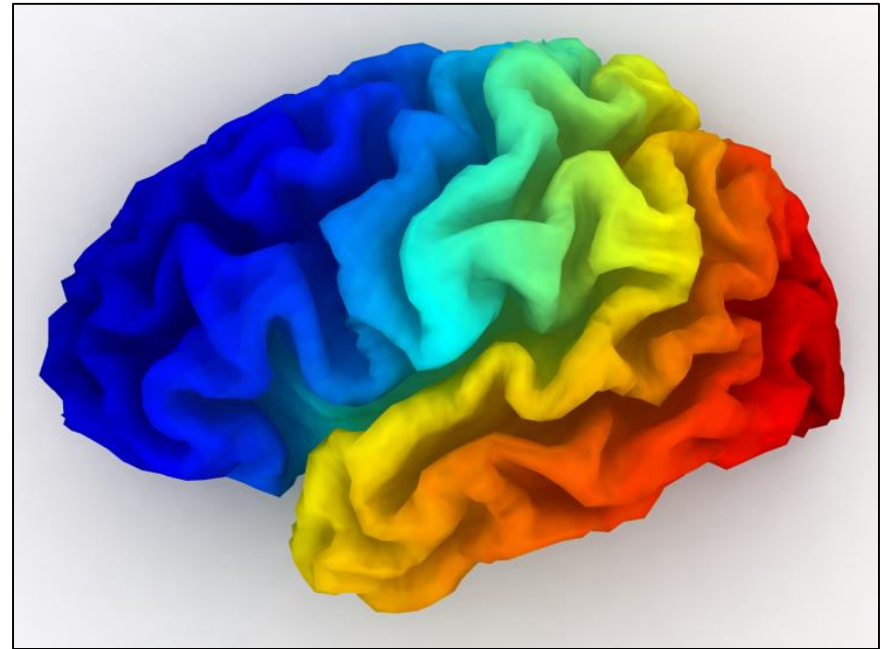
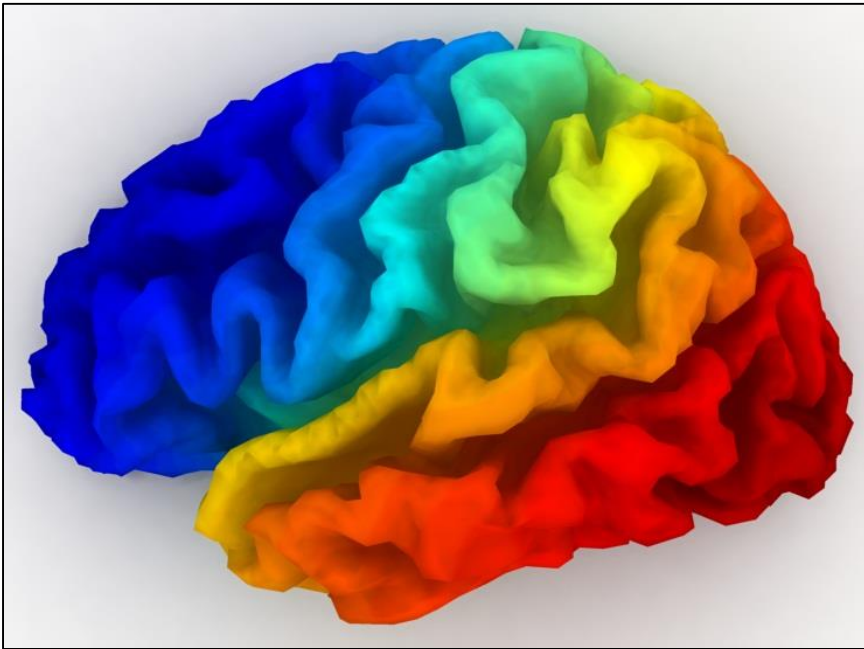


Surface Correspondence

Justin Solomon
MIT, Spring 2017



Correspondence Problems



Which points on one object correspond to points on another?



How is this different
from **registration**?

Typical Distinction

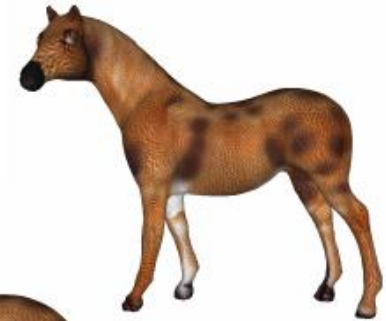
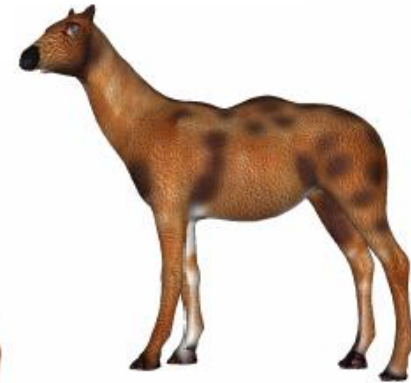
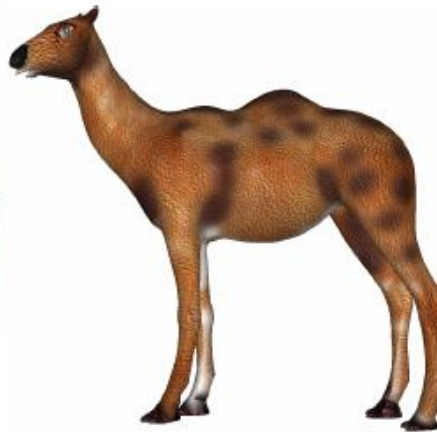
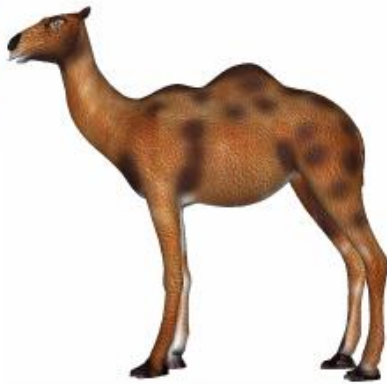
Seek **shared structure**
instead of alignment



Applications



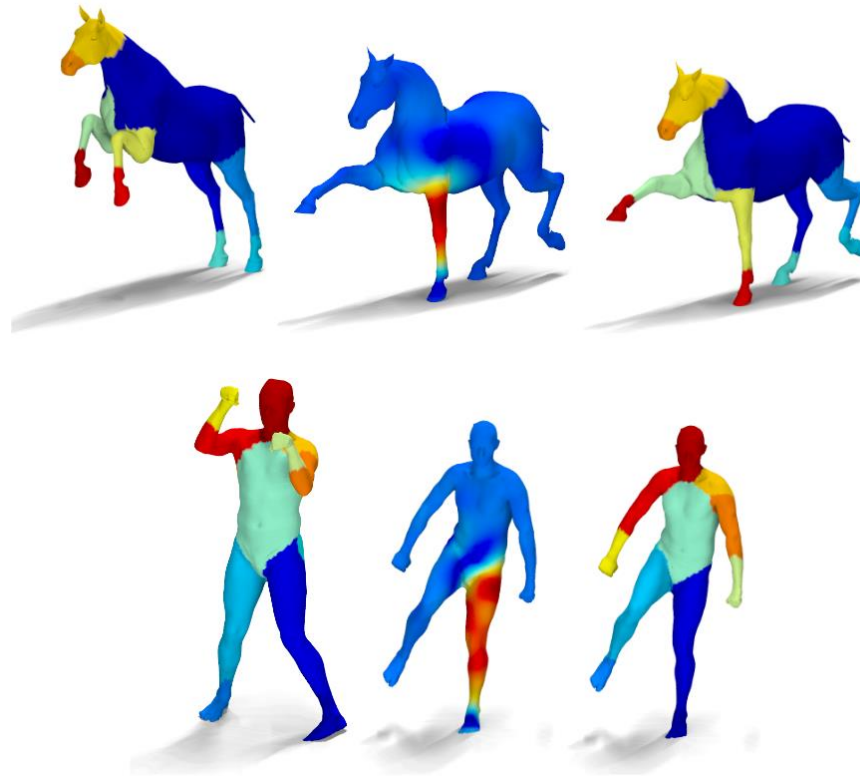
Praun et al. 2001



Kraevoy and Sheffer 2004

Texture transfer

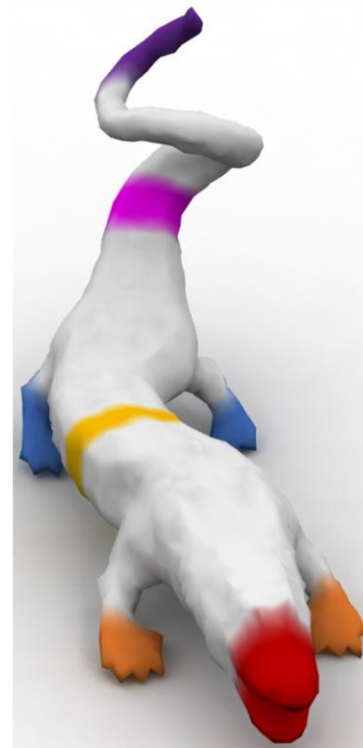
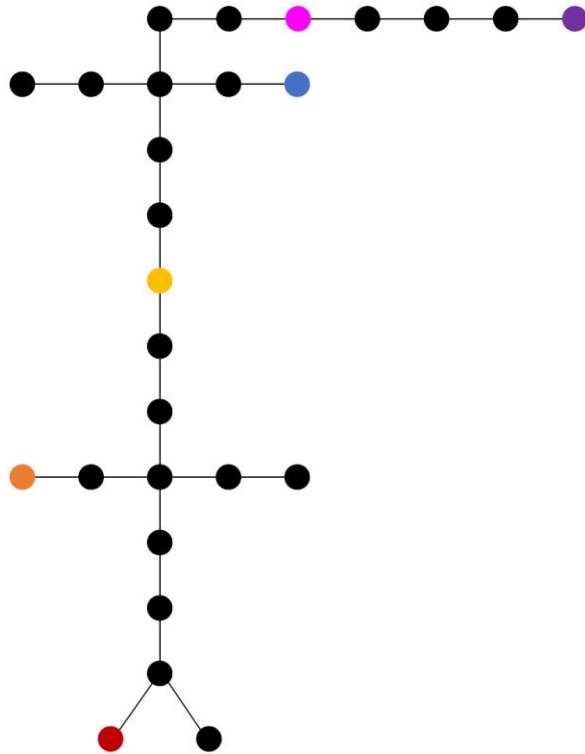
Applications



Ovsjanikov et al. 2012

Segmentation transfer

Applications



Solomon et al. 2016

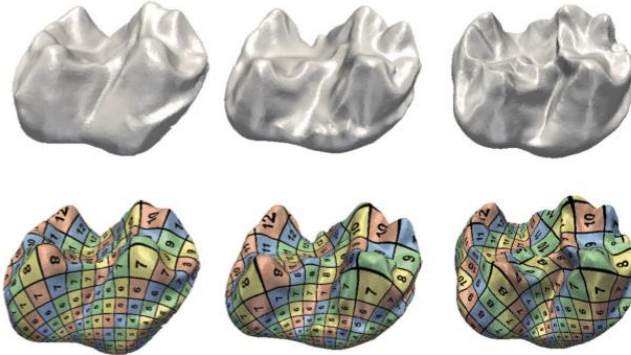
Layout

Applications

P. tricuspidens

NMB Bru4

Plat daubrei



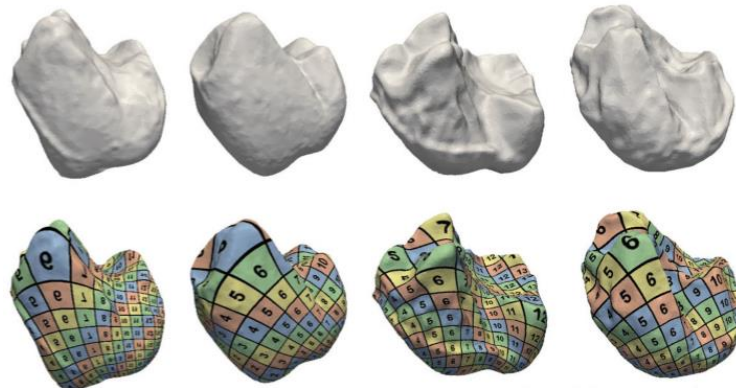
A. Second lower molars

P. tricuspidens

P. cookei

NMB Bru4

Plat daubrei



B. Fourth Lower premolars

Boyer, Costeur, and Lipman 2012

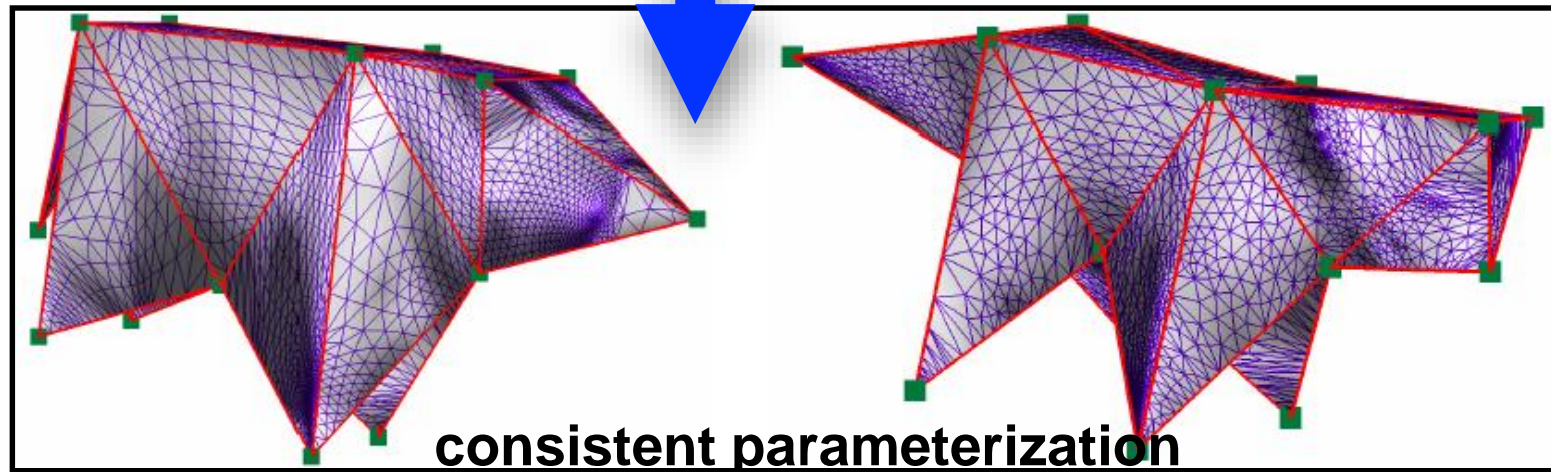
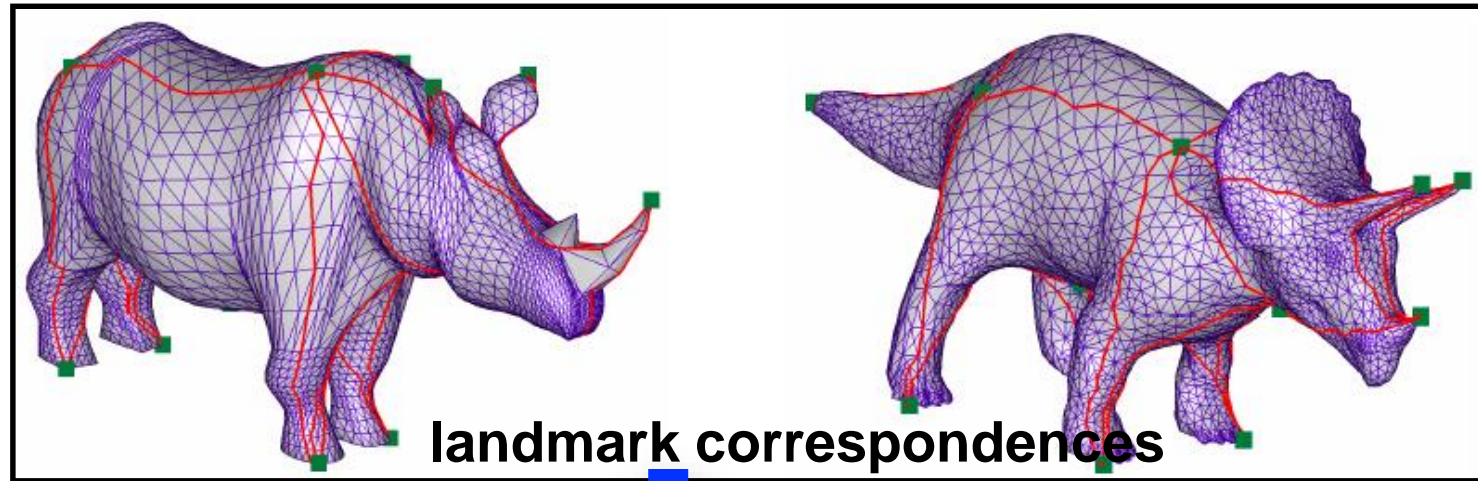
Paleontology

