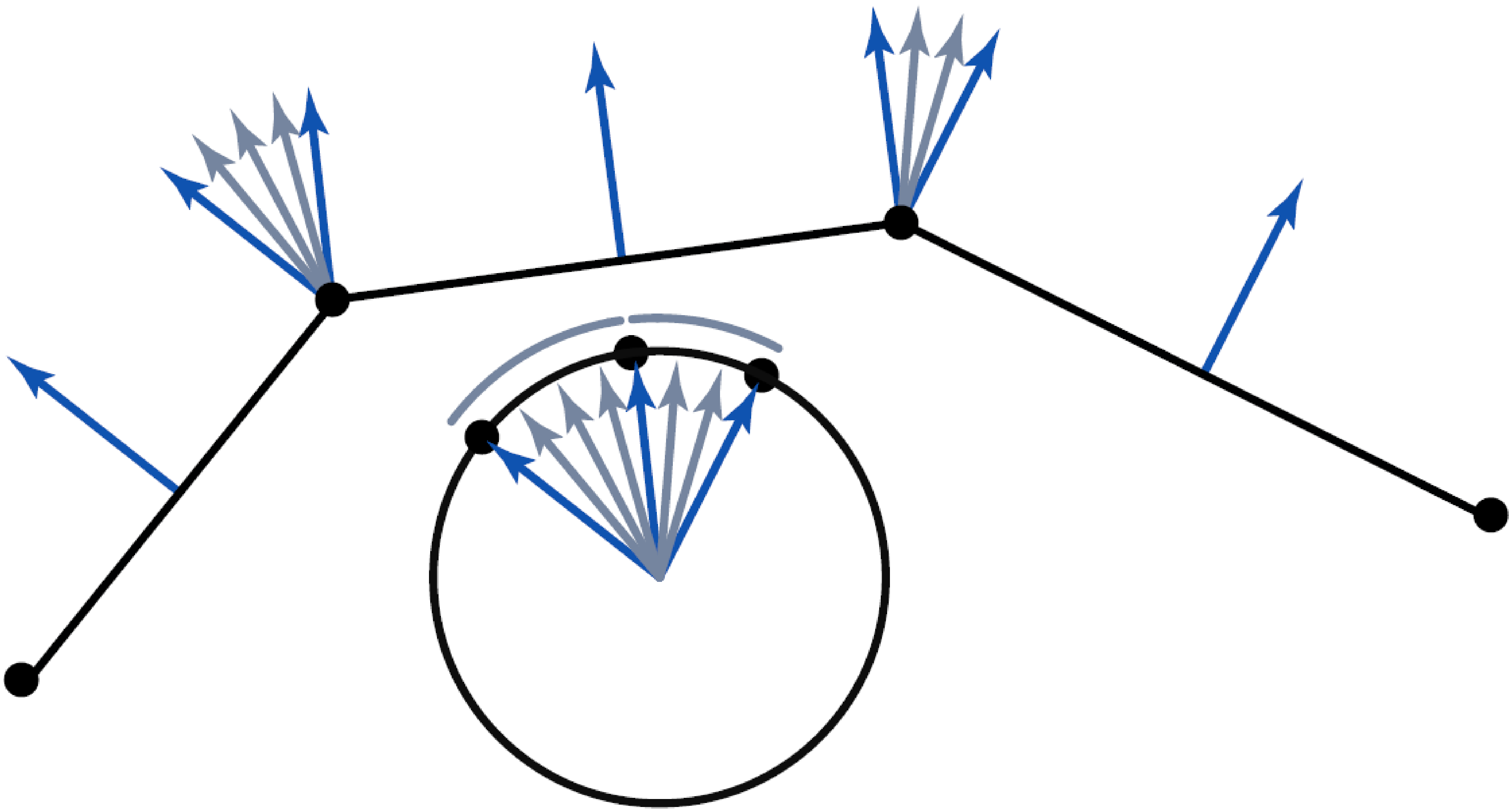
$$\Delta\theta = \int_{s_0}^{s_1} \kappa(s)\, ds$$

Turning Number Theorem

$$\int_{\Omega} \kappa(s)\, ds = 2\pi k$$

A “global” theorem!

Discrete Gauss Map



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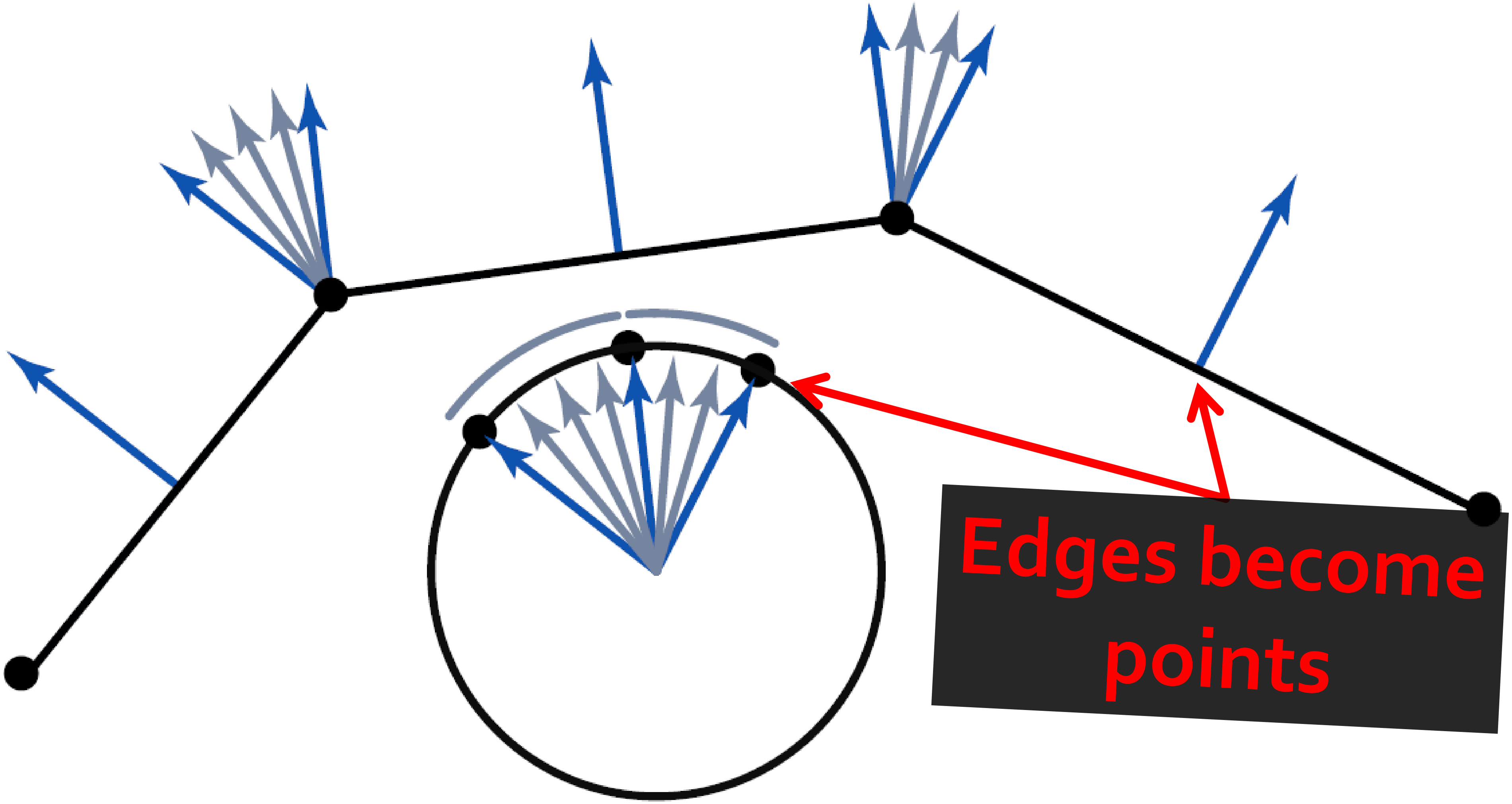
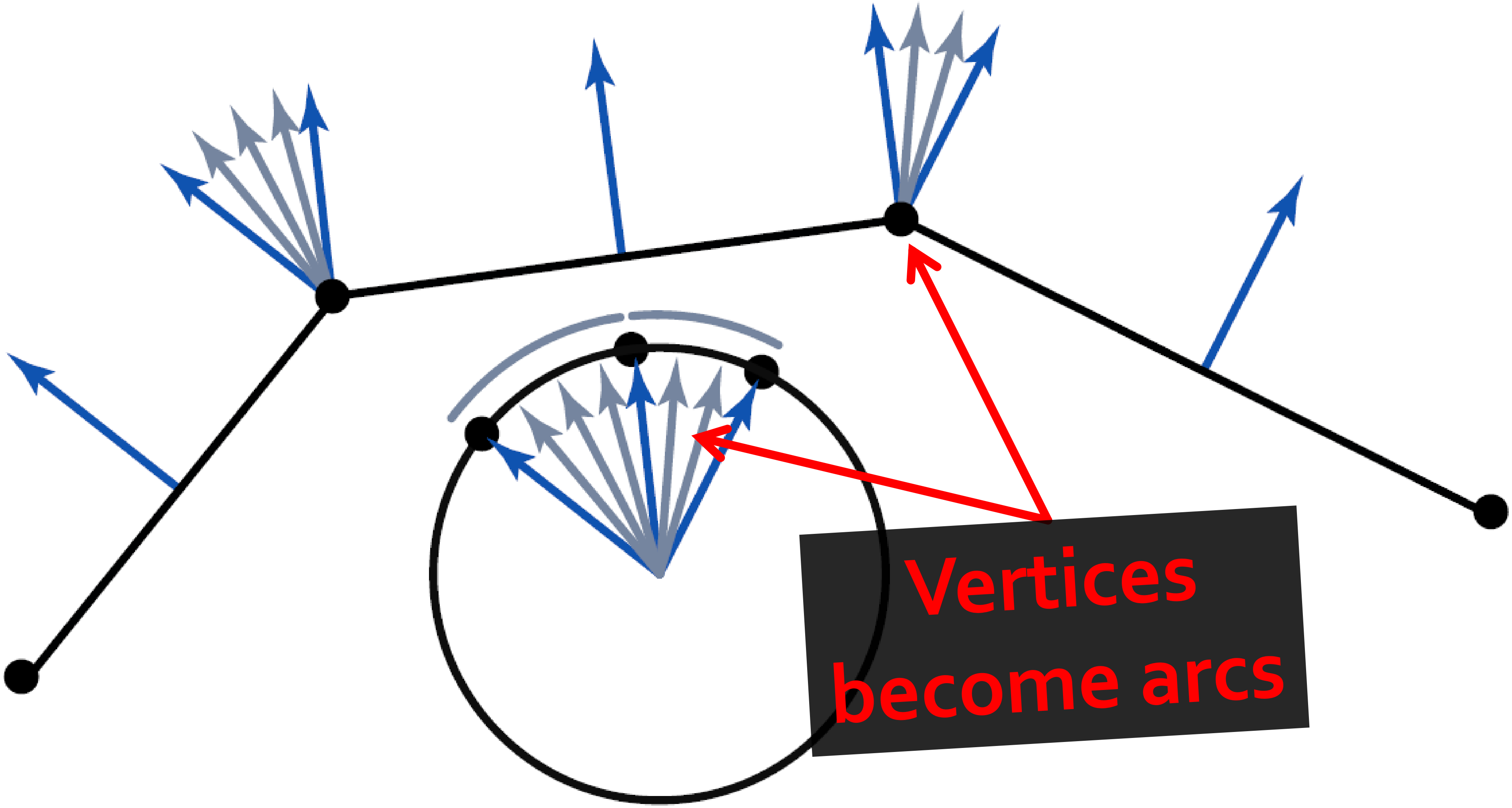
Discrete Gauss Map



