Coordinating Control and Planning for Navigation on Simplicial Complex Attractors

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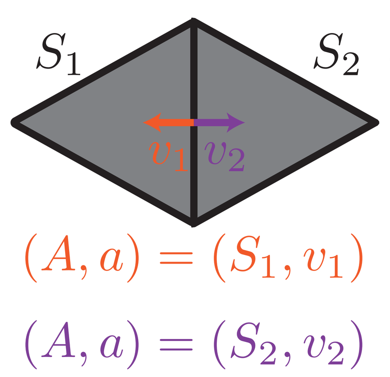
Introduction

Animals and humans navigate complex spaces—both physical and conceptual—using internal cognitive maps, thought to be encoded as neural manifolds [1]. However, **how the brain represents and navigates manifolds with complex geometry** remains unclear.

We propose that neural circuits approximate these manifolds by **stitching together piecewise linear patches**, known as **simplicial complexes (SCs)**. A local feedback **control mechanism enables navigation within a single patch**, while a global **hierarchical planning system computes efficient paths between patches**, supporting flexible navigation.

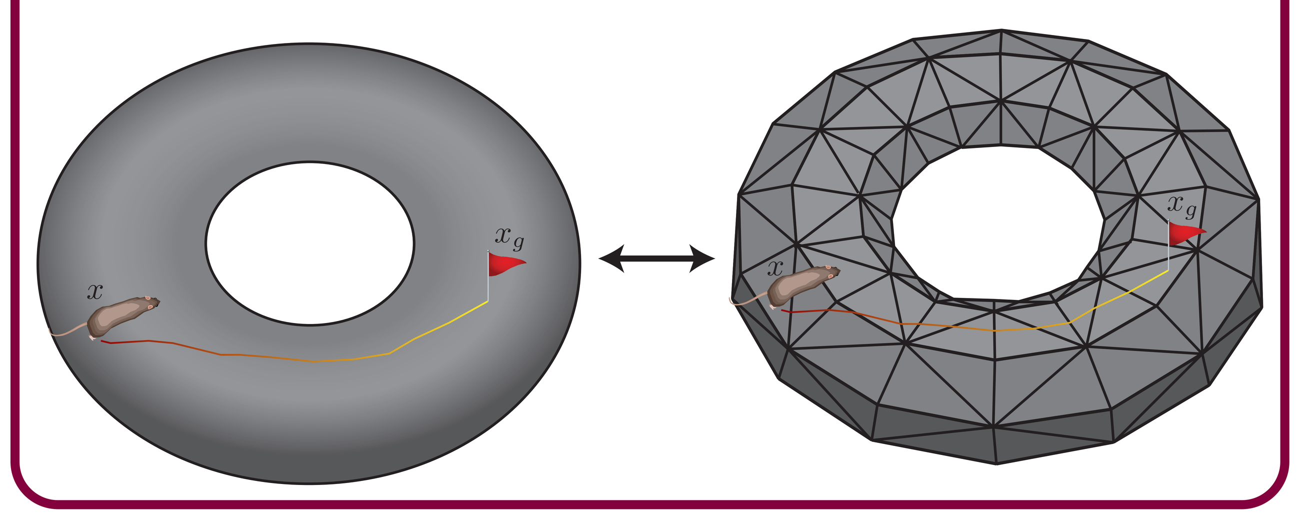
Navigation on simplicial complexes

An n-**simplicial complex (SC)** S is a collection Σ of m-simplexes, where $m \le n$, **glued together** in such a way that **the intersection of any two simplexes is also a simplex**.

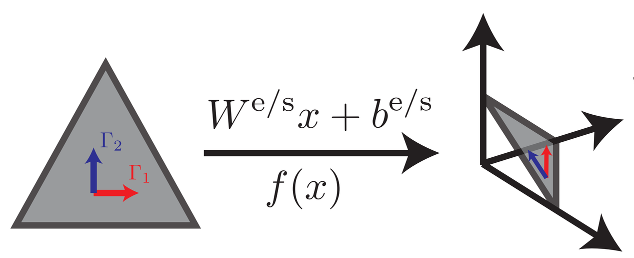


Given a basis $\Gamma^\sigma \in \mathbb{R}^{\dim(\sigma)\times\dim(\sigma)}$ for each simplex $\sigma \in \Sigma$, the action dynamics on the SC are described as: $\tau_e \dot{x} = \sum_{\sigma\in\Sigma} \mathbf{1}_{x\in\sigma,\sigma=A}\Gamma^{(\sigma)} a_{1:\dim(\sigma)}$, where $a : \mathbb{R}^n \times \mathbb{R}$ and $A : \Sigma \times \mathbb{R}$ represent the time-varying actions and the simplex to act on.

