RAMBO II: Rapidly Reconfigurable Atomic Memory for Dynamic Networks\textsuperscript{1}

by

Seth Gilbert

Submitted to the Department of Electrical Engineering and Computer Science
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Abstract

Future civilian rescue and military operations will depend on a complex system of communicating devices that can operate in highly dynamic environments. In order to present a consistent view of a complex world, these devices will need to maintain data objects with atomic (linearizable) read/write semantics.

Lynch and Shvartsman have recently developed a reconfigurable atomic read/write memory algorithm for such environments [12, 13] This algorithm, called RAMBO, guarantees atomicity for arbitrary patterns of asynchrony, message loss, and node crashes. RAMBO installs new configurations lazily, transferring data from old configurations to new configurations using a background information transfer task. That task handles configurations sequentially, transferring information from each configuration to the next.

This paper presents a new algorithm, RAMBO II, that implements a radically different approach to installing new configurations: instead of operating sequentially, the new algorithm reconfigures “aggressively”, transferring information from old configurations in parallel. This improvement substantially reduces the time necessary to remove obsolete configurations, which in turn substantially increases the fault-tolerance. This paper presents a formal specification of the new algorithm, a correctness proof, and a conditional analysis of its performance. Preliminary empirical studies performed using LAN implementations of RAMBO and the new algorithm illustrate the advantages of the new algorithm.

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Chapter 1

Introduction

Future large scale civilian rescue and military deployment operations will involve large numbers of communication and computing devices operating in highly dynamic network substrates. Successful coordination and marshaling of human resources and equipment involves collecting information about a complex real-world situation using sensors and input devices, gathering the information in survivable repositories, and providing appropriate and coherent information to the stakeholders.

Data objects with atomic (linearizable) read/write semantics commonly occur in such settings. Replication of objects is a prerequisite for fault-tolerance and availability, and with replication comes the need to maintain consistency. Additionally, in dynamic settings where participants may join and leave the environment, may fail, and where the physical objects migrate, one needs to be able to effectively move the corresponding data objects from one set of data owners to another.

Lynch and Shvartsman developed a reconfigurable atomic read/write memory algorithm for dynamic networks [12, 13]. The algorithm, called RAMBO, guarantees atomicity for arbitrary patterns of asynchrony, message loss, and node crashes. Conditional performance analysis of the algorithm shows that when the environment timing stabilizes, when failures are within specific parameters, and when the reconfigurations are not frequent and not bursty, then read and write operations have small latency bounded in terms of the maximum message delay and the periodic gossip interval. However when the reconfigurations are frequent or
bursty, this algorithm may perform poorly because of the inherently sequential processing of the new configurations once they become determined by the algorithm. In particular, the number of configurations maintained by the algorithm may grow without bound, leading to the unbounded number of messages necessary in processing the read and write operations. Such situations may arise due to failures or asynchrony, yet these are not the only reasons. Even in synchronous failure-free environments the world dynamics may require that frequent reconfigurations are performed to keep track of the rapidly moving physical objects or rapidly changing set of stakeholders.

This thesis presents a new algorithm, RAMBO II, integrated with RAMBO, that implements a radically different approach to installing new configurations: instead of operating sequentially, the new algorithm reconfigures “aggressively”, transferring information from old configurations in parallel. This improvement substantially reduces the time necessary to process new configurations and to remove obsolete configurations from the system, which in turn substantially increases fault-tolerance. This is due to the fact that once a configuration is removed, the system no longer depends on it, and as soon as the configuration is removed, it is allowed to fail. The process executing the new algorithm achieves a linear speed-up in the number of old configurations known to the process. For example, our conditional performance analysis shows that if a process knows about a sequence of \( h \) configurations, then the it can eliminates all but one of these configurations in time \( O(1) \), as compared to the original RAMBO, where this takes \( \Theta(h) \) time. Additionally, the new algorithm reduces the number of messages necessary to process these configurations.

This thesis presents a formal specification of the new algorithm, a correctness proof, and a conditional analysis of its performance. Preliminary empirical studies performed using LAN implementations of RAMBO and the new algorithm illustrate the advantages of the new algorithm.

**Background.** Starting with the work of Gifford [6] and Thomas [18], intersecting collections of sets found use in several algorithms providing consistent data in distributed settings. Depending on the algorithm and its setting, such collections of sets, called quorums when any two have non-empty intersection, represent either sets of processors or their knowledge. Up-

In long-lived systems where processors may dynamically join and leave the system, it is important to reconfigure a quorum system to adapt it to the new set of processors [8, 4, 7, 17]. Prior approaches required that the new quorum system include processors from the old quorum system. This is stated as a static constraint on the quorum system that needs to be satisfied during or even before the reconfiguration. In our work on reconfigurable atomic memory [15, 5, 12] we replace the space-domain requirement on successive quorum system intersections with the time-domain requirement that some quorums from the old and the new system are involved in the reconfiguration algorithm. Such systems are more dynamic because they allow for more choices of new quorum systems and do not require that successive configurations intersect.

Reconfiguration in Highly Dynamic Settings. Lynch and Shvartsman’s earlier algorithms [15, 5] allowed a single distinguished process to act as the quorum system reconfigurer. The advantage of the single-reconfigurer approach is its relative simplicity and efficiency: any process maintains at most two configurations, the current configuration and the proposed new configuration. The disadvantage of the single reconfigurer is that it is a single point of failure – no further reconfiguration is possible if the reconfigurer fails.

The RAMBO algorithm [12, 13] removed the requirement of having a single reconfigurer, thus enabling any process within its own current configuration to begin reconfiguration to a new quorum system supplied by the environment. The algorithm implements atomic shared memory suitable for use in highly dynamic settings, and it guarantees atomicity in any asynchronous execution and in the presence of arbitrary process and network failures. However the multiple-reconfigurer approach introduces the problem of maintaining multi-
ple configurations and removing old configurations from the system. RAMBO implements a sequential “garbage-collection” algorithm where processes remove obsolete configurations one-at-a-time. Configuration removal requires that information is propagated from the earliest known configuration to its successor. Since arbitrarily many new configurations may be introduced this leads to an unbounded number of old configurations that need to be sequentially removed.

The environment may introduce new configurations for several reasons: (i) due to failures and network instability that endanger installed configurations, (ii) due to the mobility of the physical objects represented by the abstract memory objects and the mobility of the processes maintaining the object replicas, and (iii) due to the need to rebalance loads on processes within installed configurations. Frequent or bursty reconfiguration can substantially increase the number of installed configurations and, since a process performing a read or a write operation potentially needs to contact quorums in all configurations known to it, this leads to the corresponding increase in the number of messages needed to perform the operation.

The New Algorithm. The primary contribution of this thesis is a new algorithm for reconfigurable atomic memory, based on the original RAMBO, that implements an aggressive configuration-replacement protocol where any locally-known contiguous sequence of configurations is replaced by the last configuration in the sequence. The removal of the old configurations is done in parallel, while preserving all other properties of the original RAMBO. Specifically, we maintain a loose coupling between the reconfiguration algorithms and the original RAMBO algorithms implementing the read and write operations.

In order to achieve availability in the presence of failures, the objects are replicated at several network locations. In order to maintain memory consistency in the presence of small and transient changes, the algorithm uses configurations, each of which consists of a set of members plus sets of read-quorums and write-quorums. In order to accommodate larger and more permanent changes, the algorithm supports reconfiguration, by which the set of members and the sets of quorums are modified. Such changes do not cause violations of atomicity. Any quorum configuration may be installed at any time—no intersection requirement is imposed on the sets of members or on the quorums of distinct configurations.
The algorithm is composed of a *main algorithm*, which handles reading, writing, and replacement of old configurations with a successor configuration, and a global configuration announcement service, *Recon*, which provides the main algorithm with a consistent sequence of configurations. Several configurations may be known to the algorithm at one time, and read and write operations can use them all without any harm.

The main algorithm performs read and write operations requested by clients using a two-phase strategy, where the first phase gathers information from read-quorums of active configurations and the second phase propagates information to write-quorums of active configurations. This communication is carried out using background gossiping, which allows the algorithm to maintain only a small amount of protocol state information. Each phase is terminated by a *fixed point* condition that involves a quorum from each active configuration. Different read and write operations may execute concurrently; the restricted semantics of reads and writes permit the effects of this concurrency to be sorted out afterward.

The main algorithm provides a new configuration-replacement algorithm that removes old configurations while ensuring that their use is no longer necessary for maintaining consistency. Configuration-replacement also uses a two-phase strategy, where the first phase communicates in parallel with all old configurations being removed and the second phase communicates with a new configuration. A configuration-replacement operation ensures that both a read-quorum and a write-quorum of each old configuration learn about the new configuration, and that the latest value from all old configurations is conveyed to a write-quorum of the new configuration. The strength of the new algorithm is that it proceeds aggressively in parallel. An arbitrary number of old configurations can be replaced in constant time (assuming bounded message latency and non-failure of active configurations).

The configuration announcement service is implemented by a distributed algorithm that uses distributed consensus to agree on the successive configurations. Any member of the latest configuration \( c \) may propose a new configuration at any time; different proposals are reconciled by an execution of consensus among the members of \( c \). Consensus is, in turn, implemented using a version of the Paxos algorithm [9], as described formally in [3]. Although such consensus executions may be slow—in fact, in some situations, they may not even terminate—they do not cause any delays for read and write operations.
All services and algorithms, and their interactions, are specified using I/O automata. We show correctness (atomicity) of the algorithm for arbitrary patterns of asynchrony and failures. On the other hand, we analyze performance conditionally, based on certain failure and timing assumptions. For example, assuming that gossip and configuration-replacement occur periodically, and that quorums of active configurations do not fail, we show that read and write operations complete within time $8d$, where $d$ is the maximum message latency. Note that the original RAMBO algorithm also had to assume also that garbage-collection is able to keep up—this assumption is not necessary in the new algorithm due to the new configuration replacement algorithm. For the configuration replacement algorithm we show that any number of configurations can be replaced by their successor in constant time.

At the same time, all the performance results of the original RAMBO algorithm still hold; in instances where the network is reliable and timely throughout the execution, the bounds described in the previous RAMBO papers [12, 13] still hold.

Implementations of RAMBO and RAMBO II on a LAN are currently being completed [16]. Preliminary empirical studies performed using this implementation illustrate the advantages of the new algorithm over the previous one.

**Document Structure.** In Chapter 2 we describe the original RAMBO algorithm of Lynch and Shvartsman, and then in Chapter 3 present and discuss the formal specification of RAMBO II. In Chapter 4 we present some notation, and restate some basic lemmas, only slightly modified from RAMBO. In Chapter 5 we prove that the new algorithm guarantees atomic consistency. In Chapter 6 we present the reconfiguration service. In Chapter 7 we analyze the performance of RAMBO II, and discuss in detail the areas in which this algorithm improves over the original RAMBO algorithm. In Chapter 8 we discuss the preliminary performance results. Finally, in Chapter 9 we summarize the results, and areas for future research.
Chapter 2

The Original Rambo Algorithm

In this chapter, we present the original RAMBO algorithm, on which the new algorithm RAMBO II is based. RAMBO is an algorithm designed to support read/write operations on an atomic shared memory.

In order to achieve fault tolerance and availability, RAMBO replicates data at several network locations. The algorithm uses configurations to maintain consistency in the presence of small and transient changes. Each configuration consists of a set of members plus sets of read-quorums and write-quorums. The quorum intersection property requires that every read-quorum intersect every write-quorum. Read and write operations are implemented as a two-phase protocol, in which each phase accesses a set of read or write quorums.

RAMBO supports reconfiguration, which modifies the set of members and the sets of quorums, thereby accommodating larger and more permanent changes without violating atomicity. In this way, failed nodes can be removed from active quorums, and newly joined nodes can be integrated into the system. Any quorum configuration may be installed at any time – no intersection requirement is imposed on the sets of members or on the quorums of distinct configurations.

The RAMBO algorithm consists of three kinds of automata:

- **Joiner** automata, which handle join requests,
- **Recon** automata, which handle reconfiguration requests, and generate a totally ordered sequence of configurations, and
• Reader-Writer automata, which handle read and write requests, manage garbage collection, and send and receive gossip messages.

In this thesis, we discuss only the Reader-Writer automaton. The Joiner automaton is quite simple; it sends a join message when node \( i \) joins, and sends a join-ack message in response to join messages. The Recon automaton depends on a consensus service, implemented using Paxos [9], to agree on a total ordering of configurations. However, we assume that this total ordering exists, and therefore need not discuss this automaton any further. For more details of these two automata, see the original RAMBO paper [12, 13].

The complete implementation \( \mathcal{S} \) is the composition of all the automata described above—the Joiner, Reader-Writer, and Recon automata for all \( i \), and all the channels, with all the actions that are not external actions of the RAMBO specification hidden.

<table>
<thead>
<tr>
<th>Input:</th>
<th>Output:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{join}(\text{rambo}, J)_{x,i}, J \text{ a finite subset of } I - {i}, x \in X, i \in I, )</td>
<td>( \text{join-ack}(\text{rambo})_{x,i}, x \in X, i \in I )</td>
</tr>
<tr>
<td>such that if ( i = (i_0)_x ) then ( J = \emptyset )</td>
<td>( \text{read-ack}(v)_{x,i}, v \in V_x, x \in X, i \in I )</td>
</tr>
<tr>
<td>( \text{read}_{x,i}, x \in X, i \in I )</td>
<td>( \text{write-ack}_{x,i}, x \in X, i \in I )</td>
</tr>
<tr>
<td>( \text{write}(v)_{x,i}, v \in V_x, x \in X, i \in I )</td>
<td>( \text{recon-ack}(b)_{x,i}, b \in {\text{ok,nok}}, x \in X, i \in I )</td>
</tr>
<tr>
<td>( \text{recon}(c, d)_{x,i}, c, d \in C, i \in \text{members}(c), x \in X, i \in I )</td>
<td>( \text{report}(c)_{x,i}, c \in C, c \in X, i \in I )</td>
</tr>
<tr>
<td>( \text{fail}_{x,i}, i \in I )</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2-1: RAMBO\((x)\): External signature

The external signature for RAMBO appears in Figure 2-1. The algorithm is specified for a single memory location, and extended to implement a complete shared memory. A client uses the join\(_i\) action to join the system. After receiving a join-ack\(_i\), the client can issue read\(_i\) and write\(_i\) requests, which results in read-ack\(_i\) and write-ack\(_i\) responses. The client can issue a recon\(_i\) request to propose a new configuration. Finally, the fail\(_i\) action is used to model node \( i \) failing.

