Abstract

Let $T$ be a hamiltonian tournament with $n$ vertices and $\gamma$ a hamiltonian cycle of $T$. In this paper we start the study of the following question: What is the maximum intersection with $\gamma$ of a cycle of length $k$? This number is denoted $f(n, k)$. We prove that for $k$ in the range, $3 \leq k \leq \frac{n+1}{2}$, $f(n, k) \geq k - 3$, and that the result is best possible; in fact, a characterization of the values of $n, k$, for which $f(n, k) = k - 3$ is presented.

In a forthcoming paper we study $f(n, k)$ for the case of cycles of length $k > \frac{n+4}{2}$.

1 Introduction

The subject of pancyclicity in tournaments has been studied by several authors (e.g., [1],[2]). Two types of pancyclicity have been considered. A tournament $T$ is vertex-pancyclic if given any vertex $v$ there are cycles of every length containing $v$. Similarly, a tournament $T$ is arc-pancyclic if given any arc $e$ there are cycles of every length containing $e$. It is well known that a hamiltonian tournament is vertex-pancyclic, but not necessarily arc-pancyclic. In this paper we introduce the concept of cycle-pancyclicity to study questions such as the following. Given a cycle $C$, what is the maximum number of arcs which a cycle of length $k$ contained in $T$ has in common with $C$? Clearly, to study this kind of question it is sufficient to consider a hamiltonian tournament where $C$ is a hamiltonian cycle of $T$.

Let $T$ be a tournament with vertex set $V = \{0, 1, \ldots, n-1\}$ and arc set $A$. Assume without loss of generality that $\gamma = (0, 1, \ldots, n-1, 0)$ is a hamiltonian cycle of $T$. Let $C_k$ denote a directed cycle of length $k$. For a cycle $C_k$ we denote $\mathcal{I}_{\gamma}(C_k) = |A(\gamma) \cap A(C_k)|$, or simply $\mathcal{I}(C_k)$ when $\gamma$ is understood. Let $f(n, k, T) = \max \{\mathcal{I}_{\gamma}(C_k) | C_k \subseteq T\}$ and $f(n, k) = \min \{f(n, k, T) | T$ is a hamiltonian tournament with $n$ vertices$\}$. This paper is the first part of a study of $f(n, k)$;
it is devoted to $k$ in the range $3 \leq k \leq \frac{n+4}{2}$. It is proved that $f(n, k) = k - 3$ if and only if $n \geq 2k - 4$, and $n \not\equiv k \pmod{k-2}$. Also, $f(n, k) \geq k - 2$ if and only if $n \equiv k \pmod{k-2}$.

In a forthcoming paper we study $f(n, k)$ for $k > \frac{n+4}{2}$.

The rest of this paper is organized as follows. In Section 2 some notation and basic results needed in the rest of the paper are introduced. The proof of the main result, i.e., that $f(n, k) \geq k - 3$ for $n \geq 2k - 4$ appears in Section 3, 4, 5 and 6. Sections 3 through 5 contain special cases (for particular values of $n$ and $k$). The general case is left to Section 6, where it is proved that $f(n, k) \geq k - 3$. In Section 7 it is proved that $f(n, k) \leq k - 3$, when $n \not\equiv k \pmod{k-2}$, and that the results are best possible; namely, for $n < 2k - 4$, $f(n, k) < k - 3$. Thus, a characterization is presented of the values of $n, k$ for which $f(n, k) = k - 3$ and for which $f(n, k) = k - 2$.

2 Preliminaries

A chord of a cycle $C$ is an arc not in $C$ with both terminal vertices in $C$. The length of a chord $f = (u, v)$ of $C$, denoted $l(f)$, is equal to the length of $(u, C, v)$, where $(u, C, v)$ denotes the $uv$-directed path contained in $C$. We say that $f$ is a $c$-chord if $l(f) = c$ and $f = (u, v)$ is a $-c$-chord if $l(v, C, u) = c$. Observe that if $f$ is a $c$-chord then it is also a $-(n-c)$-chord.

In what follows all notation is taken modulo $n$.

For any $a$, $2 \leq a \leq n - 2$, denote by $t_a$ the largest integer such that $a + t_a(k - 2) < n - 1$. The important case of $t_{k-1}$ is denoted by $t$ in the rest of the paper. Let $r$ be defined as follows: $r = n - \lfloor k - 1 + t(k - 2) \rfloor$.

Notice the following facts.

- If $a \leq b$, then $t_a \geq t_b$.
- $t \geq 0$.
- $2 \leq r \leq k - 1$.

Lemma 2.1 If the $a$-chord with initial vertex 0 (recall that 0 is an arbitrary vertex of $T$) is in $A$, then at least one of the two following properties holds.