The objective of navigation is to find actions $(A(x,x_g), a(x,x_g))$ such that, for a given goal $x_g \in S$, x converges to x_g efficiently.



Learning to represent a simplex attractor



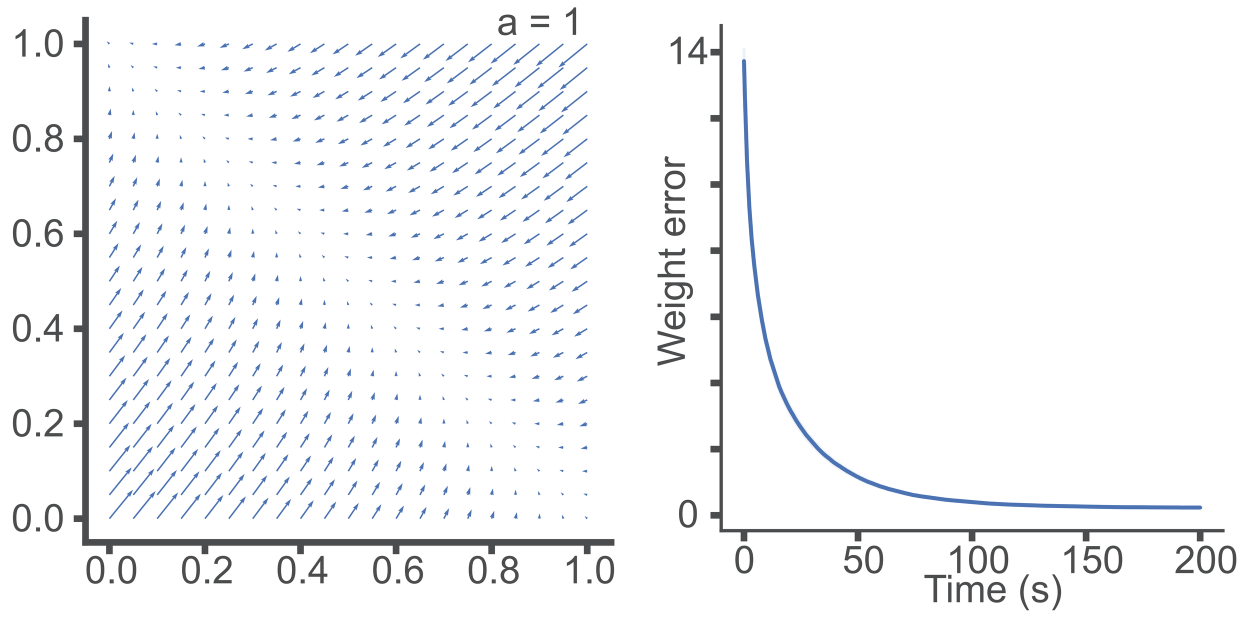
Consider an n-simplex Δ_n with action basis Γ and a linear input function $f(x) = W^{e/s}x + b^{e/s}$.

State neurons y^s represent the simplex. It receives inputs from motor neurons y^m and its dynamic is:

$$\tau_n \dot{y}^s = -y^s + \left[W^s y^s + W^{m/s} y^m + b^s\right]_+ .$$

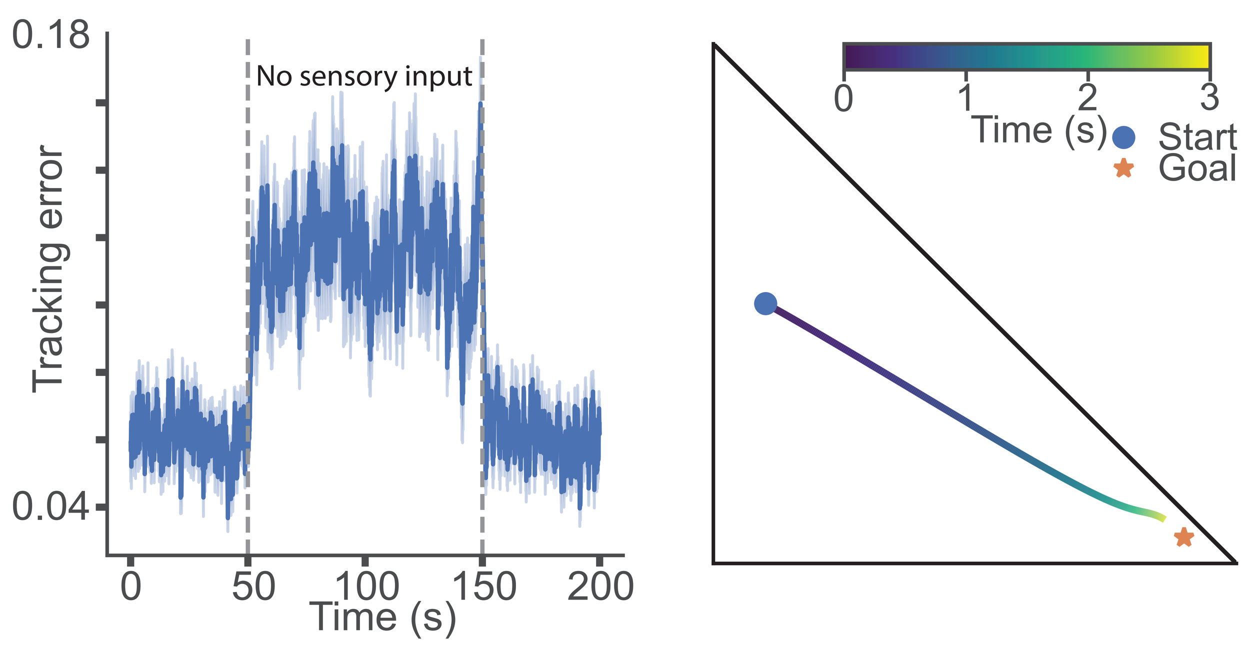
We can prove the following theorem:

Let $W^s = -(b^s - \hat{b}^{e/s})(\hat{b}^{e/s})^\dagger + W^{e/s}(W^{e/s})^\dagger$, $W^{m/s} = \frac{\tau_n}{\tau_e} W^{e/s}\Gamma$, and $a = y^m$. Here, $\hat{b}^{e/s} = (I - W^{e/s}(W^{e/s})^\dagger)b^{e/s}$. Then, the **dynamics of y^s corresponds to the pushforward of dynamics of the simplex** $f(\Delta_n)$. When $y^m = 0$, y^s in particular forms an **attractor of the simplex** $f(\Delta_n)$.



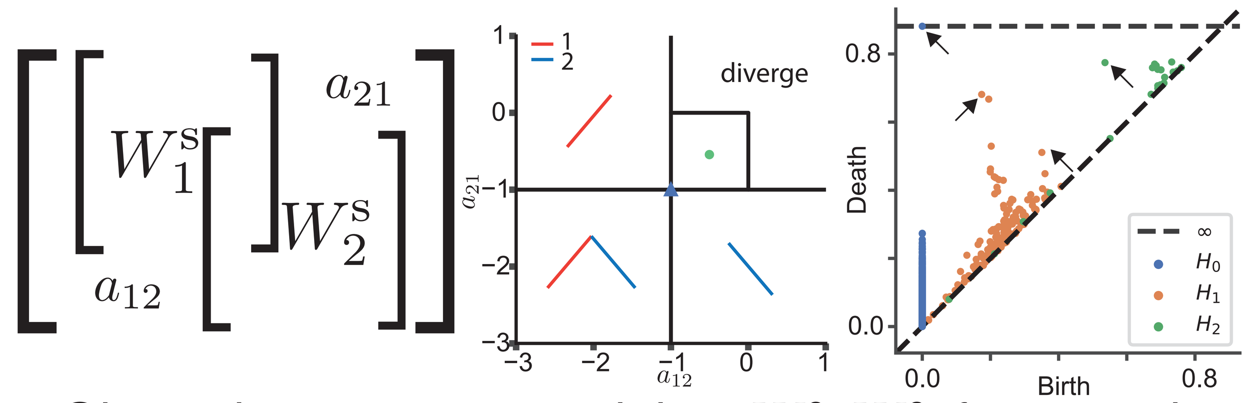
To learn W^s, we show that the **following learning rule can exponentially converge to the desired attractor weight**: $\tau_w \dot{W}^s = -e^s(e^{e/s})^\top + e^{e/s}(e^{e/s})^\top - \kappa W^s$ where $e^s, e^{e/s}$ is the eligibility trace of $y^s, f(x)$.