The signature and state for the Reader-Writer automata is presented in Figure 2-2. The code for the Reader-Writer automata is presented in Figure 2-3. All three operations, read, write, and garbage-collect, are implemented using gossip messages. Unlike in many other algorithms, there are no directed messages specified in this algorithm; at no point does a given node, say \( i \), decide to send a message specifically to node \( j \). Instead, at regular intervals node \( i \) will non-deterministically send all of its public state to other nodes. Progress
Signature:

Input:
read_i,
write(v)_i, v ∈ V
new-config(c, k)_i, c ∈ C, k ∈ \mathbb{N}^+
recv(join)_i, j ∈ I - \{i\}
recv(m)_j, m ∈ M, j ∈ I
join(rw)_i
fail,

Output:
join-ack(rw)_i
read-ack(v)_i, v ∈ V
write-ack_i
send(m)_i,j, m ∈ M, j ∈ I

State:

status ∈ \{idle, joining, active, failed\}, initially idle
world, a finite subset of I, initially \emptyset
value ∈ V, initially v_0
tag ∈ T, initially (0, i_0)
cmap ∈ CMap, initially cmap(0) = c_0,
cmap(k) = ⊥ for k ≥ 1
pnum1 ∈ \mathbb{N}, initially 0
pnum2, a mapping from I to \mathbb{N}, initially everywhere 0
failed, a Boolean, initially false

Internal:
query-fix_i
prop-fix_i
gc(k)_i, k ∈ \mathbb{N}
gc-query-fix(k)_i, k ∈ \mathbb{N}
gc-prop-fix(k)_i, k ∈ \mathbb{N}
gc-ack(k)_i, k ∈ \mathbb{N}

op, a record with fields:
type ∈ \{read, write\}
phase ∈ \{idle, query, prop, done\}, initially idle
pnum ∈ \mathbb{N}
cmap ∈ CMap
acc, a finite subset of I
value ∈ V

gc, a record with fields:
phase ∈ \{idle, query, prop\}, initially idle
pnum ∈ \mathbb{N}
acc, a finite subset of I
cmap ∈ CMap
index ∈ \mathbb{N}

Figure 2-2: Reader-Writer_i: Signature and state

in an operation occurs when enough information has been exchanged. After initiating an operation, the automaton waits until it can be sure that it has shared state with enough other nodes (using gossip messages), and then declares the operation complete. The phase numbering regime, implemented using pnum1 and pnum2 is used to determine when enough communication has completed.

Every node maintains a tag and a value for the data object. Every new value is assigned a unique tag, with ties broken by process-ids. These tags are used to determine an ordering of the write operations, and therefore determine the value that a read operation should return.

Read and write operations require two phases, a query phase and a propagation phase,
Output send((W, v, t, cm, pns, pnr))_i,j
Precondition:
\(-\text{failed}\)
\(\text{status} = \text{active}\)
\(j \in \text{world}\)
\(\langle W, v, t, cm, pns, pnr \rangle = \langle \text{world}, \text{value}, \text{tag}, \text{cmap}, \text{pnum1}, \text{pnum2}(j) \rangle\)
Effect:
none

Input recv((W, v, t, cm, pns, pnr))_j,i
Effect:
if \(-\text{failed}\) then
if \(\text{status} \neq \text{idle}\) then
\(\text{status} \leftarrow \text{active}\)
\(\text{world} \leftarrow \text{world} \cup W\)
if \(t > \text{tag}\) then \((\text{value}, \text{tag}) \leftarrow (v, t)\)
\(\text{cmap} \leftarrow \text{update(cmap, cm)}\)
\(\text{pnum2}(j) \leftarrow \max(\text{pnum2}(j), \text{pns})\)
if \(\text{op.phase} \in \{\text{query, prop}\} \text{ and } \text{pnr} \geq \text{op.pnum}\) then
\(\text{op.cmap} \leftarrow \text{extend(\text{op.cmap, truncate(cmap)})}\)
if \(\text{op.cmap} \in \text{Truncated}\) then
\(\text{op.acc} \leftarrow \text{op.acc} \cup \{j\}\)
else
\(\text{op.acc} \leftarrow \emptyset\)
\(\text{op.cmap} \leftarrow \text{truncate(cmap)}\)
if \(\text{gc.phase} \in \{\text{query, prop}\} \text{ and } \text{pnr} \geq \text{gc.pnum}\) then
\(\text{gc.acc} \leftarrow \text{gc.acc} \cup \{j\}\)

Input new-config(c, k)_i
Effect:
if \(-\text{failed}\) then
if \(\text{status} \neq \text{idle}\) then
\(\text{cmap}(k) \leftarrow \text{update(cmap, k)}\)

Input read_i
Effect:
if \(-\text{failed}\) then
if \(\text{status} \neq \text{idle}\) then
\(\text{pnum1} \leftarrow \text{pnum1} + 1\)
\(\langle \text{op.pnum}, \text{op.type}, \text{op.phase}, \text{op.cmap}, \text{op.acc} \rangle \leftarrow \langle \text{pnum1}, \text{read}, \text{query}, \text{truncate(cmap)}, \emptyset \rangle\)

Input write(v)_i
Effect:
if \(-\text{failed}\) then
if \(\text{status} \neq \text{idle}\) then
\(\text{pnum1} \leftarrow \text{pnum1} + 1\)
\(\langle \text{op.pnum}, \text{op.type}, \text{op.phase}, \text{op.cmap}, \text{op.acc}, \text{op.value} \rangle \leftarrow \langle \text{pnum1}, \text{write}, \text{query}, \text{truncate(cmap)}, \emptyset, v \rangle\)

Internal query-fix_i
Precondition:
\(-\text{failed}\)
\(\text{status} = \text{active}\)
\(\text{op.type} \in \{\text{read, write}\}\)
\(\text{op.phase} = \text{query}\)
\(\forall k \in \mathbb{N}, c \in C : \text{op.cmap}(k) = c\)
\(\Rightarrow \exists R \in \text{read-quorums}(c) : R \subseteq \text{op.acc}\)
Effect:
if \(\text{op.type} = \text{read}\) then
\(\text{op.value} \leftarrow \text{value}\)
else
\(\text{value} \leftarrow \text{op.value}\)
\(\text{tag} \leftarrow (\text{tag.seq} + 1, i)\)
\(\text{pnum1} \leftarrow \text{pnum1} + 1\)
\(\text{op.pnum} \leftarrow \text{pnum1}\)
\(\text{op.phase} \leftarrow \text{prop}\)
\(\text{op.cmap} \leftarrow \text{truncate(cmap)}\)
\(\text{op.acc} \leftarrow \emptyset\)

Internal prop-fix_i
Precondition:
\(-\text{failed}\)
\(\text{status} = \text{active}\)
\(\text{op.type} \in \{\text{read, write}\}\)
\(\text{op.phase} = \text{prop}\)
\(\forall k \in \mathbb{N}, c \in C : \text{op.cmap}(k) = c\)
\(\Rightarrow \exists W \in \text{write-quorums}(c) : W \subseteq \text{op.acc}\)
Effect:
\(\text{op.phase} = \text{done}\)

Output read-ack(v)_i
Precondition:
\(-\text{failed}\)
\(\text{status} = \text{active}\)
\(\text{op.type} = \text{read}\)
\(\text{op.phase} = \text{done}\)
\(v = \text{op.value}\)
Effect:
\(\text{op.phase} = \text{idle}\)

Output write-ack_i
Precondition:
\(-\text{failed}\)
\(\text{status} = \text{active}\)
\(\text{op.type} = \text{write}\)
\(\text{op.phase} = \text{done}\)
Effect:
\(\text{op.phase} = \text{idle}\)

Figure 2-3: Reader-Writer: Read/write transitions
Internal gc($k$);
Precondition:
  \[-failed\]
  status = active
gc.phase = idle
cmap($k$) \in C
k = 0 or cmap($k - 1$) = \pm
Effect:
  pnum1 \leftarrow pnum1 + 1
gc.pnum \leftarrow pnum1
gc.phase \leftarrow query
gc.acc \leftarrow \emptyset
gc.index \leftarrow k

Internal gc-query-fix($k$);
Precondition:
  \[-failed\]
  status = active
gc.phase = query
gc.index = k
cmap($k$) \neq \pm
\exists R \in \text{read-quorums}(\text{cmap}(k)):
  \exists W \in \text{write-quorums}(\text{cmap}(k)):
    R \cup W \subseteq \text{gc.acc}
Effect:
  pnum1 \leftarrow pnum1 + 1
gc.pnum \leftarrow pnum1
gc.phase \leftarrow prop
gc.acc \leftarrow \emptyset

Internal gc-prop-fix($k$);
Precondition:
  \[-failed\]
  status = active
gc.phase = prop
gc.index = k
\exists W \in \text{write-quorums}(\text{cmap}(k + 1)) : W \subseteq \text{gc.acc}
Effect:
  cmap($k$) \leftarrow \pm

Internal gc-ack($k$);
Precondition:
  \[-failed\]
  status = active
gc.index = k
cmap($k$) = \pm
Effect:
  gc.phase = idle

Figure 2-4: Reader-Writer$_i$: Rambo Garbage-collection transitions

Each of which accesses certain quorums of replicas. Assume the operation is initiated at node $i$. See Figure 2-5 for a summary of the two phases. First, in the query phase, node $i$ contacts read quorums to determine the most recent available tag and value. Then, in the propagation phase, node $i$ contacts write quorums. If the operation is a read operation, the second phase propagates the largest tag discovered in the query phase, and its associated value. If the operation is a write operation, node $i$ chooses a new tag, strictly larger than every tag discovered in the query phase and propagates the new tag and the new value to the write quorums. Note that every operation accesses both read and write quorums.

During a phase of an operation, whenever node $i$ receives a gossip message from node $j$, it compares the largest phase number $j$ has received from $i$ (by examining pns) to the local
Operation initiated by read, or write(v)_i

Phase 1:

- Node i communicates with a read-quorum from each configuration in op.cmap in order to determine the largest value/tag pair.

Phase 2:

- Node i communicates with a write-quorum from each configuration in in op.cmap to notify it of the current largest value/tag pair (or the new value/tag pair, if it is a write operation).

Figure 2-5: Summary of two phase read or write operation

phase number when the operation began. If j initiated the gossip message after receiving a message from i sent after the phase began, then i adds j to the acc set. In effect, there has been a round-trip message sent from i to j back to i. Also, i then updates its op.cmap if necessary.

Garbage collection operations remove old configurations from the system. A garbage collection operation involves two configurations: the old configuration being removed and the new configuration being established. See Figure 2-6 for a summary of the two phases. A garbage collection operation requires two phases, a query phase and a propagation phase. The first phase contacts a read-quorum and a write-quorum from the old configuration, and the second phase contacts a write-quorum from the new configuration.

Note that, unlike a read or write operation, the first phase of the garbage-collection operation must contact two types of quorums: a read-quorum and a write-quorum for the

Operation initiated by gc(k)_i

Phase 1:

- Node i communicates with a read-quorum from configuration c(k) in order to determine the largest value/tag pair.
- Node i communicates with a write-quorum from configuration c(k) in order to notify it of configuration k + 1.

Phase 2:

- Node i communicates with a write-quorum from configuration c(k + 1) to notify it of the current largest value/tag pair.

Figure 2-6: Summary of two phase garbage-collection operation
configuration being garbage-collected. This ensures that enough nodes are aware of the new configurations, and ensures that any ongoing read/write operations will include the new, larger, configuration.

The $cmap$ is a mapping from integer indices to configurations $\cup\{\perp, \pm\}$, that initially maps every index to $\perp$. The $cmap$ tracks which configurations are active, which are not defined, indicated by $\perp$, and which are removed, indicated by $\pm$. The total ordering on configurations determined by the $Recon$ automata ensures that all nodes agree on which configuration is stored in each position in the array. We define $c(k)$ to be the configuration associated with index $k$.

The record $op$ stores information about the current phase of an ongoing read or write operation, while $gc$ stores information about an ongoing garbage collection operation. (A node can process read and write operations even when a garbage collection operation is ongoing.) The $op.cmap$ subfield records the configuration map for an operation. This consists of the node’s $cmap$ when a phase begins, augmented by any new configurations discovered during the phase. A phase can complete only when the initiator has exchanged information with quorums from every non-removed configuration in $op.cmap$. The $pnum$ subfield records the phase number when the phase begins, allowing the initiator to determine which responses correspond to the current phase. The $acc$ subfield records which nodes from which quorums have responded during the current phase.

In Rambo, configurations go through three phases: proposal, installation, and upgrade. First, a configuration is proposed by a recon event. Next, if the proposal is successful, the Recon service achieves consensus on the new configuration, and notifies participants with decide events. When every non-failed member of the previous configuration has been notified, the configuration is installed. The configuration is upgraded when every configuration with a smaller index has been removed at some process in the system. Once a configuration has been upgraded, it is responsible for maintaining the data.
Chapter 3

Formal Specification of RAMBO II

In this chapter we present the new algorithm in detail, and discuss how it differs from the RAMBO algorithm. The complete implementation, $S$, is the composition of all the automata described—the $Joiner_i$ and $Recon_i$ automata described in RAMBO, the new $Reader-Writer_i$ automaton described here, for all $i$, and all the channels – with all the actions that are not external actions of the RAMBO II specification hidden.