http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

Discrete Gauss Map



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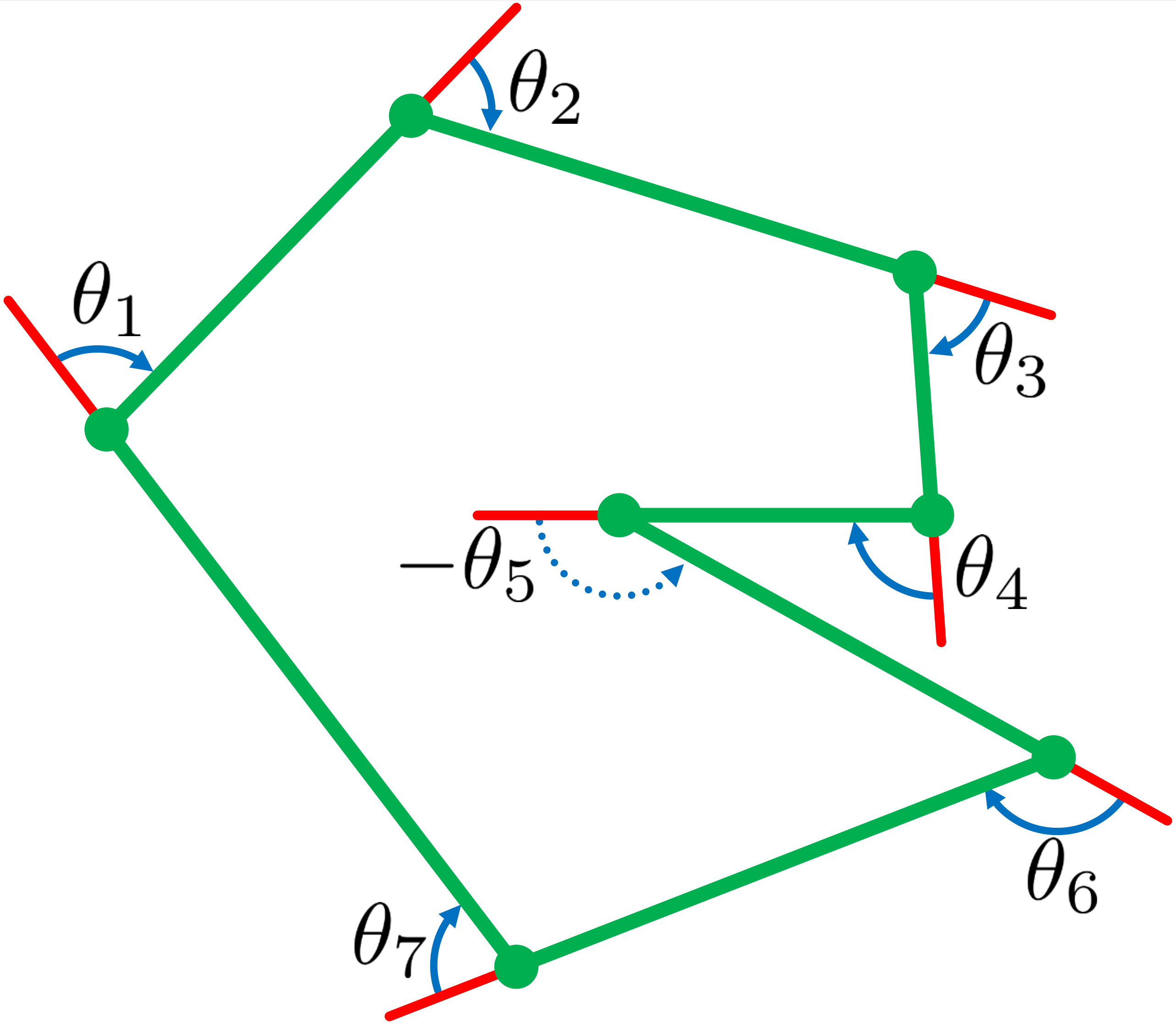
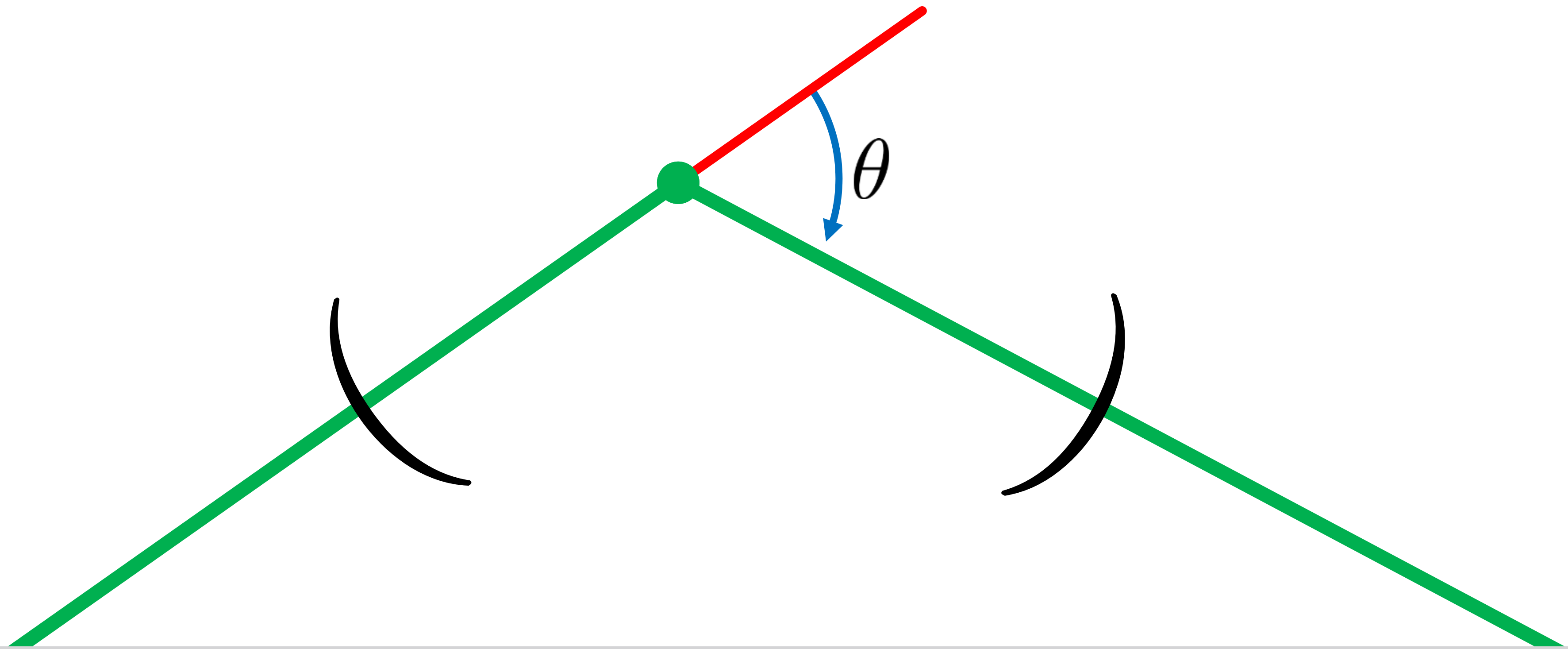
Key Observation



$$\sum_i \theta_i = 2\pi k$$

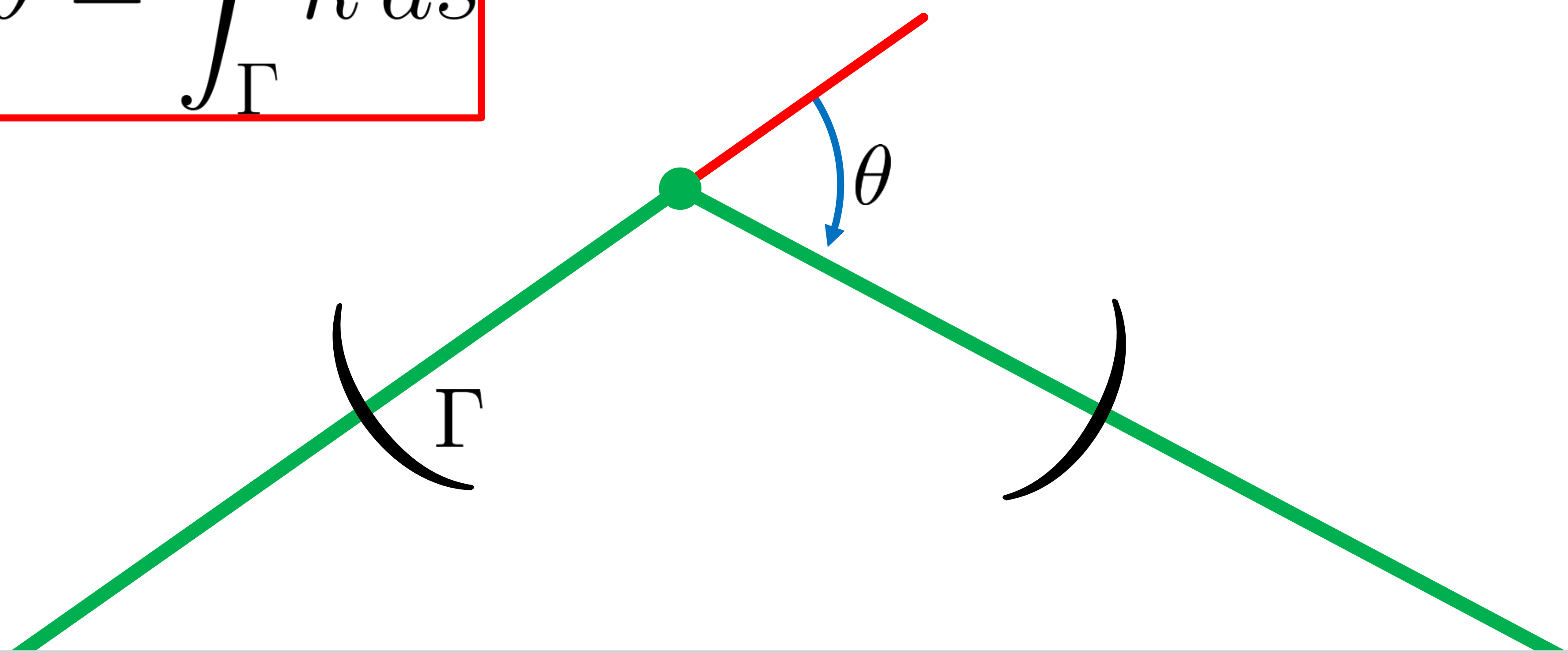
What's Going On?



Total change in curvature

What's Going On?

$$\theta = \int_\Gamma \kappa \, ds$$

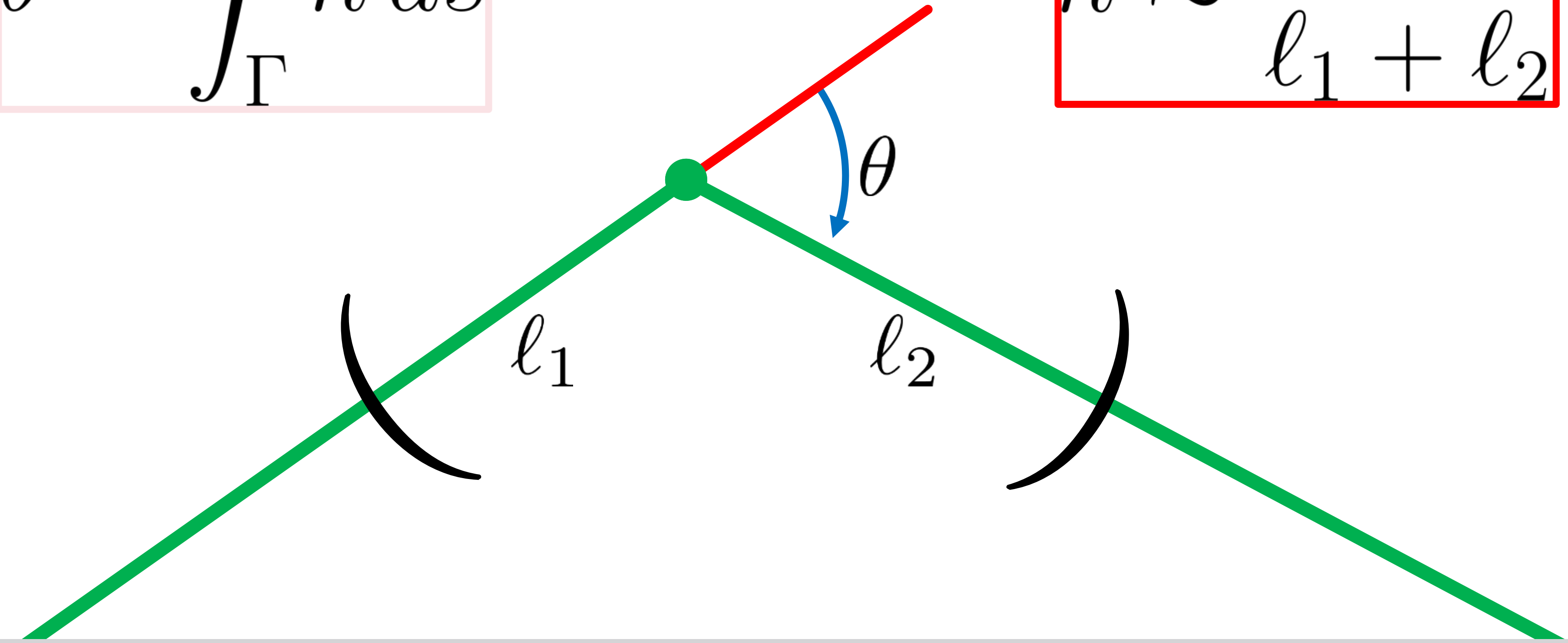


Total change in curvature

What's Going On?