Navigation on a simplex via control



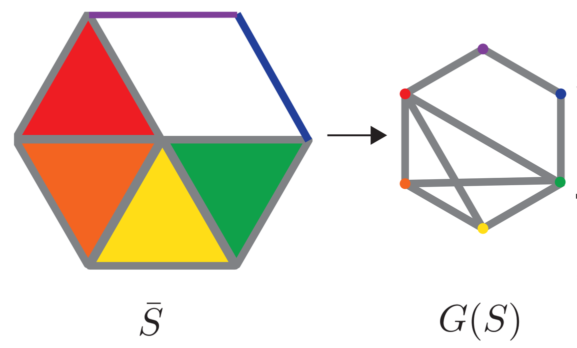
Motor neurons y^m receive inputs from both state neurons y^s and goal neurons y^g and their connections can be learned such that $y^m = \tau_e(W^{e/s}\Gamma)^\dagger(y^g - y^s)$. In this case, the **dynamics of simplex follow** $\tau_e \dot{x} = \tau_e(x_g - x)$, **driving** x **exponentially fast toward** x_g.

Stitching simplex together to form SC



- Given the recurrent weights W_1^s, W_2^s for two simplexes, if they **share neurons with coincident connections**, we can form an union weight $W_{1\cup 2}^s$.
- If the **mutual inhibition between simplexes are large, then one can stitch them together**
- We represent a torus by stitching triangles together. The persistent topology of the neural data reveals **one connected component** (H_0), **two loops** (H_1), and **one void** (H_2), **capturing toroidal topology**.

