Leader Election in SINR Model with Arbitrary Power Control*

Magnús M. Halldórsson[1], Stephan Holzer[2], and Evangelia Anna Markatou[2]

[1] ICE-TCS, School of Computer Science, Reykjavik University, Iceland. mmh@ru.is
[2] TDS Group, Massachusetts Institute of Technology, USA.
{holzer,markatou}@mit.edu

Abstract. In this article, we study the Leader Election Problem in the Signal-to-Interference-plus-Noise-Ratio (SINR) model where nodes can adjust their transmission power. We show that in this setting it is possible to solve the leader election problem in two communication rounds, with high probability. Previously, it was known that $\Omega(\log n)$ rounds were sufficient and necessary when using uniform power, where n is the number of nodes in the network.
We then examine how much power control is needed to achieve fast leader election. We show that any 2-round leader election algorithm in the SINR model running correctly w.h.p. requires a power range $2^{\Omega(n)}$ even when n is known. We match this with an algorithm that uses power range $2^{\tilde{O}(n)}$, when n is known and $2^{\tilde{O}(n^{1.5})}$, when n is not known. We also explore tradeoffs between time and power used, and show that to elect a leader in t rounds, a power range $exp(n^{1/\Theta(t)})$ is sufficient and necessary.

Keywords: SINR, leader election, power control, capture effect

1 Introduction

In this article we discuss what we can accomplish in a Signal-to-Interference-plus-Noise-Ratio (SINR) network using *power control*, the ability of nodes to transmit with variable transmission power, and the *capture effect*, a property of SINR networks, where a transmission can be successful while other transmissions within the communication range occur in the same round.

We study the leader election problem as a vehicle to explore this frontier. Leader election, the problem of determining a *unique leader* among the nodes in a network, is one of the oldest and most studied problems in distributed computing. It provides a strong form of breaking symmetry within radio networks in an initially unknown system, and is frequently used as a preliminary step in more complex communication tasks.

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The leader election problem was originally introduced in the 1970s, with the publication of the ALOHA radio network paper [1]. In the following years, many variations of the leader election problem have been extensively studied under a variety of models and algorithmic constraints such as with collision detection in the multiple access channel model [12], no collision detection in the SINR model [7], or under a colored graph in the LOCAL model [6].

We treat the leader election problem in SINR networks, first studied by Gupta and Kumar [9] for algorithmic purposes. In the SINR model, nodes operate in synchronous rounds. In each round a node either broadcasts a message to its neighbors or listens. A node v receives a message from node u depending on the distance between u and v, the transmission power of u, and the interference generated by other broadcasting nodes, as defined in Section 2.

The best solution known for this problem in an SINR network achieves $O(\log n)$ runtime with high probability (w.h.p.) using uniform transmission power [7]. In the classical radio network model, the leader election problem requires $\Theta(\log^2 n)$ rounds w.h.p. [11]. Fineman et al. [7] show that $O(\log n)$ rounds suffice to elect a leader in SINR networks without power control, and show that $\Omega(\log n)$ are also necessary when using uniform power. They suggest that improved bounds may be possible using power control. Indeed, we show that power control can provide the ultimate speedup.

Our Contributions: We present an algorithm that solves the leader election problem in two rounds w.h.p.. We also present a multi-round leader election algorithm that uses limited transmission power. Our work is complemented by nearly matching lower bounds on the transmission power range for both two round and multi-round leader election algorithms.

1.1 Related Work

The leader election problem was first studied with the publication of the ALOHA radio network in the 1970s [1], and plenty of work considering this problem was published in the following decade. Gallager [8] presents a good survey of early work on leader election. Starting in the 1990s there was an increased interest in the *radio network models* [4]. Under radio network models concurrent transmissions are lost due to collisions, and nodes do not know whether or not their message was successfully received. In this model, the leader election problem can be solved in $\Theta(\log^2 n)$ rounds w.h.p. [11] where n is the number of nodes in the network. This bound can be improved to $\Theta(\log n)$ w.h.p. assuming that nodes can detect collisions [11], and to $O(\log n_u)$ expected rounds assuming an upper bound n_u of n [2].