The key problem that prevents rapid stabilization in the original algorithm is the sequential nature of the configuration upgrade mechanism: in RAMBO, configurations are upgraded one at a time, in order. (Recall that in RAMBO, a configuration is upgraded when every configuration with a smaller index has been garbage collected.) Configuration $c(k)$ can be upgraded only if configuration $c(k-1)$ has previously been upgraded. This requirement arises from the need to ensure that information is preserved as configurations are changed. As in RAMBO, a configuration in RAMBO II is upgraded when every configuration with a smaller index has been removed at some process in the system. RAMBO II, however, implements a new reconfiguration protocol that can upgrade any configuration, even if configurations with smaller indices have not been upgraded. Unlike in RAMBO, then, there may be configurations that are not upgraded until they themselves are removed, at the same instant that some configuration with a larger index is upgraded.

After RAMBO II completes an upgrade operation for some configuration, all configurations with smaller indices can be removed. Thus a single upgrade operation in RAMBO II
Signature:
As in RAMBO, with the following modifications:
Internal:
cfg-upgrade(k), k ∈ N^0
cfg-upg-query-fix(k), k ∈ N^0
cfg-upg-prop-fix(k), k ∈ N^0
cfg-upgrade-ack(k), k ∈ N^0

Configuration Management Transitions:

Internal cfg-upgrade(k)_i
Precondition:
¬failed
status = active
upg.phase = idle
(A) cmap(k) ∈ C

(B) ∀ℓ ∈ N, ℓ < k : cmap(ℓ) ≠ ⊥
Effect:
pnumI ← pnumI + 1
(C) upg ← (query, pnumI, cmap, ∅, k)

Internal cfg-upg-query-fix(k)_i
Precondition:
¬failed
status = active
upg.phase = query
upg.target = k
(D) ∀ℓ ∈ N, ℓ < k : upg.cmap(ℓ) ∈ C
⇒ ∃R ∈ read-quorums(upg.cmap(ℓ)) :

(E) R ∪ W ⊆ upg.acc
Effect:
pnumI ← pnumI + 1
(F) upg.pnum ← pnumI
upg.phase ← prop
(G) upg.acc ← ∅

Internal cfg-upg-prop-fix(k)_i
Precondition:
¬failed
status = active
upg.phase = prop
upg.target = k
(H) ∃W ∈ write-quorums(upg.cmap(k)) : W ⊆ upg.acc
Effect:

(I) for ℓ ∈ N : ℓ < k do
(J) cmap(ℓ) ← ±

Configuration Management State:
As in RAMBO, with the following replacing the gc record:
upg, a record with fields:

phase ∈ {idle, query, prop}, initially idle
pnum ∈ N
cmap ∈ CMap,
acc, a finite subset of I

target ∈ N

Internal cfg-upgrade-ack(k)_i
Precondition:
¬failed
status = active
upg.target = k
∀ℓ ∈ N, ℓ < k : cmap(ℓ) = ±
Effect:
upg.phase = idle

Output send((W, v, t, cm, pns, pnr))_{i,j}
Precondition:
¬failed
status = active
j ∈ world
(W, v, t, cm, pns, pnr) =

(world, value, tag, cmap, pnum1, pnum2(j))
Effect:
none

Input recv((W, v, t, cm, pns, pnr))_{j,i}
Effect:

if ¬failed then
  if status ≠ idle then
    status ← active
    world ← world ∪ W
    if t > tag then (value, tag) ← (v, t)
    cmap ← update(cmap, cm)
    pnum2(j) ← max(pnum2(j), pns)
    if op.phase ∈ {query, prop} and pnr ≥ op.pnum then
      op.cmap ← extend(op.cmap, truncate(cm))
      if op.cmap ∈ Truncated then
        op.acc ← op.acc ∪ {j}
      else
        op.acc ← ∅
        op.cmap ← truncate(cm)
      if upg.phase ∈ {query, prop} and pnr ≥ upg.pnum then
        upg.acc ← upg.acc ∪ {j}

Figure 3-1: Reader-Writer_i: Configuration Management transitions
potentially has the effect of many garbage collection operations in RAMBO, each of which can only remove a single configuration. The name has been changed to emphasize the operation’s active role in configuration management: configuration upgrade is an inherent part of preparing a configuration to assume responsibility for the data. The code for the new configuration management mechanism appears in Figure 3-1. All labeled lines in this section refer to the code therein.

We now describe in more detail the configuration upgrade operation, which is at the heart of RAMBO II. A configuration upgrade is a two-phase operation, much like the garbage-collection operation in RAMBO. See Figure 3-2 for a summary of the two phases. An upgrade operation is initiated at node $i$ with a $\text{cfg-upgrade}(k)$ event. When this happens, $\text{cmap}(k)$ must be defined, that is, must be a valid configuration $\in C$ (line A). Additionally, for every configuration $\ell < k$, $\text{cmap}(\ell)$ must be either $\in C$ or removed, that is, $\pm$ (line B).

We refer to configuration $c(k)$ as the target of the upgrade operation, and we refer to the set of configurations to be removed, \{c($\ell$) : $\ell < k \land \text{upg.cmap}(\ell) \in C$\}, as the removal-set of the configuration upgrade operation. The configuration management mechanism guarantees that the removal-set consists of configurations with a contiguous set of indices.

As a result of the $\text{cfg-upgrade}$ event, node $i$ initializes its $\text{upg}$ state (line C), and begins the query phase of the upgrade operation. In particular, node $i$ stores its current $\text{cmap}$ in $\text{upg.cmap}$, which records the configurations that are currently active. Only these configurations (and, in fact, only those with index smaller than $k$) matter during the operation; new configurations are ignored.

The query phase continues until node $i$ receives responses from enough nodes. In particular, for every configuration $c(\ell)$ with index less than $k$ in $\text{upg.cmap}$, there must exist a read-quorum, $R$, of configuration $c(\ell)$, and a write-quorum, $W$, of configuration $c(\ell)$ such that $i$ has received a response (that is, a recent gossip message) from every node in $R \cup W$ (lines D–E).

When the query phase completes, a $\text{cfg-upg-query-fix}$ event occurs. When this event

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1In the conference version of the thesis, this line was omitted. The removal of this line has no detrimental effect on the algorithm, since the operation then completes in zero time. However for clarity sake it is included.
Operation initiated by $\text{cfg-upgrade}(k)$:

Phase 1:
- Node $i$ communicates with a read-quorum from each configuration being removed in order to determine the largest value/tag pair.
- Node $i$ communicates with a write-quorum from each configuration being removed to notify it of the new, active configuration.

Phase 2:
- Node $i$ communicates with a write-quorum from the target configuration being upgraded, to notify it of the current largest value/tag pair.

Figure 3-2: Summary of two phase configuration upgrade operation

occurs, node $i$ then has the most recent tag and value discovered by operations using configurations with index smaller than $k$. Further, all configurations with indices smaller than $k$ have been notified of configuration $c(k)$. Node $i$ then reinitializes $\text{upg}$ to begin the propagation phase (lines F–G).

The propagation phase continues until node $i$ receives responses from a write-quorum in configuration $c(k)$. In particular, there must exist a write-quorum, $W$, of configuration $c(k)$, such that $i$ has received a response from every node in $W$ (line H).

When the propagation phase completes, a $\text{cfg-upg-prop-fix}$ event occurs, which verifies the termination condition. At this point node $i$ has ensured that configuration $c(k)$ has received the most recent value known to $i$, which, as a result of the query phase, is itself a recent value. At this point, the configurations with index $< k$ are no longer needed, and node $i$ removes these configurations from its local $cmap$, setting $cmap(\ell) = \pm$ for all $\ell < k$ (line I–J). Gossip messages may eventually notify other processes that these configurations have been removed.

Finally, a $\text{cfg-upgrade-ack}(k)$ event notifies the client that configuration $c(k)$ has been successfully upgraded.

Notice that the algorithm allows a nondeterministic choice of which configuration to upgrade – and therefore which configurations to remove. Therefore it is possible to restrict the algorithm so that it removes only the smallest configuration, upgrading the configurations one at a time. In this case the algorithm progresses exactly as the original RAMBO
algorithm. Therefore it is clearly possible, by restricting the nondeterminism appropriately, to implement RAMBO II in such a way as to guarantee equivalent performance as RAMBO. However we will show that by allowing greater flexibility we can achieve equivalent safety results and improved performance.

The new algorithm introduces several difficulties not present in RAMBO. Consider, for example, a nice property guaranteed by the sequential garbage collection algorithm in RAMBO: every configuration is upgraded before it is removed. In RAMBO II, on the other hand, some configurations never receive up to date information; a configuration may be upgraded at the same instant it is removed.

As a result of this fact, a number of plausible improvements fail. Assume that during an ongoing upgrade operation for configuration $c(k)$ initiated by node $i$, node $i$ receives a message indicating that configuration $c(k')$ has been removed, for some $k' < k$. In RAMBO II, node $i$ sets $cmap(k') = \pm$, but does not change $upg.cmap$. Consider the following incorrect modification to the configuration management mechanism. When node $i$ receives such a message, it sets $upg.cmap(k')$ to $\pm$. Since the configuration has been removed, it seems plausible that the configuration upgrade operation can safely ignore it, thus completing more quickly. It turns out, however, that this improvement results in a race condition that can lead to data loss. The configuration upgrade operation that removes configuration $c(k')$ might occur concurrently with the operation at node $i$ upgrading configuration $c(k)$. This concurrency might result in data being propagated from configuration $c(k')$ to a configuration $c(k'') : k' < k'' < k$ that has already been processed by the upgrade operation at node $i$. The data thus propagated might then be lost.
Chapter 4

Notation and Basic Lemmas

This chapter is, to a large extent, a restatement of notation and results from the original RAMBO paper [13]. Some of the notation in the proofs has been slightly modified to account for the new configuration management mechanism, and some of the proofs have therefore been updated, but the results are essentially unchanged. Much of this chapter is taken directly from [13].

4.1 Good Executions

Throughout the rest of this thesis, we will talk about “good” executions of the algorithm. In this section, we present a set of environment assumptions that define a “good” execution. In general, the assumptions we will present require well-formed requests: clients follow the protocol to join and to initiate reconfigurations; clients initiate only one operation at a time; clients wait for appropriate acknowledgments before proceeding.

We consider executions of \( S \) (recall that \( S \) is the entire system combining Reader-Writer, Recon and Joiner automata) whose traces satisfy certain assumptions about the environment. We call these good executions. In particular, an “invariant” is a statement that is true of all states that are reachable in good executions of \( S \). The environment assumptions are simple “well-formedness” conditions:

- **Well-formedness for Reader-Writer:**
- For every \( x \) and \( i \):
  
  * No \( \text{join}(\text{rambo}, \ast, x, i) \), \( \text{read}_{x,i} \), \( \text{write}(\ast, x, i) \), or \( \text{recon}(\ast, \ast, x, i) \) event is preceded by a \text{fail}_i event.
  
  * At most one \( \text{join}(\text{rambo}, \ast, x, i) \) event occurs.

  * Any \( \text{read}_{x,i} \), \( \text{write}(\ast, x, i) \), or \( \text{recon}(\ast, \ast, x, i) \) event is preceded by a \( \text{join-ack}(\text{rambo}, x, i) \) event.

  * Any \( \text{read}_{x,i} \), \( \text{write}(\ast, x, i) \), or \( \text{recon}(\ast, \ast, x, i) \) event is preceded by an \text{-ack} event for any preceding event of any of these kinds.

- For every \( x \) and \( c \), at most one \( \text{recon}(\ast, c, x, \ast) \) event occurs. (This says that configuration identifiers that are proposed in \( \text{recon} \) events are unique. It does not say that the membership and/or quorum sets are unique—just the identifiers. The same membership and quorum sets may be associated with different configuration identifiers.) Uniqueness of configuration identifiers is achievable using local process identifiers and sequence numbers.

- For every \( c, c', x, \) and \( i \), if a \( \text{recon}(c, c', x, i) \) event occurs, then it is preceded by:

  * A \( \text{report}(c, x, i) \) event, and

  * A \( \text{join-ack}(\text{rambo}, x, j) \) event for every \( j \in \text{members}(c') \).

- **Well-formedness for Recon:**

  - For every \( i \):

    * No \( \text{join}(\text{recon})_i \) or \( \text{recon}(\ast, \ast)_i \) event is preceded by a \text{fail}_i event.

    * At most one \( \text{join}(\text{recon})_i \) event occurs.

    * Any \( \text{recon}(\ast, \ast)_i \) event is preceded by a \( \text{join-ack}(\text{recon})_i \) event.

    * Any \( \text{recon}(\ast, \ast)_i \) event is preceded by an \text{-ack} for any preceding \( \text{recon}(\ast, \ast)_i \) event.

  - For every \( c \), at most one \( \text{recon}(\ast, c, \ast) \) event occurs.

\(^1\)The following properties appear in Chapter 6, but we repeat them here for completeness.
For every $c$, $c'$, $x$, and $i$, if a \text{recon}(c, c')_i$ event occurs, then it is preceded by:

* A \text{report}(c)_i event, and

* A \text{join-ack}(\text{recon})_j for every $j \in \text{members}(c')$.

### 4.2 Notational conventions

In this section, we introduce some definitions and notational conventions, and we add certain history variables to the global state of the system $S$.

Definitions:

- \textit{update}, a binary function on $C_{\pm}$, defined by $\text{update}(c, c') = \max(c, c')$ if $c$ and $c'$ are comparable (in the augmented partial ordering of $C_{\pm}$), $\text{update}(c, c') = c$ otherwise.

- \textit{extend}, a binary function on $C_{\pm}$, defined by $\text{extend}(c, c') = c'$ if $c = \bot$ and $c' \in C$, and $\text{extend}(c, c') = c$ otherwise.