(i) $f(n, k, T) \geq k - 2$.

(ii) For every $0 \leq i \leq t_a$, the $a + i(k - 2)$-chord with initial vertex 0 is in $A$.

Proof: Suppose that (ii) in the lemma is false, and let

$$j = \min\{i \in \{1, 2, \ldots, t_a\} \mid (a + i(k - 2), 0) \in A\},$$


then
\[ C_k = (0, a + (j-1)(k-2)) \cup (a + (j-1)(k-2), \gamma, a + j(k-2)) \cup (a + j(k-2), 0) \]
is a cycle such that \( I(C_k) = k - 2 \) and hence (i) in the lemma is true.

The following is a consequence of Lemma 2.1.

**Corollary 2.2** At least one of the two following properties holds.

(i) \( f(n, k, T) \geq k - 2 \).

(ii) For every \( 0 \leq i \leq t \), every \((k-1) + i(k-2)\)-chord is in \( A \).

**Proof:** Clearly, for any vertex \( 0, (0, k-1) \in A \) since otherwise \((k-1, 0) \in A \) and \( C_k = (0,1,\ldots,k-1,0) \) is a cycle with \( I(C_k) = k - 1 \) and thus (i) holds.

Now applying Lemma 2.1 with \( a = k - 1 \) we have that (i) or (ii) hold.

### 3 The Cases \( k = 3, 4, 5 \)

**Theorem 3.1** \( f(n, 3) \geq 1 \).

**Proof:** Let \( i = \min\{j \in V | (j, 0) \in A \} \). Observe that \( i \) is well defined since \((n-1,0) \in A \). Clearly \( i \neq 1 \), so \( i - 1 > 0 \) and then \((0, i-1, i, 0) \) is a cycle \( C_3 \) with \( I(C_3) \geq 1 \).

**Theorem 3.2** \( f(n, 4) \geq 1 \).

**Proof:** We proceed by contradiction. Taking \( a = 3 \) and \( x_0 = 0 \) in Lemma 2.1 we get that for each \( i, 0 \leq i \leq t_a \), the \((3+2i)\)-chord \((0, 3+2i)\) is in \( A \). Recall that \( t_a \) is the greatest integer such that \( 3 + 2t_a < n - 1 \).

When \( n \) is even, it holds that \( t_a = (n - 4)/2 - 1, (0, 3 + 2t_a) \in A \). That is, \((0, n-3) \in A \) and \( C_4 = (0, n-3, n-2, n-1, 0) \) is a cycle with \( I(C_4) = 3 \). When \( n \) is odd, it holds that \( t_a = \lceil \frac{n-4}{2} \rceil \) and \((0, 3 + 2t_a) \in A \), namely \((0, n-2) \in A \).

Now, we may assume that \((n-3, 0) \in A \), because otherwise the cycle \( C_4 = (0, n-3, n-2, n-1, 0) \) satisfies \( I(C_4) = 3 \). If \((n-1, n-3) \in A \) then \( C_4 = (n-1, n-3, 0, n-2, n-1) \) is a cycle with \( I(C_4) = 1 \). Else, \((n-3, n-1) \in A \) and \( C_4 = (n-3, n-1, 0, n-4, n-3) \) is a cycle with \( I(C_4) = 1 \).

**Theorem 3.3** \( f(n, 5) \geq 2 \).
Proof: We consider the three cases \( n \equiv 0 \pmod{3}, n \equiv 1 \pmod{3}, n \equiv 2 \pmod{3} \).

Case \( n \equiv 2 \pmod{3} \). Taking \( a = 4 \) in Lemma 2.1, we get that \( (0, n-4) \in A \) and \( C_5 = (0, n-4, n-3, n-2, n-1, 0) \) is a cycle with \( I(C_5) = 4 \).

Case \( n \equiv 1 \pmod{3} \). Taking \( a = 4 \) in Lemma 2.1, we get that \( 4 + 3t_4 = n - 3 \). Hence \( (0, n-3) \in A \) and \( (0, n-6) \in A \). Observe that \( (n-4,0) \in A \). Otherwise \( (0, n-4) \in A \) and \( C_5 = (0, n-4, n-3, n-2, n-1, 0) \) is a cycle with \( I(C_5) = 4 \).

Now, if \( (n-2, n-5) \in A \) then \( C_5 = (n-2, n-5, n-4, 0, n-3, n-2) \) is a cycle with \( I(C_5) = 2 \). Else \( (n-5, n-2) \in A \) and \( C_5 = (0, n-6, n-5, n-2, n-1, 0) \) is a cycle with \( I(C_5) = 3 \).