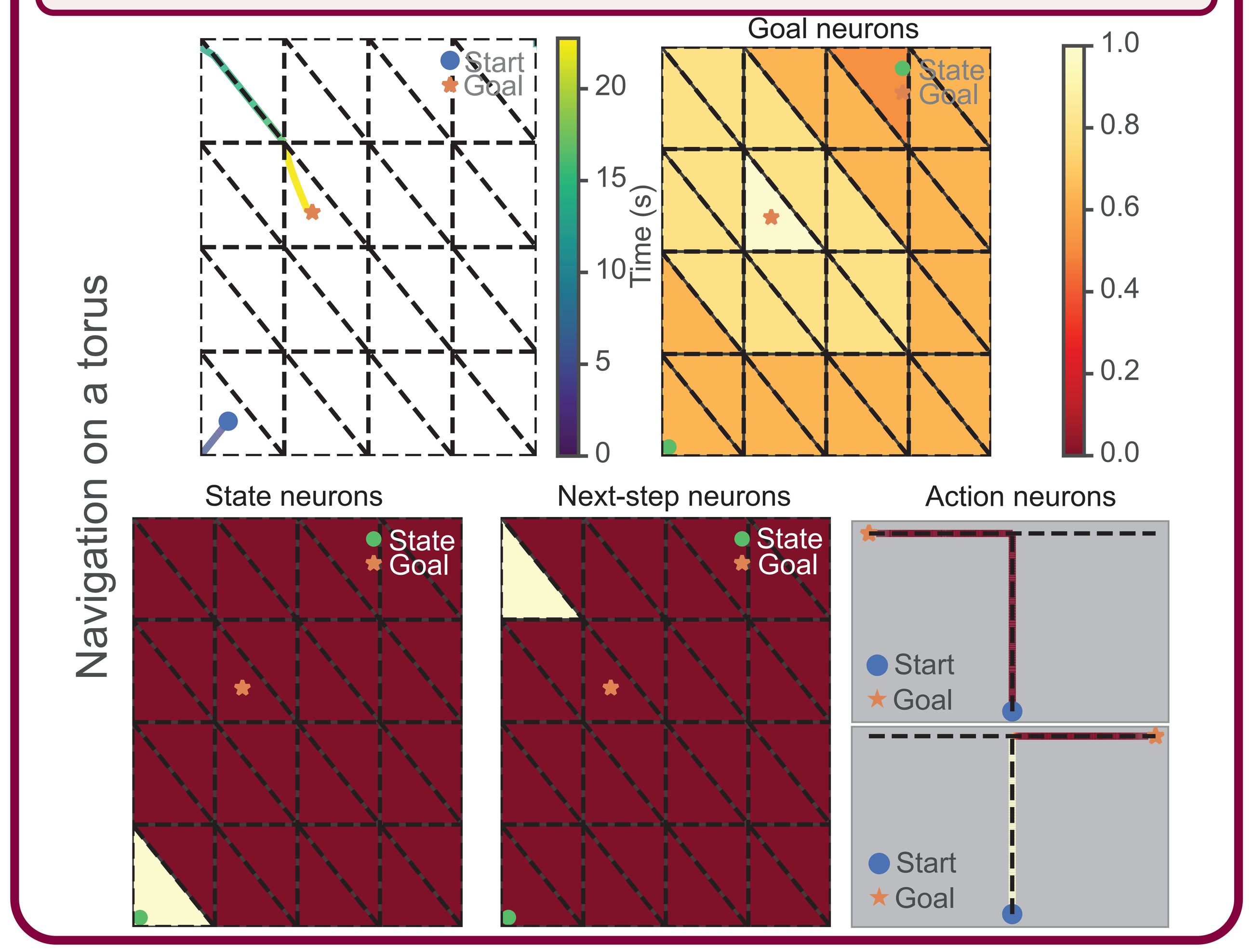
Planning to navigate across simplex



For planning, we treat the **simplicial complex as a hypergraph, where each vertex represents a simplex**, and an edge exists if two simplexes share a vertex

High-level logic of the circuit:

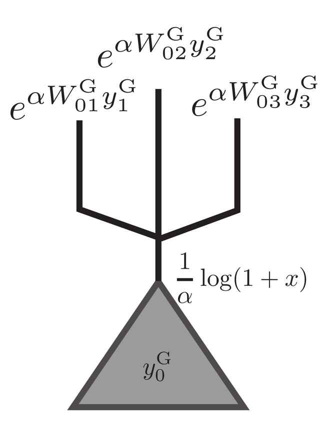
- **State neurons** y^S: receive input from SC attractors and indicate the current vertex.
- **Goal neurons** y^G: computes the shortest path so that $y_i^G = \gamma^{\text{dist}(i,G)}$ via recurrent dynamics.
- **Next-step neurons** y^N: receive inputs from y^G and parent vertexes in y^S and select the closest neighboring vertex to the goal via WTA.
- **Action neurons** y^A: integrate y^N, y^S to select the action leading to the next vertex in y^N.



Dynamics to compute the shortest path

To compute the shortest path, the goal neurons learn the adjacency matrix of $G(S)$ as its weight W^G, receive one-hot goal inputs I^G and evolve as follows:

$$\forall i \in G(S),\ \tau_n \dot{y}_i^G = -y_i^G + \gamma \max\left\{W_{ij}^G y_j^G, I_i^G\right\}$$



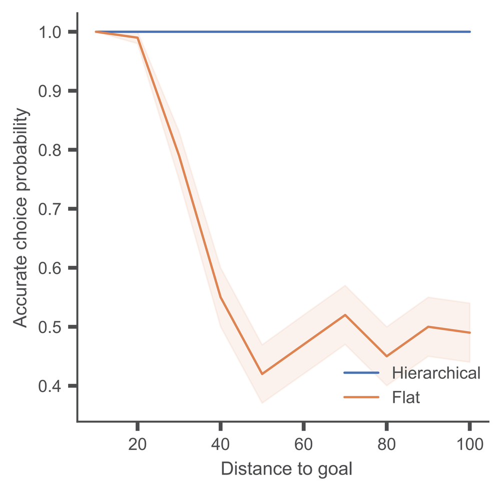
It can be proved that the **dynamic converges to** $y_i^G = \gamma^{\text{dist}(i,G)}$, **analogous to Dijkstra's algorithm**. However, it is unclear if a neuron can compute WTA on synaptic inputs. Therefore, we consider the following approximation where $u_{ij} = \exp\left(\alpha W_{ij}^G y_j^G\right), u_i^I = \exp(\alpha I_i)$:

$$\forall i \in G(S),\ \tau_n \dot{y}_i^G = -y_i^G + \frac{\gamma}{\alpha}\log\left(\sum u_{ij} + u_i^I + 1\right)$$

Here, u_{ij} is dendritic supralinear activation and $\log(1+x)/\alpha$ is somatic nonlinearity. **When α is large, the neuron approximates WTA on synaptic inputs**.