- \textit{CMap}, the set of configuration maps, defined as the set of mappings from $\mathbb{N}$ to $C_{\pm}$. The \textit{update} and \textit{extend} operators are extended element-wise to binary operations on \textit{CMap}.

- \textit{truncate}, a unary function on \textit{CMap}, defined by $\text{truncate}(cm)(k) = \bot$ if there exists $\ell \leq k$ such that $cm(\ell) = \bot$, $\text{truncate}(cm)(k) = cm(k)$ otherwise. This truncates configuration map $cm$ by removing all the configuration identifiers that follow a $\bot$.

- \textit{Truncated}, the subset of \textit{CMap} such that $cm \in \text{Truncated}$ if and only if $\text{truncate}(cm) = cm$.

- \textit{Usable}, the subset of \textit{CMap} such that $cm \in \text{Usable}$ iff the pattern occurring in $cm$ consists of a prefix of finitely many $\pm$s, followed by an element of $C$, followed by an infinite sequence of elements of $\mathbb{C} \cup \{\bot\}$ in which all but finitely many elements are $\bot$.

An \textit{operation} is a pair $(n, i)$ consisting of a natural number $n$ and an index $i \in I$. Here, $i$ is the index of the process running the operation, and $n$ is the value of $\text{pnum1}_i$ just after the \texttt{read}, \texttt{write}, or \texttt{cfg-upgrade} event of the operation occurs.
We introduce the following history variables:

- **in-transit**, a set of messages, initially \( \emptyset \).
  A message is added to the set when it is sent by any \textit{Reader-Writer}_i to any \textit{Reader-Writer}_j.
  No message is ever removed from this set.

- For every \( k \in \mathbb{N} \):
  1. \( c(k) \in C \); initially undefined.
     This is set when the first \textit{new-config}(c, k)_i occurs, for some \( c \) and \( i \). It is set to the \( c \) that appears as the first argument of this action.

- For every operation \( \pi \):
  1. \( \text{tag}(\pi) \in T \); initially undefined.
     This is set to the value of \textit{tag} at the process running \( \pi \), at the point right after \( \pi \)’s \textit{query-fix} or \textit{cfg-upg-query-fix} event occurs. If \( \pi \) is a read or configuration upgrade operation, this is the highest tag that it encounters during the query phase. If \( \pi \) is a write operation, this is the new tag that is selected for performing the write.

- For every read or write operation \( \pi \):
  1. \( \text{query-cmap}(\pi) \), a \textit{CMap}, initially undefined.
     This is set in the \textit{query-fix} step of \( \pi \), to the value of \textit{op.cmap} in the pre-state.
  2. \( R(\pi, k) \), for \( k \in \mathbb{N} \) a subset of \( I \), initially undefined.
     This is set in the \textit{query-fix} step of \( \pi \), for each \( k \) such that \textit{query-cmap}(\( \pi \))(k) \( \in C \).
     It is set to an arbitrary \( R \in \text{read-quorums}(c(k)) \) such that \( R \subseteq \text{op.acc} \) in the pre-state.
  3. \( \text{prop-cmap}(\pi) \), a \textit{CMap}, initially undefined.
     This is set in the \textit{prop-fix} step of \( \pi \), to the value of \textit{op.cmap} in the pre-state.
  4. \( W(\pi, k) \), for \( k \in \mathbb{N} \) a subset of \( I \), initially undefined.
     This is set in the \textit{prop-fix} step of \( \pi \), for each \( k \) such that \textit{prop-cmap}(\( \pi \))(k) \( \in C \).
     It is set to an arbitrary \( W \in \text{write-quorums}(c(k)) \) such that \( W \subseteq \text{op.acc} \) in the pre-state.
• For every configuration upgrade operation \( \gamma \) for \( k \):

1. \( \text{removal-set}(\gamma) \), a subset of \( \mathbb{N} \), initially undefined.
   This is set in the \( \text{cfg-upgrade} \) step of \( \gamma \), to the set \( \{ \ell : \ell < k, \ cmap(\ell) \neq \pm \} \).

2. \( R(\gamma, \ell) \), for \( \ell \in \mathbb{N} \), a subset of \( I \), initially undefined.
   This is set in the \( \text{cfg-upg-query-fix} \) step of \( \gamma \), for each \( \ell \in \text{removal-set}(\gamma) \), to an arbitrary \( R \in \text{read-quorums}(c(\ell)) \) such that \( R \subseteq \text{upg.acc} \) in the pre-state.

3. \( W_1(\gamma, \ell) \), for \( \ell \in \mathbb{N} \), a subset of \( I \), initially undefined.
   This is set in the \( \text{cfg-upg-query-fix} \) step of \( \gamma \), for each \( \ell \in \text{removal-set}(\gamma) \), to an arbitrary \( W \in \text{write-quorums}(c(\ell)) \) such that \( W \subseteq \text{upg.acc} \) in the pre-state.

4. \( W_2(\gamma) \), a subset of \( I \), initially undefined.
   This is set in the \( \text{cfg-upg-prop-fix} \) step of \( \gamma \), to an arbitrary \( W \in \text{write-quorums}(c(k)) \) such that \( W \subseteq \text{upg.acc} \) in the pre-state.

In any good execution \( \alpha \), we define the following events (more precisely, we are giving additional names to some existing events):

1. For every read or write operation \( \pi \):

   (a) \( \text{query-phase-start}(\pi) \), initially undefined.
   This is defined in the \( \text{query-fix} \) step of \( \pi \), to be the unique earlier event at which the collection of query results was started and not subsequently restarted. This is either a \( \text{read} \), \( \text{write} \), or \( \text{recv} \) event.

   (b) \( \text{prop-phase-start}(\pi) \), initially undefined.
   This is defined in the \( \text{prop-fix} \) step of \( \pi \), to be the unique earlier event at which the collection of propagation results was started and not subsequently restarted. This is either a \( \text{query-fix} \) or \( \text{recv} \) event.

### 4.3 Configuration map invariants

In this section, we give invariants describing the kinds of configuration maps that may appear in various places in the state of \( \mathcal{S} \). We begin with a lemma saying that various operations
yield or preserve the “usable” property:

**Lemma 4.3.1** 1. If $cm, cm' \in Usable$ then $update(cm, cm') \in Usable$.

2. If $cm \in Usable$, $k \in N$, $c \in C$, and $cm'$ is identical to $cm$ except that $cm'(k) = update(cm(k), c)$, then $cm' \in Usable$.

3. If $cm, cm' \in Usable$ then $extend(cm, cm') \in Usable$.

4. If $cm \in Usable$ then $truncate(cm) \in Usable$.

**Proof.** Part 1 is shown using a case analysis based on which of $cm$ and $cm'$ has a longer prefix of $\pm$'s. Part 2 uses a case analysis based on where $k$ is with respect to the prefix of $\pm$'s. Part 3 and Part 4 are also straightforward. □

The next invariant (recall from Section 4.1 that this means a property of all states that arise in good executions of $S$) describes some properties of $cmap_i$ that hold while $Reader-Writer_i$ is conducting a configuration upgrade operation:

**Invariant 4.3.2** If $upg.phase_i \neq idle$ and $upg.target_i = k$, then:

1. $\forall \ell : \ell \leq k \Rightarrow cmap(\ell)_i \in C \cup \{\pm\}$.

2. If $k_1 = \min\{\ell : \ell \leq k \text{ and } upg.cmap(\ell) \neq \pm\}$ then $k_1 = 0$ or $cmap(k_1 - 1)_i = \pm$.

**Proof.** By the precondition of $cfg\text{-}upgrade(k)_i$ and monotonicity of all the changes to $cmap_i$. □

We next proceed to describe the patterns of $C$, $\bot$, and $\pm$ values that may occur in configuration maps in various places in the system state.

**Invariant 4.3.3** Let $cm$ be a CMap that appears as one of the following:

1. The $cm$ component of some message in in-transit.

2. $cmap_i$ for any $i \in I$.

3. $op\text{-}cmap_i$ for some $i \in I$ for which $op\text{-}phase \neq idle$. 

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4. \( \text{query-cmap}(\pi) \) or \( \text{prop-cmap}(\pi) \) for any operation \( \pi \).

5. \( \text{upg-cmap}_i \) for some \( i \in I \) for which \( \text{upg-phase} \neq \text{idle} \).

Then \( cm \in \text{Usable} \).

In the following proof and elsewhere, we use dot notation to indicate components of a state, for example, \( s.cmap_i \) indicates the value of \( cmap_i \) in state \( s \).

**Proof.** By induction on the length of a finite good execution.

*Base:* Part 1 holds because initially, \( \text{in-transit} \) is empty. Part 2 holds because initially, for every \( i \), \( cmap(0)_i = c_0 \) and \( cmap(k)_i = \bot \); the resulting \( \text{CMap} \) is in \( \text{Usable} \). Part 3 and Part 5 hold vacuously, because in the initial state, all \( \text{op-phase} \) and \( \text{upg-phase} \) values are \( \text{idle} \). Part 4 also holds vacuously, because in the initial state, all \( \text{query-cmap} \) and \( \text{prop-cmap} \) variables are undefined.

*Inductive step:* Let \( s \) and \( s' \) be the states before and after the new event, respectively. We consider Parts 1–5 one by one.

For Part 1, the interesting case is a \( \text{send}_i \) event that puts a message containing \( cm \) in \( \text{in-transit} \). The precondition on the \( \text{send} \) action implies that \( cm \) is set to \( s.cmap_i \). The inductive hypothesis, Part 2, implies that \( s.cmap_i \in \text{Usable} \), which suffices.

For Part 2, fix \( i \). The interesting cases are those that may change \( cmap_i \), namely, \( \text{new-config}_i \), \( \text{recv}_i \) for a gossip (non-join) message, and \( \text{cfg-upg-prop-fix}_i \). The latter case is the only one modified from the original RAMBO algorithm.

1. \( \text{new-config}(c, *)_i \).

   By inductive hypothesis, \( s.cmap_i \in \text{Usable} \). The only change this can make is changing a \( \bot \) to \( c \). Then Lemma 4.3.1, Part 2, implies that \( s'.cmap_i \in \text{Usable} \).

2. \( \text{recv}((*, *, cm, *, *))_i \).

   By inductive hypothesis, \( cm \in \text{Usable} \) and \( s.cmap_i \in \text{Usable} \). The step sets \( s'.cmap_i \) to \( \text{update}(s.cmap_i, cm) \). Lemma 4.3.1, Part 1, then implies that \( s'.cmap_i \in \text{Usable} \).
3. \texttt{cfg-upg-prop-fix}(k)_i.

This sets \( cmap(\ell)_i \) to \( \pm \) for all \( \ell < k \). By the definition of this step, \( s'.cmap(\ell)_i = \pm \) for \( \ell < k \).

If \( s.cmap(k-1)_i = \pm \), then the operation has no effect, and \( s'.cmap_i = s.cmap_i \in \text{Usable} \). Assume, then, that \( s.cmap(k-1)_i \in C \cup \{\bot\} \). This implies, by the inductive hypothesis showing \( s.cmap_i \in \text{Usable} \), that \( s.cmap(\ell)_i \in C \cup \{\bot\} \) for all \( \ell \geq k-1 \).

By Invariant 4.3.2, we know that \( s.cmap(k)_i \in C \cup \{\pm\} \), and therefore \( s.cmap(k)_i \in C \). Therefore \( s'.cmap(k)_i \in C \) and \( s'.cmap(\ell)_i \in C \cup \{\bot\} \) for all \( \ell > k \), since the \texttt{cfg-upg-prop-fix} does not change entries in the \( cmap \) larger than \( k-1 \). Further, there are only finitely many entries in \( s.cmap_i \) that are in \( C \) (by the inductive hypothesis), and so there are still only finitely many entries in \( s'.cmap_i \). Therefore, \( s'.cmap_i \in \text{Usable} \).

For Part 3, the interesting actions to consider are those that modify \( op.cmap \), namely, \texttt{read}_i, \texttt{write}_i, \texttt{recv}_i, and \texttt{query-fix}_i.

1. \texttt{read}_i, \texttt{write}_i, or \texttt{query-fix}_i.

By inductive hypothesis, \( s.cmap_i \in \text{Usable} \). The new step sets \( s'.op.cmap_i \) to \texttt{truncate}(\( s.cmap_i \)); since \( s.cmap_i \in \text{Usable} \), Lemma 4.3.1, Part 4, implies that this is also usable.

2. \texttt{recv}((*, *, cm, *, *))_i.

This step may alter \( op.cmap_i \) only if \( s.op.phase \in \{\text{query, prop}\} \), and then in only two ways: by setting it either to \texttt{extend}(\( s.op.cmap_i, \text{truncate}(cm) \)) or to \texttt{truncate}(\( \text{update}(s.cmap_i, cm) \)).

The inductive hypothesis implies that \( s.op.cmap_i, cmap_i, \) and \( cm \) are all in \( \text{Usable} \).

Lemma 4.3.1 implies that \texttt{truncate}, \texttt{extend}, and \texttt{update} all preserve usability. Therefore, \( s'.op.cmap_i \in \text{Usable} \).

For Part 4, the actions to consider are \texttt{query-fix}_i and \texttt{prop-fix}_i.

1. \texttt{query-fix}_i.

This sets \( s'.query-cmap_i \) to the value of \( s.op.cmap_i \). Since by inductive hypothesis the latter is usable, so is \( s'.query-cmap_i \).
2. \( \text{prop-fix}_i \).

This sets \( s'.\text{prop-cmap}_i \) to the value of \( s.\text{op-cmap}_i \). Since by inductive hypothesis, the latter is usable, so is \( s'.\text{prop-cmap}_i \).