$$\theta = \int_\Gamma \kappa \, ds$$

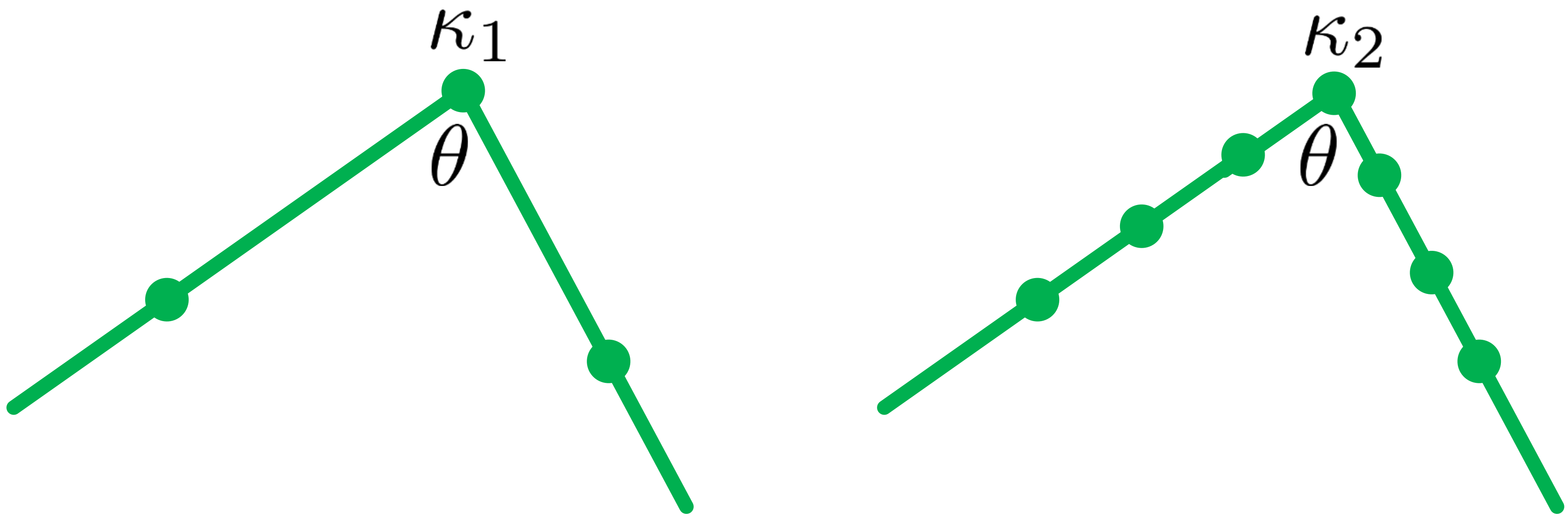
$$\kappa \approx \frac{\theta}{\ell_1 + \ell_2}$$