In the beginning of the new millennium came a renewed interest in *fading radio networks*, captured with the SINR model, which claim to capture the real behavior of systems better than previous models, as they take interference into account in a more realistic way. Moscibroda and Wattenhofer [10] showed that algorithms on the fading radio networks model can achieve better runtimes than algorithms for the radio networks model on certain problems, as SINR allows for better spatial reuse.

In the SINR model the most efficient currently published leader election protocol is by Fineman et al. [7]. The authors present an algorithm that achieves $O(\log n + \log R)$ runtime w.h.p. in a single-hop network using uniform transmission power, where n is the number of nodes and $R = O(poly(n))$ is the ratio between the longest and shortest link. Fineman et al. suggest that it may be possible to achieve better performance using power control. Indeed, for problems like link scheduling and connectivity, power control has been shown to give much better performance [10]. Power control has also been used in the SINR setting to solve the link scheduling problem while conserving energy, e.g. [3], [5].

To our knowledge, there has been no published work using power control to optimize the runtime of the leader election problem, or examining the trade-offs between the required communication complexity and power range of a leader election algorithm.

2 Model and Problem Statement

Let V be a set of n nodes, deployed in a single-hop network, that represent wireless devices. Every node can communicate with any other node using transmission power P, in absence of interference from other nodes. Time is divided into synchronous rounds. In each round, a node v can either transmit a message of size $O(\log n)$ with some power P_v, or listen. Node $v \in V$ can receive a message transmitted by node $u \in V$, iff v is listening and

$$SINR(u,v,I) = \frac{\frac{P_u}{d(u,v)^\alpha}}{N + \sum_{w \in I} \frac{P_w}{d(w,v)^\alpha}} \geq \beta, \qquad (1)$$

where I is the set of other nodes transmitting simultaneously. $d(u,v)$ is the distance between nodes v and u, and α, β, N are constants. Specifically, α is the path-loss exponent, N is the non-zero ambient noise, and β is a hardware-dependent minimum SINR threshold required for a successful message reception. Our algorithms work for any $\beta > 0$, while the lower bounds use $\beta \geq 2$.

In this paper, we consider the leader election problem.

Problem 1 (Leader Election Problem). Given n nodes in a network, eventually elect exactly one node *(called the leader)*, with all nodes knowing whether or not they were elected to be the leader.

We denote by R the ratio of the longest to shortest distance between any two nodes in the network. Similar to [7], we assume that R is bounded by a polynomial in n, $R \leq n^c$, for some $c \in \mathbb{N}$. Let γ be a constant such that $\gamma \geq \max(1, c\alpha + 1 + \log \beta)$. We assume that the nodes know or can infer (an upper bound on) γ.

The $\tilde{O}$-notation omits logarithmic factors. All logs are base 2. We consider that an event happens with high probability (w.h.p.) if it happens with probability greater than $1 - 1/n$.

We need the following version of Chernoff bounds.

Theorem 1 (Chernoff Bound). *Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables and $X = \sum_{i=1}^{n} X_i$. Then,* $\Pr[X \geq 2 \cdot \mathbb{E}[X]] \leq 2^{-0.55\mathbb{E}[X]}$.

3 2-Round Leader Election Algorithm

In this section, we present a 2-round leader election algorithm. First, we give some key ideas behind our algorithm. Then, we present a 2-round leader election algorithm that requires no knowledge of n, followed by the analysis.

3.1 The Essence of Our Algorithm

Below we present a high level description of the key ideas behind our algorithm.

(i) **Geometric random variable**: The nodes use a geometric random variable k to count the tails flipped in a sequence of coin flips before the first heads is flipped. This geometric random variable allows some nodes to approximate n with no prior knowledge of the instance. More specifically, at least one and at most $8 \log n$ nodes flip a coin more than $\log n - \log\log n - 2$ times.

(ii) **Random IDs**: Each node chooses an ID (identification number) randomly using k. The geometric random variable k ensures that exactly one node v holds the maximum ID, which allows node v to break the symmetry of the network and stand out as the leader.

(iii) **The loudest node wins:** Each broadcasting node v determines its transmission power by evaluating power function $f(ID_v) = P \cdot ID_v^{\gamma ID_v}$ using its identification number, ID_v. Transmission power function f ensures that all listening nodes receive a message exactly from the node with the largest ID, as long as that ID is unique (see (ii)).