For Part 5, the actions to consider are \( \text{cfg-upgrade}(k)_i \) and \( \text{cfg-upg-query-fix}(k)_i \). These set \( s'.\text{upg-cmap}_i \) to the value of \( s.\text{cmap}_i \). Since by the inductive hypothesis the latter is usable, so is \( s'.\text{upg-cmap}_i \). \( \square \)

We now strengthen Invariant 4.3.3 to say more about the form of the \( \text{CMaps} \) that are used for read and write operations:

**Invariant 4.3.4** Let \( \text{cm} \) be a \( \text{CMap} \) that appears as \( \text{op-cmap}_i \) for some \( i \in I \) for which \( \text{op.phase}_i \neq \text{idle} \), or as \( \text{query-cmap}(\pi) \) or \( \text{prop-cmap}(\pi) \) for any operation \( \pi \). Then:

1. \( \text{cm} \in \text{Truncated} \).

2. \( \text{cm} \) consists of finitely many \( \pm \) entries followed by finitely many \( C \) entries followed by an infinite number of \( \bot \) entries.

**Proof.** We prove that the desired properties hold for a \( \text{cm} \) that is \( \text{op-cmap}_i \). The same properties for \( \text{query-cmap}_i \) and \( \text{prop-cmap}_i \) follow by the way they are defined, from \( \text{op-cmap}_i \).

To prove Part 1 we proceed by induction. In the initial state, \( \text{op.phase}_i = \text{idle} \), which makes the claim vacuously true. For the inductive step we consider all actions that alter \( \text{op-cmap}_i \):

1. \( \text{read}_i, \text{write}_b \) or \( \text{query-fix}_i \).

   These set \( \text{op-cmap}_i \) to \( \text{truncate}(\text{cmap}_i) \), which is necessarily in \( \text{Truncated} \).

2. \( \text{recv}_i \).

   This first sets \( \text{op-cmap}_i \) to a preliminary value and then tests if the result is in \( \text{Truncated} \). If it is, we are done. If not, then this step resets \( \text{op-cmap}_i \) to \( \text{truncate}(\text{cmap}_i) \), which is in \( \text{Truncated} \).

   To see Part 2, note that \( \text{cm} \in \text{Usable} \) by Invariant 4.3.3. The fact that \( \text{cm} \in \text{Truncated} \) then follows from the definition of \( \text{Usable} \) and Part 1. \( \square \)
4.4 Phase guarantees

In this section, we present results saying what is achieved by the individual operation phases. We give four lemmas, describing the messages that must be sent and received and the information flow that must occur during the two phases of configuration-upgrades and during the two phases of read and write operations.

Note that these lemmas treat the case where $j = i$ uniformly with the case where $j \neq i$. This is because, in the Reader-Writer algorithm, communication from a location to itself is treated uniformly with communication between two different locations. We first consider the query phase of a configuration-upgrade:

**Lemma 4.4.1** Suppose that a cfg-upg-query-fix($k$)$_i$ event for configuration upgrade operation $\gamma$ occurs in $\alpha$ and $k' \in$ removal-set($\gamma$). Suppose $j \in R(\gamma, k') \cup W_1(\gamma, k')$. Then there exist messages $m$ from $i$ to $j$ and $m'$ from $j$ to $i$ such that:

1. $m$ is sent after the cfg-upgrade($k$)$_i$ event of $\gamma$.

2. $m'$ is sent after $j$ receives $m$.

3. $m'$ is received before the cfg-upg-query-fix($k$)$_i$ event of $\gamma$.

4. In any state after $j$ receives $m$, cmap($\ell$)$_j \neq \bot$ for all $\ell \leq k$.

5. tag($\gamma$) $\geq t$, where $t$ is the value of tag$_j$ in any state before $j$ sends message $m'$.

**Proof.** The phase number discipline implies the existence of the claimed messages $m$ and $m'$.

For Part 4, the precondition of cfg-upgrade($k$) implies that, when the cfg-upgrade($k$)$_i$ event of $\gamma$ occurs, cmap($\ell$)$_i \neq \bot$ for all $\ell \leq k$. Therefore, $j$ sets cmap($\ell$)$_j \neq \bot$ for all $\ell \leq k$ when it receives $m$. Monotonicity of cmap$_j$ ensures that this property persists forever.

For Part 5, let $t$ be the value of tag$_j$ in any state before $j$ sends message $m'$. Let $t'$ be the value of tag$_j$ in the state just before $j$ sends $m'$. Then $t \leq t'$, by monotonicity. The tag component of $m'$ is equal to $t'$, by the code for send. Since $i$ receives this message before the cfg-upg-query-fix($k$), it follows that tag($\gamma$) is set by $i$ to a value $\geq t$. 

□
Next, we consider the propagation phase of a configuration upgrade:

**Lemma 4.4.2** Suppose that a \( \text{cfg-upg-prop-fix}(k)_i \) event for a configuration upgrade operation \( \gamma \) occurs in \( \alpha \). Suppose that \( j \in W_2(\gamma) \).

Then there exist messages \( m \) from \( i \) to \( j \) and \( m' \) from \( j \) to \( i \) such that:

1. \( m \) is sent after the \( \text{cfg-upg-query-fix}(k)_i \) event of \( \gamma \).
2. \( m' \) is sent after \( j \) receives \( m \).
3. \( m' \) is received before the \( \text{cfg-upg-prop-fix}(k)_i \) event of \( \gamma \).
4. In any state after \( j \) receives \( m \), \( \text{tag}_j \geq \text{tag}(\gamma) \).

**Proof.** The phase number discipline implies the existence of the claimed messages \( m \) and \( m' \).

For Part 4, when \( j \) receives \( m \), it sets \( \text{tag}_j \) to be \( \geq \text{tag}(\gamma) \). Monotonicity of \( \text{tag}_j \) ensures that this property persists in later states.

Next, we consider the query phase of read and write operations:

**Lemma 4.4.3** Suppose that a \( \text{query-fix}_i \) event for a read or write operation \( \pi \) occurs in \( \alpha \).

Let \( k, k' \in \mathbb{N} \). Suppose \( \text{query-cmap}(\pi)(k) \in C \) and \( j \in R(\pi, k) \).

Then there exist messages \( m \) from \( i \) to \( j \) and \( m' \) from \( j \) to \( i \) such that:

1. \( m \) is sent after the \( \text{query-phase-start}(\pi) \) event.
2. \( m' \) is sent after \( j \) receives \( m \).
3. \( m' \) is received before the \( \text{query-fix} \) event of \( \pi \).
4. If \( t \) is the value of \( \text{tag}_j \) in any state before \( j \) sends \( m' \), then:
   
   (a) \( \text{tag}(\pi) \geq t \).
   
   (b) If \( \pi \) is a write operation then \( \text{tag}(\pi) > t \).
5. If \( \text{cmap}(t)_j \neq \bot \) for all \( t \leq k' \) in any state before \( j \) sends \( m' \), then \( \text{query-cmap}(\pi)(t) \in C \) for some \( t \geq k' \).
Proof. The phase number discipline implies the existence of the claimed messages $m$ and $m'$.

For Part 4, the tag component of message $m'$ is $\geq t$, so $i$ receives a tag that is $\geq t$ during the query phase of $\pi$. Therefore, $\text{tag}(\pi) \geq t$. Also, if $\pi$ is a write, the effects of the query-fix imply that $\text{tag}(\pi) > t$.

Finally, we show Part 5. In the cm component of message $m'$, $\text{cm}(\ell) \neq \bot$ for all $\ell \leq k'$. Therefore, $\text{truncate}(\text{cm})(\ell) = \text{cm}(\ell)$ for all $\ell \leq k'$, so $\text{truncate}(\text{cm})(\ell) \neq \bot$ for all $\ell \leq k'$.

Let $cm'$ be the configuration map $\text{extend}(\text{op.cmap}_i, \text{truncate}(\text{cm}))$ computed by $i$ during the effects of the recv event for $m'$. Since $i$ does not reset $\text{op.acc}$ to $\emptyset$ in this step, by definition of the query-phase-start event, it follows that $cm' \in \text{Truncated}$, and $cm'$ is the value of $\text{op.cmap}_i$ just after the recv step.

Fix $\ell, 0 \leq \ell \leq k'$. We claim that $cm'(\ell) \neq \bot$. We consider cases:

1. $\text{op.cmap}(\ell)_i \neq \bot$ just before the recv step.

Then the definition of $\text{extend}$ implies that $cm'(\ell) \neq \bot$, as needed.

2. $\text{op.cmap}(\ell)_i = \bot$ just before the recv step and $\text{truncate}(\text{cm})(\ell) \in C$.

Then the definition of $\text{extend}$ implies that $cm'(\ell) \in C$, which implies that $cm'(\ell) \neq \bot$, as needed.

3. $\text{op.cmap}(\ell)_i = \bot$ just before the recv step and $\text{truncate}(\text{cm})(\ell) \notin C$.

Since $\text{truncate}(\text{cm})(\ell) \neq \bot$, it follows that $\text{truncate}(\text{cm})(\ell) = \pm$. Since $\text{truncate}(\text{cm})(\ell) = \pm$ and $\text{truncate}(\text{cm}) \in \text{Usable}$, it follows that, for some $\ell' > \ell$, $\text{truncate}(\text{cm})(\ell') \in C$.

By the case assumption, $\text{op.cmap}(\ell)_i = \bot$ just before the recv step. Since, by Invariant 4.3.4, $\text{op.cmap}_i \in \text{Truncated}$, it follows that $\text{op.cmap}(\ell')_i = \bot$ before the recv step.

Then by definition of $\text{extend}$, we have that $cm'(\ell) = \bot$ while $cm'(\ell') \in C$. This implies that $cm' \notin \text{Truncated}$, which contradicts the fact, already shown, that $cm' \notin \text{Truncated}$, So this case cannot arise.
Since this argument holds for all $\ell$, $0 \leq \ell \leq k'$, it follows that $cm'(\ell) \neq \perp$ for all $\ell \leq k'$. Since $cm'(\ell) \neq \perp$ for all $\ell \leq k'$, Invariant 4.3.3 implies that $cm' \in \text{Usable}$, which implies by definition of $\text{Usable}$ that $cm'(\ell) \in C$ for some $\ell \geq k'$. That is, $\text{op-cmap}_i(\ell) \in C$ for some $\ell \geq k'$ immediately after the $\text{recv}$ step. This implies that $\text{query-cmap}(\pi)(\ell) \in C$ for some $\ell \geq k'$, as needed. \qed

And finally, we consider the propagation phase of read and write operations:

**Lemma 4.4.4** Suppose that a prop-fix, event for a read or write operation $\pi$ occurs in $\alpha$. Suppose $\text{prop-cmap}(\pi)(k) \in C$ and $j \in W(\pi, k)$.

Then there exist messages $m$ from $i$ to $j$ and $m'$ from $j$ to $i$ such that:

1. $m$ is sent after the $\text{prop-phase-start}(\pi)$ event.

2. $m'$ is sent after $j$ receives $m$.

3. $m'$ is received before the $\text{prop-fix}$ event of $\pi$.

4. In any state after $j$ receives $m$, $\text{tag}_j \geq \text{tag}(\pi)$.

5. If $\text{cmap}(\ell)_j \neq \perp$ for all $\ell \leq k'$ in any state before $j$ sends $m'$, then $\text{prop-cmap}(\pi)(\ell) \in C$ for some $\ell \geq k'$.

**Proof.** The phase number discipline implies the existence of the claimed messages $m$ and $m'$.

For Part 4, let $m.\text{tag}$ be the $\text{tag}$ field of message $m$. Since $m$ is sent after the $\text{prop-phase-start}$ event, which is not earlier than the $\text{query-fix}$, it must be that $m.\text{tag} \geq \text{tag}(\pi)$. Therefore, by the effects of the $\text{recv}$, just after $j$ receives $m$, $\text{tag}_j \geq m.\text{tag} \geq \text{tag}(\pi)$. Then monotonicity of $\text{tag}_j$ implies that $\text{tag}_j \geq \text{tag}(\pi)$ in any state after $j$ receives $m$.

For Part 5, the proof is analogous to the proof of Part 5 of Lemma 4.4.3. In fact, it is identical except for the final conclusion, which now says that $\text{prop-cmap}(\pi)(\ell) \in C$ for some $\ell \geq k'$.
Chapter 5

Atomic Consistency

This section contains the proof of atomic consistency. The proof is carried out in several stages. First in Section 5.1 we present some lemmas about the new configuration management mechanism, describing the relationship between configuration upgrade operations. Section 5.2 describes the relationship between read/write operations and configuration upgrade operations. Section 5.3 then considers two read or write operations, and culminates in Lemma 5.3.3, which says that tags are monotonic with respect to non-concurrent read or write operations. Finally, Section 5.4 uses the tags to define a partial order on operations and verifies the four properties required for atomicity.

5.1 Behavior of configuration upgrade

This section presents the key new technical lemmas on which the proof of atomicity is based. Specifically, we present lemmas describing information flow between configuration upgrade operations. These lemmas assert the existence of a sequence of configuration upgrade operations on which we can make certain necessary guarantees. In particular, the key property is that the tags are monotonically increasing with respect to the specific sequence of upgrade operations, guaranteeing that value/tag information is propagated to newer configurations.

The first lemma shows that if all configuration upgrade operations remove two particular configurations together, then those two configuration are always in the same state in all
Lemma 5.1.1 Suppose that \( k > 0 \), and \( \alpha \) is an execution in which no \( \text{cfg-upg-prop-fix}(k) \) event occurs in \( \alpha \). Suppose that \( cm \) is a CMap that appears as one of the following in any state in \( \alpha \):

1. The \( cm \) component of some message in in-transit.
2. \( cmap_i \) for any \( i \in I \).

If \( cm(k - 1) = \pm \) then \( cm(k) = \pm \).

Proof. Fix some \( \alpha \) and \( k > 0 \) such that no \( \text{cfg-upg-prop-fix}(k) \) event occurs in \( \alpha \). We proceed by induction on the length of a finite prefix of \( \alpha \): for every action in \( \alpha \), if before the action \( cm(k - 1) = \pm \) \( \Rightarrow \) \( cm(k) = \pm \), then the same implication holds after the action.