Total change in curvature

Interesting Distinction

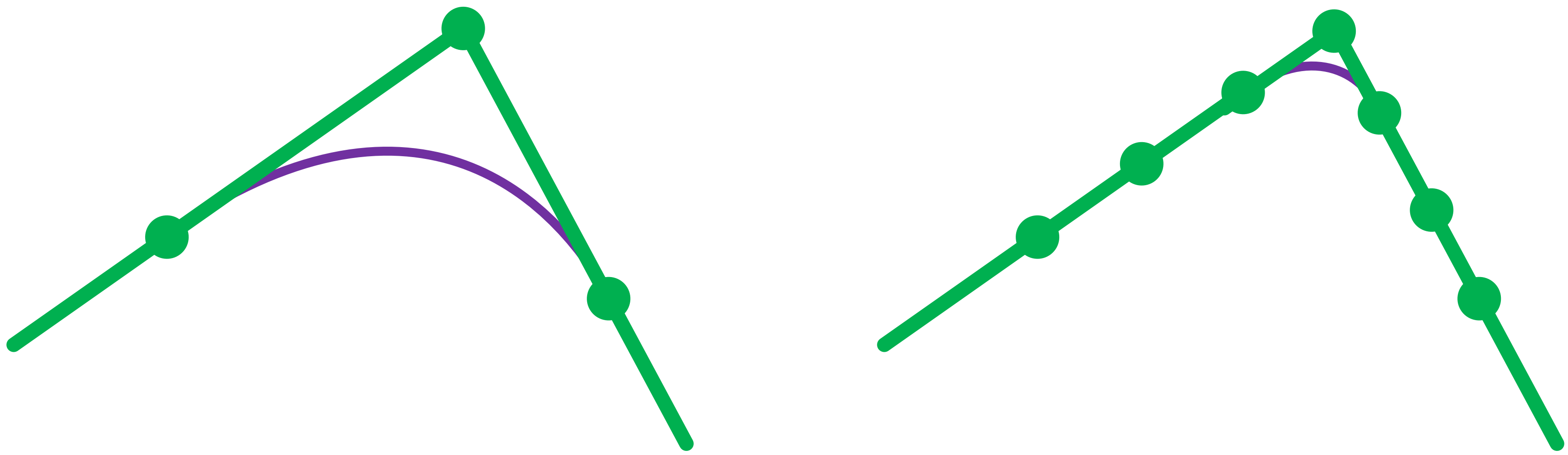
$$\kappa_1 \neq \kappa_2$$



Same integrated curvature

Interesting Distinction

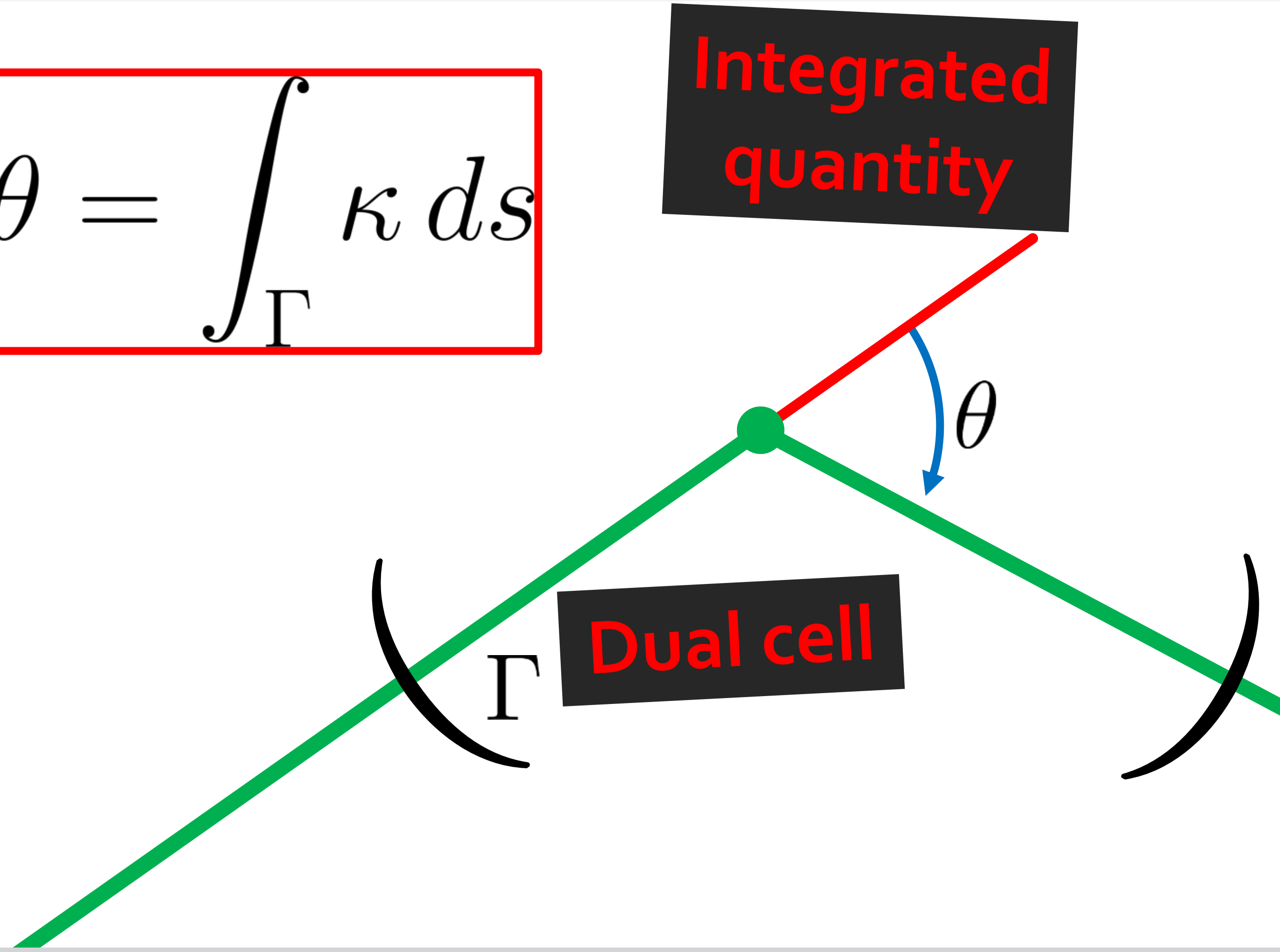
$$\kappa_1 \neq \kappa_2$$



Same integrated curvature

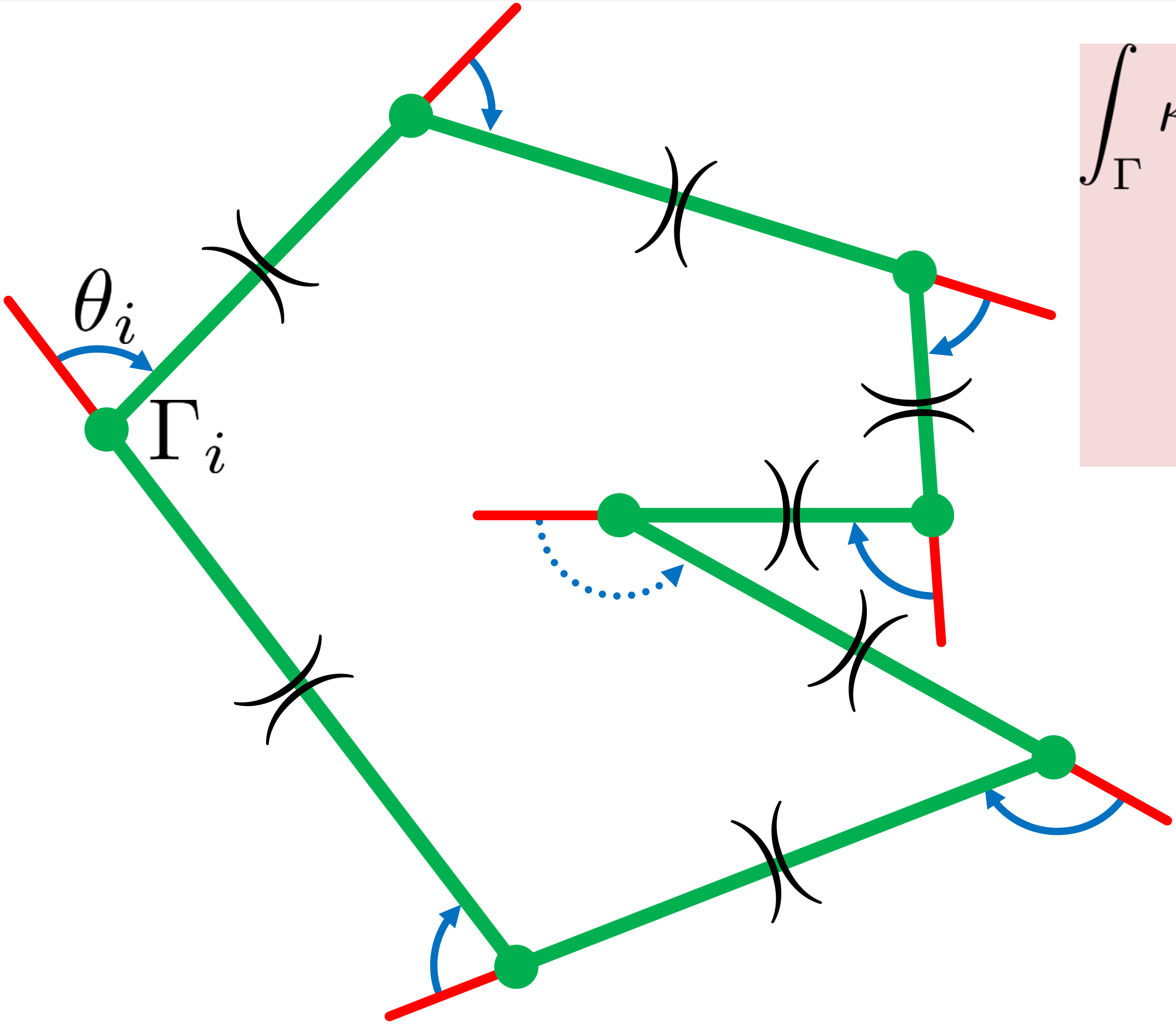
What's Going On?

$$\theta = \int_\Gamma \kappa \, ds$$



Total change in curvature

Discrete Turning Angle Theorem

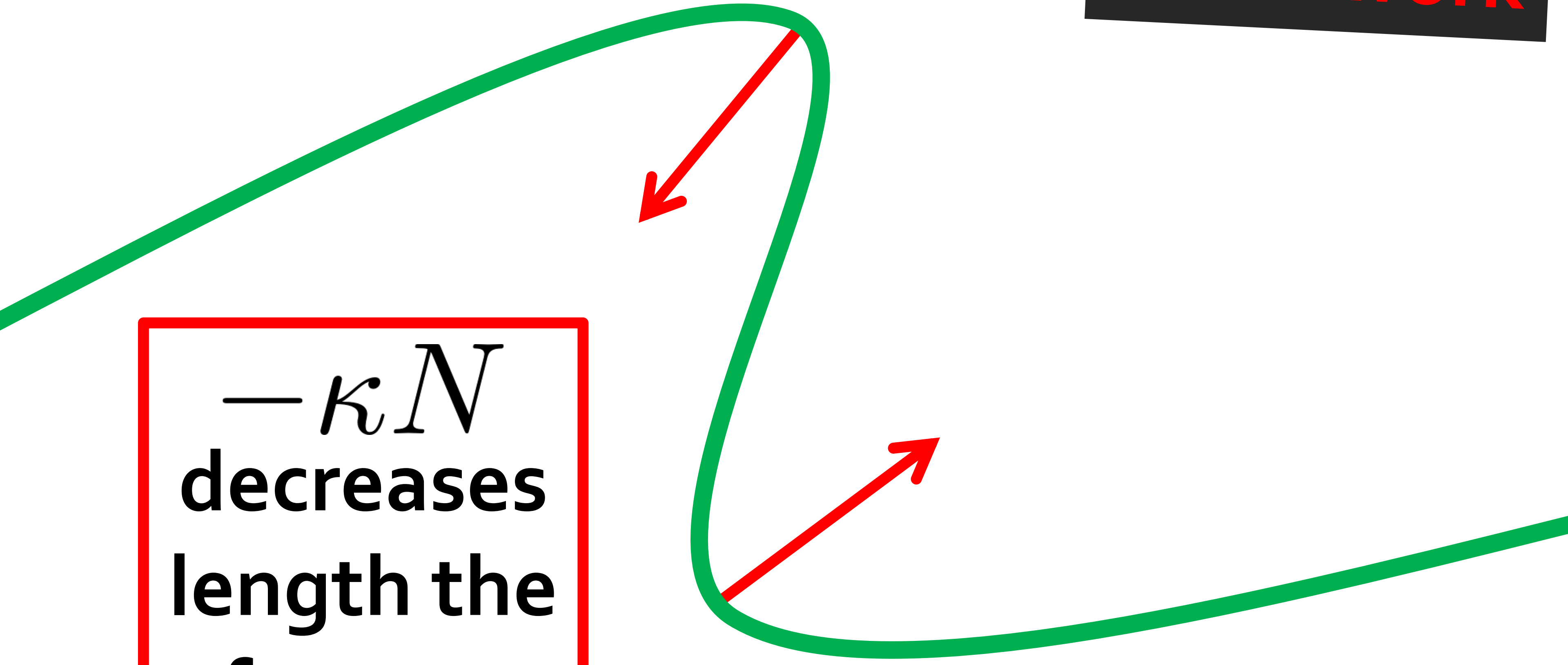


$$\int_\Gamma \kappa\, ds = \sum_i \int_{\Gamma_i} \kappa\, ds$$
$$= \sum_i \theta_i$$
$$= 2\pi k$$

Preserved structure!

Alternative Definition

Homework



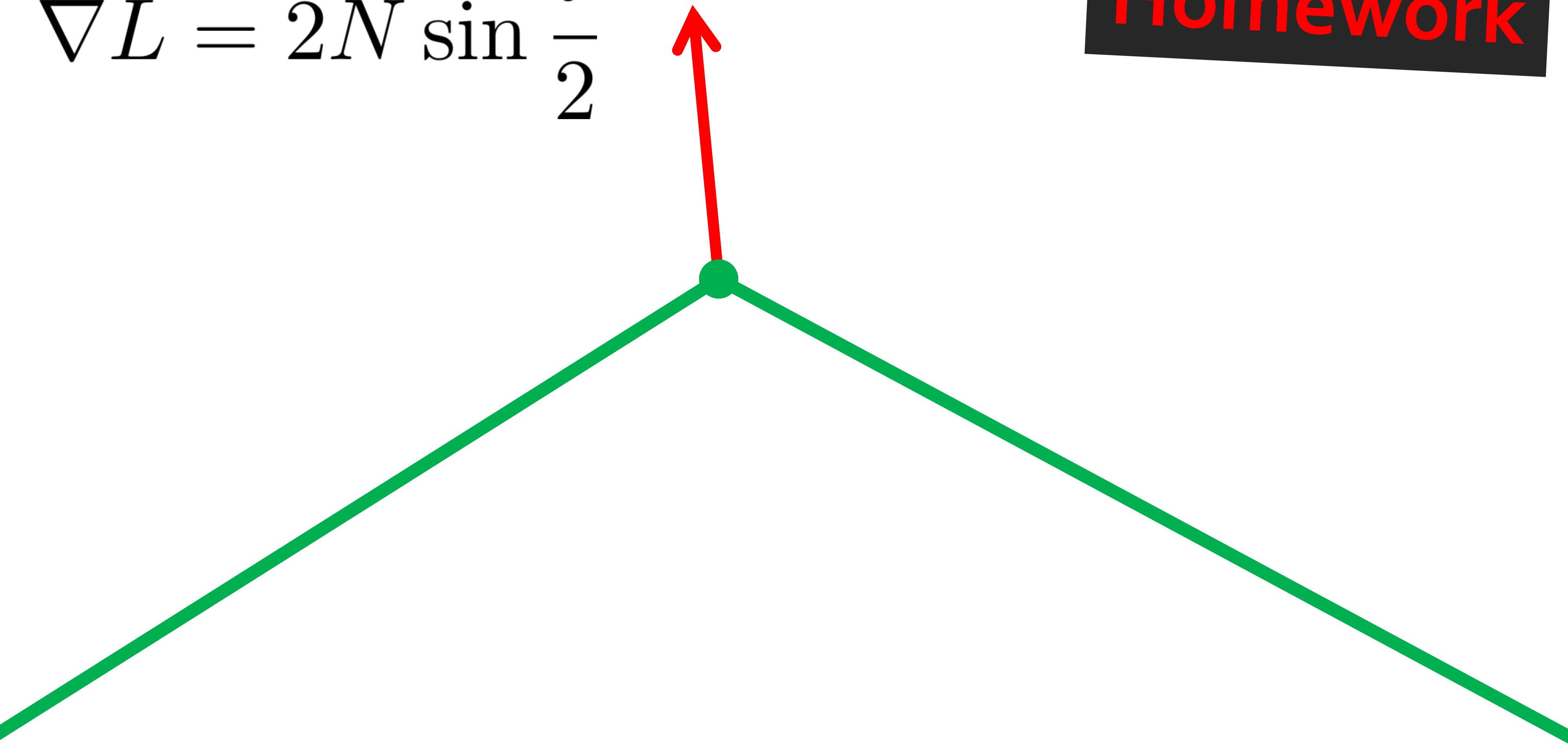
$-\kappa N$

decreases length the fastest.

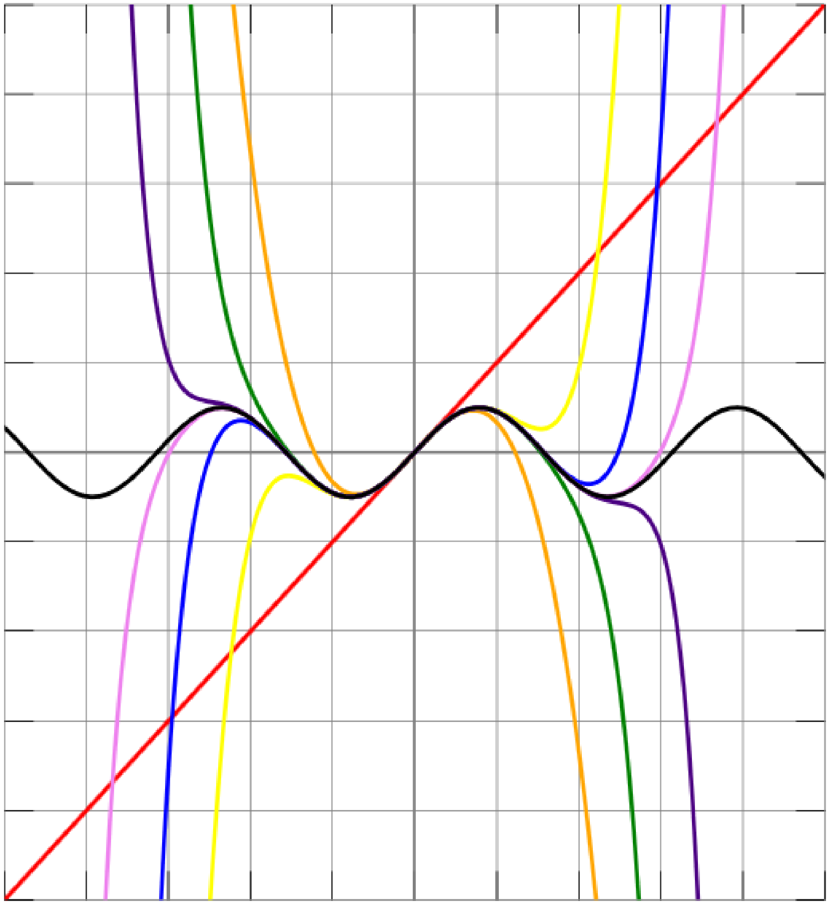
Discrete Case

$$\nabla L = 2N \sin \frac{\theta}{2}$$

Homework



For Small θ



http://en.wikipedia.org/wiki/Taylor_series

$$2\sin\frac{\theta}{2} \approx 2\cdot\frac{\theta}{2}$$

$$= \theta$$

Same behavior in the limit

Remaining Question

Does discrete curvature converge in limit?