(iv) **Feedback**: In order to inform all nodes of the leader node v, we split the set of nodes V into listeners and competitors. The competitors compete for the leader position during the first round. The listeners inform the competitors of the winner during the second round. Both rounds use the same protocol with different message contents.

In summary, a *geometric random variable* allows the nodes to approximate n with no prior knowledge of the instance, *random IDs* ensure that the node v with the maximum ID stands out, arbitrary transmission power allows the *loudest node* v to inform the other nodes it is the leader, and *feedback* makes sure that all nodes know who the leader node is.

3.2 Leader Election Algorithm

The algorithm proceeds as follows. Initially, each node v flips a coin (a Bernoulli random variable) to determine its *role*, which is a *competitor* if heads are flipped, and *listener* if tails. It then computes a geometric random variable (r.v.) k_v, which counts the tails flipped in a sequence of coin flips before the first heads is flipped. The ID of the node, ID_v, is an integer selected uniformly at random from the range $[J, 2 \cdot J]$, where $J = g(k_v) := 2^{k_v} k_v^4$. Finally, the power P_v that v uses for broadcast is given by $f(ID_v) := P \cdot ID_v^{\gamma ID_v}$, where P is the minimum power needed to reach all nodes in the network (overcoming the ambient noise).

During round 1, competitors transmit their ID using the assigned power P_v, which is to be received by the listeners. In round 2, the roles are reversed, as the listeners report back the ID of the purported leader that they received.

We shall argue that, with high probability, a unique competitor succeeds in transmitting to all the listeners, and a unique listener succeeds in reporting back to all the competitors. The leader is then that successful competitor.

Algorithm 1 2-Round Leader Election Algorithm for node v

$Role_v$, a boolean $Bernoulli(\frac{1}{2})$ random variable {'competitor' if heads, 'listener' if tails}
k_v, a $Geometric(\frac{1}{2})$ random variable, $k_v \in \mathbb{Z}_{\geq 0}$
ID_v, chosen uniformly at random from $[J, 2 \cdot J)]$, where $J = g(k_v) := 2^{k_v} k_v^4$, $ID_v \in \mathbb{Z}_{\geq 0}$
P_v, the transmission power, $P_v = f(ID_v) := P \cdot (ID_v)^{\gamma ID_v}$, $P_v \in \mathbb{Z}_{\geq 0}$
$Leader_v$, a string denoting the identity of the leader, initially empty
Round 1:
if $Role_v =$ competitor **then**
    Broadcast ID_v using power P_v
else
    Receive $Leader_v$
Round 2:
if $Role_v =$ competitor **then**
    Receive $Leader_v$
else
    Broadcast $Leader_v$ using power P_v

3.3 Analysis

We proceed by showing that the highest power used by a competitor is sufficient to overpower all the other competitors, ensuring that this competitor is heard by all the listeners. Identical arguments hold for the reporting back in round 2.

To this end, we first show that there is a competitor whose geometric r.v. is nearly $\log n$, and at most a logarithmic number of competitors have that large value. We then show that all the $O(\log n)$ IDs at the high end of the spectrum are unique, i.e., selected by a single node. The difference in power used by nodes with different ID ensures that the competitor with highest ID will overpower all the other competitors and be heard by all the listeners.

Lemma 1. *Let $k_1 := \log n - \log\log n - 2$. For at least one and at most $8\log n$ competitors v does it hold that $k_v \geq k_1$, with probability greater than $1 - \frac{1}{8n}$.*

Proof. Let $t = \lceil k_1 \rceil = \lceil \log n - \log\log n - 2 \rceil$. Let A_v be the event that a given node v is a competitor and has $k_v \geq t$. The probability of A_v is $\Pr[A_v] = 2^{-1-t} = 2^{-1-\lceil k_1 \rceil}$. Thus,

$$\frac{2\log n}{n} = 2^{-1-k_1} \leq \Pr[A_v] \leq 2^{-k_1} = \frac{4\log n}{n} \ .$$

The probability that no node satisfies A_v is then at most

$$\Pr\left[\bigwedge_v \overline{A_v}\right] \leq \left(1 - \frac{2\log n}{n}\right)^n \leq e^{-2\log n} \leq n^{-2.88} \leq \frac{1}{16n} \ ,$$

for n sufficiently large, establishing the first part of the claim.