Base: For Part 1, the conclusion follows vacuously because initially in-transit is empty.
For Part 2, the conclusion again follows vacuously because initially \( cmap_i(\ell) \neq \pm \) for all \( i \) and \( \ell \).

Inductive step: Let \( s \) and \( s' \) be the states before and after the new event, respectively. We consider Parts 1 and 2 separately.
For Part 1, the interesting case is a \( \text{send}_i \) event that puts a message containing \( cm \) in in-transit. The precondition on the \( \text{send} \) action implies that \( cm \) is set to \( s.cmap_i \). The inductive hypothesis, Part 2, implies that if \( s.cmap(k - 1) = \pm \), then \( s.cmap(k) = \pm \). Therefore in state \( s' \), the same holds for \( cm \), which has been added to in-transit.
For Part 2, fix \( i \). The interesting cases are those that may change \( cmap_i \), namely, \( \text{new-config}_i \), \( \text{recv}_i \) for a gossip message, and \( \text{cfg-upg-prop-fix}_i \).

1. \( \text{new-config}(c,*),_i \).
   If \( s'.cmap(k - 1)_i = \pm \), then \( s.cmap(k - 1)_i = \pm \), since installing a new configuration does not set any entry to \( \pm \). Then by the inductive hypothesis \( s.cmap(k)_i = \pm \), which implies that \( s'.cmap(k)_i = \pm \), since this action cannot modify an entry that is already \( \pm \).
2. \( \text{recv}((\ast, \ast, cm, \ast, \ast)) \).

First, if \( cm(0) \neq \pm \), then the message does not cause any entry in \( s.cmap \) to be set to \( \pm \), and as in Case 1 the desired property still holds. Also, if \( s.cmap(0) \neq \pm \), then for all \( \ell \), \( s'.cmap(\ell) = \pm \) if and only if \( cm(\ell) = \pm \). By the inductive hypothesis \( cm(k-1) = \pm \implies cm(k) = \pm \), so the desired conclusion follows. For the rest of this case, we will assume that \( cm(0) = \pm \) and \( s.cmap(0) = \pm \).

By Invariant 4.3.3, \( cm \in Usable \). Therefore we can define \( k_{msg-max} \) such that \( cm(\ell) = \pm \) for all \( \ell \leq k_{msg-max} \) and \( cm(\ell) \neq \pm \) for all \( \ell > k_{msg-max} \). Similarly, we can define \( k_{max} \) such that \( s.cmap(\ell)_i = \pm \) for all \( \ell \leq k_{max} \) and \( s.cmap(\ell)_i \neq \pm \) for all \( \ell > k_{max} \). Define \( k'_{max} \) in the same way for the poststate, \( s' \).

There are two cases. First, assume \( k_{max} \geq k_{msg-max} \). Then \( k'_{max} = k_{max} \), by the monotonicity of \( CMap \). By our inductive hypothesis \( s.cmap(k-1) = \pm \implies s.cmap(k) = \pm \); it follows, then, that if \( k-1 \leq k_{max} \) then \( k \leq k_{max} \). Therefore if \( k-1 \leq k'_{max} \), then \( k \leq k'_{max} \). Finally, then, if \( s'.cmap(k-1) = \pm \), then \( s'.cmap(k) = \pm \).

Assume, then, that \( k_{msg-max} > k_{max} \). Then after the update operation, \( k'_{max} = k_{msg-max} \). By our inductive hypothesis, \( cm(k-1) = \pm \implies cm(k) = \pm \); it follows, then, that if \( k-1 \leq k_{msg-max} \) then \( k \leq k_{msg-max} \). Therefore if \( k-1 \leq k'_{max} \), then \( k \leq k'_{max} \). Finally, then, \( s'.cmap(k-1) = \pm \) implies that \( s'.cmap(k) = \pm \).

3. \( \text{cfg-upg-prop-fix}(k') \).

By assumption, \( k \neq k' \). If \( k < k' \), then this operation sets both \( s'.cmap(k-1)_i = \pm \) and \( s'.cmap(k)_i = \pm \). If \( k > k' \), then this operation has no effect on \( cmap(k)_i \) or \( cmap(k-1)_i \), and the desired property still holds.

\[\square\]

The following corollary says that if a \( \text{cfg-upgrade}(k) \) event for an upgrade operation \( \gamma \) occurs in an execution, then there is some previous configuration upgrade operation \( \gamma' \) (that completes before the upgrade event) where the target of \( \gamma' \) is the configuration with the smallest index removed by \( \gamma \).
Corollary 5.1.2  Let \( \gamma \) be a configuration upgrade operation, initiated by a \( \text{cfg-upgrade}(k)_i \) event in \( \alpha \), and let \( k_1 = \min\{\text{removal-set}(\gamma)\} \). That is, \( k_1 \) is the smallest element such that \( \text{upg-cmap}(\gamma)(k_1) \in C \). Assume \( k_1 > 0 \). Then a \( \text{cfg-upg-prop-fix}(k_1)_j \) event for some configuration upgrade operation \( \gamma' \) occurs in \( \alpha \) for some \( j \) such that the \( \text{cfg-upg-prop-fix}_j \) event of \( \gamma' \) precedes the \( \text{cfg-upgrade}(k)_i \) event in \( \alpha \).

Proof. By the definition of \( k_1 \), we know that in the state just after the \( \text{cfg-upgrade} \) event, \( \text{upg.cmap}(k_1 - 1)_i = \pm \) and \( \text{upg.cmap}(k_1)_i \neq \pm \). Since \( \text{upg.cmap}_i \) is set by the \( \text{cfg-upgrade} \) event to \( \text{cmap}_i \) in the state just prior to the \( \text{cfg-upgrade} \) event, we know that \( \text{cmap}(k_1 - 1)_i = \pm \) and \( \text{cmap}(k_1)_i \neq \pm \) in the state just prior to the \( \text{cfg-upgrade} \) event. Lemma 5.1.1, then, implies that some \( \text{cfg-upgrade-prop-fix}(k_1) \) event for some operation \( \gamma' \) occurs in \( \alpha \) preceding the \( \text{cfg-upgrade} \) event. \( \square \)

The next lemma says that for a given configuration upgrade operation \( \gamma \), there exists a sequence of preceding upgrade operations satisfying certain properties. The lemma begins by assuming that some configuration with index \( k \) is removed by the specified upgrade operation. For every configuration with an index smaller than \( k \), we choose a single upgrade operation – that removes that configuration – to add to the sequence. Therefore the constructed sequence may well contain the same configuration upgrade operation multiple times, if the operation has removed multiple configurations. If two elements in the sequence are distinct upgrade operations, then the earlier operation in the sequence completes before the later operation in the sequence is initiated. Also, the target of an upgrade operation in the sequence is removed by the next distinct upgrade operation in the sequence. As a result of these properties, the configuration upgrade process obeys a sequential discipline.

Lemma 5.1.3  If a \( \text{cfg-upgrade}_i \) event for upgrade operation \( \gamma \) occurs in \( \alpha \) such that \( k \in \text{removal-set}(\gamma) \), then there exists a sequence (possibly containing repeated elements) of configuration upgrade operations \( \gamma_0, \gamma_1, \ldots, \gamma_k \) with the following properties:

1. \( \forall \ s : 0 \leq s \leq k \), \( s \in \text{removal-set}(\gamma_s) \),

2. \( \forall \ s : 0 \leq s < k \), if \( \gamma_s \neq \gamma_{s+1} \), then the \( \text{cfg-upg-prop-fix} \) event of \( \gamma_s \) occurs in \( \alpha \) and the
\textbf{cfg-upgrade} event of $\gamma_{s+1}$ occurs in $\alpha$, and the \textbf{cfg-upg-prop-fix} event of $\gamma_s$ precedes the \textbf{cfg-upgrade} event of $\gamma_{s+1}$, and

3. $\forall s : 0 \leq s < k$, if $\gamma_s \neq \gamma_{s+1}$, then $\text{target}(\gamma_s) \in \text{removal-set}(\gamma_{s+1})$.

\textbf{Proof.} We construct the sequence in reverse order, first defining $\gamma_k$, and then at each step defining the preceding element. We prove the lemma by backward induction on $\ell$, for $\ell = k$ down to $\ell = 0$, maintaining the following three properties at each step of the induction:

$1'. \forall s : \ell \leq s \leq k$, $s \in \text{removal-set}(\gamma_s)$,

$2'. \forall s : \ell \leq s < k$, if $\gamma_s \neq \gamma_{s+1}$, then the \textbf{cfg-upg-prop-fix} event of $\gamma_s$ occurs in $\alpha$ and the \textbf{cfg-upgrade} event of $\gamma_{s+1}$ occurs in $\alpha$, and the \textbf{cfg-upg-prop-fix} event of $\gamma_s$ precedes the \textbf{cfg-upgrade} event of $\gamma_{s+1}$, and

$3'. \forall s : \ell \leq s < k$, if $\gamma_s \neq \gamma_{s+1}$, then $\text{target}(\gamma_s) \in \text{removal-set}(\gamma_{s+1})$.

To begin the induction, we first examine the base case, where $\ell = k$. Define $\gamma_k = \gamma$. Property 1' holds by assumption, and Property 2' and Property 3' are vacuously true.

For the inductive step, we assume that $\gamma_\ell$ has been defined and that properties 1'–3' hold. If $\ell = 0$, then $\gamma_0$ has been defined, and we are done. Otherwise, we need to define $\gamma_{\ell-1}$. If $\ell - 1 \in \text{removal-set}(\gamma_\ell)$, then let $\gamma_{\ell-1} = \gamma_\ell$, and all the properties still hold.

Otherwise, $\ell - 1 \notin \text{removal-set}(\gamma_\ell)$ and $\ell \in \text{removal-set}(\gamma_\ell)$, which implies that $\ell = \min\{\text{removal-set}(\gamma_\ell)\}$ because each configuration upgrade operates on a consecutive sequence of configurations. Then by Corollary 5.1.2, there occurs in $\alpha$ a configuration upgrade operation, that we label $\gamma_{\ell-1}$, with the following properties: (i) the \textbf{cfg-upg-prop-fix} event of $\gamma_{\ell-1}$ precedes the \textbf{cfg-upgrade} event of $\gamma_\ell$, and (ii) $\text{target}(\gamma_{\ell-1}) = \min\{k' : k' \in \text{removal-set}(\gamma_\ell)\}$.

Recall that $\ell = \min\{\text{removal-set}(\gamma_\ell)\}$. Therefore, by Property (ii) of $\gamma_{\ell-1}$, $\text{target}(\gamma_{\ell-1}) = \ell$. Since $\text{removal-set}(\gamma_{\ell-1}) \neq \emptyset$, this implies that $\ell - 1 \in \text{removal-set}(\gamma_{\ell-1})$, proving Property 1'. Property 2' follows from Property (i) of $\gamma_{\ell-1}$. Property 3' follows from Property (ii) of $\gamma_{\ell-1}$. \hfill $\square$

The sequential nature of configuration upgrade has a nice consequence for propagation of tags: for any sequence of upgrade operations like that in Lemma 5.1.3, $\text{tag}(\gamma_s)$ is nondecreasing in $s$. 

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Lemma 5.1.4 Let $\gamma_1, \ldots, \gamma_k$ be a sequence of configuration upgrade operations such that:

1. $\forall s : 0 \leq s \leq k, \ s \in \text{removal-set}(\gamma_s)$,

2. $\forall s : 0 \leq s < k$, if $\gamma_s \neq \gamma_{s+1}$, then the \text{cfg-upg-prop-fix} event of $\gamma_s$ occurs in $\alpha$ and the \text{cfg-upgrade} event of $\gamma_{s+1}$ occurs in $\alpha$, and the \text{cfg-upg-prop-fix} event of $\gamma_s$ precedes the \text{cfg-upgrade} event of $\gamma_{s+1}$, and

3. $\forall s : 0 \leq s < k$, if $\gamma_s \neq \gamma_{s+1}$, then $\text{target}(\gamma_s) \in \text{removal-set}(\gamma_{s+1})$.

Then $\forall s : 0 \leq s < k, \ \text{tag}(\gamma_s) \leq \text{tag}(\gamma_{s+1})$.

Proof. If $\gamma_s = \gamma_{s+1}$, then it is trivially true that $\text{tag}(\gamma_s) \leq \text{tag}(\gamma_{s+1})$. Therefore assume that $\gamma_s \neq \gamma_{s+1}$; this implies that the \text{cfg-upg-prop-fix} event of $\gamma_s$ precedes the \text{cfg-upgrade} event of $\gamma_{s+1}$. Let $k_2$ be the largest element in $\text{removal-set}(\gamma_s)$. We know by assumption that $k_2 + 1 \in \text{removal-set}(\gamma_{s+1})$. Therefore, $W_2(\gamma_s)$, a write-quorum of configuration $c(k_2 + 1)$, has at least one element in common with $R(\gamma_{s+1}, k_2 + 1)$; label this node $j$. By Lemma 4.4.2, and the monotonicity of $\text{tag}_j$, after the \text{cfg-upg-prop-fix} event of $\gamma_s$ we know that $\text{tag}_j \geq \text{tag}(\gamma_s)$. Then by Lemma 4.4.1 $\text{tag}(\gamma_{s+1}) \geq \text{tag}_j$. Therefore $\text{tag}(\gamma_s) \leq \text{tag}(\gamma_{s+1})$. \qed

Corollary 5.1.5 Let $\gamma_1, \ldots, \gamma_k$ be a sequence of configuration upgrade operations such that:

1. $\forall s : 0 \leq s \leq k, \ s \in \text{removal-set}(\gamma_s)$,

2. $\forall s : 0 \leq s < k$, if $\gamma_s \neq \gamma_{s+1}$, then the \text{cfg-upg-prop-fix} event of $\gamma_s$ occurs in $\alpha$ and the \text{cfg-upgrade} event of $\gamma_{s+1}$ occurs in $\alpha$, and the \text{cfg-upg-prop-fix} event of $\gamma_s$ precedes the \text{cfg-upgrade} event of $\gamma_{s+1}$, and

3. $\forall s : 0 \leq s < k$, if $\gamma_s \neq \gamma_{s+1}$, then $\text{target}(\gamma_s) \in \text{removal-set}(\gamma_{s+1})$.