Yes!

Remaining Question

Does discrete curvature converge in limit?

Yes!

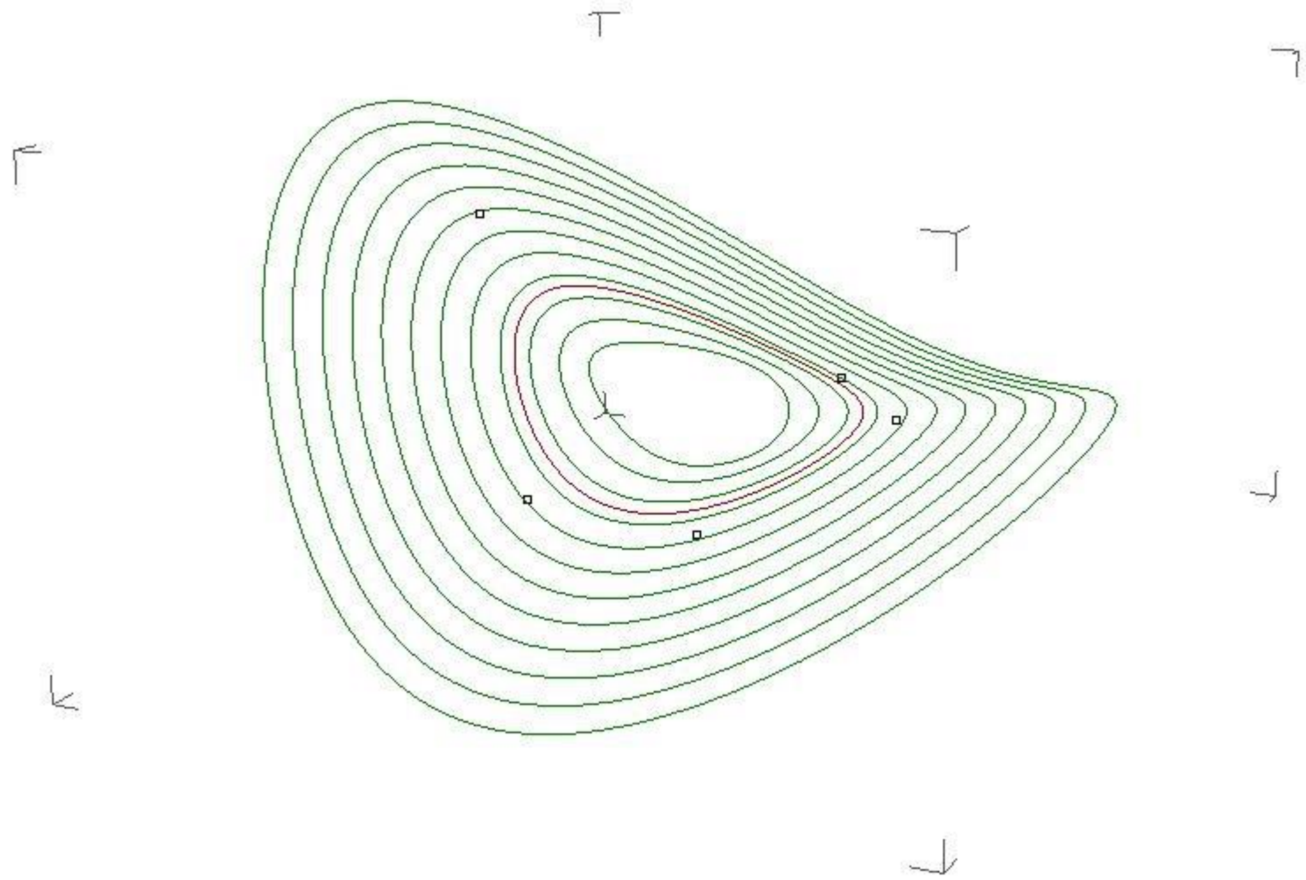
Questions:

- Type of convergence?
- Sampling?
- Class of curves?

Discrete Differential Geometry

- Different discrete behavior
- Same convergence

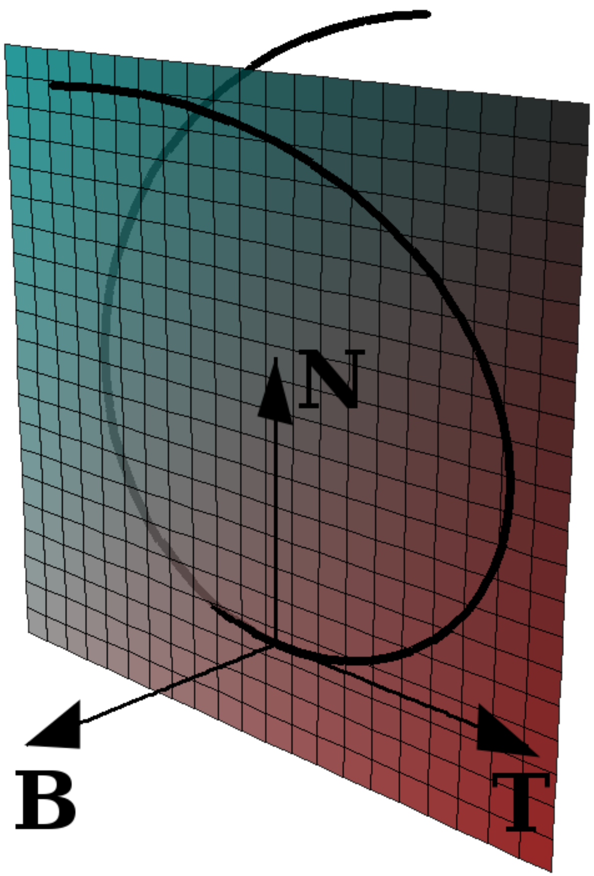
Next



http://www.grasshopper3d.com/forum/topics/offseting-3d-curves-component

Curves in 3D

Frenet Frame

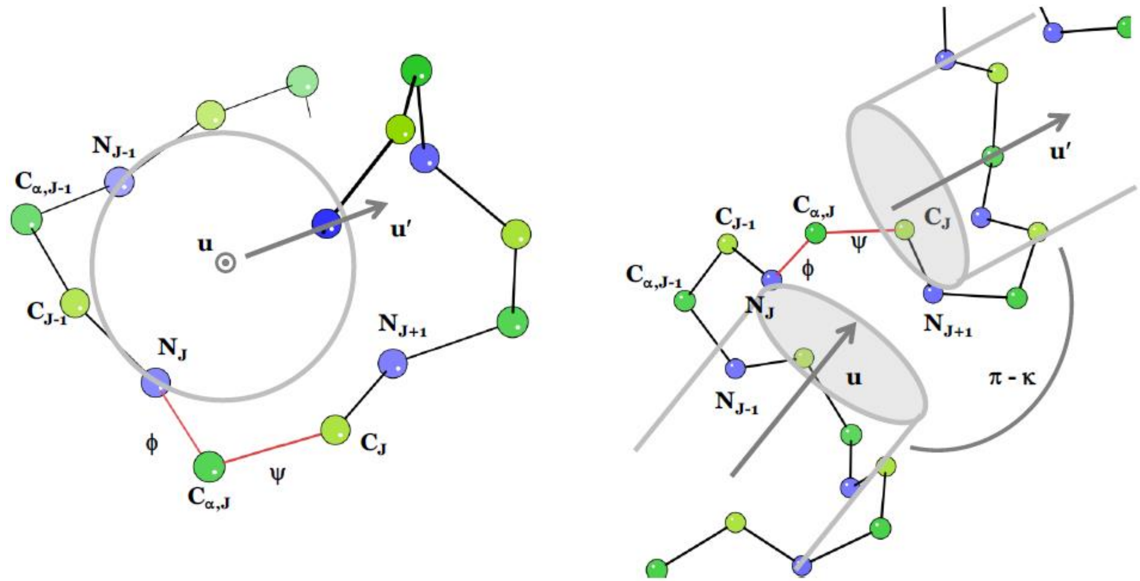


$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

http://upload.wikimedia.org/wikipedia/commons/6/6f/Frenet.png

Application

NMR scanner



Kinked alpha helix

Structure Determination of Membrane Proteins Using Discrete Frenet Frame and Solid State NMR Restraints

Achuthan and Quine

Discrete Mathematics and its Applications, ed. M. Sethumadhavan (2006)

Potential Discretization

$$T_j = \frac{p_{j+1} - p_j}{\|p_{j+1} - p_j\|}$$

$$B_j = t_{j-1} \times t_j$$

$$N_j = b_j \times t_j$$

Discrete Frenet frame

$$T_k = R(B_k, \theta_k) T_{k-1}$$

$$B_{k+1} = R(T_k, \phi_k) B_k$$

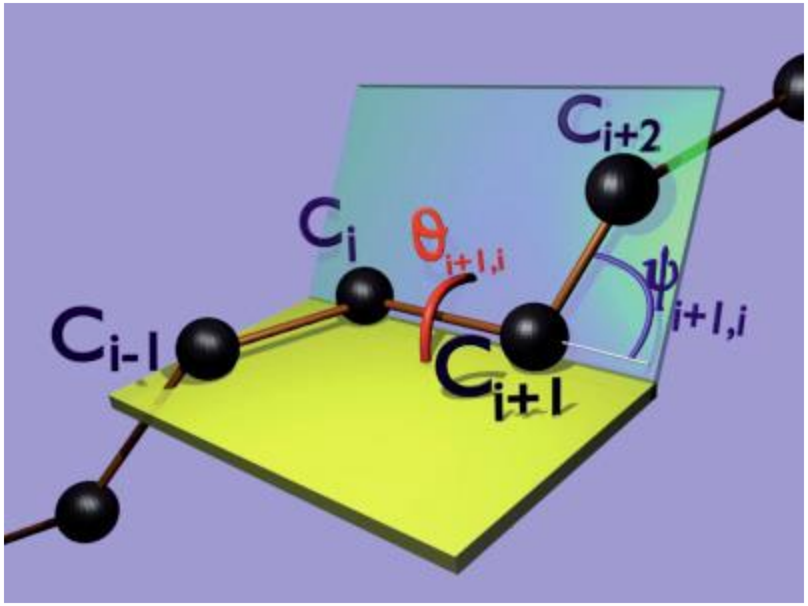
"Bond and torsion angles"

(derivatives converge to κ and τ, resp.)

Discrete frame introduced in:
The resultant electric moment of complex molecules
Eyring, Physical Review, 39(4):746—748, 1932.

Transfer Matrix

$$\begin{pmatrix} T_{i+1} \\ N_{i+1} \\ B_{i+1} \end{pmatrix} = R_{i+1,i} \begin{pmatrix} T_i \\ N_i \\ B_i \end{pmatrix}$$



<u>Discrete</u> construction that works for fractal curves and converges in continuum limit.