Case \( n \equiv 0 \pmod{3} \). If \( (0,3) \in A \) then taking \( a = 3 \) in Lemma 2.1, we obtain that \( (0, n-6) \in A \) and \( (0, n-3) \in A \). The proof proceeds exactly as in the proof for the case \( n \equiv 1 \pmod{3} \). Hence, let us assume that \( (3,0) \in A \).

Observe that \( (5,0) \in A \), because otherwise \( (0,5) \in A \) and taking \( a = 5 \) in Lemma 2.1, we get that \( (0, n-4) \in A \) and \( C_5 = (0, n-4, n-3, n-2, n-1, 0) \) is a cycle with \( I(C_5) = 4 \).

Therefore we have that \( (5,0) \in A \) and \( (3,0) \in A \). Considering the cycle \( (0,1,2,3,4,5,0) \) it is easy to check that \( (5,3) \in A \) and \((1,5) \in A \) (or else the proof follows). Analyzing the direction of the arc joining 2 and 5 we see that in any case there is a cycle \( C_5 \) with \( I(C_5) = 2 \): If \( (5,2) \in A \) then the cycle is \( C_5 = (3,0,1,5,2,3) \), else, if \( (2,5) \in A \) then the cycle is \( C_5 = (3,0,1,2,5,3) \).

4 The case of \( n = 2k - 4 \)

In this section it is proved that if \( n = 2k - 4 \) then \( f(n,k) \geq k - 3 \).

Theorem 4.1 If \( n = 2k - 4 \) then \( f(n,k) \geq k - 3 \).

Proof: Let \( x \) and \( y \) be two vertices of \( T \) such that \( l(x, y) = l(y, x) = k - 2 \). Without loss of generality we can assume that \( x = 0 \), \( y = k-2 \) and \( (0, k-2) \in A \). Hence \((k-1,2)\) is a \((k-1)\)-chord, \( l(2,\gamma,k-1) = k-3 \), \((1,k)\) is a \((k-1)\)-chord and \( l(2,\gamma,k+1) = k-1 \).

- \((k,2)\in A\). Otherwise \((2,k)\in A\) and then \( C_k = (k-2,k-1,k) \cup \langle k, \gamma, 0 \rangle \cup (0, k-2) \) is a cycle with \( I(C_k) = k - 3 \).

- \((1,k-1)\in A\). Otherwise \((k-1,1)\in A\) and then \( C_k = (k-1,1,k)\cup\langle k, \gamma, 0 \rangle\cup(0, k-2, k-1) \) is a cycle with \( I(C_k) = k - 3 \).

Therefore, since \((k,2)\in A\) and \((1,k-1)\in A\) then \( C_k = (1, k-1, k, 2, k+1) \cup \langle k+1, \gamma, 1 \rangle \) is a cycle with \( I(C_k) = k - 3 \).

4
5 The case of $r = k - 1$ and $r = k - 2$

In this section it is proved that if $r = k - 1$ or $r = k - 2$ then $f(n, k) \geq k - 3$.

**Theorem 5.1** If $r = k - 1$ or $r = k - 2$ then $f(n, k) \geq k - 3$.

**Proof:** Assume $r = k - 1$. By Corollary 2.2 (taking $i = 0$) either $f(n, k, T) \geq k - 2$ or $(0, k - 1) \in A$. In the latter case we have that $(k - 1 + r(k - 2), \gamma, 0) \cup (0, k - 1 + r(k - 2))$ is a cycle of length $k$ intersecting $\gamma$ in $k - 1$ arcs. Thus, in both cases, $f(n, k, T) \geq k - 2$.

Now, assume $r = k - 2$ and $f(n, k, T) < k - 3$.

We consider the vertices $x = k - 1 + r(k - 2)$, $y = k - 1 + (t - 1)(k - 2)$. Observe that when $t = 0$ we obtain $y = 1$.

(i) $(0, x) \in A$. It follows from Corollary 2.2.

(ii) $(y, x + 1) \in A$. If $(y, x + 1) \notin A$ then $(x + 1, y) \in A$ and $(x + 1, y) \cup (y, \gamma, x + 1)$ is a cycle of length $k$ which intersects $\gamma$ in $k - 1$ arcs.

(iii) $(x, y) \in A$. If $(x, y) \notin A$ then $(y, x) \in A$ and $(y, x) \cup (x, \gamma, 0) \cup (0, y)$ (Corollary 2.2 implies $(0, y) \in A$) is a cycle of length $k$ intersecting $\gamma$ in at least $k - 2$ arcs.

It follows from (i), (ii) and (iii) that $(x, y) \cup (y, x + 1) \cup (x + 1, \gamma, 0) \cup (0, x)$ is a cycle of length $k$ which intersects $\gamma$ in at least $k - 3$ arcs. A contradiction. 