Hierarchical graph navigation via BTSP

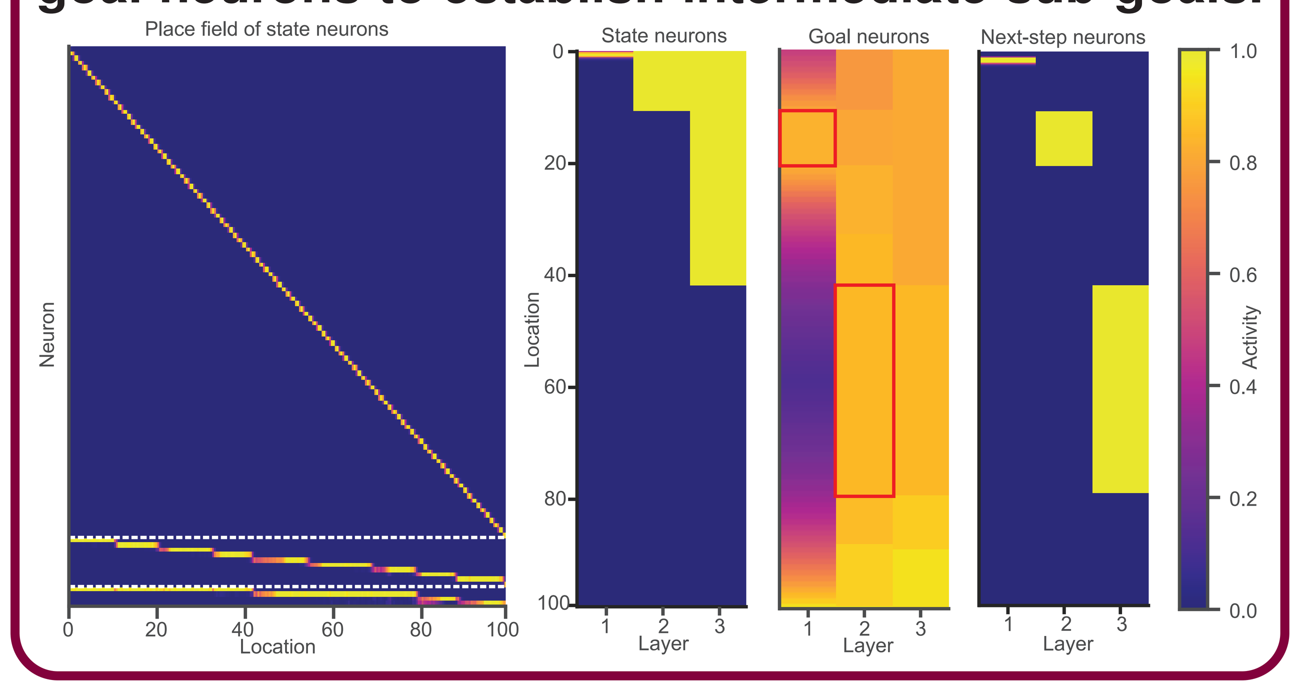
In large graphs, $\gamma^{\text{dist}(i,G)} \approx 0$ for distant nodes, making next-step WTA selection unreliable.



To combat this, we introduce **behavioral timescale plasticity (BTSP)** [2] to learn **hierarchical graphs and sub-goals**, where W^{SS_i} is feedforward weights from layer i, u_{i+1} the synaptic input to layer $i+1$, and e_i the eligibility trace of layer i.

$$\tau_w W^{SS_i} = (2 - W^{SS_i}) \cdot u_{i+1} e_i^\top - W^{SS_i} \cdot u_{i+1} e_i^\top$$

The circuit then functions as follows: the higher-level goal neurons help select **the higher-level next-step**, which **serves as an input to lower-level goal neurons to establish intermediate sub-goals.**



KEY REFERENCES

[1] Niel, E. H. et al. Nature. 2021
[2] Bittner, K. C. et al. Science. 2017

CONTACTS

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Open to collaboration and discussion!