Then $\forall s, s' : 0 \leq s \leq s' \leq k, \ \text{tag}(\gamma_s) \leq \text{tag}(\gamma_{s'})$.

Proof. This follows immediately from Lemma 5.1.4 by induction. \qed
5.2 Behavior of a read or a write following a configuration upgrade

Now we describe the relationship between an upgrade operation and a following read or write operation. These three lemmas relate the removal-set of a preceding configuration upgrade operation with the query-cmap of a later read or write operation.

The first lemma shows that if, for some read or write operation, $k$ is the smallest index such that $\text{query-cmap}(k) \in C$, then some configuration upgrade operation with target $k$ precedes the read or write operation.

**Lemma 5.2.1** Let $\pi$ be a read or write operation whose query-fix event occurs in $\alpha$. Let $k$ be the smallest element such that $\text{query-cmap}(\pi)(k) \in C$. Assume $k > 0$. Then there must exist a configuration upgrade operation $\gamma$ such that $k = \text{target}(\gamma)$, and the cfg-upg-prop-fix event of $\gamma$ precedes the query-phase-start($\pi$) event.

**Proof.** This follows from Lemma 5.1.1. Let $s$ be the state just before the query-phase-start($\pi$) event. By definition, $\text{query-cmap}(\pi) = s.cmap_i$. Since $s.cmap(k-1)_i = \pm$ and $s.cmap(k)_i \neq \pm$, there must exist such a configuration upgrade operation for $k$ by the contrapositive of Lemma 5.1.1.  

Second, if some upgrade removing $k$ does complete before the query-phase-start event of a read or write operation, then some configuration with index $\geq k + 1$ must be included in the query-cmap of a later read or write operation.

**Lemma 5.2.2** Let $\gamma$ be a configuration upgrade operation such that $k \in \text{removal-set}(\gamma)$. Let $\pi$ be a read or write operation whose query-fix event occurs in $\alpha$. Suppose that the cfg-upg-prop-fix event of $\gamma$ precedes the query-phase-start($\pi$) event in $\alpha$. Then $\text{query-cmap}(\pi)(\ell) \in C$ for some $\ell \geq k + 1$.

**Proof.** Suppose for the sake of contradiction that $\text{query-cmap}(\pi)(\ell) \notin C$ for all $\ell \geq k + 1$. Fix $k' = \max(\{\ell' : \text{query-cmap}(\pi)(\ell') \in C\})$. Then $k' \leq k$.  

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Let \( \gamma_0, \ldots, \gamma_k \) be the sequence of upgrade operations whose existence is asserted by Lemma 5.1.3, where \( \gamma_k = \gamma \). Then, by this construction, \( k' \in \text{removal-set}(\gamma_{k'}) \), and the \text{cfg-upg-prop-fix} event of \( \gamma_{k'} \) does not come after the \text{cfg-upg-prop-fix} event of \( \gamma \) in \( \alpha \). By assumption, the \text{cfg-upg-prop-fix} event of \( \gamma \) precedes the \text{query-phase-start}(\pi) event in \( \alpha \). Therefore the \text{cfg-upg-prop-fix} event of \( \gamma_{k'} \) precedes the \text{query-phase-start}(\pi) event in \( \alpha \).

Then, since \( k' \in \text{removal-set}(\gamma_{k'}) \), write-quorum \( W_1(\gamma_{k'}, k') \) is defined. Since \( \text{query-cmap}(k') \in C \), the read-quorum \( R(\pi, k') \) is defined. Choose \( j \in W_1(\gamma_{k'}, k') \cap R(\pi, k') \). Assume that \( k_t = \text{target}(\gamma_{k'}) \). Notice that \( k' < k_t \). Then Lemma 4.4.1 and monotonicity of \text{cmap} imply that, in the state just prior to the \text{cfg-upg-query} event of \( \gamma_{k'} \), \( \text{cmap}(\ell)_j \neq \bot \) for all \( \ell \leq k_t \). Then Lemma 4.4.3 implies that \( \text{query-cmap}(\pi)(\ell) \in C \) for some \( \ell \geq k_t \). But this contradicts the choice of \( k' \).

The next lemma describes propagation of \text{tag} information from a configuration upgrade operation to a following read or write operation. For this lemma, we assume that \( \text{query-cmap}(k) \in C \), where \( k \) is the target of the upgrade operation.

**Lemma 5.2.3** Let \( \gamma \) be a configuration upgrade operation. Assume that \( k = \text{target}(\gamma) \). Let \( \pi \) be a read or write operation whose \text{query-fix} event occurs in \( \alpha \). Suppose that the \text{cfg-upg-prop-fix} event of \( \gamma \) precedes the \text{query-phase-start}(\pi) event in execution \( \alpha \). Suppose also that \( \text{query-cmap}(\pi)(k) \in C \). Then:

1. \( \text{tag}(\gamma) \leq \text{tag}(\pi) \).

2. If \( \pi \) is a write operation then \( \text{tag}(\gamma) < \text{tag}(\pi) \).

**Proof.** The propagation phase of \( \gamma \) accesses write-quorum \( W_2(\gamma) \) of \( c(k) \), whereas the query phase of \( \pi \) accesses read-quorum \( R(\pi, k) \). Since both are quorums of configuration \( c(k) \), they have a nonempty intersection; choose \( j \in W_2(\gamma) \cap R(\pi, k) \).

Lemma 4.4.2 implies that, in any state after the \text{cfg-upg-prop-fix} event for \( \gamma \), \( \text{tag}_j \geq \text{tag}(\gamma) \).

Since the \text{cfg-upg-prop-fix} event of \( \gamma \) precedes the \text{query-phase-start}(\pi) event, we have that \( t \geq \text{tag}(\gamma) \), where \( t \) is defined to be the value of \( \text{tag}_j \) just before the \text{query-phase-start}(\pi) event. Then Lemma 4.4.3 implies that \( \text{tag}(\pi) \geq t \), and if \( \pi \) is a write operation, then \( \text{tag}(\pi) > t \). Combining the inequalities yields both conclusions of the lemma. \[ \Box \]
5.3 Behavior of sequential reads and writes

Read or write operations that originate at different locations may proceed concurrently. However, in the special case where they execute sequentially, we can prove some relationships between their query-cmaps, prop-cmaps, and tags. The first lemma says that, when two read or write operations execute sequentially, the smallest configuration index used in the propagation phase of the first operation is less than or equal to the largest index used in the query phase of the second. In other words, we cannot have a situation in which the second operation’s query phase executes using only configurations with indices that are strictly less than any used in the first operation’s propagation phase.

Lemma 5.3.1 Assume \( \pi_1 \) and \( \pi_2 \) are two read or write operations, such that:

1. The prop-fix event of \( \pi_1 \) occurs in \( \alpha \).
2. The query-fix event of \( \pi_2 \) occurs in \( \alpha \).
3. The prop-fix event of \( \pi_1 \) precedes the query-phase-start\((\pi_2)\) event.

Then \( \min(\{ \ell : \text{prop-cmap}(\pi_1)(\ell) \in C \}) \leq \max(\{ \ell : \text{query-cmap}(\pi_2)(\ell) \in C \}) \).

Proof. Suppose for the sake of contradiction that \( \min(\{ \ell : \text{prop-cmap}(\pi_1)(\ell) \in C \}) > k \), where \( k \) is defined to be \( \max(\{ \ell : \text{query-cmap}(\pi_2)(\ell) \in C \}) \). Then in particular, \( \text{prop-cmap}(\pi_1)(k) \notin C \). The form of \( \text{prop-cmap}(\pi_1) \), as expressed in Invariant 4.3.4, implies that \( \text{prop-cmap}(\pi_1)(k) = \pm \).

This implies that some cfg-upg-prop-fix event for some upgrade operation \( \gamma \) such that \( k \in \text{removal-set}(\gamma) \) occurs prior to the prop-fix of \( \pi_1 \), and hence prior to the query-phase-start\((\pi_2)\) event. Lemma 5.2.2 then implies that \( \text{query-cmap}(\pi_2)(\ell) \in C \) for some \( \ell \geq k + 1 \). But this contradicts the choice of \( k \). \( \square \)

The next lemma describes propagation of tag information, in the case where the propagation phase of the first operation and the query phase of the second operation share a configuration.

Lemma 5.3.2 Assume \( \pi_1 \) and \( \pi_2 \) are two read or write operations, and \( k \in \mathbb{N} \), such that:
1. The prop-fix event of $\pi_1$ occurs in $\alpha$.

2. The query-fix event of $\pi_2$ occurs in $\alpha$.

3. The prop-fix event of $\pi_1$ precedes the query-phase-start($\pi_2$) event.

4. prop-cmap($\pi_1$)(k) and query-cmap($\pi_2$)(k) are both in C.

Then:

1. $\text{tag}(\pi_1) \leq \text{tag}(\pi_2)$.

2. If $\pi_2$ is a write then $\text{tag}(\pi_1) < \text{tag}(\pi_2)$.

**Proof.** The hypotheses imply that prop-cmap($\pi_1$)(k) = query-cmap($\pi_2$)(k) = c(k). Then $W(\pi_1, k)$ and $R(\pi_2, k)$ are both defined in $\alpha$. Since they are both quorums of configuration c(k), they have a nonempty intersection; choose $j \in W(\pi_1, k) \cap R(\pi_2, k)$.

Lemma 4.4.4 implies that, in any state after the prop-fix event of $\pi_1$, $\text{tag}_j \geq \text{tag}(\pi_1)$. Since the prop-fix event of $\pi_1$ precedes the query-phase-start($\pi_2$) event, we have that $t \geq \text{tag}(\pi_1)$, where t is defined to be the value of $\text{tag}_j$ just before the query-phase-start($\pi_2$) event. Then Lemma 4.4.3 implies that $\text{tag}(\pi_2) \geq t$, and if $\pi_2$ is a write operation, then $\text{tag}(\pi_2) > t$. Combining the inequalities yields both conclusions. □

The final lemma is similar to the previous one, but it does not assume that the propagation phase of the first operation and the query phase of the second operation share a configuration. The main focus of the proof is on the situation where all the configuration indices used in the query phase of the second operation are greater than those used in the propagation phase of the first operation.

**Lemma 5.3.3** Assume $\pi_1$ and $\pi_2$ are two read or write operations, such that:

1. The prop-fix of $\pi_1$ occurs in $\alpha$.

2. The query-fix of $\pi_2$ occurs in $\alpha$.

3. The prop-fix event of $\pi_1$ precedes the query-phase-start($\pi_2$) event.
Then:

1. \( \text{tag}(\pi_1) \leq \text{tag}(\pi_2) \).

2. If \( \pi_2 \) is a write then \( \text{tag}(\pi_1) < \text{tag}(\pi_2) \).

**Proof.**  Let \( i_1 \) and \( i_2 \) be the indices of the processes that run operations \( \pi_1 \) and \( \pi_2 \), respectively. Let \( cm_1 = \text{prop-cmap}(\pi_1) \) and \( cm_2 = \text{query-cmap}(\pi_2) \). If there exists \( k \) such that \( cm_1(k) \in C \) and \( cm_2(k) \in C \), then Lemma 5.3.2 implies the conclusions of the lemma. So from now on, we assume that no such \( k \) exists.

Lemma 5.3.1 implies that \( \min(\{ \ell : cm_1(\ell) \in C \}) \leq \max(\{ \ell : cm_2(\ell) \in C \}) \). Invariant 4.3.4 implies that the set of indices used in each phase consists of consecutive integers. Since the intervals have no indices in common, it follows that \( s_1 < s_2 \), where \( s_1 \) is defined to be \( \max(\{ \ell : cm_1(\ell) \in C \}) \) and \( s_2 \) is defined to be \( \min(\{ \ell : cm_2(\ell) \in C \}) \).

Lemma 5.2.1 implies that there exists a configuration upgrade operation that we will call \( \gamma_{s_2-1} \) such that \( s_2 = \text{target}(\gamma_{s_2-1}) \), and the \textit{cfg-upg-prop-fix} of \( \gamma_{s_2-1} \) precedes the \textit{query-phase-start}(\( \pi_2 \)) event. Then by Lemma 5.2.3, \( \text{tag}(\gamma_{s_2-1}) \leq \text{tag}(\pi_2) \), and if \( \pi_2 \) is a write operation then \( \text{tag}(\gamma_{s_2-1}) < \text{tag}(\pi_2) \).

Next we will demonstrate a chain of configuration upgrade operations with non-decreasing tags. Lemma 5.1.3, in conjunction with the already defined \( \gamma_{s_2-1} \), implies the existence of a sequence of configuration upgrade operations \( \gamma_0, \ldots, \gamma_{s_2-1} \) such that:

1. \( \forall s : 0 \leq s \leq s_2 - 1, s \in \text{removal-set}(\gamma_s) \).

2. \( \forall s : 0 \leq s < s_2 - 1, \text{if } \gamma_s \neq \gamma_{s+1}, \text{then the } \text{cfg-upg-prop-fix} \text{ event of } \gamma_s \text{ precedes the } \text{cfg-upgrade} \text{ event of } \gamma_{s+1} \) in \( \alpha \).

3. \( \forall s : 0 \leq s < s_2 - 1, \text{if } \gamma_s \neq \gamma_{s+1}, \text{then } \text{target}(\gamma_s) \in \text{removal-set}(\gamma_{s+1}) \).

As a special case of Property 1, since \( s_1 \leq s_2 - 1 \), we know that \( s_1 \in \text{removal-set}(\gamma_{s_1}) \). Then Corollary 5.1.5 implies that \( \text{tag}(\gamma_{s_1}) \leq \text{tag}(\gamma_{s_2-1}) \).