Desirable Properties

Given two (or more) shapes
Find a map f , that is:

- Automatic
- Fast to compute
 - Bijective
(if we expect global correspondence)
- Low-distortion

Example: Consistent Remeshing

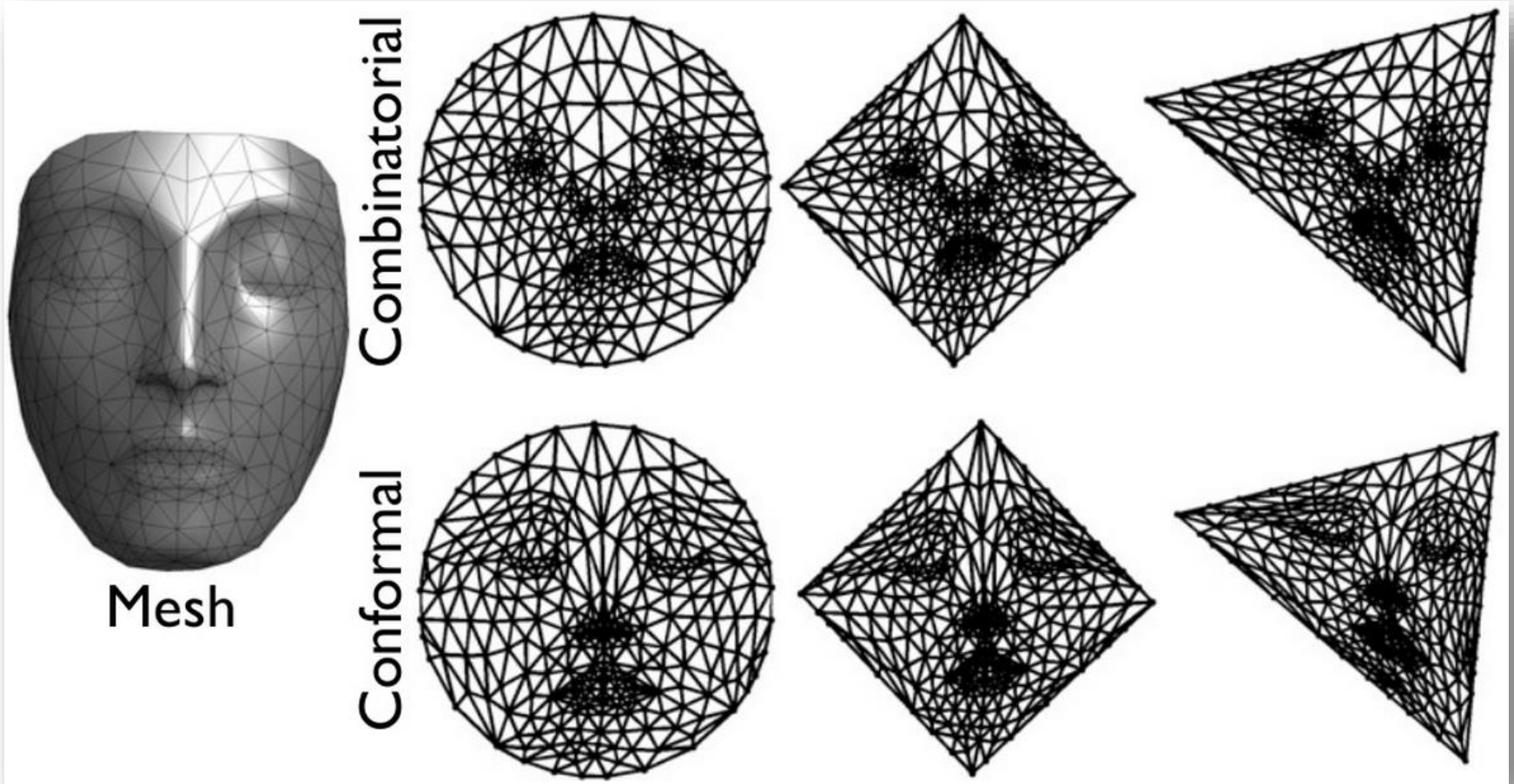


Kraevoy 2004

Adapted from slides by Q. Huang, V. Kim

Recall:

Example: Mesh Embedding



Recall:

Linear Solve for Embedding

$$\begin{aligned} \min_{x_1, \dots, x_{|V|}} \quad & \sum_{(i,j) \in E} w_{ij} \|x_i - x_j\|_2^2 \\ \text{s.t.} \quad & x_v \text{ fixed } \forall v \in V_0 \end{aligned}$$

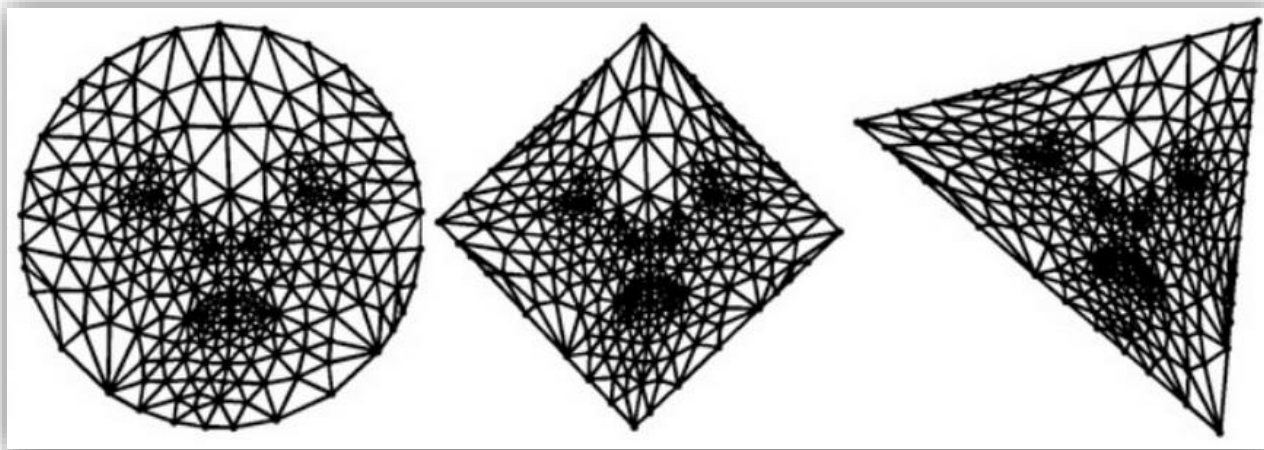
- **$w_{ij} \equiv 1$:** Tutte embedding
- **w_{ij} from mesh:** Harmonic embedding

Assumption: w symmetric.

Tutte Embedding Theorem

$$\begin{aligned} \min_{x_1, \dots, x_{|V|}} & \sum_{(i,j) \in E} w_{ij} \|x_i - x_j\|_2^2 \\ \text{s.t.} & x_v \text{ fixed } \forall v \in V_0 \end{aligned}$$

Tutte embedding **bijjective** if w nonnegative and boundary mapped to a convex polygon.



“How to draw a graph” (Proc. London Mathematical Society; Tutte, 1963)

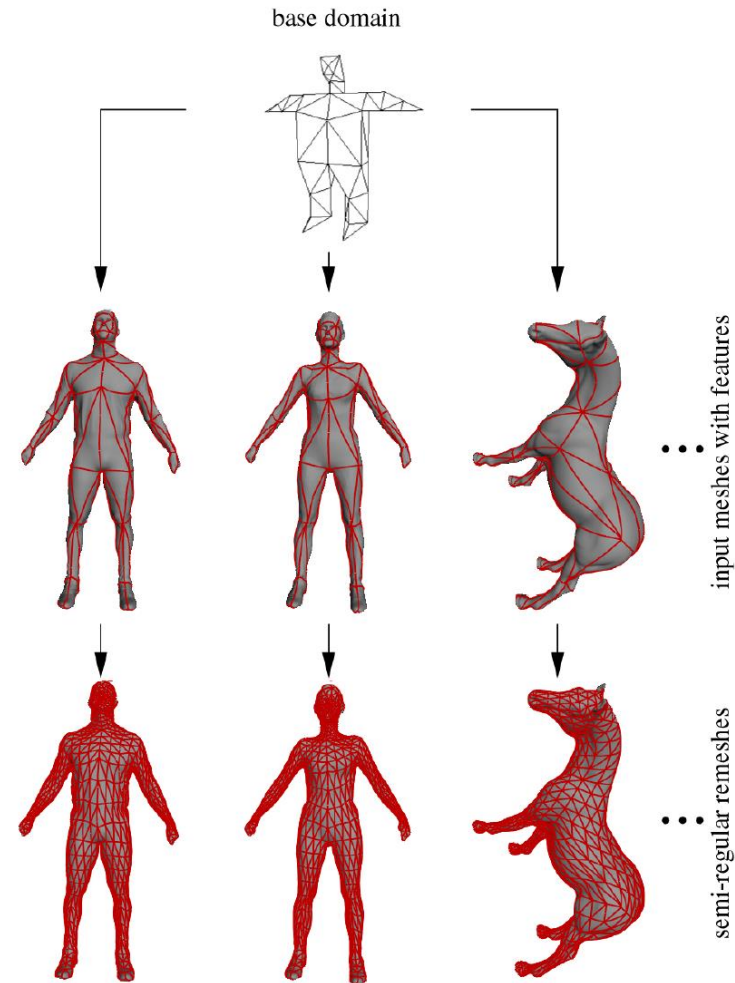
Tradeoff: Consistent Remeshing

- **Pros:**

- Easy
- Straightforward applications

- **Cons:**

- Need manual landmarks
- Hard to minimize distortion



Praun et al. 2001

Recently Revisited

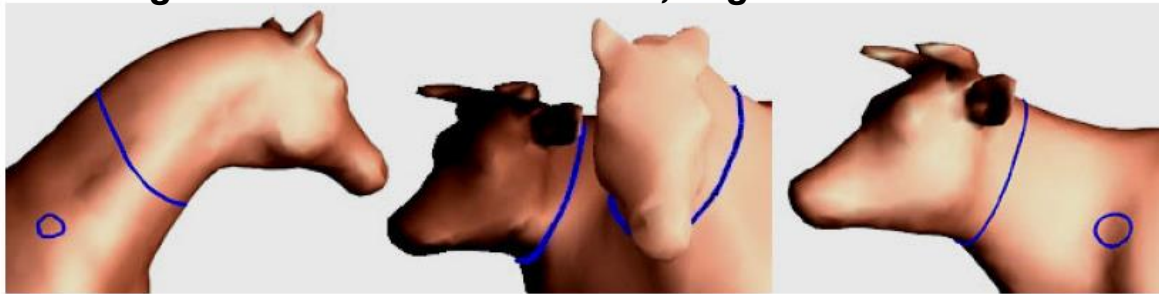


“Orbifold Tutte Embeddings” (Aigerman and Lipman, SIGGRAPH Asia 2015)

Automatic Landmarks

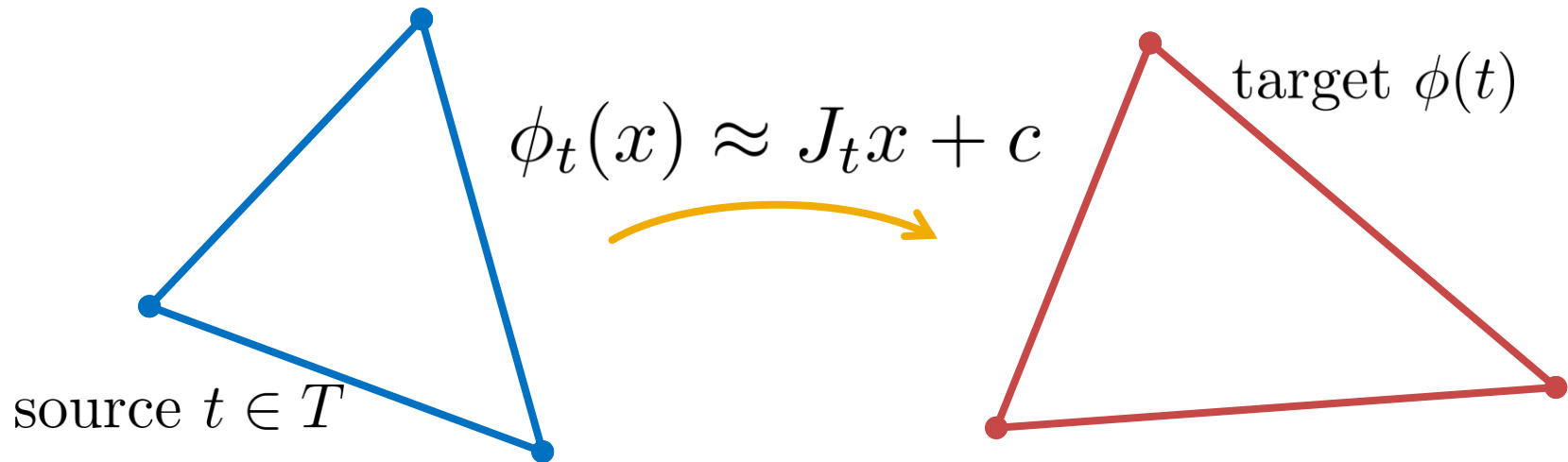
- **Simple algorithm:**
 - Set landmarks
 - Measure energy
 - Repeat
- **Possible metrics**
 - Conformality
 - Area preservation
 - Stretch

E.g. small conformal distortion, large area distortion:



Schreiner et al. 2004

Local Distortion Measure



$$\text{Distortion} := \sum_{t \in T} A_t \mathcal{D}(J_t)$$

Triangle distortion measure



How do you measure
distortion of a triangle?

Typical Distortion Measures

Name	$\mathcal{D}(\mathbf{J})$	$\mathcal{D}(\sigma)$
Symmetric Dirichlet	$\ \mathbf{J}\ _F^2 + \ \mathbf{J}^{-1}\ _F^2$	$\sum_{i=1}^n (\sigma_i^2 + \sigma_i^{-2})$
Exponential Symmetric Dirichlet	$\exp(s(\ \mathbf{J}\ _F^2 + \ \mathbf{J}^{-1}\ _F^2))$	$\exp(s \sum_{i=1}^n (\sigma_i^2 + \sigma_i^{-2}))$
Hencky strain	$\ \log \mathbf{J}^\top \mathbf{J}\ _F^2$	$\sum_{i=1}^n (\log^2 \sigma_i)$
AMIPS	$\exp(s \cdot \frac{1}{2} (\frac{\text{tr}(\mathbf{J}^\top \mathbf{J})}{\det(\mathbf{J})} + \frac{1}{2} (\det(\mathbf{J}) + \det(\mathbf{J}^{-1}))))$	$\exp(s (\frac{1}{2} (\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}) + \frac{1}{4} (\sigma_1 \sigma_2 + \frac{1}{\sigma_1 \sigma_2})))$
Conformal AMIPS 2D	$\frac{\text{tr}(\mathbf{J}^\top \mathbf{J})}{\det(\mathbf{J})}$	$\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 \sigma_2}$
Conformal AMIPS 3D	$\frac{\text{tr}(\mathbf{J}^\top \mathbf{J})}{\det(\mathbf{J})^{\frac{2}{3}}}$	$\frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{(\sigma_1 \sigma_2 \sigma_3)^{\frac{2}{3}}}$

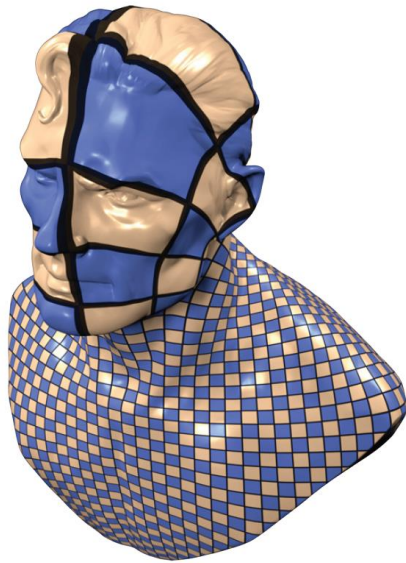
Open challenge:
**Optimize
directly**

Table from "Scalable Locally Injective Mappings" (Rabinovich et al., 2017)

Related Problem

Mapping specifically
into the plane

Initialization



10 iterations



20 iterations



Converged (196)



Image from "Scalable Locally Injective Mappings" (Rabinovich et al., 2017)

Parameterization

New Idea

Not all calculations have to be at the triangle level!