Let X be the number of nodes v for which A_v holds. Then $2\log n \leq \mathbb{E}[X] \leq 4\log n$ and by Chernoff bound (Thm. 1),

$$\Pr[X \geq 8\log n] \leq \Pr[X \geq 2\mathbb{E}[X]] \leq 2^{-0.55\mathbb{E}[X]} < 2^{-2.2\log n} = n^{-2.2} \leq \frac{1}{16n} \ ,$$

for n large enough. I.e., at most $8\log n$ nodes satisfy A_v, with probability greater than $1 - \frac{1}{16n}$.

Combined, with probability at least $1 - \frac{1}{8n}$, both of these claimed events hold.

The range from which the IDs are chosen is $[J, 2J]$, for $J \geq g(k_1)$, with high probability. Observe that $g(k_1) = 2^{k_1} k_1^4 \geq \frac{n \cdot \log^3 n}{8}$, for sufficiently large values of n.

Lemma 2. *A sole competitor receives the highest ID with probability greater than $1 - \frac{1}{8n}$, given that at least one node calculated $k_v \geq k_1$.*

Proof. The ranges of IDs assigned to nodes of different k_v values are disjoint. The competitor receiving the highest ID will therefore necessarily be one with a highest k_v value, which we denote by K. Let Z be the set of competitors with $k_v = K \geq k_1 (= \log n - \log\log n - 2)$. By Lemma 1, Z is non-empty and contains at most $8\log n$ nodes.

The probability that a given pair of nodes in Z receive the same ID is inversely proportional to the range of IDs sampled from, or $1/J \leq \frac{1}{g(k_1)} \leq \frac{8}{n \cdot \log^3 n}$. The probability that some pair of nodes in Z are assigned the same ID is then, by the union bound, at most

$$\frac{\binom{|Z|}{2}}{J} \leq \frac{(8\log n)^2}{\frac{n\cdot\log^3 n}{8}} = \frac{512}{n\log n} < \frac{1}{8n},$$

for large enough n. In particular, all nodes in Z receive different IDs with probability greater than $1 - \frac{1}{8n}$.

The highest ID received, ID_w, is at least $g(k_1) \geq n$, for sufficiently large values of n.

Lemma 3. *If a sole competitor receives the highest ID, then its transmission is received by all the listeners.*

Proof. Let w be the sole competitor with the highest ID. For any other competitor v it then holds that

$$\frac{P_w}{P_v} \geq \frac{f(ID_w)}{f(ID_w - 1)} \geq ID_w^\gamma \geq n^\gamma \geq \beta n^{c\alpha+1} \ . \tag{2}$$

Let u be a listener. We bound the noise and interference received by u in terms of the signal $S_u := P_w/d(w,u)^\alpha$ it receives from w. Recall that $d(w,u) \leq R \cdot d(v,u) \leq n^c \cdot d(v,u)$, and thus $d(w,u)^\alpha \leq n^{c\alpha} \cdot d(v,u)^\alpha$, for any competitor v. Hence, applying (2), the interference received from a competitor v is bounded by

$$I_v := \frac{P_v}{d(v,u)^\alpha} \leq \frac{P_w \cdot n^{c\alpha}}{\beta n^{c\alpha+1} \cdot d(w,u)^\alpha} = \frac{S_u}{\beta n} \ . \tag{3}$$

The definition of minimum power P ensures that $\frac{P/d(w,u)^\alpha}{N} \geq \beta$. Thus, we can use (2) to bound the noise term by

$$N \leq \frac{P}{d(w,u)^\alpha \cdot \beta} \leq \frac{P_w}{d(w,u)^\alpha \cdot n^\gamma \cdot \beta} = \frac{S_u}{\beta n^\gamma} \leq \frac{S_u}{\beta n} \ . \tag{4}$$

Combining (3) and (4), we get that the SINR of w's signal at receiver u is bounded below by

$$\frac{S_u}{N + \sum_{v\in X} I_v} \geq \frac{\beta n}{1 + |X|} \geq \beta \ ,$$

where X is the set of competitors other than w. Thus, w overpowers all other competitors at all the listeners.