**Corollary 5.2** If $t = 0$ then $f(n, k) \geq k - 3$.

**Proof:** If $t = 0$ then $n = k - 1 + r$, where $k - 3 \leq r \leq k - 1$ since $n \geq 2k - 4$. When $r = k - 1$ or $r = k - 2$, Theorem 5.1 implies that $f(n, k) \geq k - 3$. If $r = k - 3$ then $n = 2k - 4$ and Theorem 4.1 implies that $f(n, k) \geq k - 3$.

6 The General Case

In this section we assume that $r \leq k - 3$, since the case $r > k - 3$ has been considered in Theorem 5.1, and that $t \geq 1$, since the case of $t = 0$ has been considered in Corollary 5.2. The next lemma follows directly from Lemma 2.1.

**Lemma 6.1** If the $k - 1 + \alpha$-chord, $0 \leq \alpha \leq r$, with initial vertex $0$ is in $A$, then at least one of the two following properties holds.

(i) $f(n, k, T) \geq k - 2$. 

(ii) For every $0 \leq i \leq t - 1$, the $k - 1 + \alpha + i(k - 2)$-chord with initial vertex $0$ is in $A$.

**Lemma 6.2** At least one of the two following properties holds.

(i) $f(n, k, T) \geq k - 3$.

(ii) All the following chords are in $A$.
   
   (a) Every $(k - 1)$-chord.
   
   (b) Every $(r)$-chord.
   
   (c) Every $(k - 2)$-chord.
   
   (d) Every $-(r + 1)$-chord.

**Proof:** The proof of (a) follows directly from Corollary 2.2.

The proof of (b) follows from Corollary 2.2, observing that $n - r = k - 1 + t(k - 2)$.

To prove (c) assume that there is a $-(k - 2)$-chord, say $f = (y, x)$. Consider the vertex $x - 1$. It follows from (a) that $(x - 1, y)$ is in $A$, and it follows from (b) that $(x - 1 + r, x - 1)$ is in $A$. Therefore, there exists a vertex $z$ in $(x - 1 + r, y - 1)$ such that $(z, x - 1)$ and $(x - 1, z + 1)$ are in $A$. Then $C_k = (y, x) \cup \langle x, \gamma, z \rangle \cup \langle z, x - 1 \rangle \cup \langle x - 1, z + 1 \rangle \cup \langle z + 1, \gamma, y \rangle$ is a cycle with $I(C_k) = k - 3$, and (i) holds.

Finally, to prove (d) let $(y, x)$ be a $(r + 1)$-chord. It follows from (c) and Lemma 2.1 that every $t(k - 2)$-chord is in $A$. In particular, $(x + k - 2, x + (t + 1)(k - 2))$ is in $A$. Observe that $y = x + (t + 1)(k - 2)$ since $n = (k - 1) + t(k - 2) + r$. It follows that $C_k = (y, x) \cup \langle x, \gamma, x + k - 2 \rangle \cup \langle x + k - 2, y \rangle$ is a cycle with $I(C_k) = k - 2$. Hence (i) holds.

**Lemma 6.3** Let $-1 \leq i \leq r$. If all the $-r$-chords, $-(r + 1)$-chords, $(k - 2 + i)$-chords and $(k - 1 + i)$-chords are in $T$ then at least one of the following properties holds.

(i) $f(n, k, T) \geq k - 3$.

(ii) All the $-(2r - i + 1)$-chords, $-(2r - i + 2)$-chords and $-(2r - i + 3)$-chords are in $T$.

**Proof:** Assume that the hypothesis of the lemma holds and (i) is false. Let us prove that (ii) holds.

Since all the $[(k - 2) + i]$-chords and all the $[(k - 1) + i]$-chords are in $T$, it follows from Lemma 6.1 (taking $\alpha = i - 1$) that every $[k - 2 + i + (t - 1)(k - 2)]$-chord is in $I$, and that (taking $\alpha = i$) every $[k - 1 + i + (t - 1)(k - 2)]$-chord is in $I$. Thus the following arcs are in $T$: $(r, 0), (r + 1, 0), (0, k - 1 + (t - 1)(k - 2) + i), (0, k - 1 + (t - 1)(k - 2) + i - 1)$.

Let $x_1 = r, x_2 = r + 1, x_3 = k - 1 + (t - 1)(k - 2) + i - 1, x_4 = x_3 + 1, x_5 = x_4 + k - 2, x_6 = x_5 + 1, x_7 = x_5 - 1$ and $x_8 = x_7 - 1$. Therefore $(0, x_4)$ and $(0, x_3)$ are in $A$.