Long-distance interactions
can stabilize geometric computations.

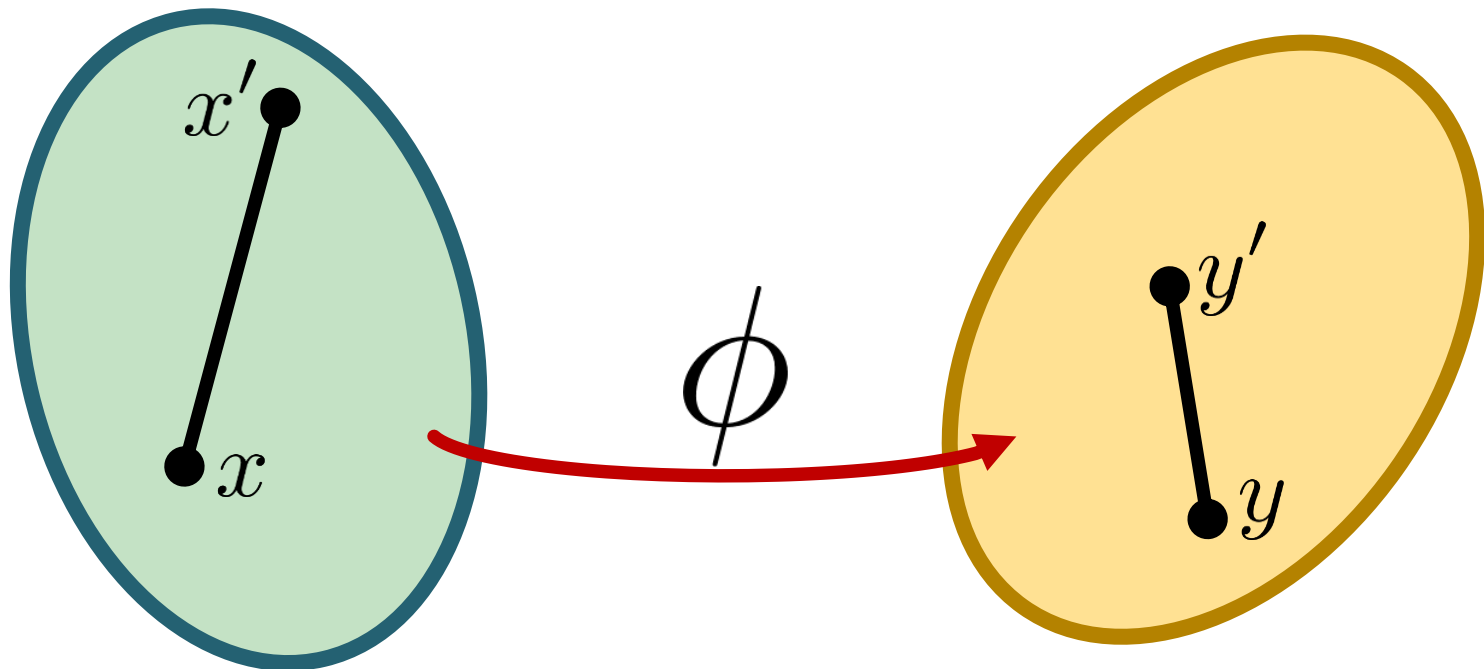
Gromov-Hausdorff Distance

Distance between metric spaces X, Y

$$d_{\text{GH}}(X, Y) := \inf_{\phi: X \rightarrow Y} \sup_{x, x' \in X} |d_X(x, x') - d_Y(\phi(x), \phi(x'))|$$

Best map

Worst distortion



Recall:

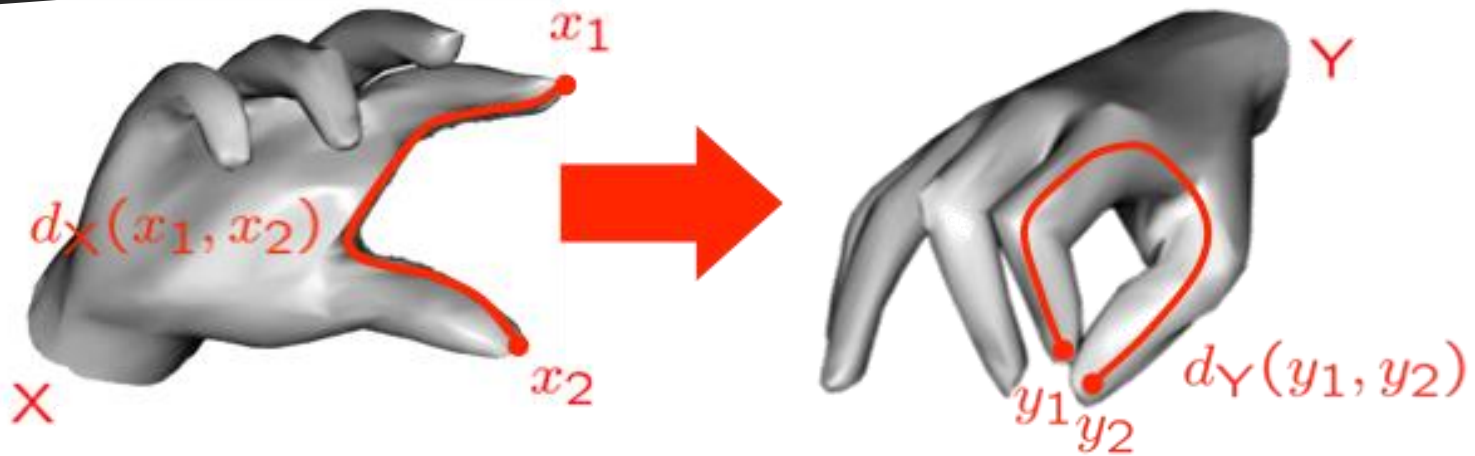
Classical Multidimensional Scaling

1. Double centering: $B := -\frac{1}{2}JDJ$
Centering matrix $J := I - \frac{1}{n}\mathbf{1}\mathbf{1}^\top$
2. Find m largest eigenvalues/eigenvectors
3. $X = E_m\Lambda_m^{1/2}$

"MDS"

Generalized MDS

Search for a permutation!

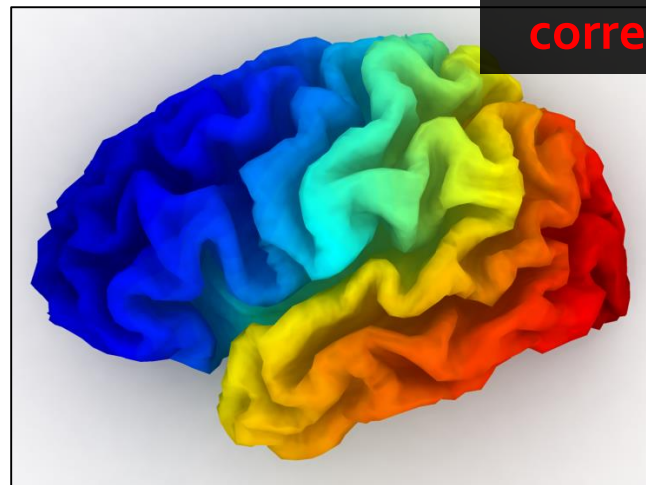


$$d_{\text{int}}(X, Y) := \min_{\{y_1, \dots, y_n\} \subset Y} \|d_X(x_i, x_j) - d_Y(y_i, y_j)\|$$

Problem: Quadratic Assignment

$$\begin{aligned} \min_T \quad & \langle M_0 T, T M_1 \rangle \\ \text{s.t.} \quad & T \in \{0, 1\}^{n \times n} \\ & T \mathbf{1} = p_0 \\ & T^\top \mathbf{1} = p_1 \end{aligned}$$

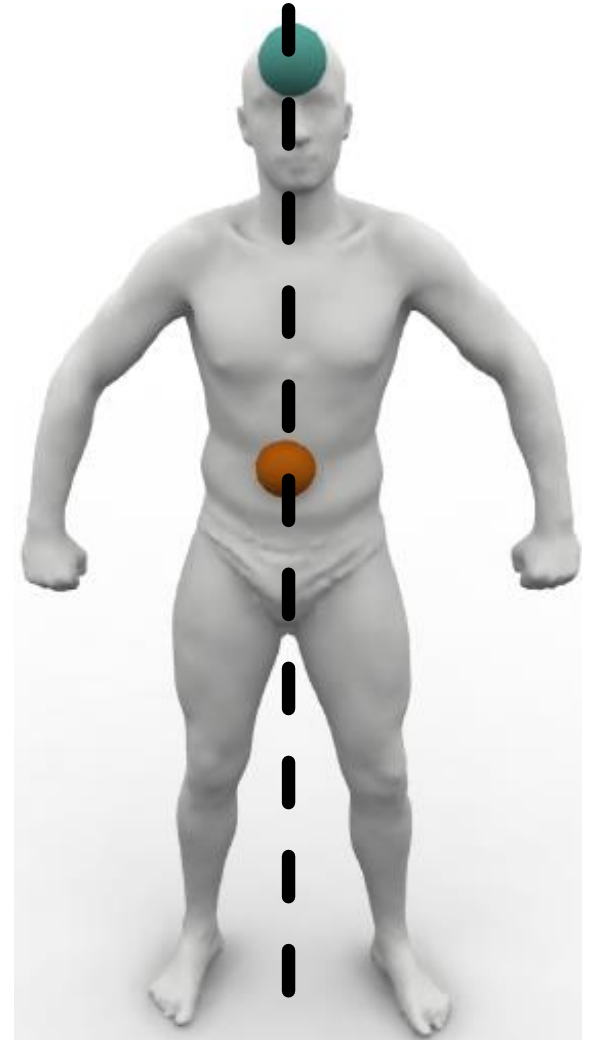
Nonconvex quadratic program!
NP-hard!



General notion of
correspondence

What's Wrong?

- **Hard to optimize**
- **Multiple optima**



Tradeoff: GMDS

- **Pros:**
 - Good distance for non-isometric metric spaces
- **Cons:**
 - Non-convex
 - HUGE search space (i.e. permutations)

GMDS in Practice

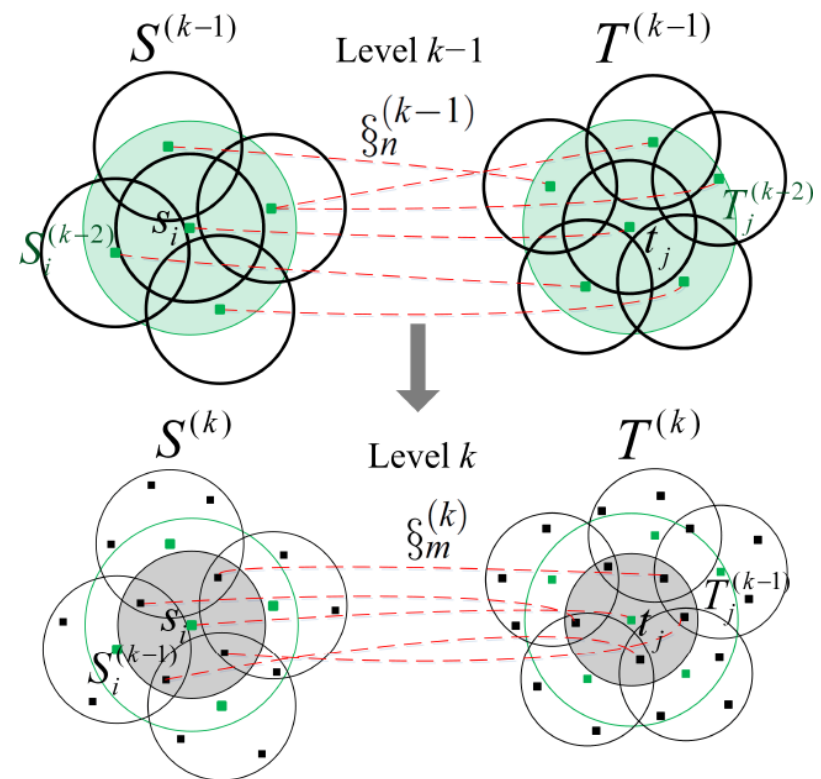
- Heuristics to explore the permutations
 - **Solve at a very coarse scale and interpolate**
 - Coarse-to-fine
 - Partial matching



Bronstein'o8

GMDS in Practice

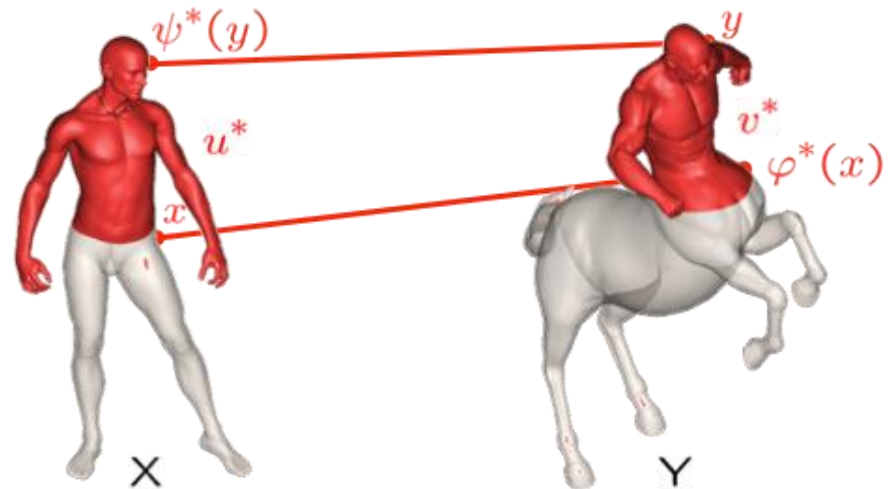
- Heuristics to explore the permutations
 - Solve at a very coarse scale and interpolate
 - **Coarse-to-fine**
 - Partial matching



Sahillioglu'12

GMDS in Practice

- Heuristics to explore the permutations
 - Solve at a very coarse scale and interpolate
 - Coarse-to-fine
 - **Partial matching**



- Find correspondence φ^*, ψ^* minimizing distortion between current parts u^*, v^*
- Select parts u^*, v^* minimizing the distortion with current correspondence φ^*, ψ^* subject to $\lambda(u^*, v^*) \leq \lambda_0$

Returning to Desirable Properties

Given two (or more) shapes

Find a map f , that is:

- Automatic
- ~~Fast to compute~~
- ~~Bijjective~~

(if we expect global correspondence)

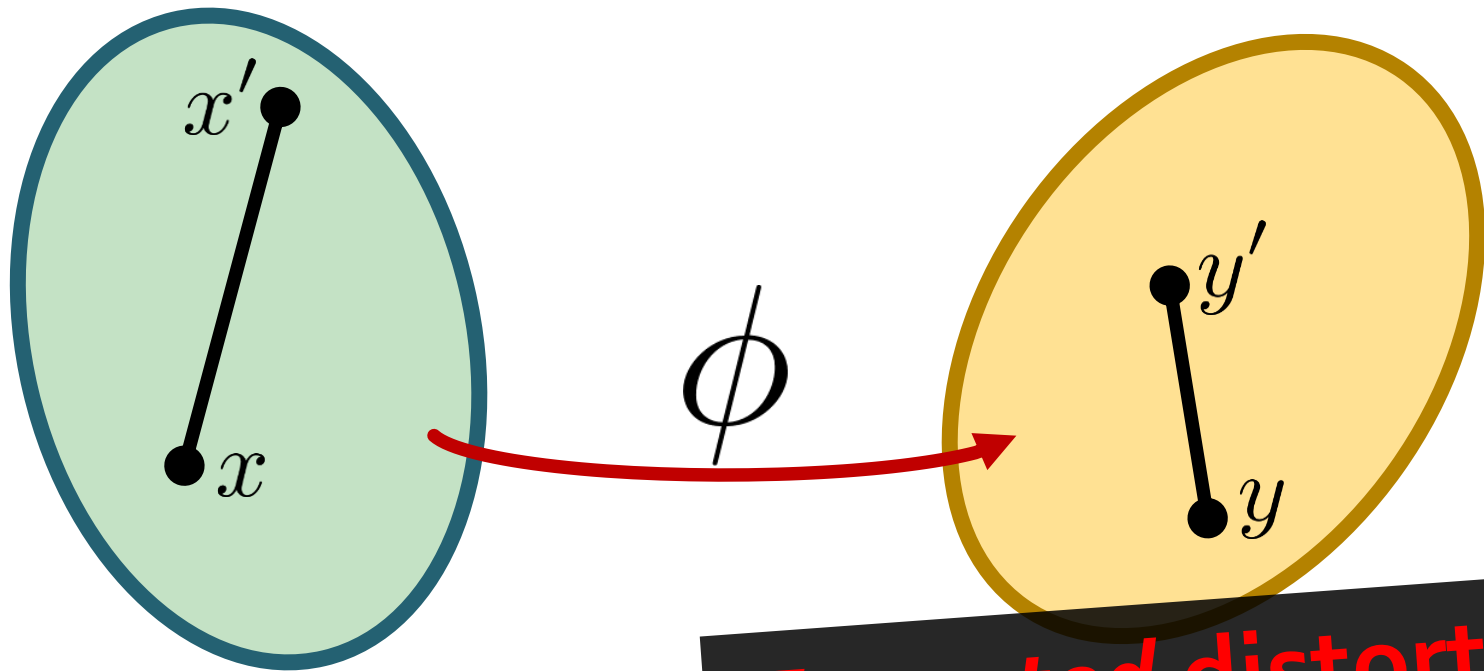
- Low-distortion

(unless local optimum is bad)