Theorem 2. *The 2-round leader election algorithm terminates with all nodes agreeing on a common leader, w.h.p.*

Proof. Adding up the error probabilities of Lemmas 1 and 2, we find that a sole competitor w receives the highest ID, with probability at least $1 - \frac{1}{4n}$. By Lemma 3, w then successfully informs all the receivers. All three lemmas work identically for the reporting process in round 2. Hence, with probability at least $1 - \frac{1}{2n}$, the algorithm succeeds.

Remark 1. Leader election can be achieved in a single round if simultaneous transmission and reception is possible. Such *full-duplex* radios operate by subtracting the transmitted signal from the received one. While they are still rare, being hard to implement, such technology has been progressing significantly in recent years and may well become a commodity feature. With full-duplex, our arguments apply unchanged to the success of reception by the other competitors, thus succeeding after only a single round.

4 Range of Power Needed For a 2-Round Leader Election

Power control is the essential feature that allows our algorithms to work. That begs the question *how much* power control is needed?

We say that an algorithm uses a *power range* X if the powers assigned fall in the range $[P, \ldots, X \cdot P]$. The basic question is then how the power range must grow as a function of n for leader election to work correctly.

4.1 Upper Bound

Theorem 3. *Our 2-round leader election algorithm can be made to work correctly with a power range of* $2^{\tilde{O}(n^{1.5})}$*, w.h.p.*

Proof. The algorithm as is may select power assignments inducing a range of $2^{\tilde{O}(n^2)}$, since k_v is no larger than $2 \log n + 2$, with probability greater than $1 - \frac{1}{2n}$. However, if the range is bounded, we may assume that the nodes know the upper bound of the range, P_{max}. Thus, the algorithm would automatically truncate the power assigned to be at most P_{max}. We observe that this truncation can occur for at most one vertex, for the node with the highest ID to succeed. Namely, the probability that two or more nodes select a k_v value greater than $1.5 \log n$ is at most

$$\binom{n}{2} 2^{-3 \log n} \leq \frac{1}{2n} \ .$$

The bound on the maximum power now follows immediately.

If nodes know n, we can work with a smaller power range as follows: We can first sample the nodes with probability $\Theta(\log n / n)$, and have each selected node select ID uniformly at random from the range $[J, 2J]$, where $J = n \log^2 n$. The power used is $f(ID_v)$ as before, and the arguments are otherwise the same. This results in a power range of at most $2(n \log^2 n)^{n \log^2 n} = 2^{\tilde{O}(n)}$.

Proposition 1. *When nodes know* n*, a power range of* $2^{\tilde{O}(n)}$ *suffices.*

4.2 Lower bound

We show that an exponential-size power range is actually necessary for any leader election protocol running in (at most) two rounds.

Theorem 4. *Every 2-round leader election algorithm in the SINR model running correctly w.h.p. requires a power range $2^{\Omega(n)}$. This holds even if the nodes know n, the number of nodes in the network, and if the nodes are located in a unit metric space (where all distances are equal).*

Proof. Consider n nodes located in a unit metric. In the unit metric, either a single message is received by all the listeners or none of them hear anything (assuming $\beta \geq 1$). Since the nodes don't operate full-duplex, two rounds are needed to inform the transmitting nodes of the winner, and the winner must be heard by all listeners in the first round.

We divide the available range of power into subranges, each within factor 2. Namely, if P_{max} is the maximum power available, then the i-th highest subrange is $[P_{max}/2^i, P_{max}/2^{i-1}]$. If the highest range used is used by two or more nodes, then the algorithm fails (assuming $\beta \geq 2$). We shall bound from below the probability that exactly two nodes use the highest subrange in use; this is clearly a lower bound on the failure probability of the algorithm.

Let X_i^v be the event that node v transmits in the first round using the i-th highest subrange. Since the nodes are identical, the same probability holds for them all, so let $p_i = \Pr[X_i^v]$. Observe that the probability that no node transmits in the round is at least $1 - n\sum_i p_i$, and since that can hold with probability at most $1/n$, it follows that $\sum_i p_i \geq \frac{1}{n}(1 - \frac{1}{n})$. Let q be the largest number such that

$$\sum_{i=1}^{q} p_i \leq \frac{1}{2n} \ . \qquad (5)$$

So, a subrange of rank at least $q+1$ is in use.