Observe that:
• It follows from $x_5 = k - 1 + l(k - 2) + i$, and $n = k - 1 + l(k - 2) + r$ that $l(x_5, \gamma, 0) = n - x_5 = r - i$.
• $l(x_6, \gamma, 0) = r - i - 1$.
• $l(x_6, \gamma, x_1) = 2r - i - 1$.
• $l(x_7, \gamma, x_1) = 2r - i + 1$.
• $l(x_7, \gamma, x_2) = 2r - i + 2$.
• $l(x_8, \gamma, x_2) = 2r - i + 3$.
• $l(x_9, \gamma, x_7) = k - 3$.
• $l(x_{10}, \gamma, x_8) = k - 3$.

We first prove that every $-(2r - i + 1)$-chord is in $T$. Suppose that there exists a $(2r - i + 1)$-chord. We can assume w.l.o.g. that $(x_7, x_1)$ is such a chord. Hence $C_k = (x_7, x_1, 0, x_4) \cup (x_4, \gamma, x_7)$ is a cycle with $I(C_k) = k - 3$.

Now we prove that every $-(2r - i + 2)$-chord is in $T$. Assume the contrary and let $(x_7, x_2)$ be a $(2r - i + 2)$-chord. Then $C_k = (x_7, x_2, 0, x_4) \cup (x_4, \gamma, x_7)$ is a cycle with $I(C_k) = k - 3$.

Finally we show that every $-(2r - i + 3)$-chord is in $T$. Assuming the opposite let $(x_8, x_2)$ be a $(2r - i + 3)$-chord. Then $C_k = (x_8, x_2, 0, x_3) \cup (x_3, \gamma, x_8)$ is a cycle with $I(C_k) = k - 3$.

Lemma 6.4 At least one of the following properties holds.

(i) $f(n, k, T) \geq k - 2$.

(ii) For any vertex $x$, there exist at most $k - 3$ consecutive vertices in $\gamma$ which are in-neighbors of $x$.

Proof: Assume that (i) does not hold. Assume without loss of generality that $x = 0$. The vertices $k - 1 + i(k - 2)$, for $0 \leq i \leq t$, are not in-neighbors of $0$. This follows from Lemma 6.2 part (a), and Lemma 2.1. Thus, there are at most $k - 3$ consecutive vertices in $(k - 1, \gamma, 0)$ which are in-neighbors of $0$. Since $(0, 1) \in A$, also in $(0, \gamma, k - 1)$ there are at most $k - 3$ consecutive in-neighbors of $0$.

Observe that in the Lemma 6.4 the general assumption of this section that $n \geq 2k - 4$ is not needed. The following corollary is a direct consequence of this lemma.

Corollary 6.5 Let $T$ be a tournament with $n$ vertices and $\gamma$ a Hamiltonian cycle of $T$. For each vertex $x$ of $T$ such that the number of consecutive in-neighbors of $x$ in $\gamma$ is at least $k - 2$, $3 \leq k \leq n$, there exists a cycle $C_k$ containing the vertex $x$, with $I(C_k) \geq k - 2$.
Lemma 6.6 If every $k$-chord and every $(-r)$-chord is in $A$ then at least one of the two following properties holds.

(i) $f(n, k, T) \geq k - 3$.

(ii) For every $0 < ar < k$, every $-(a + 1)r$-chord is in $A$.

Proof: Assume that (ii) does not hold; we show that (i) holds. Let $a$ be the least integer for which an $(a + 1)r$-chord is in $A$, and let $(x_2, x_1)$ be an $(a + 1)r$-chord.

Let $x_0 \in V$ such that $l(x_2, \gamma, x_0) = r$. It follows that $(x_1, x_0) \in A$ because it is an $-ar$-chord. Let $x_3 \in V$ such that $l(x_3, \gamma, x_2) = k + (t - 1)(k - 2)$. Observe that $x_3 \in \langle x_1, \gamma, x_0 \rangle$ because $ar < k$ and $t \geq 1$.

Lemma 6.1 and the fact that every $k$-chord is in $A$ imply that either $f(n, k, T) \geq k - 2$ or every $k + (t - 1)(k - 2)$-chord is in $A$. In the latter case $(x_0, x_3) \in A$ and $l(x_3, \gamma, x_2) = k - 3$. We conclude that $C_k = \langle x_3, \gamma, x_2 \rangle \cup \langle x_2, x_1, x_0, x_3 \rangle$ is a cycle with $I(C_k) = k - 3$, and hence $f(n, k, T) \geq k - 3$.

Lemma 6.7 At least one of the following properties holds.

(i) $f(n, k, T) \geq k - 3$.

(ii) For $-1 \leq i \leq r$, every $-(2r + 1 - i)$-chord and every $(k - 1 + i)$-chord is in $A$.

Proof: Suppose that $f(n, k, T) < k - 3$. We shall prove that property (ii) holds by induction on $i$. We start with $i = -1$ and $i = 0$, namely, we prove that the following chords are in $A$:

(a) Every $(k - 2)$-chord.