Recent idea:

Gromov-Wasserstein Distance

[Mémoli 2007]

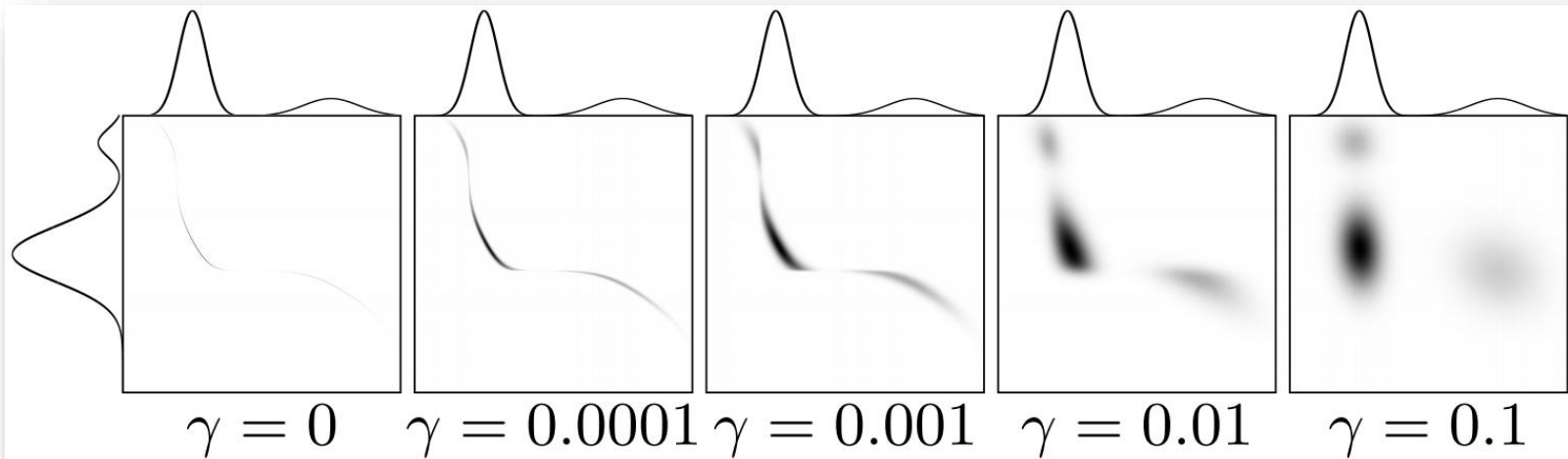


$$\text{GW}_2^2((\mu_0, d_0), (\mu, d)) :=$$

$$\min_{\gamma \in \mathcal{M}(\mu_0, \mu)} \iint_{\Sigma_0 \times \Sigma} [d_0(x, x') - d(y, y')]^2 d\gamma(x, y) d\gamma(x', y')$$

Recall:

Entropic Regularization



$$\min_T \quad \sum_{ij} T_{ij} d(x_i, x_j) - \gamma H(T)$$

$$\text{s.t.} \quad \sum_j T_{ij} = p_i$$

$$\sum_i T_{ij} = q_j$$

$$T \geq 0$$

$$H(T) := - \sum_{ij} T_{ij} \log T_{ij}$$

Gromov-Wasserstein Plus Entropy

Entropic Metric Alignment for Correspondence Problems

Justin Solomon*
MIT

Gabriel Peyré
CNRS & Univ. Paris-Dauphine

Vladimir G. Kim
Adobe Research

Suvrit Sra
MIT

Abstract

Many shape and image processing tools rely on computation of correspondences between geometric domains. Efficient methods that stably extract “soft” matches in the presence of diverse geometric structures have proven to be valuable for shape retrieval and transfer of labels or semantic information. With these applications in mind, we present an algorithm for probabilistic correspondence that optimizes an entropy-regularized Gromov-Wasserstein (GW) objective. Built upon recent developments in numerical optimal transportation, our algorithm is compact, provably convergent, and applicable to any geometric domain expressible as a metric measure matrix. We provide comprehensive experiments illustrating the convergence and applicability of our algorithm to a variety of graphics tasks. Furthermore, we expand entropic GW correspondence to a framework for other matching problems, incorporating partial distance matrices, user guidance, shape exploration, symmetry detection, and joint analysis of more than two domains. These applications expand the scope of entropic GW correspondence to major shape analysis problems and are stable to distortion and noise.

Keywords: Gromov-Wasserstein, matching, entropy

Concepts: •Computing methodologies → Shape analysis;

1 Introduction

A basic component of the geometry processing toolbox is a tool for *mapping* or *correspondence*, the problem of finding which points on a target domain correspond to points on a source. Many variations of this problem have been considered in the graphics literature, e.g.

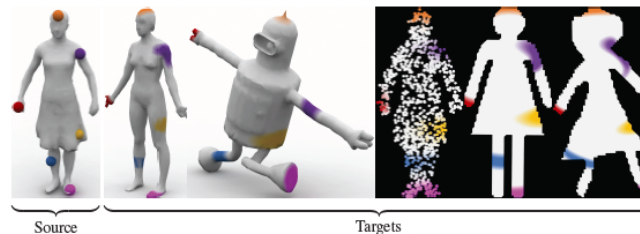


Figure 1: Entropic GW can find correspondences between a 3D surface (left) and a surface with similar shared semantic structure, a noisy 3D point cloud (right). Each fuzzy map was computed by solving a local optimization problem.

are violated these algorithms suffer from local minima. We address this by incorporating local elastic terms into a single global matching problem.

In this paper, we propose a new correspondence problem that minimizes distortion of long- and short-range correspondences. We study an entropically-regularized version of the Gromov-Wasserstein (GW) mapping objective function from [Solomon et al. 2012] that minimizes the distortion of geodesic distances. The correspondence is expressed as a “fuzzy” map of [Kim et al. 2012; Solomon et al. 2012] that finds the correspondence via the weight of an entropic regularization.

Although [Mémoli 2011] and subsequent work have shown the utility of using GW distances for geometric matching, these challenges hampered their practical use. To address these challenges, we build upon recent methods for optimal transportation introduced in [Benard et al. 2015]. While optimal transportation is a well-posed optimization problem from regularized GW computation (linear

```
function GROMOV-WASSERSTEIN( $\mu_0, \mathbf{D}_0, \mu, \mathbf{D}, \alpha, \eta$ )  
    // Computes a local minimizer  $\Gamma$  of (6)  
     $\Gamma \leftarrow \text{ONES}(n_0 \times n)$   
    for  $i = 1, 2, 3, \dots$   
         $\mathbf{K} \leftarrow \exp(\mathbf{D}_0[\mu_0]\Gamma[\mu]\mathbf{D}^\top/\alpha)$   
         $\Gamma \leftarrow \text{SINKHORN-PROJECTION}(\mathbf{K}^{\wedge \eta} \otimes \Gamma^{\wedge(1-\eta)}; \mu_0, \mu)$   
    return  $\Gamma$ 
```

```
function SINKHORN-PROJECTION( $\mathbf{K}; \mu_0, \mu$ )  
    // Finds  $\Gamma$  minimizing  $\text{KL}(\Gamma|\mathbf{K})$  subject to  $\Gamma \in \overline{\mathcal{M}}(\mu_0, \mu)$   
     $\mathbf{v}, \mathbf{w} \leftarrow \mathbf{1}$   
    for  $j = 1, 2, 3, \dots$   
         $\mathbf{v} \leftarrow \mathbf{1} \otimes \mathbf{K}(\mathbf{w} \otimes \mu)$   
         $\mathbf{w} \leftarrow \mathbf{1} \otimes \mathbf{K}^\top(\mathbf{v} \otimes \mu_0)$   
    return  $[\mathbf{v}]\mathbf{K}[\mathbf{w}]$ 
```

Algorithm 1: Iteration for finding regularized Gromov-Wasserstein distances. \otimes, \oslash denote elementwise multiplication and division.

Convex Relaxation

Tight Relaxation of Quadratic Matching

Itay Kezurer[†]

Shahar Z. Kovalsky[†]

Ronen Basri

Yaron Lipman

Weizmann Institute of Science

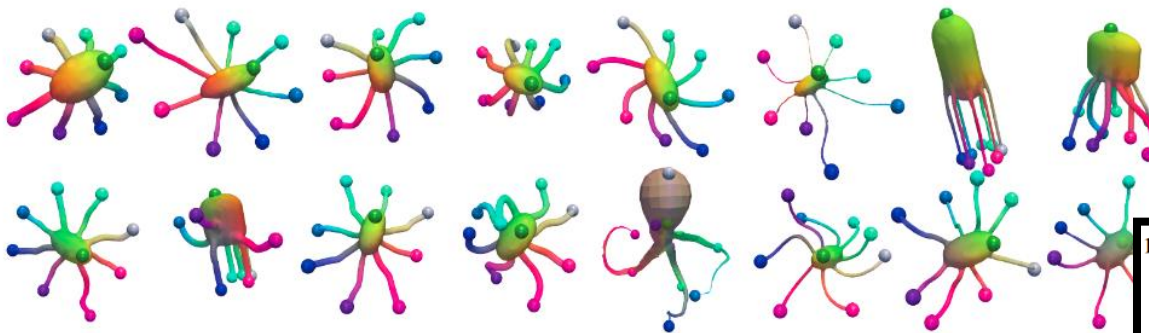


Figure 1: Consistent Collection Matching. Results of the proposed one-stage procedure for finding consistent correspondences between shapes in a collection showing strong variability and non-rigid deformations.

Abstract

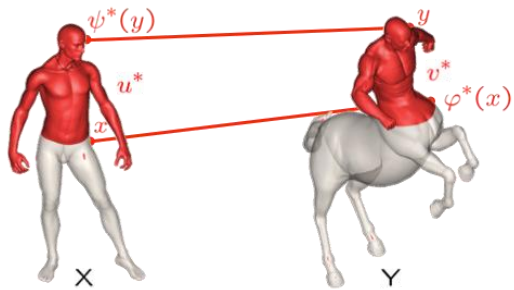
Establishing point correspondences between shapes is extremely challenging as it involves both finding semantically persistent feature points, as well as their combinatorial matching. We focus on the latter and consider the Quadratic Assignment Matching (QAM) model. We suggest a novel convex relaxation for this NP-hard problem that builds upon a rank-one reformulation of the problem in a higher dimension, followed by relaxation in semidefinite program (SDP). Our method is shown to be a certain hybrid of the popular spectral and do stochastic relaxations of QAM and in particular we prove that it is tighter than both. Experimental evaluation shows that the proposed relaxation is extremely tight: in the majority of our experiments it achieved the certified global optimum solution for the problem, while other relaxations tend to produce sub-optimal solutions. This, however, comes at the price of solving an SDP in a higher dimension.

$$\begin{aligned} \max_Y \quad & \text{tr}(WY) \\ \text{s.t.} \quad & Y \succeq [X][X]^T \\ & X \in \text{conv} \Pi_n^k \\ & \text{tr}Y = k \\ & Y \geq 0 \\ & \sum_{qrst} Y_{qrst} = k^2 \\ & Y_{qrst} \leq \begin{cases} 0, & \text{if } q = s, r \neq t \\ 0, & \text{if } r = t, q \neq s \\ \min\{X_{qr}, X_{st}\}, & \text{otherwise} \end{cases} \end{aligned}$$

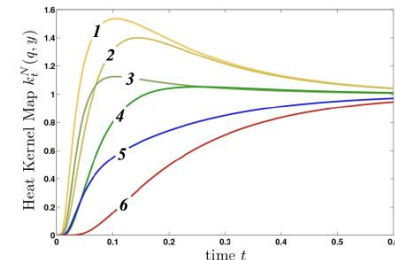
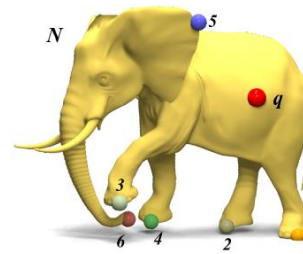
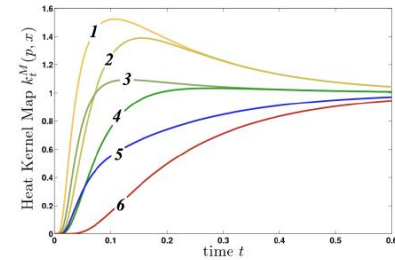
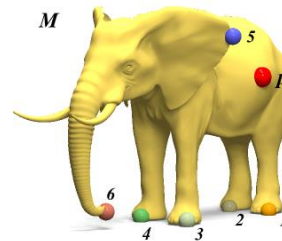
Continuum

Weak assumptions

Strong assumptions



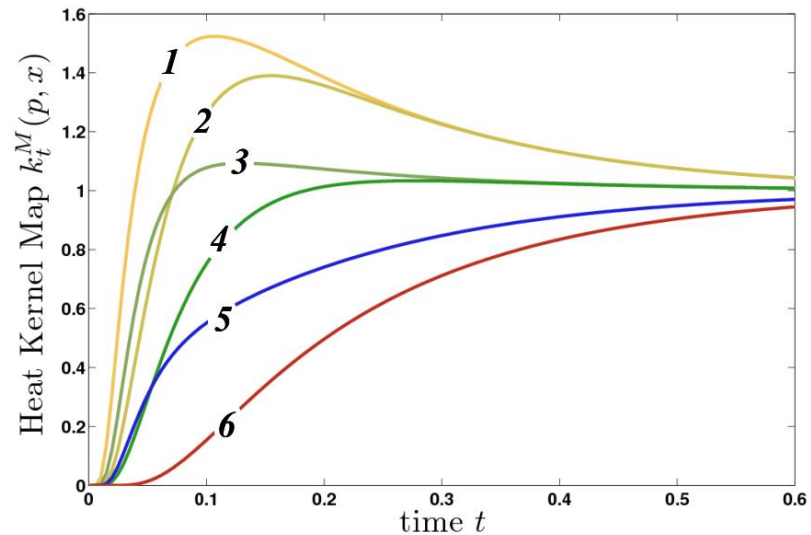
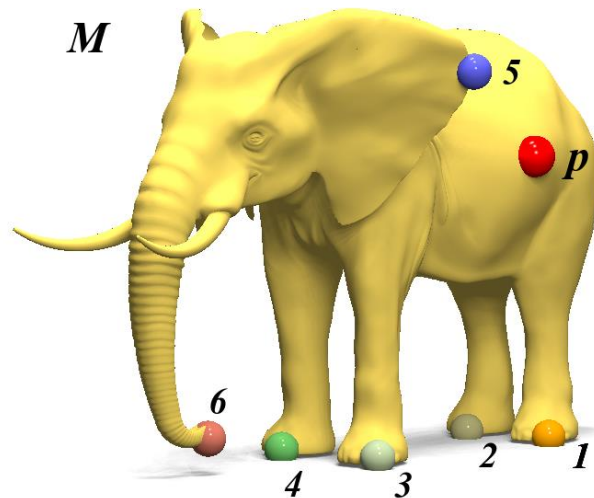
Low-distortion



Isometry

Recall:

Heat Kernel Map



$$\text{HKM}_p(x, t) := k_t(p, x)$$

Theorem: Only have to match one point!