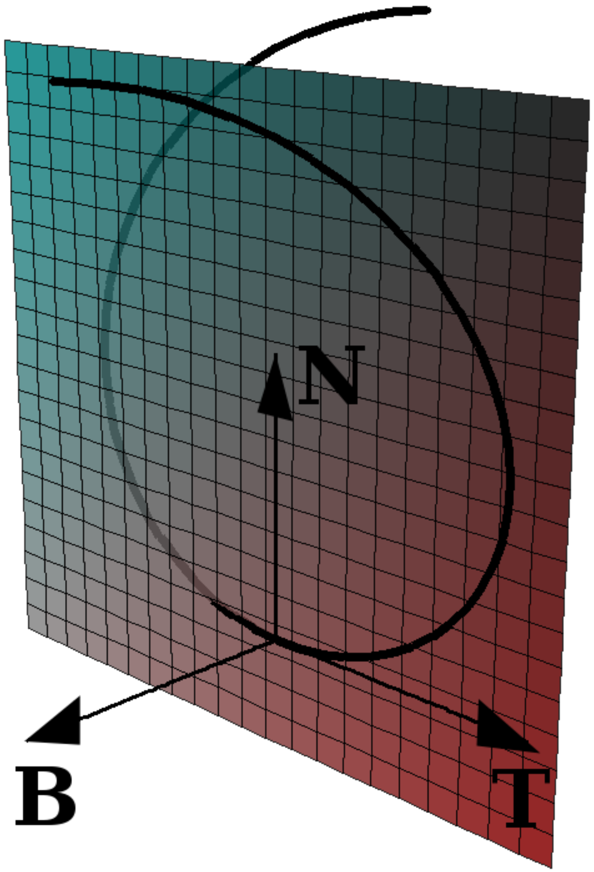
Discrete Frenet Frame, Inflection Point Solitons, and Curve Visualization with Applications to Folded Proteins

Hu, Lundgren, and Niemi

Physical Review E 83 (2011)

Frenet Frame: Issue

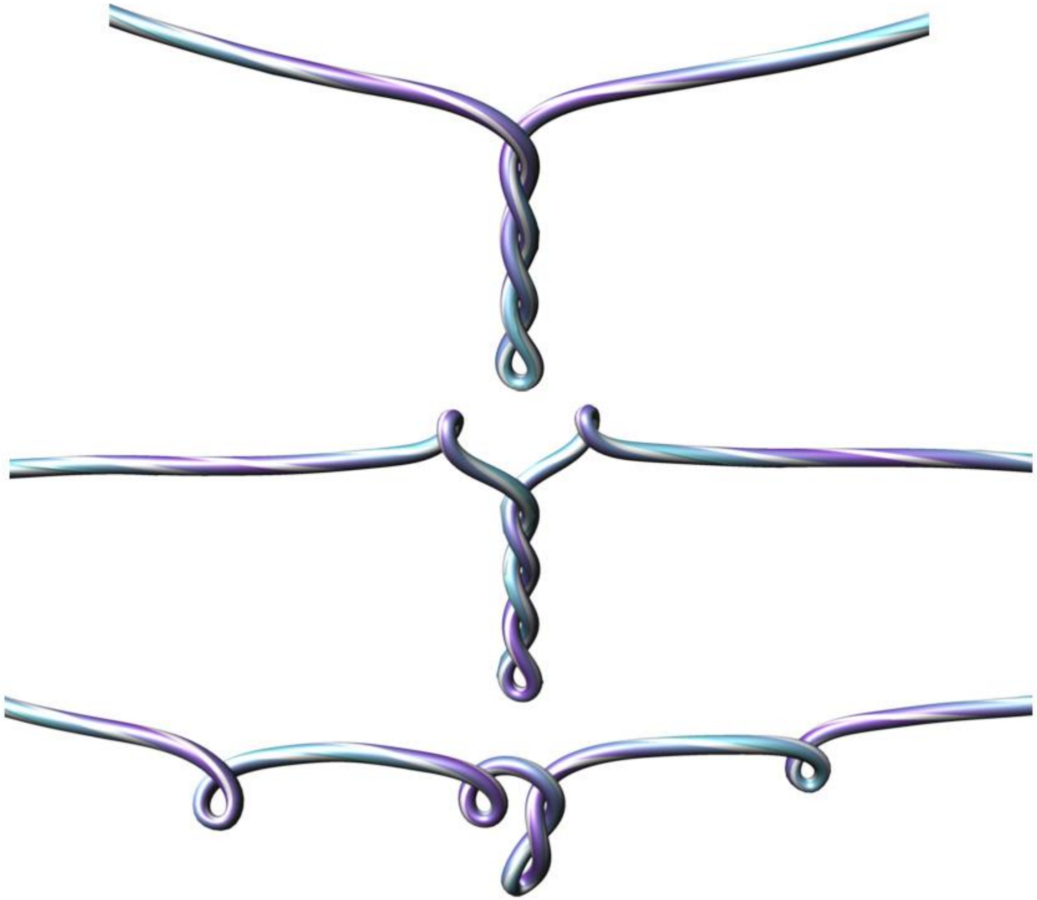
$\kappa = 0$?



$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \boxed{\kappa} & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

http://upload.wikimedia.org/wikipedia/commons/6/6f/Frenet.png

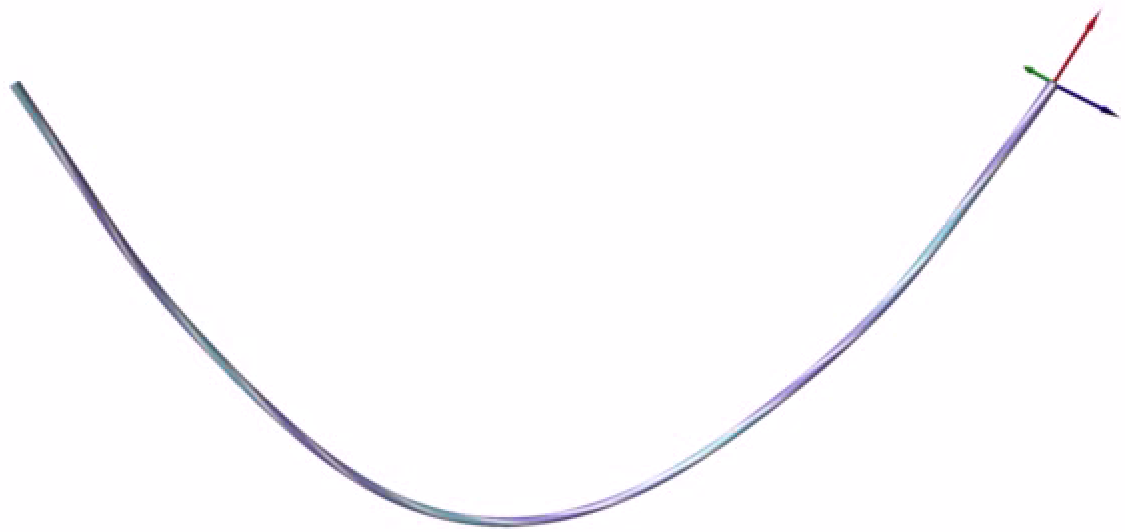
Segments Not Always Enough



Discrete Elastic Rods
Bergou, Wardetzky, Robinson, Audoly, and Grinspun
SIGGRAPH 2008

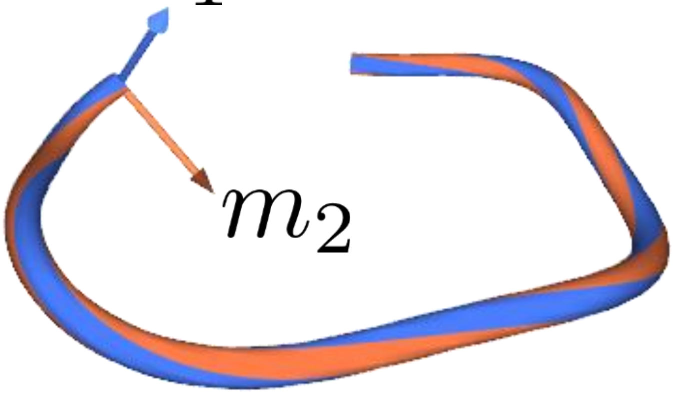
http://www.cs.columbia.edu/cg/rods/

Simulation Goal



http://www.cs.columbia.edu/cg/rods/

Adapted Framed Curve



$$\Gamma = \{\gamma(s); T, m_1, m_2\}$$

Material frame

http://www.cs.columbia.edu/cg/rods/

Normal part encodes twist

Bending Energy

$$E_{\text{bend}}(\Gamma) := \frac{1}{2}\int_\Gamma \alpha\kappa^2\, ds$$

Punish turning the steering wheel

$$\begin{aligned}\kappa N &= T' \\ &= (T' \cdot T)T + (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \\ &= (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \\ &:= \omega_1 m_1 + \omega_2 m_2\end{aligned}$$

Bending Energy

$$E_{\text{bend}}(\Gamma) := \frac{1}{2}\int_{\Gamma} \alpha(\omega_1^2 + \omega_2^2)\, ds$$

Punish turning the steering wheel

$$\begin{aligned}
\kappa N &= T' \\
&= (T' \cdot T)T + (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \\
&= (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \\
&:= \omega_1 m_1 + \omega_2 m_2
\end{aligned}$$

Twisting Energy

$$E_{\text{twist}}(\Gamma) := \frac{1}{2}\int_\Gamma \beta m^2 \, ds$$

Punish non-tangent change in material frame

$$\begin{aligned} m &:= m_1' \cdot m_2 \\ &= \frac{d}{dt}(m_1 \cdot m_2) - m_1 \cdot m_2' \\ &= -m_1 \cdot m_2' \end{aligned}$$

Twisting Energy

$$E_{\text{twist}}(\Gamma) := \frac{1}{2}\int_\Gamma \beta m^2 \, ds$$

Punish non-tangent change in material frame

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Swapping m_1 and m_2 does not affect E_{twist}!

Which Basis to Use

THERE IS MORE THAN ONE WAY TO FRAME A CURVE

RICHARD L. BISHOP

The Frenet frame of a 3-times continuously differentiable (that is, C^3) nondegenerate curve in euclidean space has long been the standard vehicle for analysing properties of the curve invariant under euclidean motions. For arbitrary moving frames, that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the curve parameter in terms of the frame itself, and due to orthonormality the coefficient matrix is always skew-symmetric. Thus it generally has three nonzero entries. The Frenet frame gains part of its special significance from the fact that one of the three derivatives is always zero. Another feature of the Frenet frame is that it is *adapted* to the curve: the members are either tangent to or perpendicular to the curve. It is the purpose of this paper to show that there are other frames which have these same advantages and to compare them with the Frenet frame.

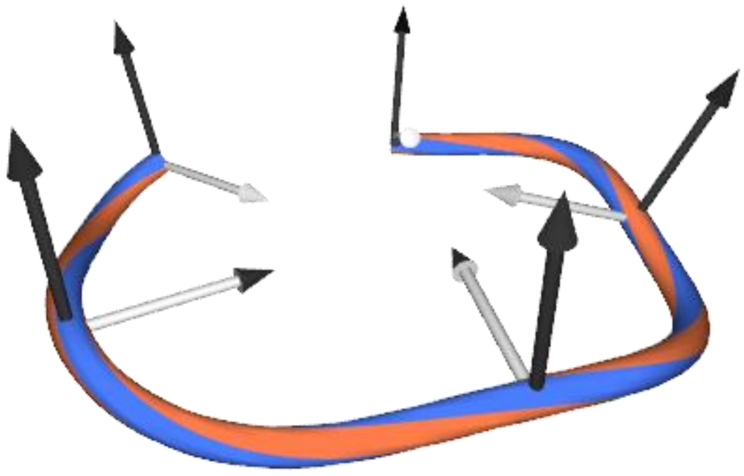
1. Relatively parallel fields. We say that a normal vector field M along a curve is *relatively parallel* if its derivative is tangential. Such a field turns only whatever amount is necessary for it to remain normal, so it is as close to being parallel as possible without losing normality. Since its derivative is perpendicular to it, a relatively parallel normal field has constant length. Such fields occur classically in