Let A_i be the event that at least two nodes use the i-th highest subrange, B_i be the event that no node transmits at subranges $1, 2, \ldots, i-1$, and $C_i = A_i \cap B_i$ be the event that both A_i and B_i occur, for $i = 1, 2, \ldots$. Then, $C = \bigcup_i C_i$ is the event that at least two nodes use the highest subrange in use. Observe that $\Pr[A_i|B_i] \geq \Pr[A_i]$, since the non-use of the $i-1$ highest subranges only makes the event A_i more likely. Then,

$$\Pr[C_i] = \Pr[A_i \cap B_i] = \Pr[A_i|B_i]\Pr[B_i] \geq \Pr[A_i]\Pr[B_i] \ .$$

We bound the probability of A_i, $i \leq q$, by the first term of the binomial expansion:

$$\Pr[A_i] > \binom{n}{2} p_i^2 (1-p_i)^{n-2} > \frac{n^2}{3} p_i^2 \left(1 - \frac{1}{2n}\right)^{n-2} > \frac{n^2}{3e} p_i^2 \ .$$

Also, applying (5),

$$\Pr[B_i] \geq 1 - n\sum_{j=1}^{i-1} p_i \geq \frac{1}{2} \ .$$

Observe that the C_i's are mutually exclusive and apply the Cauchy-Schwarz inequality followed by (5) to obtain:

$$\Pr[C] \geq \sum_{i=1}^{q} \Pr[C_i] \geq \frac{n^2}{3e} \sum_{i=1}^{q} p_i^2 \cdot \frac{1}{2} \geq \frac{n^2}{6e} \frac{(\sum_{i=1}^{q} p_i)^2}{q} \geq \frac{1}{24e \cdot q} \ .$$

The algorithm fails when C holds, and thus we may assume that $\Pr[C] \leq 1/n$, which implies that $q \geq n/(24e) = \Omega(n)$. Hence, the claim.

Observe that for the case of known n, we obtain an essentially tight bound of $2^{\tilde{\Theta}(n)}$ on the needed power range.

Remark 2. We note that a construction can be given in the Euclidean plane that achieves the same result but with slightly weaker power tradeoffs. It consists of $n/2$ well-separated node-pairs that are internally close. It, however, does not avail itself to easy generalizations to protocols with greater number of rounds, and is therefore omitted.

5 Trading Time for Power Range

In this section, we explore how much the power range can be reduced by increasing the round complexity. We present a multi-round protocol that requires limited power range and derive a lower bound on the power range required by any t-round leader election algorithm, for $t \geq 2$.

5.1 Multi-Round Protocol

When a smaller power range is available, we can give a protocol that uses a larger number of rounds.

Our multi-round algorithm simply repeats the 2-round algorithm t times, for a given number $t \geq 1$, but using a slower-growing power function. Namely, we change the ID-selection function to $g_t(k) = 2^k k^{3t+1}$, and the power function to $f_t(ID_v) = P \cdot ID_v^{\gamma(ID_v)^{1/t}}$. After each round-pair repetition, each competitor v updates its $leader_v$ value to the *largest* among those heard so far.

First, we observe that it suffices to succeed in one of the round-pairs.

Observation 1 *If, in some round-pair, all receivers hear from a particular node v, and the senders all get informed of v as a leader, then the algorithm successfully terminates with v as leader.*

Proof. After this round-pair, all nodes have $leader_v$ value set as w. Thus, all broadcasts that follow use w for the value of $leader_v$.

Let $U = n^{1/t}$. Suppose we can guarantee that the failure probability of an individual round-pair is at most $1/U$. Then, the probability that *all* t round-pairs are unsuccessful is $1/U^t = 1/n$, as desired. Thus, it suffices to ensure that the failure probability of each round be at most $1/(2U)$. Let Z be the set of competitors with the highest k_v value, and recall that $|Z| \leq 8 \log n$ with probability greater than $1 - \frac{1}{8n}$, by the same argument as in Lemma 1. Observe that for success, it suffices that one node transmits with at least $n^{c\alpha}|Z| \leq n^{c\alpha+1}$ times the power of any other transmitting node, as argued in Lemma 3. A node w with the highest ID will satisfy $ID_w^{\gamma} \geq n^{c\alpha+1}$, as $g(k_w) \geq g(k_1)$ (w.h.p.) It also holds that $ID_w^{1/t} \geq n^{1/t}$.