One Point Isometric Matching with the Heat Kernel

Ovsjanikov et al. 2010

KNN

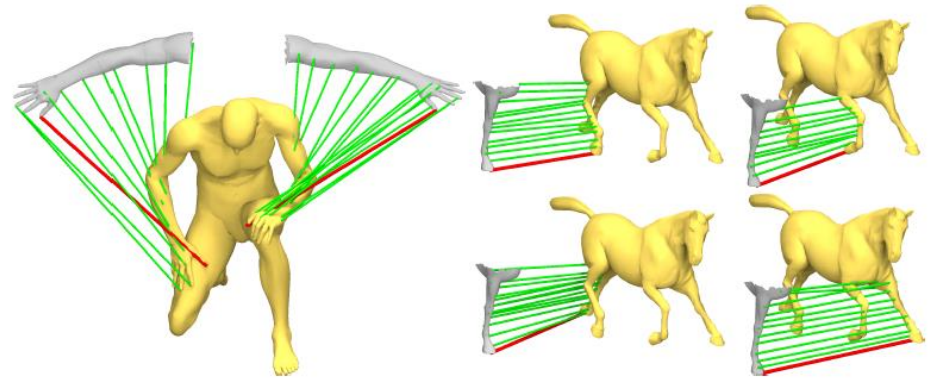
Tradeoff: Heat Kernel Map

- **Pros:**

- Tiny search space
- Some extension to partial matching

- **Cons:**

- (Extremely) sensitive to deviation from isometry

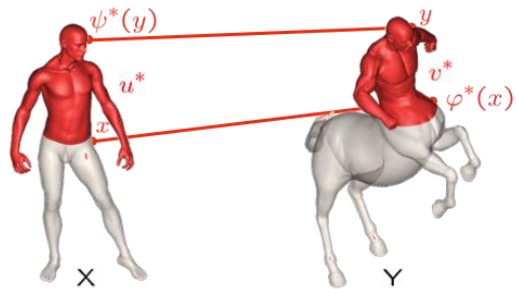


Continuum

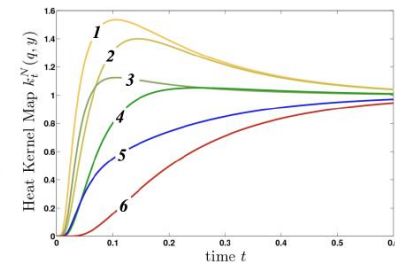
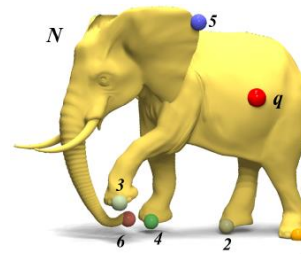
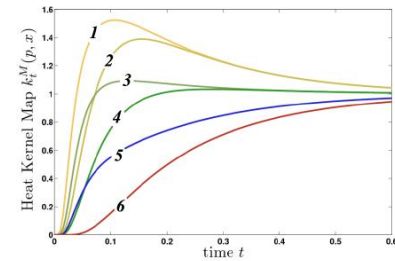
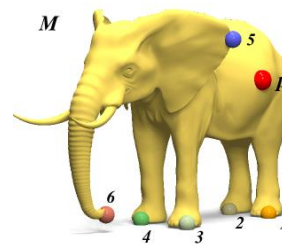
????

Weak assumptions

Strong assumptions



Low-distortion



Isometry

Observation About Mapping

Angle and area preserving

Angle preserving

isometries \subseteq conformal maps

Hard!

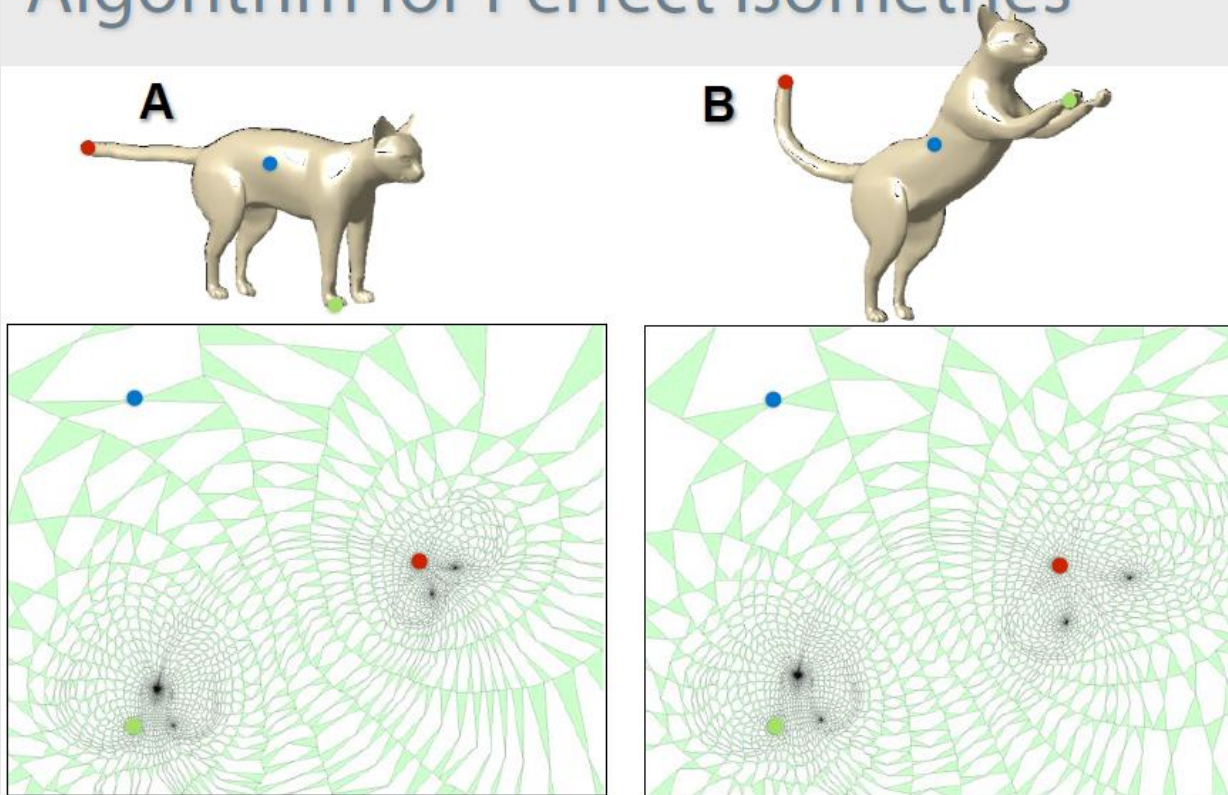
Easier

Möbius Voting for Surface Correspondence

Lipman and Funkhouser 2009

$O(n^3)$ Algorithm for Perfect Isometry

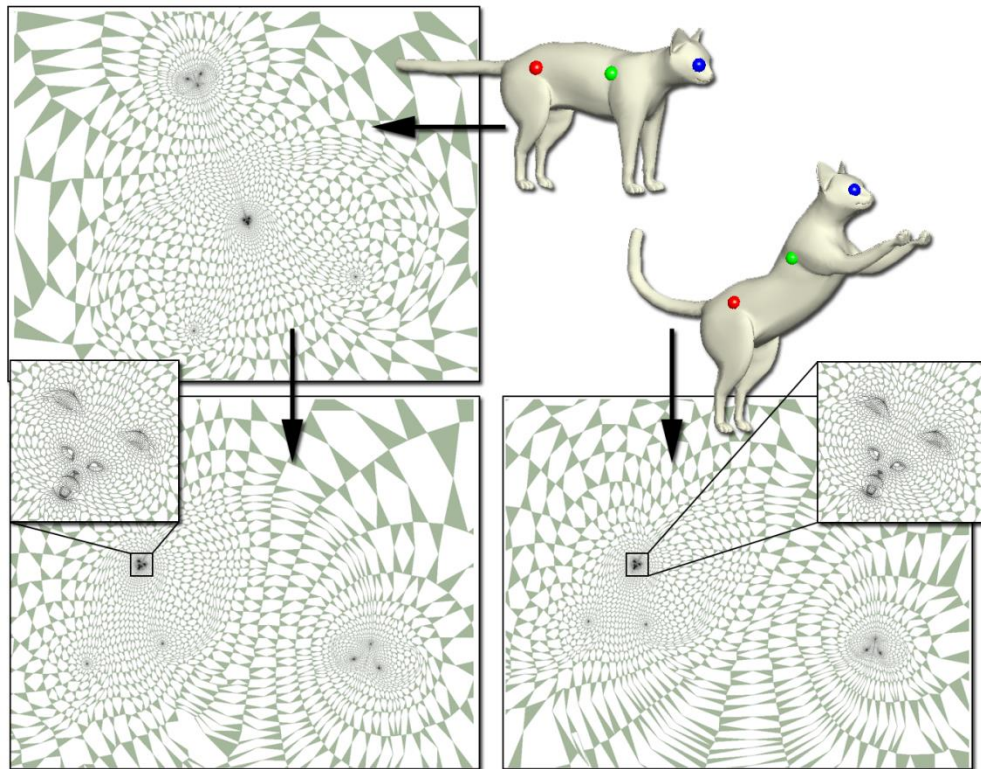
Algorithm for Perfect Isometries



http://www.mpi-inf.mpg.de/resources/deformableShapeMatching/EG2011_Tutorial/slides/4.3%20SymmetryApplications.pdf

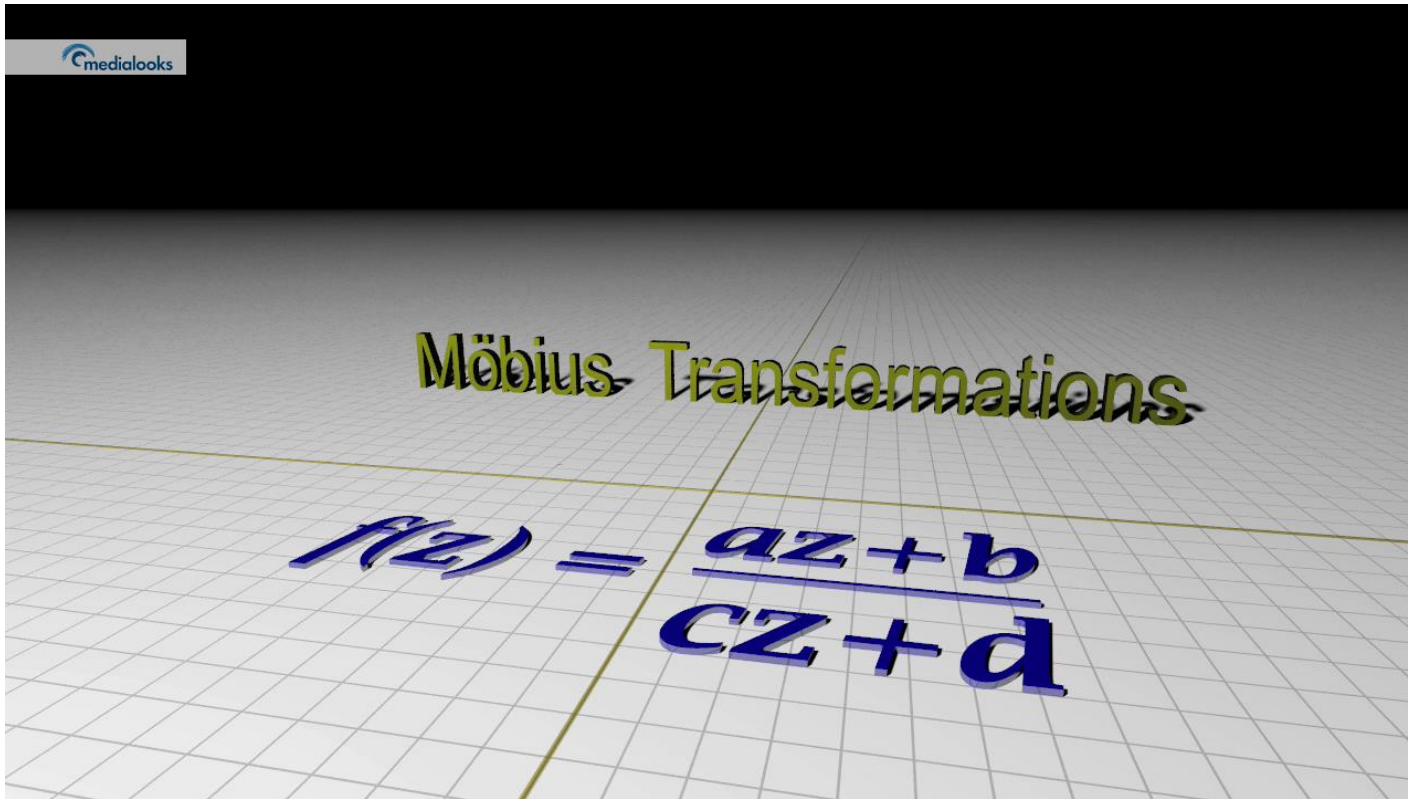
Map triplets of points

Möbius Voting



1. Map surfaces to **complex plane**
2. Select **three** points
3. **Map plane to itself** matching these points
4. **Vote** for pairings using distortion metric to weight
5. Return to 2

Möbius Transformations

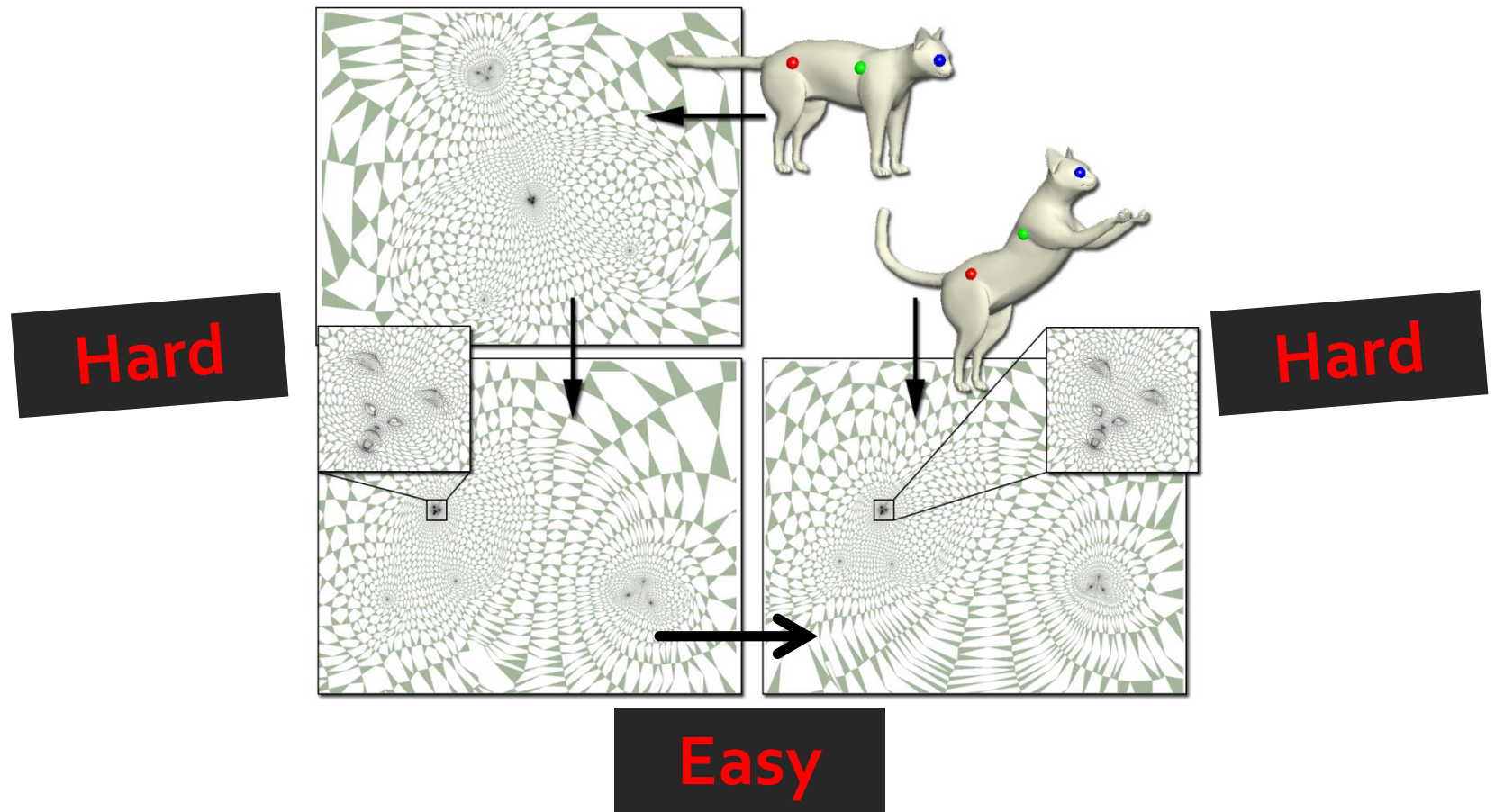


$$\frac{az + b}{cz + d}$$

<http://www.ima.umn.edu/~arnold/moebius>

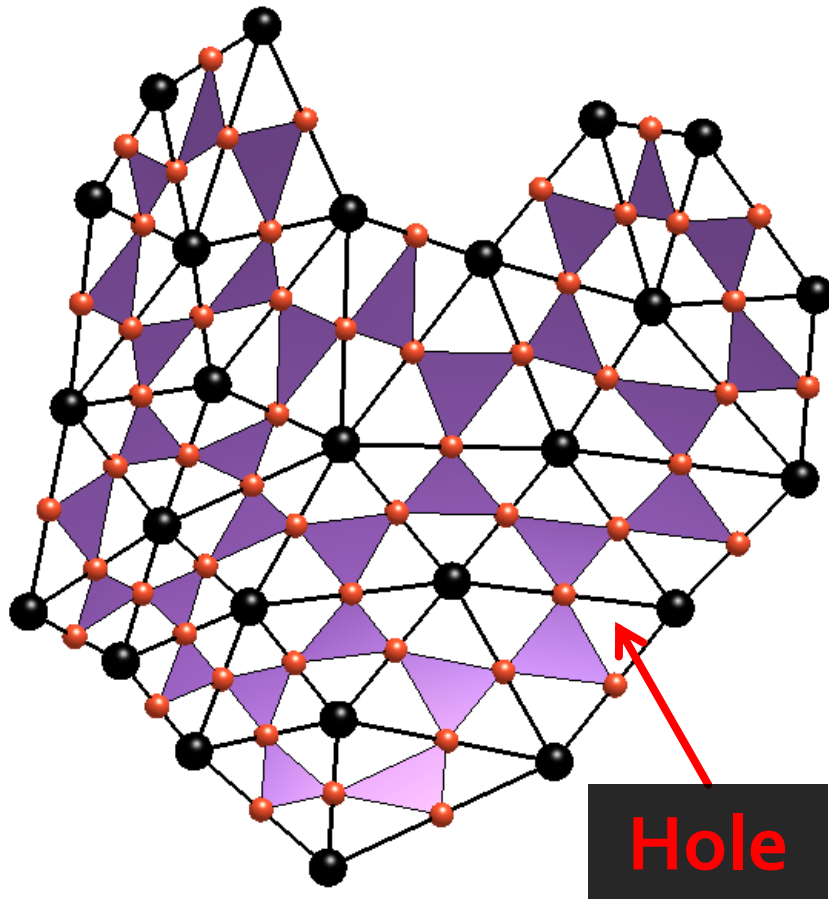
**Bijjective conformal maps of the
extended complex plane**