(b) Every $(k - 1)$-chord.

(c) Every $-(2r + 2)$-chord.

(d) Every $-(2r + 1)$-chord.

In fact we also prove that:

(e) Every $-(2r + 3)$-chord is in $A$.

The proof of (a) and (b) follows directly from Lemma 6.2.

Let $0$ be any vertex of $T$. By Lemma 6.2 (b) and (d) $(r, 0)$ and $(r + 1, 0)$ are in $A$.

It follows from Lemma 6.2 (part (a) and part (c)), and Lemma 2.1 that the following two chords, whose end-points are consecutive in $\gamma$, are in $A$: $(0, k - 1 + (t - 1)(k - 2))$ and $(0, t(k - 2))$.

Since $0$ is an arbitrary vertex of $T$, we can prove that (c), (d) and (e) hold:
• Part (c): every \(-(2r + 2)-\)chord is in \(A\). If \((n - r - 1, r + 1) \in A\) then \(C_k = (n - r - 1, r + 1) \cup (r, 0) \cup (0, k - 1 + (r - 1)(k - 2)) \cup (k - 1 + (r - 1)(k - 2), r, n - r - 1)\) is a cycle with \(I(C_k) = k - 3\), a contradiction.

• Part (d): every \(-(2r + 1)-\)chord is in \(A\). If \((n - r - 1, r) \in A\) then \(C_k = (n - r - 1, r) \cup (r, 0) \cup (0, k - 1 + (r - 1)(k - 2)) \cup (k - 1 + (r - 1)(k - 2), r, n - r - 1)\) is a cycle with \(I(C_k) = k - 3\), a contradiction.

• Part (e): every \(-(2r + 3)-\)chord is in \(A\). If \((n - r - 2, r + 1) \in A\) then \(C_k = (n - r - 2, r + 1) \cup (r + 1, 0) \cup (0, k - 2) \cup (k - 2, r, n - r - 2)\) is a cycle with \(I(C_k) = k - 3\), a contradiction.

Assume that the lemma holds for each \(i\), \(i' \leq i\) and let us prove it for \(i + 1\); namely, we prove:

(a) Every \((k + i)-\)chord is in \(A\),
(b) Every \(-(2r - i)-\)chord is in \(A\).

Proof of (a)

It follows from the inductive hypothesis that for each \(j\), \(0 \leq j \leq i\), every \((k - 1) + j-\)chord and every \((k - 2) + j-\)chord is in \(A\). Hence, by Lemmas 6.2 and 6.3, every \(-(2r - j + 1)-\)chord, \(-(2r - j + 2)-\)chord and every \(-(2r - j + 3)-\)chord is in \(A\). That is, for each \(j\), \(0 \leq j \leq i + 2\), every \(-(2r - j + 1)-\)chord is in \(A\). These are \((i + 3)-\)chords with initial vertices consecutive in \(\gamma\).

Assume for contradiction that \((x_{3b}, 0)\) is a \(-(k + i)-\)chord. Let \(x_0 = n - (2r - i - 1)\). Hence letting \(x_3 = 2\), we have that \((x_2, x_0)\) is a \(-(2r - (i - 1))-\)chord.

Let us show that \(x_0 \in \langle x_3 + 1, \gamma, n - 1 \rangle\):

\[ l\langle x_0, \gamma, 0 \rangle = 2r - i - 1, \]

\[ l\langle x_3, \gamma, x_0 \rangle = n - (k + i + 2r - i - 1) \]
\[ = k - 1 + t(k - 2) + r - (k + i + 2r - i - 1) \]
\[ \geq (k - 1) + (k - 2) + r - k - i - 2r + i + 1 = k - 2 - r. \]

Since we are assuming \(r \leq k - 3\) then \(l\langle x_3, \gamma, x_0 \rangle \geq 1\). Hence \(l\langle x_0, \gamma, 0 \rangle \geq 1\), because \(r \geq 1\).

Now, there exists an \(x \in \Gamma^+(x_0)\) such that \(x\) is in \(\langle x_2, \gamma, x_{3b} - 1 \rangle\). This is a direct consequence of Lemma 6.4 and the fact that the number of vertices in \(\langle x_2, \gamma, x_{3b} - 1 \rangle\) is at least \(k - 2\). Let \(x_4\) be the smallest (the nearest to \(0\) in \(\gamma\)) such vertex.