Bishop Frame

$$T' = \Omega \times T$$

$$u' = \Omega \times u$$

$$v' = \Omega \times v$$



$$\Omega := \kappa B \text{ (``curvature binormal'')}$$

Darboux vector

http://www.cs.columbia.edu/cg/rods/

Most relaxed frame

Bishop Frame

$$T' = \Omega \times T$$

$$u' = \Omega \times u$$

$$v' = \Omega \times v$$

$$\Omega := \kappa B \text{ (``curvature binormal'')}$$

$$u' \cdot v \equiv 0$$

No twist
("parallel transport")

http://www.cs.columbia.edu/cg/rods/

Most relaxed frame

Curve-Angle Representation

$$m_1 = u\cos\theta + v\sin\theta$$

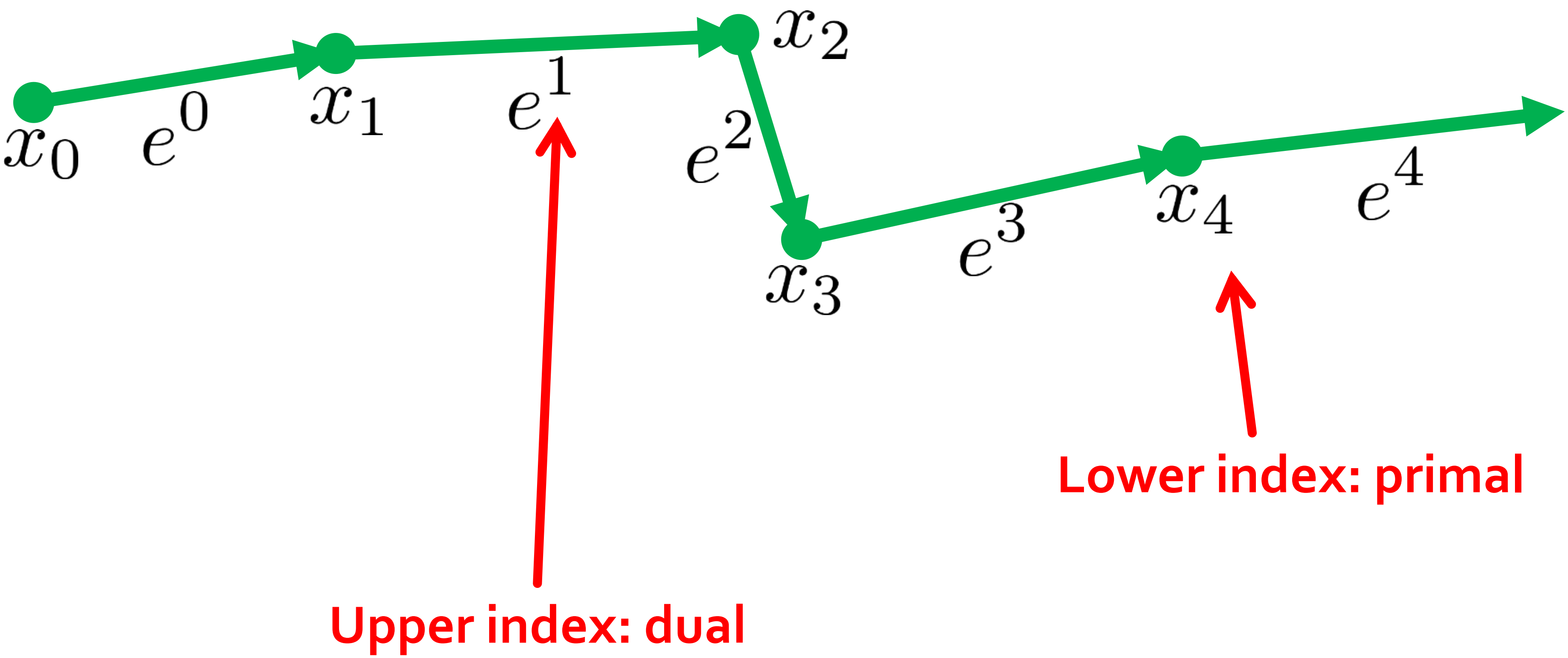
$$m_2 = -u\sin\theta + v\cos\theta$$

$$E_{\text{twist}}(\Gamma) := \frac{1}{2}\int_\Gamma \beta(\theta')^2\, ds$$

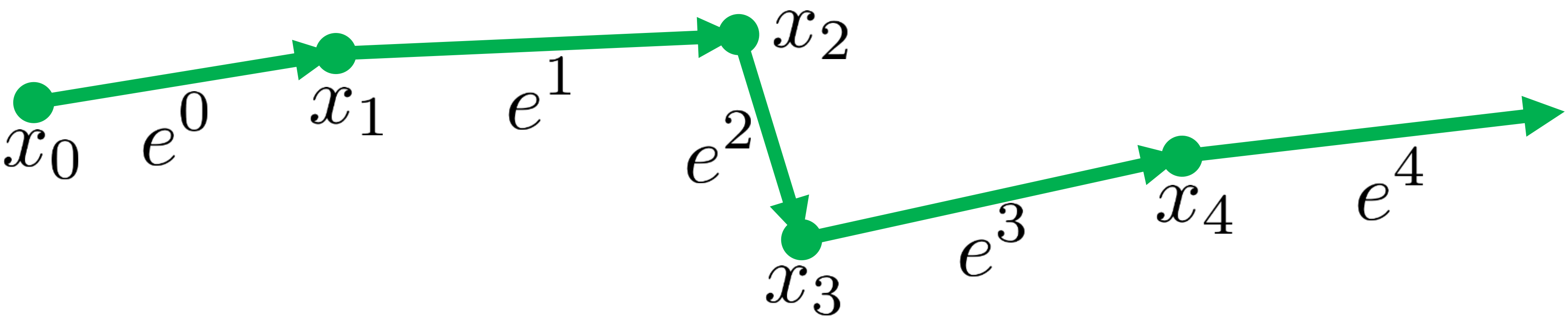
Degrees of freedom for elastic energy:

- Shape of curve
- Twist angle θ

Discrete Kirchoff Rods



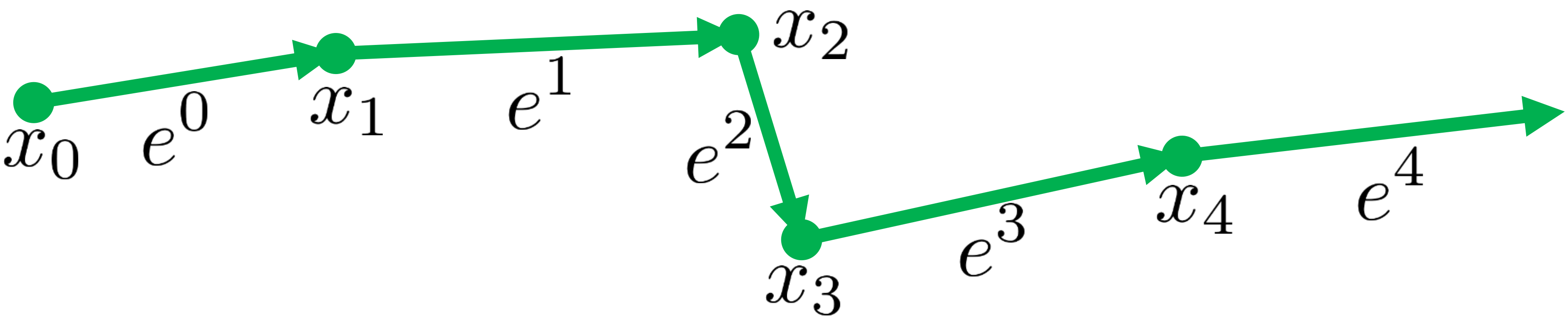
Discrete Kirchoff Rods



$$T^i := \frac{e^i}{\|e^i\|}$$

Tangent unambiguous on edge

Discrete Kirchoff Rods



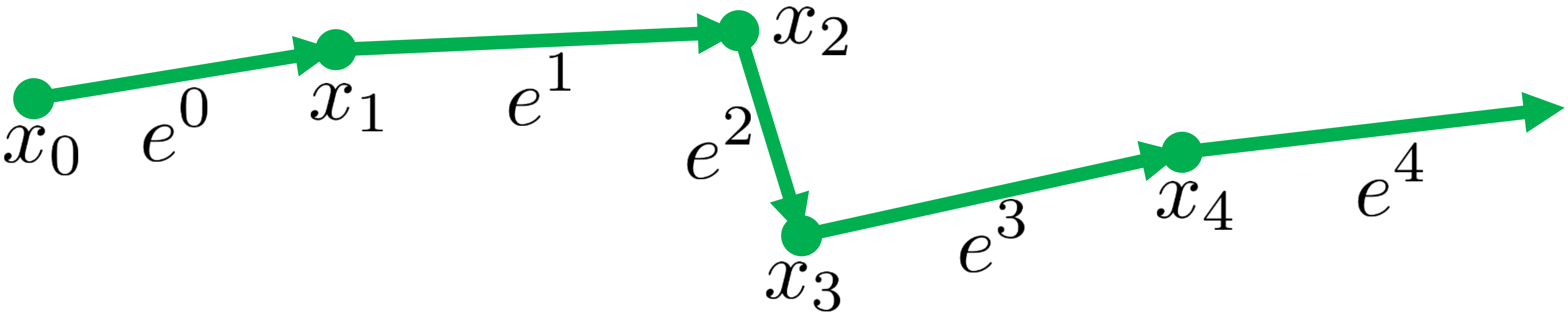
$$\kappa_i := 2\tan\frac{\phi_i}{2}$$

Turning angle

Yet another curvature!

Integrated curvature

Discrete Kirchoff Rods



$$\kappa_i := 2\tan\frac{\phi_i}{2}$$

Yet another curvature!

$$(\kappa B)_i := \frac{2e^{i-1} \times e^i}{\|e^{i-1}\|\|e^i\| + e^{i-1} \cdot e^i}$$

Orthogonal to osculating plane, norm $\boldsymbol{\kappa_i}$

Darboux vector

Bending Energy

$$E_{\text{bend}}(\Gamma) := \frac{\alpha}{2}\sum_i \left(\frac{(\kappa B)_i}{\ell_i/2}\right)^2 \frac{\ell_i}{2}$$

$$= \alpha \sum_i \frac{\|(\kappa B)_i\|^2}{\ell_i}$$

Can extend for natural bend

Convert to pointwise and integrate

Discrete Parallel Transport

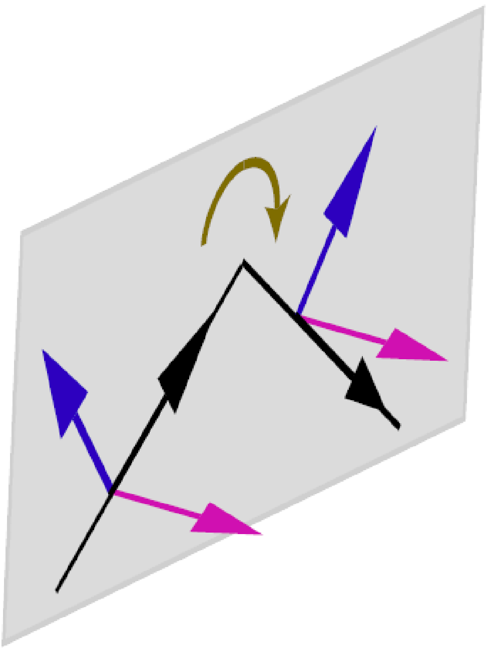
$$P_i(T^{i-1}) = T^i$$

$$P_i(T^{i-1} \times T^i) = T^{i-1} \times T^i$$

- **Map tangent to tangent**
- **Preserve binormal**
- **Orthogonal**

$$u^i = P_i(u^{i-1})$$

$$v^i = T^i \times u^i$$



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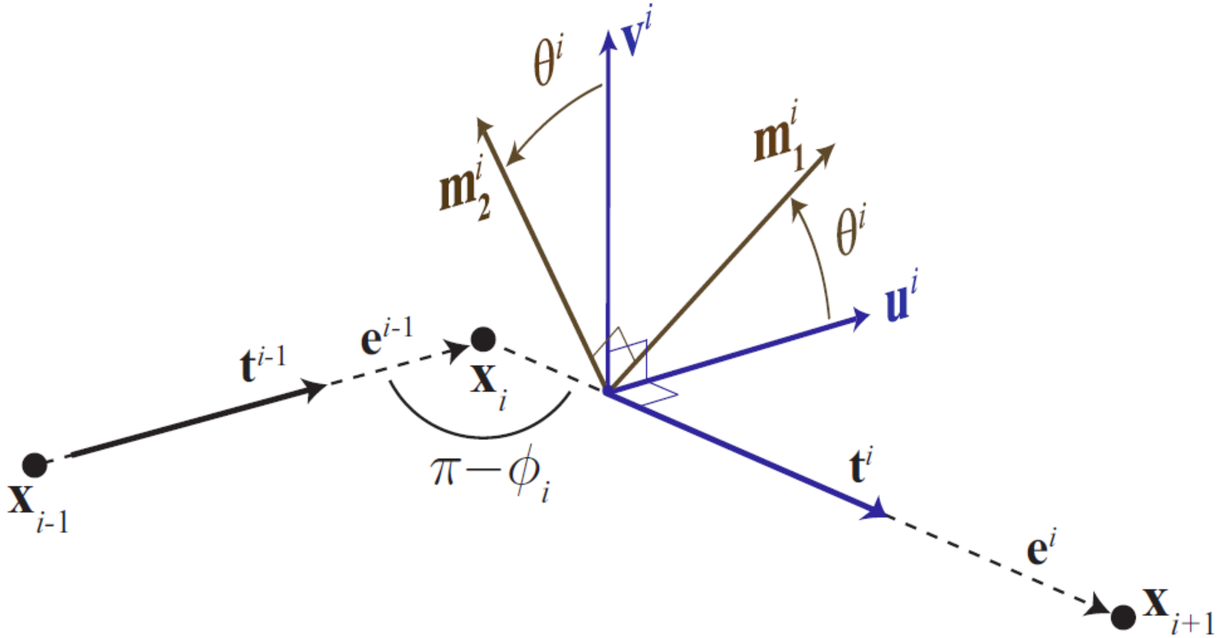
Discrete Material Frame

$$m_1^i = u^i \cos\theta^i + v^i \sin\theta^i$$

$$m_2^i = -u^i \sin\theta^i + v^i \cos\theta^i$$



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Discrete Twisting Energy

$$E_{\text{twist}}(\Gamma) := \beta \sum_i \frac{(\theta^i - \theta^{i-1})^2}{\ell_i}$$