Observation



Hard work is per-surface, not per-map

Mid-Edge Flattening



$$\Phi(v) = u(v) + iu^*(v)$$

PL,
continuous

$$\Delta u = 0$$

PL,
continuous
at midpoints

Rotate gradient
of u 90°

Cannot scale triangles to flatten

Voting Algorithm

```
Input: points  $\Sigma_1 = \{z_k\}$  and  $\Sigma_2 = \{w_\ell\}$ 
       number of iterations  $I$ 
       minimal subset size  $K$ 
Output: correspondence matrix  $C = (C_{k,\ell})$ .

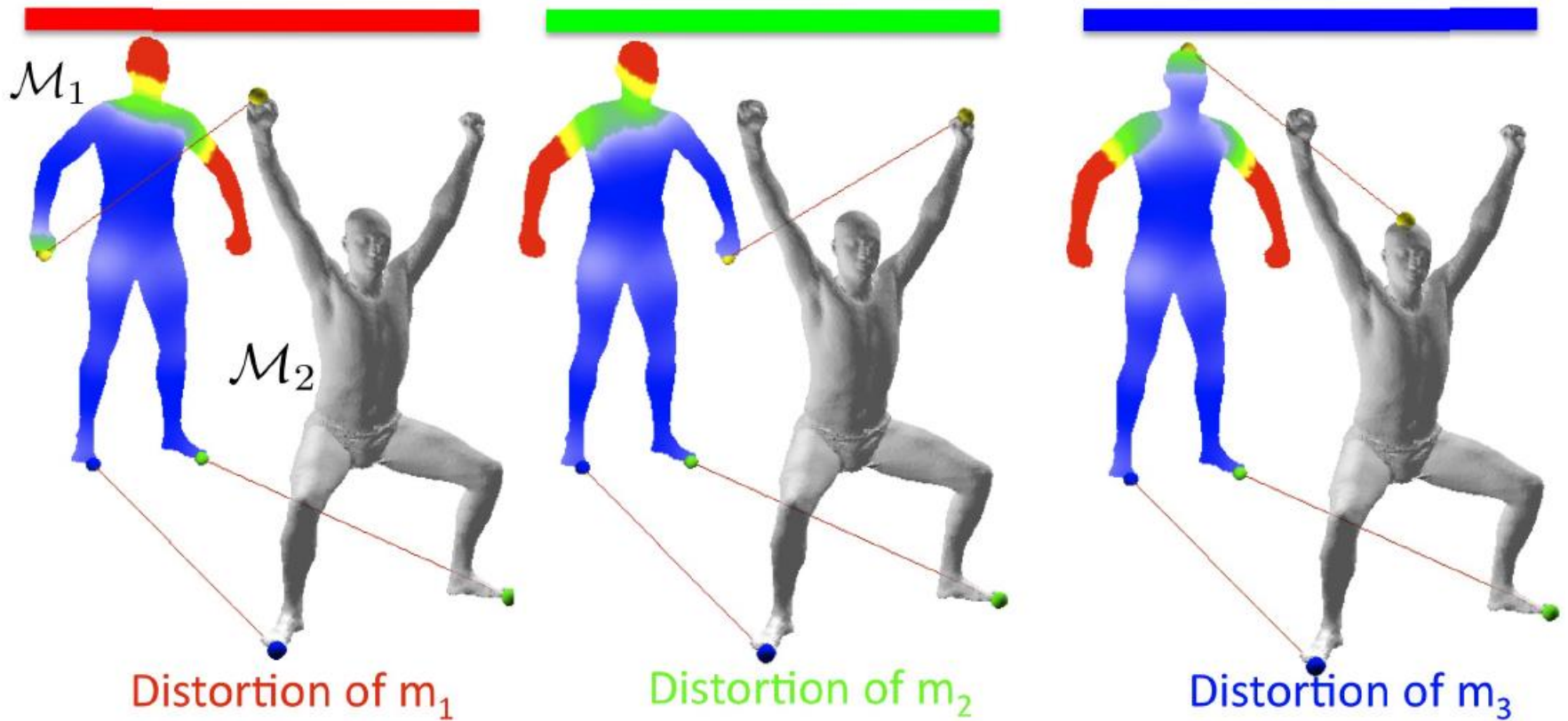
/* Möbius voting */
while number of iterations < I do
    Random  $z_1, z_2, z_3 \in \Sigma_1$ .
    Random  $w_1, w_2, w_3 \in \Sigma_2$ .
    Find the Möbius transformations  $m_1, m_2$  s.t.
         $m_1(z_j) = y_j, m_2(w_j) = y_j, j = 1, 2, 3$ .
    Apply  $m_1$  on  $\Sigma_1$  to get  $\bar{z}_k = m_1(z_k)$ .
    Apply  $m_2$  on  $\Sigma_2$  to get  $\bar{w}_\ell = m_2(w_\ell)$ .

    Find mutually nearest-neighbors  $(\bar{z}_k, \bar{w}_\ell)$  to formulate
    candidate correspondence  $c$ .
    if number of mutually closest pairs  $\geq K$  then
        Calculate the deformation energy  $E(c)$ 
        /* Vote in correspondence matrix */
        foreach  $(\bar{z}_k, \bar{w}_\ell)$  mutually nearest-neighbors do
             $C_{k,\ell} \leftarrow C_{k,\ell} + \frac{1}{\epsilon + E(c)/n}$ .
        end
    end
end
end
```

Tradeoff: Möbius Voting

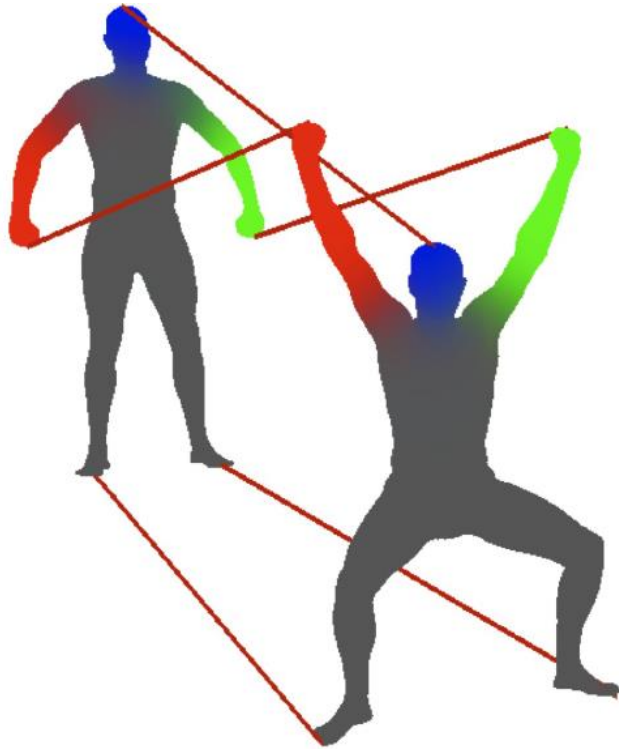
- **Pros:**
 - Efficient
 - Voting procedure handles some non-isometry
- **Cons:**
 - Does not provide smooth/continuous map
 - Does not optimize global distortion
 - Only for genus 0

Blended Intrinsic Maps

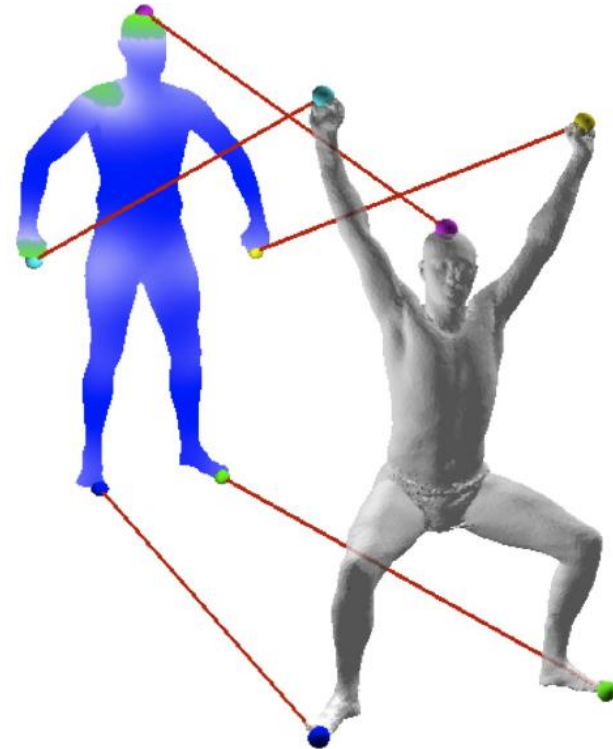


Different conformal maps distorted in different places.

Use for Dense Mapping



Blending Weights for m_1 , m_2 , and m_3



Distortion of the Blended Map

Combine good parts of different maps!

Blended Intrinsic Maps

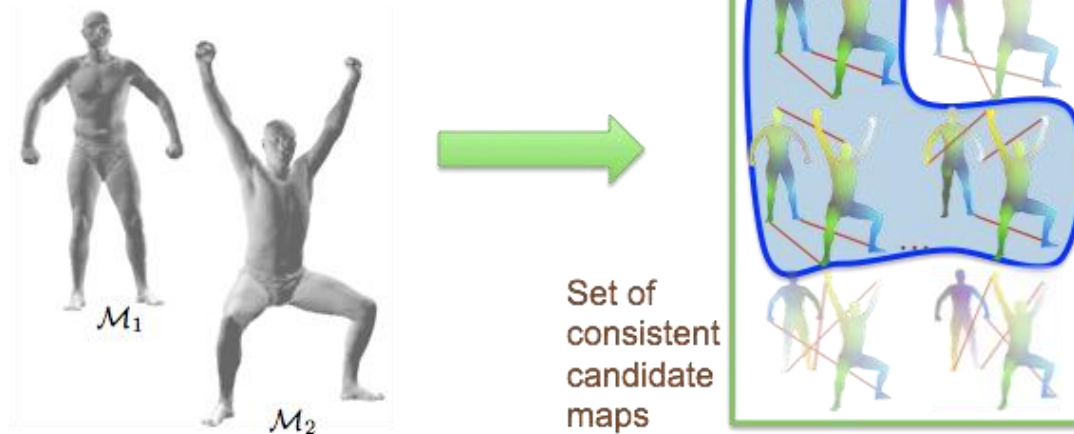
Kim, Lipman, and Funkhouser 2011

Blended Intrinsic Maps

- **Algorithm:**
 - **Generate consistent maps**
 - **Find blending weights per-point on each map**
 - **Blend maps**

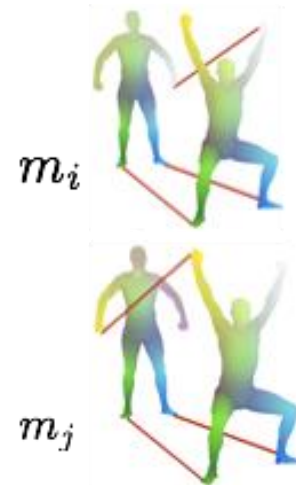
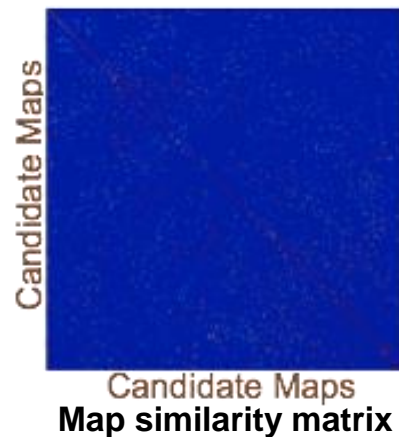
Blended Intrinsic Maps

- **Algorithm:**
 - **Generate consistent maps**
 - Find blending weights per-point on each map
 - Blend maps



Blended Intrinsic Maps

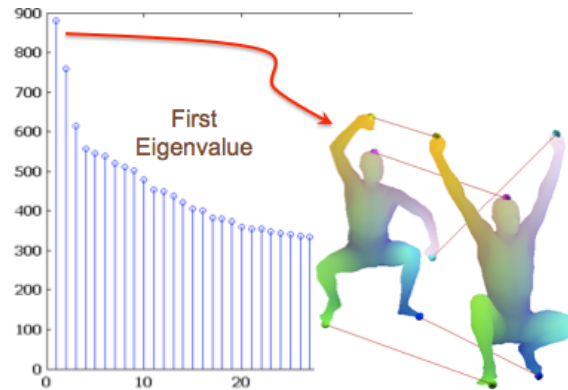
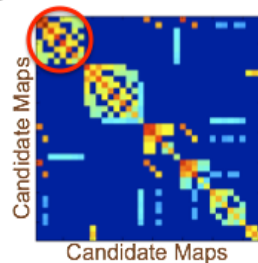
- **Algorithm:**
 - **Generate consistent maps**
 - Find blending weights per-point on each map
 - Blend maps



Blended Intrinsic Maps

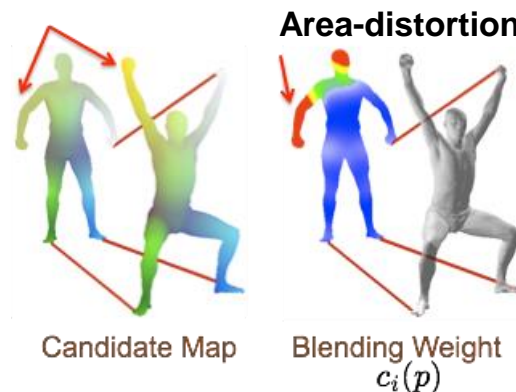
- **Algorithm:**
 - **Generate consistent maps**
 - Find blending weights per-point on each map
 - Blend maps

**Eigen-analysis:
"Block" of similar maps**



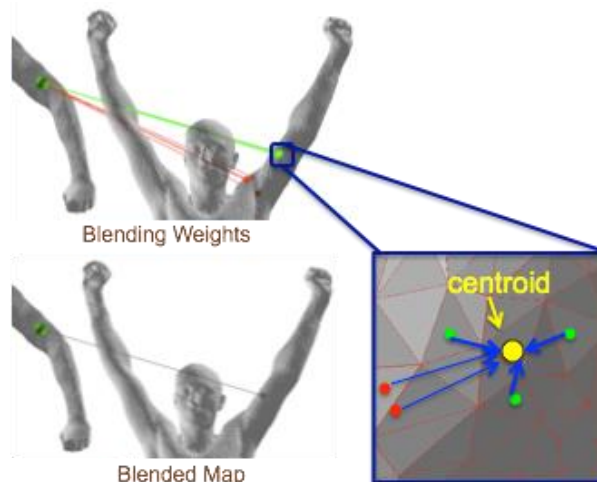
Blended Intrinsic Maps

- **Algorithm:**
 - Generate consistent maps
 - **Find blending weights per-point on each map**
 - Blend maps