Let \(x_1 = 0\). We will prove that \(x_4 - i - 3 \in \langle x_1, \gamma, x_4 - 3 \rangle\). Since for each \(j\), \(0 \leq j \leq i + 2\), every \(-(2r - j + 1)-\)chord is in \(A\), it follows that \(\{0, x_0\}, \{1, x_0\}, \{2, x_0\}, \ldots, (i + 2, x_0\} \subseteq A\). Hence, the election of \(x_4\) implies \(x_4 \geq i + 3\) and then \(x_4 - i - 3 \geq 0 = x_1\).
Finally, since $l(x_4, \gamma, x_3) + l(x_1, \gamma, x_4 - i - 3) = k - 3$ then $C_k = (x_4 - i - 3, x_8, x_4) \cup (x_4, \gamma, x_3) \cup (x_3, x_1) \cup (x_1, \gamma, x_4 - i - 3)$ is a cycle with $I(C_k) = k - 3$.

Proof of $(\beta)$

Part $(\beta)$ follows from Lemma 6.3 (taking $i + 1$ instead of $i$) and the following facts.

- Every $(k + i)$-chord is in $A$. Follows from part $(\alpha)$.
- Every $(k - 1 + i)$-chord is in $A$. Follows from the inductive hypothesis.
- Every $(-r)$-chord and every $-(r + 1)$-chord is in $A$. Follows from Lemma 6.2.

\[\text{Theorem 6.8} \quad \text{If } n \geq 2k - 4 \text{ then } f(n, k) \geq k - 3.\]

Proof: The case of $n = 2k - 4$ is considered in Section 4. Assume that $n > 2k - 4$ and assume for contradiction that $f(n, k, T) < k - 3$.

It follows from Lemma 6.7 that for each $i$, $-1 \leq i \leq r$, every $(k + i - 1)$-chord is in $A$. In particular

\[\{(0, k - 2), (0, k - 1), (0, k), \ldots, (0, k + r - 1)\} \in A. \quad (1)\]

It follows from Lemma 6.2 that every $(-r)$-chord is in $A$, and by Lemma 6.7 that every $k$-chord is in $A$. Therefore, by Lemma 6.6 for every $0 < \alpha r < k$, every $-(\alpha + 1)r$-chord is in $A$. Let $\alpha_0 = \max\{\alpha \in \mathbb{A} | \alpha r < k\}$. Clearly $\alpha_0 r < k$, and by Lemma 6.6 every $-(\alpha_0 + 1)r$-chord is in $A$. In particular $((\alpha_0 + 1)r, 0) \in A$ and $k \leq (\alpha_0 + 1)r < k + r$. Thus $y = (\alpha_0 + 1)r \in \{k - 2, k - 1, k, k + 1, \ldots, k + r - 1\}$. Therefore $(y, 0) \in A$. On the other hand, $(1)$ implies that $(0, y) \in A$. A contradiction.

\[\text{7 Upper Bounds}\]

Two upper bounds are presented in this section. The first upper bound shows that for $n, k$, such that $n \geq 2k - 4$ the lower bounds on $f(n, k)$ presented previously are tight. In fact, a characterization of tournaments with $f(n, k) \geq k - 2$ is presented.

It has been shown that for $n \geq 2k - 4$, $f(n, k) \geq k - 3$. The second upper bound shows that for $n < 2k - 4$, $f(n, k) < k - 3$.

We start the proof of the first upper bound with the following simple lemma.
Lemma 7.1 Let $C_k$ be a cycle with $I(C_k) = k - 2$. If $f_1 = (0, x_1)$, $f_2 = (y_1, y_2)$ are the arcs of $C_k$ not in $\gamma$ then $y_2 = y_1 + n - (k - 2 + x_1)$. Namely, $f_2$ is a $-(k - 2 + x_1)$-chord of $\gamma$.

Theorem 7.2 For $n \geq 5$, $k \geq 5$, such that $n \not\equiv k \pmod{k-2}$, $f(n,k) \leq k - 3$.

Proof: We prove the theorem by presenting a hamiltonian tournament $T_n$ with no cycles $C_k$ having $I(C_k) = k - 2$. We define $T_n$ as follows.

$$A(T_n) = \{(i, i + k - 1 + s(k - 2)) | i \in \{0, 1, \ldots, n-1\}, s \in \{0, 1, \ldots, t\}\} \cup \{(i + j, i) | j \in \{2, 3, \ldots, \lfloor \frac{n-1}{2} \rfloor \} - \{k - 1 + s(k - 2) | s \in \{0, 1, \ldots, t\}, i \in \{0, 1, \ldots, n-1\}\}\} \cup \{(i, i + 1) | i \in \{0, 1, \ldots, n-1\}\}.$$

($t$ was defined in Section 2) If $n$ is even it remains to define the orientation of the $n/2$-chords. These are defined as follows. For $i \in \{0, 1, \ldots, n/2 - 1\}$, the arcs

$$(i + n/2, i)$$

are in $A$.