Note θ_0 can be arbitrary

Simulation

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\omit{physics}
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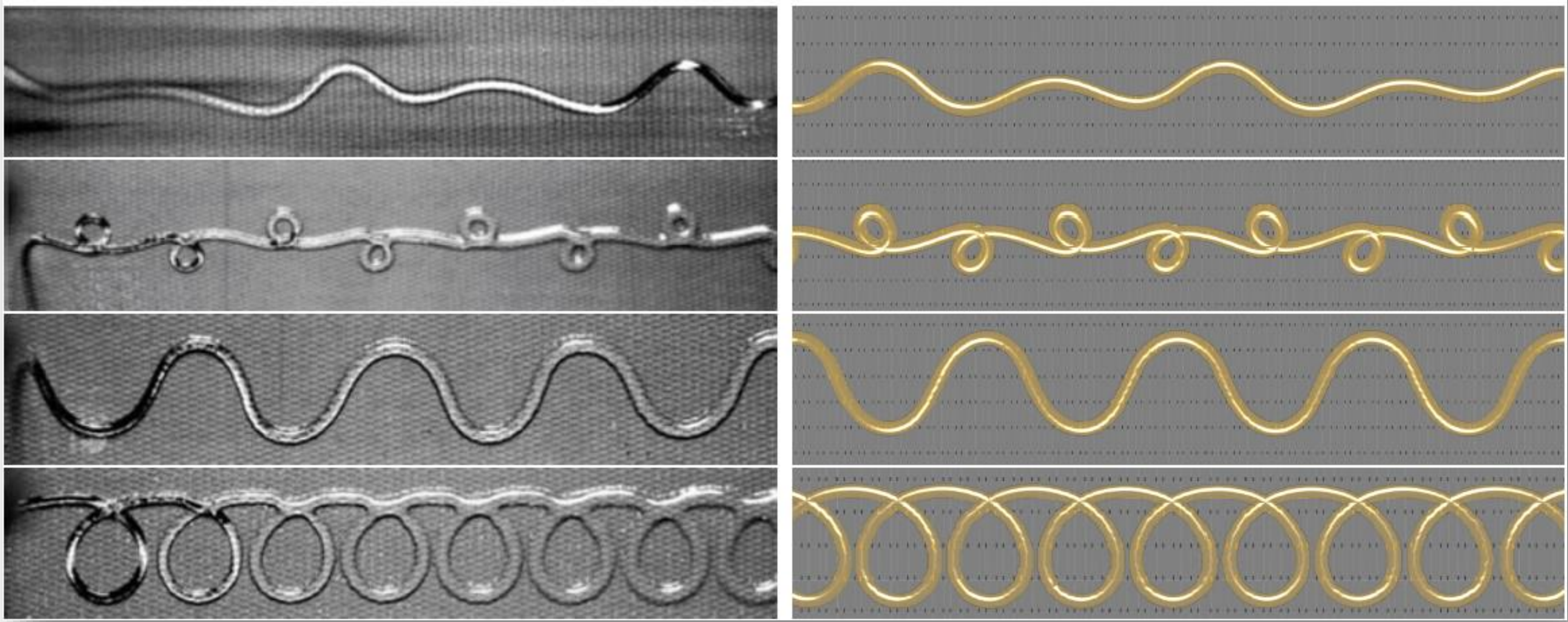
Worth reading!

Extension and Speedup

Discrete Viscous Threads

Miklós Bergou, Columbia University
Basile Audoly, UPMC Univ. Paris 06 & CNRS
Etienne Vouga, Columbia University
Max Wardetzky, Universität Göttingen
Eitan Grinspun, Columbia University



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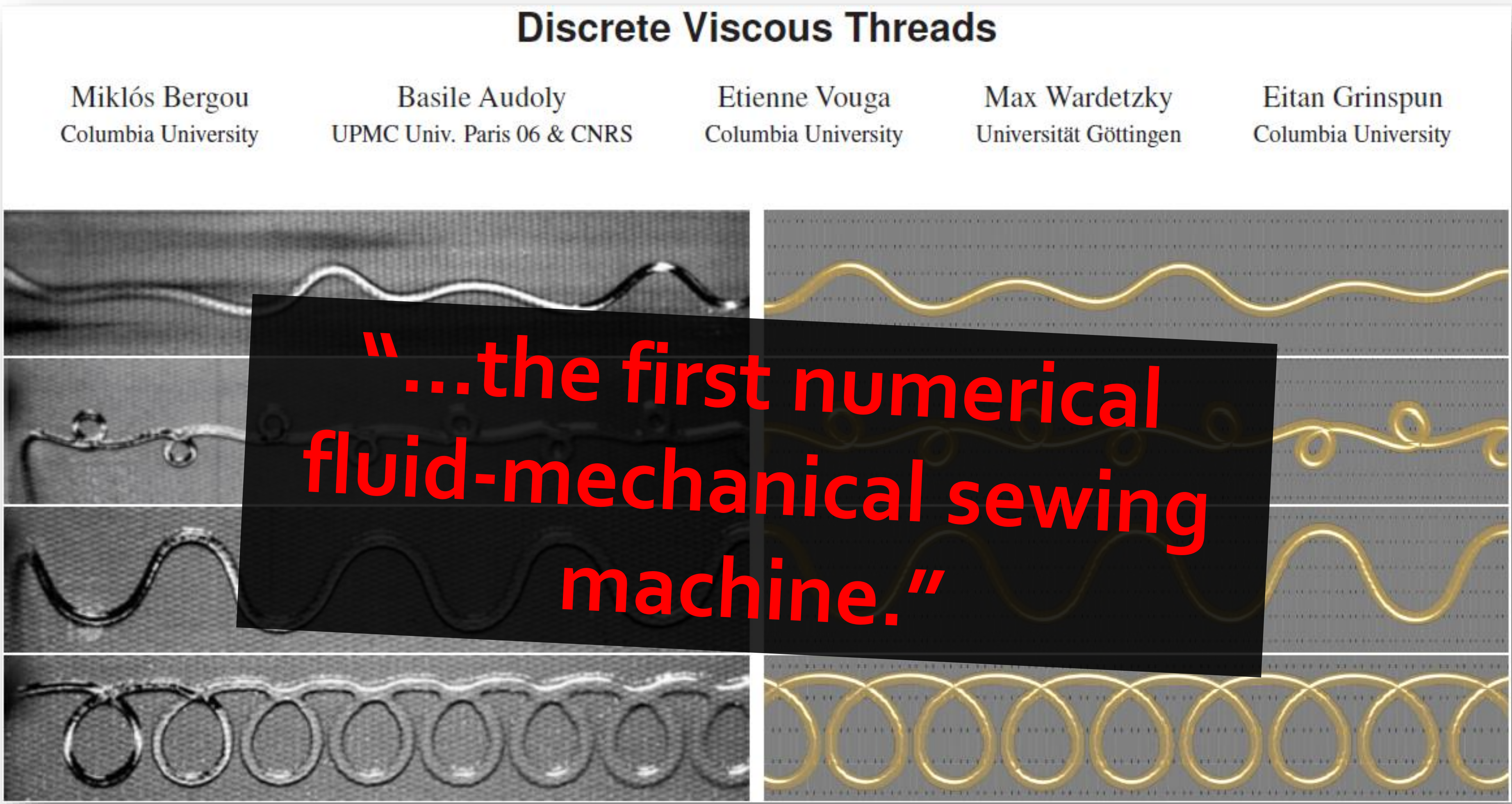
Extension and Speedup



http://www.cs.columbia.edu/cg/threads/

Morals

**One curve,
three curvatures.**

$$\theta \qquad\qquad 2\sin\frac{\theta}{2} \qquad\qquad 2\tan\frac{\theta}{2}$$

Morals

Easy theoretical object, hard to use.

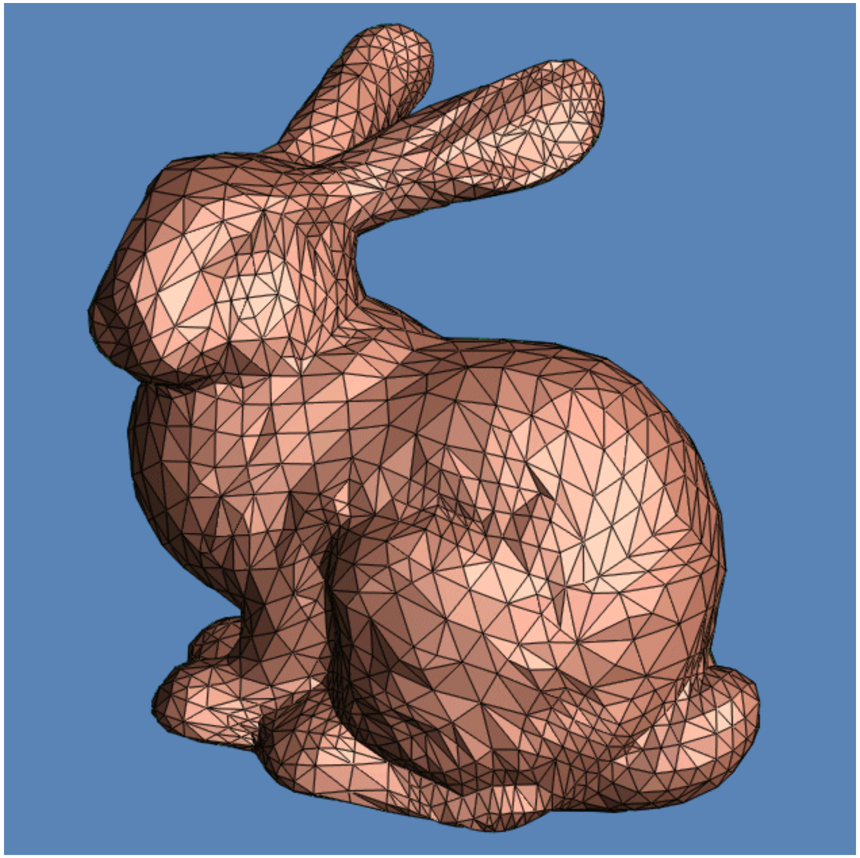
$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Morals

Proper frames and DOFs go a long way.

$$m_1^i = u^i \cos\theta^i + v^i \sin\theta^i$$
$$m_2^i = -u^i \sin\theta^i + v^i \cos\theta^i$$

Next



http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg
http://www.stat.washington.edu/wxs/images/BUNMID.gif

Surfaces

http://www.alvinomassage.com/images/knot.jpg

Curves: Continuous and Discrete

Justin Solomon
MIT, Spring 2017

MIT EECS

Some materials from Stanford CS 468, spring 2013 (Butscher & Solomon)