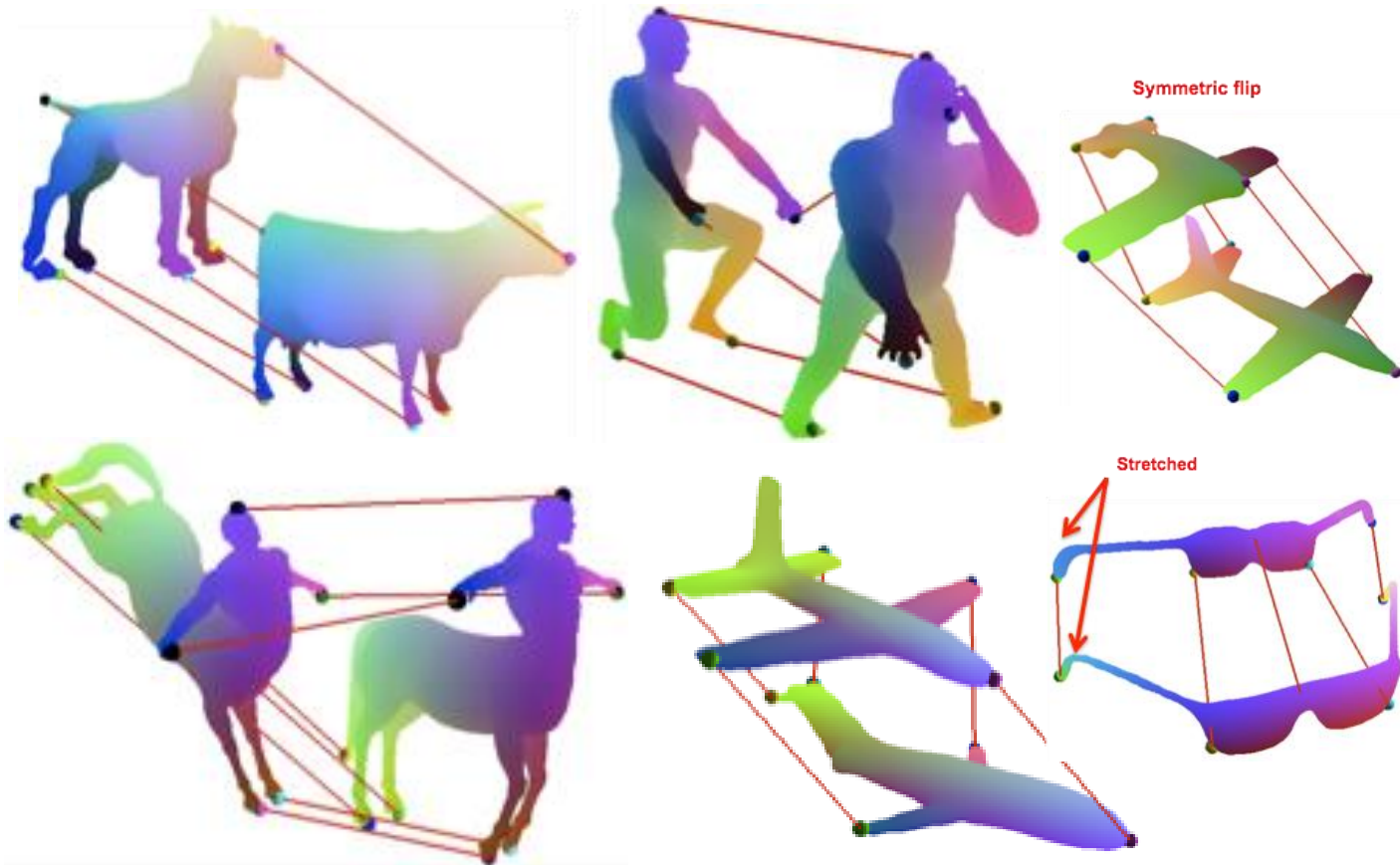


Blended Intrinsic Maps

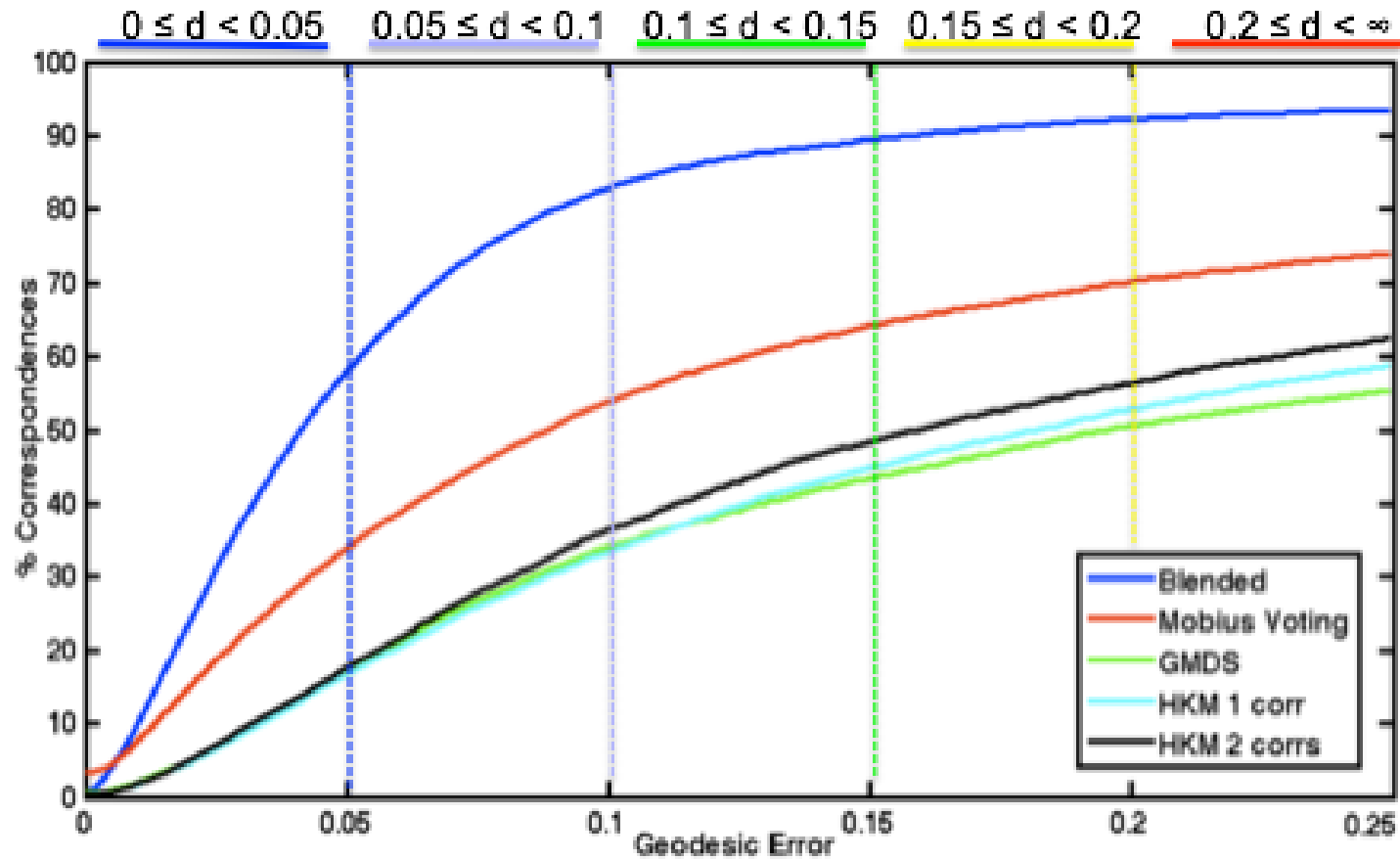
- **Algorithm:**
 - Generate consistent maps
 - Find blending weights per-point on each map
 - **Blend maps**



Some Examples



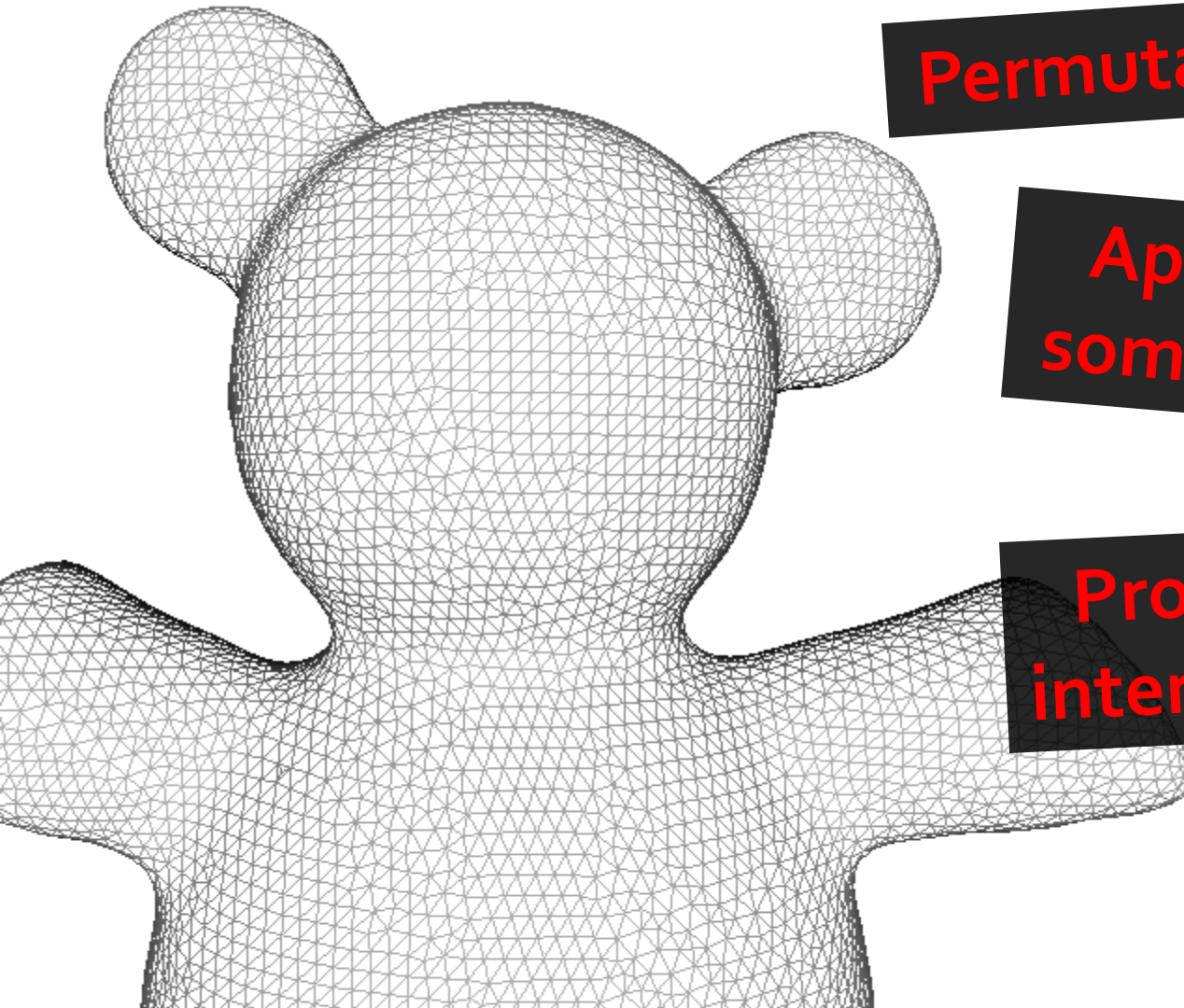
Evaluation



Tradeoff: Blended Intrinsic Maps

- **Pros:**
 - Can handle non-isometric shapes
 - Efficient
- **Cons:**
 - Lots of area distortion for some shapes
 - Genus 0 manifold surfaces

Subtlety: Representation



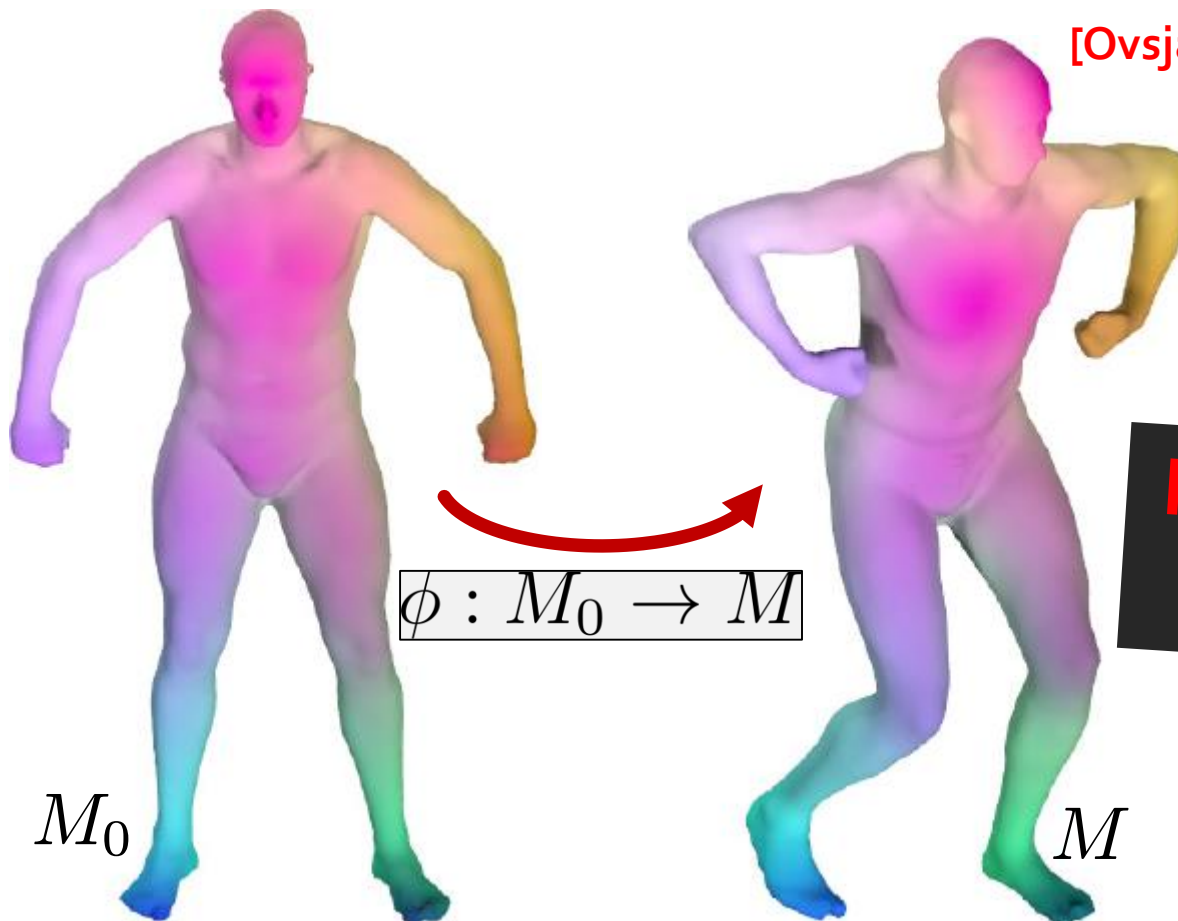
Permutation?

Approximation of something smooth?

Probabilistic interpretation?

Invertibility?