Assume for contradiction that $C_k$ is a cycle of $T_n$ with $I(C_k) = k - 2$, and let $f_1 = (0, x_1)$, $f_2 = (y_1, y_2)$ the only arcs of $C_k$ not in $\gamma$. Without loss of generality we can assume that $l(f_1) < n/2$. The definition of $T_n$ implies that $x_1 = k - 1 + s(k - 2)$, $s \in \{0, 1, \ldots, t\}$.

It follows from Lemma 7.1 that $y_2 = y_1 + n - (k - 1 + (s+1)(k - 2))$. If $s < t$ then $s+1 \leq t$ and $f_2$ is a $-(k - 1 + (s+1)(k - 2))$-chord, contradicting the definition of $T_n$.

Assume now that $s = t$. Hence $x_1 = k - 1 + t(k - 2)$, and $n = (k - 1) + t(k - 2) + r$ implies $l(x_1, \gamma, 0) = r$. On the other hand, we have that $C_k - \{(0, x_1), (y_1, y_2)\} \in \langle x_1, \gamma, 0 \rangle$. Thus $l(x_1, \gamma, 0) \geq k - 1$, and $r \geq k - 1$. The definition of $r$ implies $r \leq k - 1$. Therefore $r = k - 1$ and then $n \equiv k \pmod{k-2}$, a contradiction. $\blacksquare$

It is easy to verify that if $n \equiv k \pmod{k-2}$, then $f(n,k) \geq k - 2$. Hence as a consequence of the previous theorem we get the following characterization of $f(n,k) \geq k - 2$.

Corollary 7.3 $f(n,k) \geq k - 2$ if and only if $n \equiv k \pmod{k-2}$.

The next theorem follows from Theorem 6.8 and Theorem 7.2.

Theorem 7.4 For each $n \geq 2k - 4$, such that $n \not\equiv k \pmod{k-2}$ it holds that $f(n,k) = k - 3$.

We now present the proof of the second upper bound. The aim is to show that the range of $k$ that we have been considering $(2k - 4 \leq n)$ is as large as possible, with $f(n,k) \geq k - 3$. 11
**Theorem 7.5** For $n \geq 5$, $k \geq 5$, such that $n \leq 2k - 5$ it holds that $f(n, k) < k - 3$.

**Proof:** We prove the theorem by presenting a tournament $T_n$ with no cycles $C_k$ having $I(C_k) \geq k - 3$. We define $T_n$ as follows. If $n$ is odd then

$$A(T_n) = \{(i, i + 1)|i \in \{0, 1, \ldots, n-1\} \cup (i, i + j)|j \in \frac{n+1}{2}, \frac{n+1}{2}+1, \ldots, n-2\}.$$ 

If $n$ is even then

$$A(T_n) = \{(i, i + 1)|i \in \{0, 1, \ldots, n-1\} \cup (i, i + j)|j \in \frac{n}{2}, \frac{n}{2}+2, \ldots, n-2\} \cup (i, i + \frac{n}{2})|i \in \{0, 1, \ldots, n-1\}.$$

Consider a cycle $C_k$ of length $k$. Observe that $I(C_k) < k-2$. We prove that $I(C_k) < k-3$, by showing that for any cycle $C$ with $I(C) = k-3$, it holds that $l(C) \leq k - 1$.

Let $f_1 = (x_1, x_2)$, $f_2 = (x_3, x_4)$, and $f_3 = (x_5, x_6)$ be the three arcs of $C$ not in $\gamma$. Hence, without loss of generality,

$$C = (x_1, x_2) \cup (x_2, \gamma, x_3) \cup (x_3, x_4) \cup (x_4, \gamma, x_5) \cup (x_5, x_6) \cup (x_6, \gamma, x_1).$$

By the definition of $T_n$ it follows that $l(f_i) \geq n/2$, for each $i \in \{1, 2, 3\}$. Moreover, there exists $j \in \{1, 2, 3\}$, such that $l(f_j) > n/2$. On the other hand,

$$l(C) = l(x_2, \gamma, x_1) + l(x_6, \gamma, x_5) - l(x_3, \gamma, x_4) + 3$$

$$= n - l(f_1) + n - l(f_3) - l(f_2) + 3.$$ 

Now we proceed with the proof for $n$ even. The case of $n$ odd is analogous. Since $l(f_j) > n/2$ and $l(f_i) \geq n/2$ it follows that

$$l(C) \leq n/2 + n/2 - (n/2 + 1) + 3 = \frac{n + 4}{2}.$$ 

Therefore $l(C) \leq k - 1$, since $n \leq 2k - 5$. $lacksquare$

Finally, the complete characterization of $f(n, k) = k - 3$ is presented.

**Theorem 7.6 (Main Result)** $f(n, k) = k - 3$ if and only if $n \geq 2k - 4$, and $n \not\equiv k \pmod{k - 2}$.

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References