Thus, what remains is to argue the counterpart of Lemma 2.

Lemma 4. *In a given round, with probability at least* $1 - 1/(2U)$*, some node* w *receives an ID such that* $(ID_w)^{1/t} \geq (ID_v)^{1/t} - 1$*, for all other nodes* v*.*

Proof. Let Z be the set of competitors with the largest k_v-value. Recall that IDs are allocated uniformly at random, and for nodes in Z, the range is of size at least $g_t(k_1) = 2^{\log n - \log\log n - 2}(\log n - \log\log n - 2)^{3t+1} \geq \frac{n}{8\log n}(\frac{\log n}{2})^{3t+1} \geq \frac{1}{2^{3t+1}} n \log^{3t} n$, for large enough n. The probability that a given pair of nodes u, v in Z receive nearly equivalent IDs, with $|(ID_u)^{1/t} - (ID_v)^{1/t}| \leq 1$, is at most $g_t(k_1)^{-1/t} \leq \frac{2}{n^{1/t}\log^3 n}$. Thus, the probability that some two nodes in Z receive nearly equivalent IDs is at most

$$\frac{\binom{|Z|}{2}}{g_t(k_1)^{1/t}} \leq \frac{2^3 \cdot 8^2 \log^2 n}{n^{1/t}\log^3 n} < \frac{1}{2n^{1/t}} ,$$

for sufficiently large n.

The correctness of the algorithm follows from the above observations.

Theorem 5. *For each number* $t = O(\log n/\log\log n)$*, there is a* $2t$*-round algorithm using a power range* $2^{n^{O(1/t)}}$ *that correctly elects a leader, w.h.p.*

5.2 Lower Bound for Multi-Round Protocols

Theorem 6. *Any* t*-round leader election algorithm in the SINR model running correctly w.h.p. requires a power range* $2^{\Omega(\sqrt[t-1]{n})}$*,* $t \geq 2$*. This holds even if the nodes know* n*, the number of nodes in the network, and the nodes are located in a unit metric (where all distances are equal).*

Proof. We consider n nodes located in a unit metric space. In this setting, after any round of the algorithm either all listening nodes receive a message, or no progress is made (assuming $\beta \geq 1$). Since the nodes do not operate full-duplex, any leader election algorithm requires at least two rounds, one round for the winner to broadcast its message, and one round to be informed of the victory.

Let A be a t-round leader election algorithm in the SINR model that runs correctly with probability greater than $1 - 1/n$. Since at least two rounds of successful communication are needed, Algorithm A fails when no listening node receives a message during the first $t-1$ rounds. This happens with probability $\prod_{r=1}^{t-1} p_r$, where p_r denotes the probability that no listener receives a message in round r. Since algorithm A succeeds with probability greater than $1 - 1/n$,

$$\frac{1}{n} > \prod_{r=1}^{t-1} p_r.$$

Now, consider round r. Let q and C be as in Theorem 4. We can show by a similar argument that $Pr[C] \geq \frac{1}{12e\cdot q}$, assuming $\beta \geq 2$. No listener receives a message in round r when C holds, and thus $\Pr[C] \leq p_r$, which implies that

$$q \geq \frac{1}{12e \cdot p_r}.$$

It follows that $1/n \geq (\frac{1}{12eq})^{t-1}$, and therefore $q \geq \sqrt[t-1]{n}/(12e) = \Omega(\sqrt[t-1]{n})$. Thus, algorithm A requires a power range $2^{\Omega(\sqrt[t-1]{n})}$.

6 Conclusions and Acknowledgments

We have shown that power control can yield the ultimate speedup for leader election in the SINR model. This is thanks to the capture effect, which is the crucial property in which SINR differs from graphs-based models.

It would be exciting to see these techniques applied more widely. Multi-hop settings and more restricted power ranges are natural directions to examine, as well as problems beyond leader election. In general, the value of power control and the capture effect is still not fully understood.

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