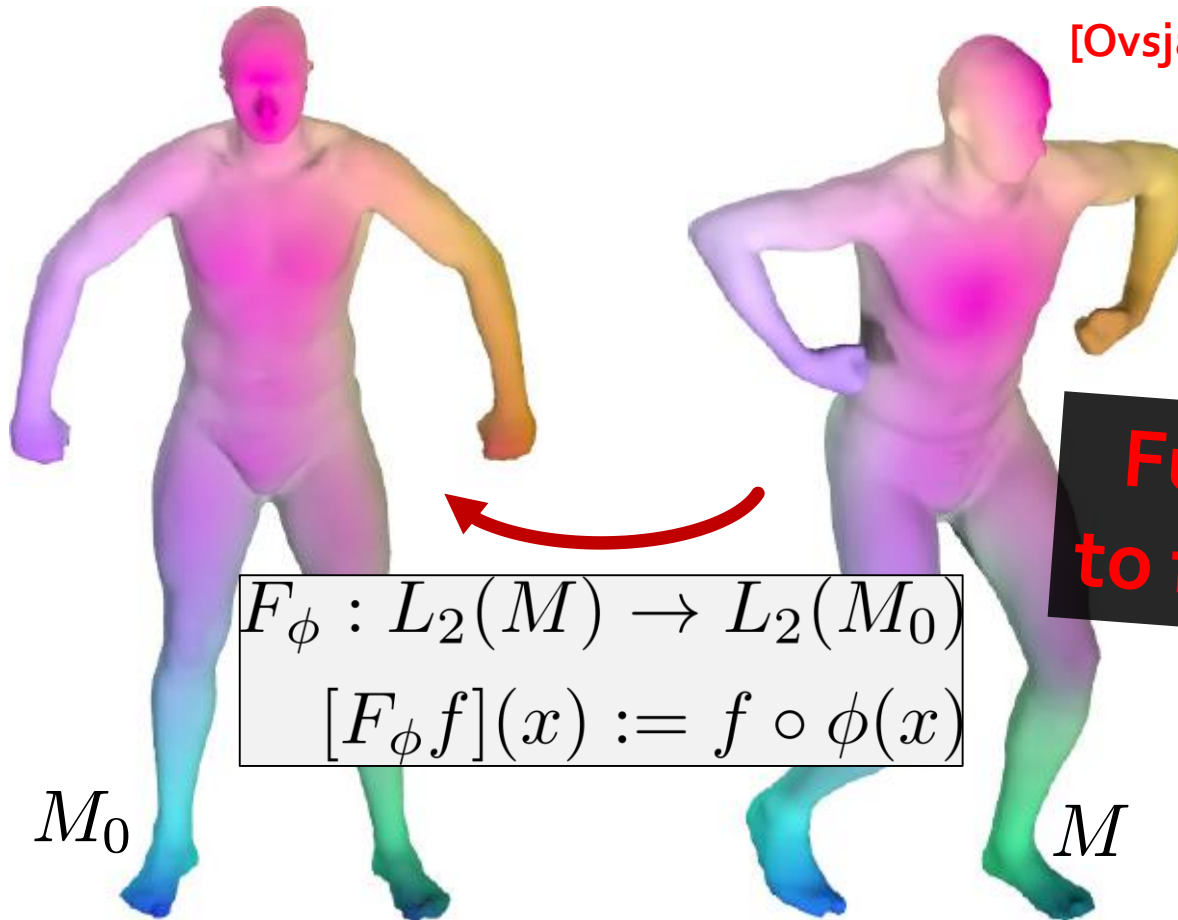
Functional Maps



Points on M_0 to points on M

Functional Maps

[Ovsjanikov et al. 2012]

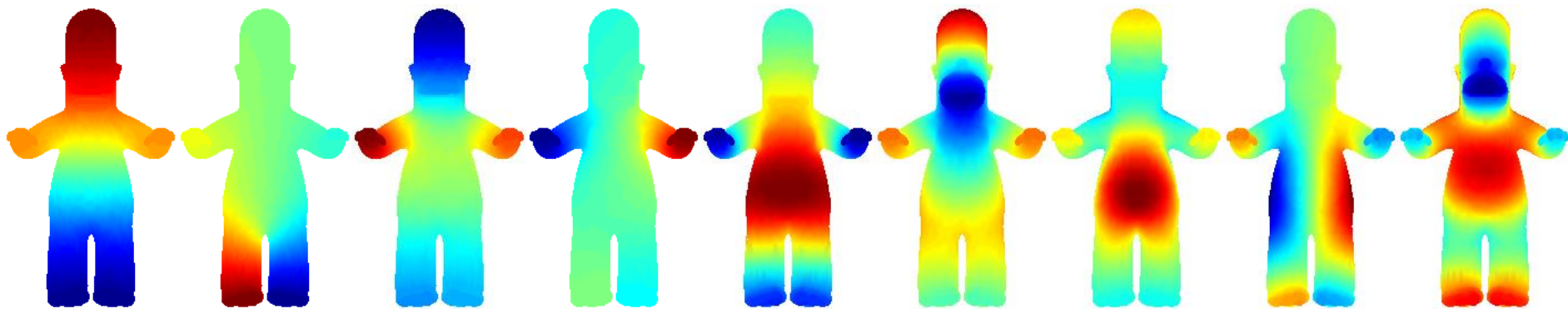


Functions
to functions

Functions on M to functions on M_0

Functional Maps

[Ovsjanikov et al. 2012]

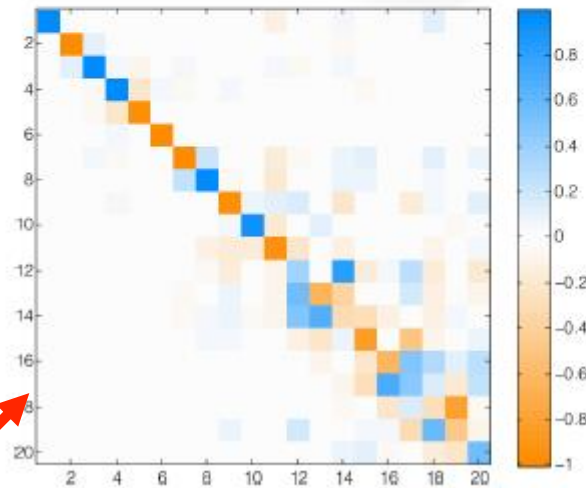


$$f(x) = \sum_i a_i \psi_i(x)$$

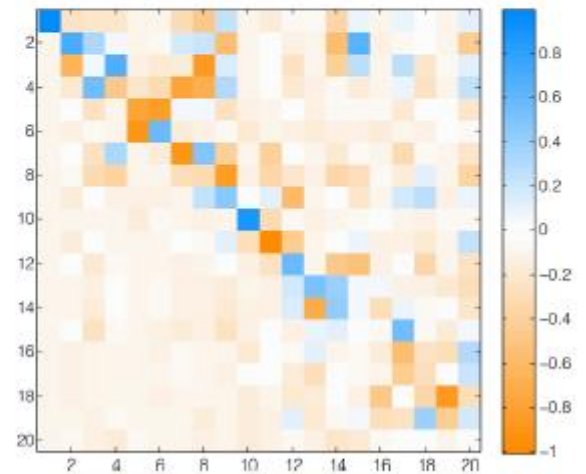
Functional map:

Matrix taking Laplace-Beltrami (Fourier) coefficients on M to coefficients on M_0

Example Maps



(c) *left to right map*



(d) *head to tail map*

**Nearly identity:
Not a mistake!**



Functional Maps

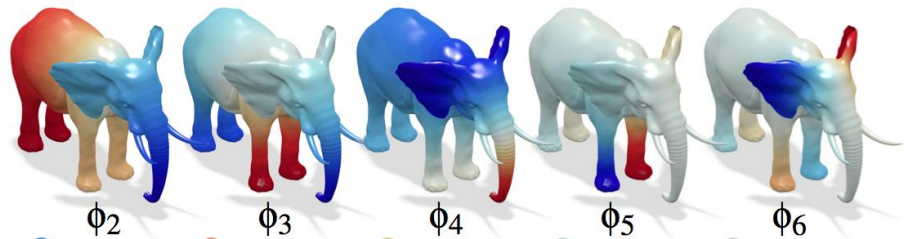
- **Simple Algorithm**
 - Compute some geometric functions to be preserved: A, B
 - Solve in least-squares sense for C : $B = C A$
- **Additional Considerations**
 - Favor commutativity
 - Favor orthonormality (if shapes are isometric)
 - Efficiently getting point-to-point correspondences

Tradeoff: Functional Maps

- **Pros:**
 - Condensed representation
 - Linear
 - Alternative perspective on mapping
 - **Many** recent papers with variations
- **Cons:**
 - Hard to handle non-isometry
Some progress in last few years!

Example extension:

Coupled Quasi-Harmonic Basis

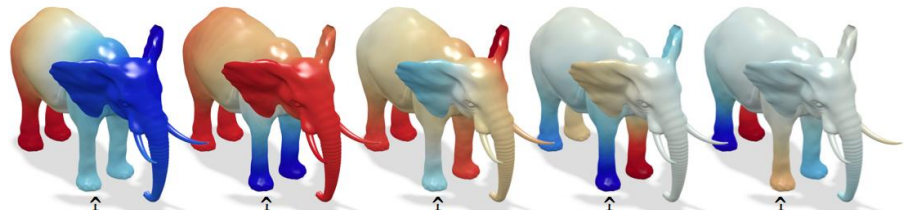


ϕ_2 ϕ_3 ϕ_4 ϕ_5 ϕ_6



ψ_2 ψ_3 ψ_4 ψ_5 ψ_6

Laplacian eigenbases



$\hat{\phi}_2$ $\hat{\phi}_3$ $\hat{\phi}_4$ $\hat{\phi}_5$ $\hat{\phi}_6$



$\hat{\psi}_2$ $\hat{\psi}_3$ $\hat{\psi}_4$ $\hat{\psi}_5$ $\hat{\psi}_6$

Coupled quasi-harmonic bases

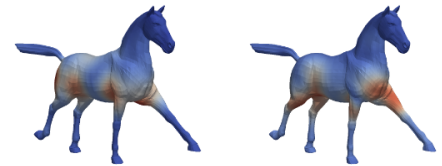
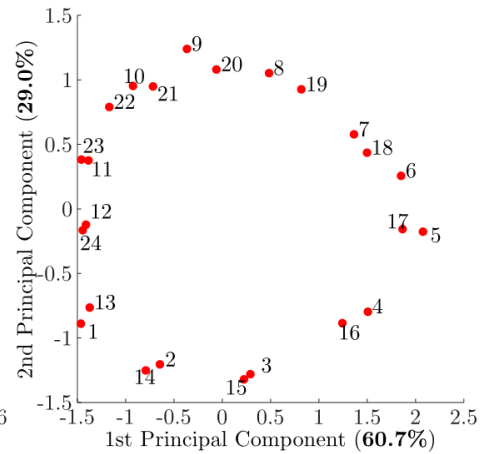
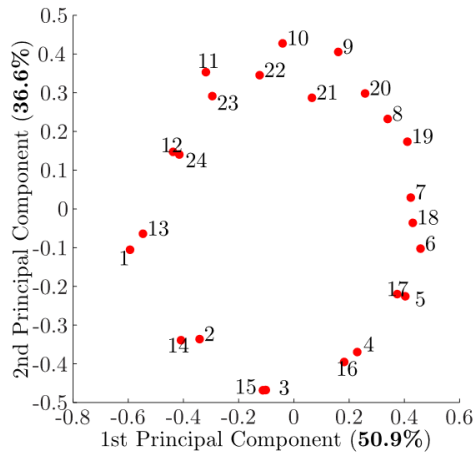
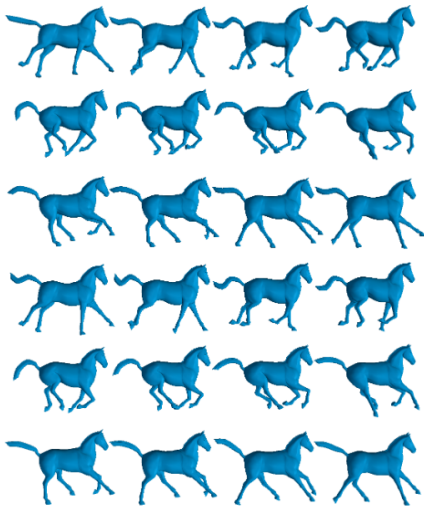
Example extension:

Leverage Symmetry



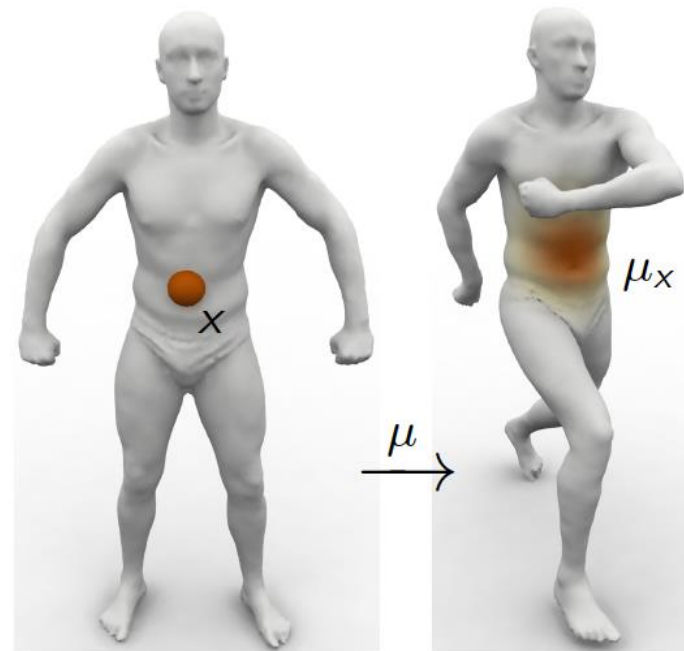
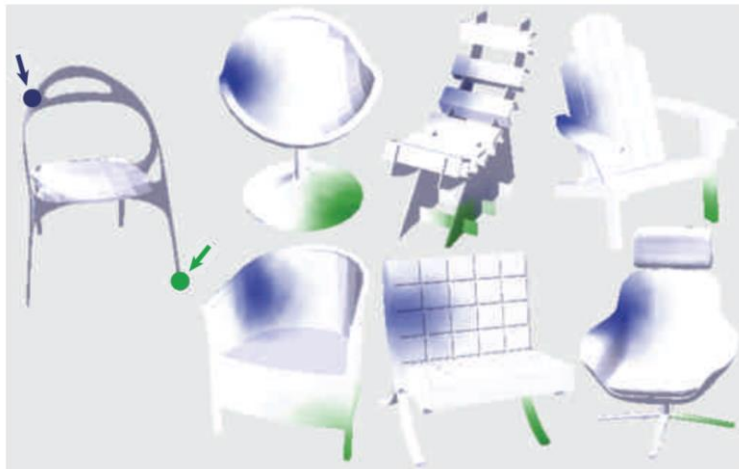
Example extension:

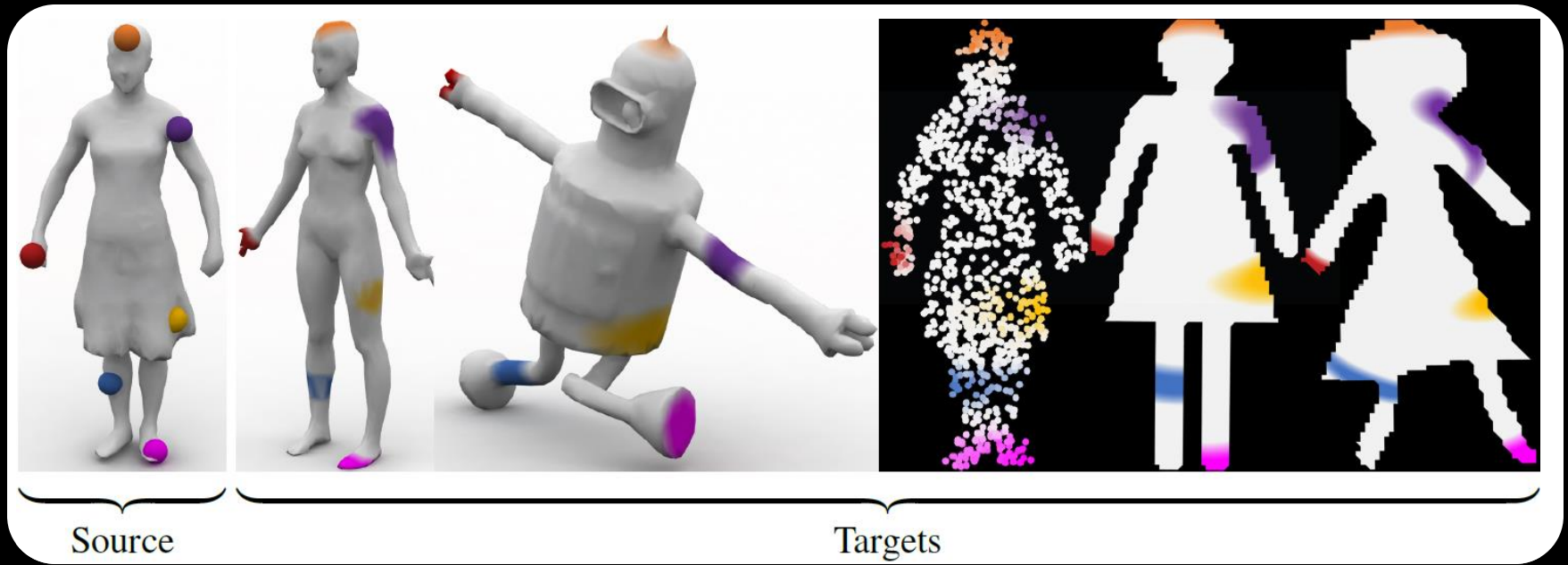
Analyze Deformation



Related method:

Soft/Fuzzy Maps





Surface Correspondence

Justin Solomon
MIT, Spring 2017