It remains to show that the tag of \( \pi_1 \) is no greater than the tag of \( \gamma_{s_1} \). Therefore we focus now on the relationship between operation \( \pi_1 \) and configuration upgrade \( \gamma_{s_1} \). The propagation phase of \( \pi_1 \) accesses write-quorum \( W(\pi_1, s_1) \) of configuration \( c(s_1) \), whereas the...
query phase of \( \gamma_{s_1} \) accesses read-quorum \( R(\gamma_{s_1}, s_1) \) of configuration \( c(s_1) \). Since \( W(\pi_1, s_1) \cap R(\gamma_{s_1}, s_1) \neq \emptyset \), we may fix some \( j \in W(\pi_1, s_1) \cap R(\gamma_{s_1}, s_1) \). Let message \( m_1 \) from \( i_1 \) to \( j \) and message \( m'_1 \) from \( j \) to \( i_1 \) be as in Lemma 4.4.4 for the propagation phase of \( \gamma_{s_1} \).

Let message \( m_2 \) from the process running \( \gamma_{s_1} \) to \( j \) and message \( m'_2 \) from \( j \) to the process running \( \gamma_{s_1} \) be the messages whose existence is asserted in Lemma 4.4.1 for the query phase of \( \gamma_{s_1} \).

We claim that \( j \) sends \( m'_1 \), its message for \( \pi_1 \), before it sends \( m'_2 \), its message for \( \gamma_{s_1} \). Suppose for the sake of contradiction that \( j \) sends \( m'_2 \) before it sends \( m'_1 \). Assume that \( s_t = \text{target}(\gamma_{s_1}) \). Notice that \( s_t > s_1 \), since \( s_1 \in \text{removal-set}(\gamma_{s_1}) \). Lemma 4.4.1 implies that in any state after \( j \) receives \( m_2 \), before \( j \) sends \( m'_2 \), \( \text{cmap}(k)_j \neq \bot \) for all \( k \leq s_t \). Since \( j \) sends \( m'_2 \) before it sends \( m'_1 \), monotonicity of \( \text{cmap} \) implies that just before \( j \) sends \( m'_1 \), \( \text{cmap}(k)_j \neq \bot \) for all \( k \leq s_t \). Then Lemma 4.4.4 implies that \( \text{prop-cmap}(\pi_1)(\ell) \in C \) for some \( \ell \geq s_t \). But this contradicts the choice of \( s_1 \), since \( s_1 < s_t \). This implies that \( j \) sends \( m'_1 \) before it sends \( m'_2 \).

Since \( j \) sends \( m'_1 \) before it sends \( m'_2 \), Lemma 4.4.4 implies that, at the time \( j \) sends \( m'_2 \), \( \text{tag}(\pi_1) \leq \text{tag}_j \). Then Lemma 4.4.1 implies that \( \text{tag}(\pi_1) \leq \text{tag}(\gamma_{s_1}) \). From above, we know that \( \text{tag}(\gamma_{s_1}) \leq \text{tag}(\gamma_{s_2-1}) \), and \( \text{tag}(\gamma_{s_2-1}) \leq \text{tag}(\pi_2) \), and if \( \pi_2 \) is a write operation then \( \text{tag}(\gamma_{s_2-1}) < \text{tag}(\pi_2) \). Combining the various inequalities then yields both conclusions. \( \square \)

### 5.4 Atomicity

In order to prove that all executions of Rambo II are atomic, we use four sufficient conditions. A memory is said to be atomic provided that the following conditions hold for all good executions:

- If all the read and write operations that are invoked complete, then the read and write operations for object \( x \) can be partially ordered by an ordering \( \prec \), so that:

  1. No operation has infinitely many other operations ordered before it.

  2. The partial order is consistent with the external order of invocations and responses, that is, there do not exist read or write operations \( \pi_1 \) and \( \pi_2 \) such that
\(\pi_1\) completes before \(\pi_2\) starts, yet \(\pi_2 \prec \pi_1\).

3. All write operations are totally ordered and every read operation is ordered with respect to all the writes.

4. Every read operation ordered after any writes returns the value of the last write preceding it in the partial order; any read operation ordered before all writes returns the initial value.

This definition is sufficient to guarantee atomicity in terms of the other common definition which is defined in terms of equivalence to a serial memory. (See, for example, Lemma 13.16 in [11].)

Let \(\beta\) be a trace of \(\mathcal{S}\), the system that implements RAMBO II (recall that this includes the \textit{Reader-Writer}, \textit{Recon} and \textit{Joiner} automata), and assume that all read and write operations complete in \(\beta\). Consider any particular good execution \(\alpha\) of \(\mathcal{S}\) whose trace is \(\beta\). We define a partial order \(\prec\) on read and write operations in \(\beta\), in terms of the operations’ tags in \(\alpha\). Namely, we totally order the writes in order of their tags, and we order each read with respect to all the writes as follows: a read with tag \(t\) is ordered after all writes with tags \(\leq t\) and before all writes with tags \(> t\).

**Lemma 5.4.1** The ordering \(\prec\) is well-defined.

**Proof.** The key is to show that no two write operations get assigned the same tag. This is obviously true for two writes that are initiated at different locations, because the low-order tiebreaker identifiers are different. For two writes at the same location, Lemma 5.3.3 implies that the tag of the second is greater than the tag of the first. This suffices. \(\Box\)

**Lemma 5.4.2** \(\prec\) satisfies the four conditions in the definition of atomicity.

**Proof.** We begin with Property 2, which as usual in such proofs, is the most interesting thing to show. Suppose for the sake of contradiction that \(\pi_1\) completes before \(\pi_2\) starts, yet \(\pi_2 \prec \pi_1\). We consider two cases:

1. \(\pi_2\) is a write operation.
Since \( \pi_1 \) completes before \( \pi_2 \) starts, Lemma 5.3.3 implies that \( \text{tag}(\pi_2) > \text{tag}(\pi_1) \). On the other hand, the fact that \( \pi_2 \prec \pi_1 \) implies that \( \text{tag}(\pi_2) \leq \text{tag}(\pi_1) \). This yields a contradiction.

2. \( \pi_2 \) is a read operation.

Since \( \pi_1 \) completes before \( \pi_2 \) starts, Lemma 5.3.3 implies that \( \text{tag}(\pi_2) \geq \text{tag}(\pi_1) \). On the other hand, the fact that \( \pi_2 \prec \pi_1 \) implies that \( \text{tag}(\pi_2) < \text{tag}(\pi_1) \). This yields a contradiction.

Since we have a contradiction in either case, Property 2 must hold.

Property 1 follows from Property 2. Properties 3 and 4 are straightforward. \( \square \)

Now we tie everything together for the proof of Theorem 5.4.3.

**Theorem 5.4.3** Let \( \beta \) be a trace of \( S \), the system that implements RAMBO II. Then \( \beta \) satisfies the atomicity guarantee.

**Proof.** Assume that all read and write operations complete in \( \beta \). Let \( \alpha \) be a good execution of \( S \) whose trace is \( \beta \). Define the ordering \( \prec \) on the read and write operations in \( \beta \) as above, using the execution \( \alpha \). Then Lemma 5.4.2 says that \( \prec \) satisfies the four conditions in the definition of atomicity. Thus, \( \beta \) satisfies the atomicity condition, as needed. \( \square \)
Chapter 6

Reconfiguration Service

In this chapter we present the specification and implementation for the reconfiguration specification. This section is a restatement of Sections 4 and 7 of the RAMBO technical report, and is taken directly from [13]. Our RAMBO implementation for each object $x$ consists of a main Reader-Writer algorithm and a reconfiguration service, $Recon(x)$; since we are suppressing mention of $x$, we write this simply as $Recon$. First, in Section 6.1, we present the specification for the $Recon$ service, as an external signature and set of traces. In Section 6.2, we present our implementation of $Recon$.

6.1 Reconfiguration Service Specification

The interface for $Recon$ appears in Figure 6-1. The client of $Recon$ at location $i$ requests to join the reconfiguration service by performing a $join(recon)_i$ input action. The service acknowledges this with a corresponding $join-ack_i$ output action. The client requests to reconfigure the object using a $recon_i$ input, which is acknowledged with a $recon-ack_i$ output action. RAMBO reports a new configuration to the client using a $report_i$ output action. Crashes are modeled using $fail$ actions.

$Recon$ also produces outputs of the form $new-config(c, k)_i$, which announce at location $i$ that $c$ is the $k^{th}$ configuration identifier for the object. These outputs are used for communication with the portion of the Reader-Writer algorithm running at location $i$. $Recon$
announces consistent information, only one configuration identifier per index in the configuration identifier sequence. It delivers information about each configuration to members of the new configuration and of the immediately preceding configuration.

<table>
<thead>
<tr>
<th>Input:</th>
<th>Output:</th>
</tr>
</thead>
<tbody>
<tr>
<td>join(recon)_i, i \in I</td>
<td>join-ack(recon)_i, i \in I</td>
</tr>
<tr>
<td>recon(c, c')_i, c, c' \in C, i \in members(c)</td>
<td>recon-ack(b)_i, b \in {ok, nak}, i \in I</td>
</tr>
<tr>
<td>fail_i, i \in I</td>
<td>report(c)_i, c \in C, i \in I</td>
</tr>
<tr>
<td></td>
<td>new-config(c, k)_i, c \in C, k \in \mathbb{N}^+, i \in I</td>
</tr>
</tbody>
</table>

Figure 6-1: Recon: External signature

Now we define the set of traces describing Recon's safety properties. Again, these are defined in terms of environment assumptions and and service guarantees. The environment assumptions are simple well-formedness conditions, consistent with the well-formedness assumptions for Rambo:

- **Well-formedness**:
  
  - For every \( i \):
    
    - No \( \text{join}(\text{recon})_i \) or \( \text{recon}(\ast, \ast)_i \) event is preceded by a \( \text{fail}_i \) event.
    
    - At most one \( \text{join}(\text{recon})_i \) event occurs.
    
    - Any \( \text{recon}(\ast, \ast)_i \) event is preceded by a \( \text{join-ack}(\text{recon})_i \) event.
    
    - Any \( \text{recon}(\ast, \ast)_i \) event is preceded by an \( \text{-ack} \) for any preceding \( \text{recon}(\ast, \ast)_i \) event.
  
  - For every \( c \), at most one \( \text{recon}(\ast, c)_* \) event occurs.
  
  - For every \( c, c', x, \) and \( i \), if a \( \text{recon}(c, c')_i \) event occurs, then it is preceded by:
    
    - A \( \text{report}(c)_i \) event, and
    
    - A \( \text{join-ack}(\text{recon})_j \) for every \( j \in \text{members}(c') \).

The safety guarantees provided by the service are as follows:

- **Well-formedness**: For every \( i \):
- No join-ack\((\text{recon})_i\), recon-ack\((\ast)_i\), report\((\ast)_i\), or new-config\((\ast, \ast)_i\) event is preceded by a fail\(_i\) event.

- Any join-ack\((\text{recon})_i\) (resp., recon-ack\((c)_i\)) event has a preceding join\((\text{recon})_i\) (resp., recon\(_i\)) event with no intervening invocation or response action for \(x\) and \(i\).

- **Agreement**: If new-config\((c, k)_i\) and new-config\((c', k)_j\) both occur, then \(c = c'\). (No disagreement arises about what the \(k^{th}\) configuration identifier is, for any \(k\).)

- **Validity**: If new-config\((c, k)_i\) occurs, then it is preceded by a recon\((\ast, c)_{i'}\) for some \(i'\) for which a matching recon-ack\((\text{nok})_{i'}\) does not occur. (Any configuration identifier that is announced was previously requested by someone who did not receive a negative acknowledgment.)

- **No duplication**: If new-config\((c, k)_i\) and new-config\((c, k')_{i'}\) both occur, then \(k = k'\).

(The same configuration identifier cannot be assigned to two different positions in the sequence of configuration identifiers.)

### 6.2 Reconfiguration Service Implementation

In this section, we describe a distributed algorithm that implements the Recon service for a particular object \(x\) (and we suppress mention of \(x\)). This algorithm is considerably simpler than the Reader-Writer algorithm. It consists of a Recon\(_i\) automaton for each location \(i\), which interacts with a collection of global consensus services Cons\((k, c)\), one for each \(k \geq 1\) and each \(c \in C\), and with a point-to-point communication service.

Cons\((k, c)\) accepts inputs from members of configuration \(c\), which it assumes to be the \(k−1^{st}\) configuration. These inputs are proposed new configurations. The decision reached by Cons\((k, c)\), which must be one of the proposed configurations, is determined to be the \(k^{th}\) configuration.

Recon\(_i\) is activated by the joining protocol. It processes reconfiguration requests using the consensus services, and records the new configurations that the consensus services determine. Recon\(_i\) also conveys information about new configurations to the members of
those configurations, and releases new configurations for use by Reader-Writer_i. It returns
acknowledgments and configuration reports to its client.

6.3 Consensus services

In this section, we specify the behavior we assume for consensus service Cons(k, c), for a fixed
k ≥ 1 and c ∈ C. This behavior can be achieved using the Paxos consensus algorithm [9], as
described formally in [14]. Fix V to be the set of consensus values. (In the implementation
of the Recon service, V will be instantiated as C.) The external signature of Cons(k, c) is
given in Figure 6-2.

<table>
<thead>
<tr>
<th>Input:</th>
<th>Output:</th>
</tr>
</thead>
<tbody>
<tr>
<td>init(v)_k,c,i, v ∈ V, i ∈ members(c)</td>
<td>decide(v)_k,c,i, v ∈ V, i ∈ members(c)</td>
</tr>
<tr>
<td>fail_i, i ∈ members(c)</td>
<td></td>
</tr>
</tbody>
</table>

Figure 6-2: Cons(k, c): External signature

We describe the safety properties of Cons(k, c) in terms of properties of a trace β of
actions in the external signature. Namely, we define the client safety assumptions:

• Well-formedness: For any i ∈ members(c):
  
  – No init(*)_k,c,i event is preceded by a fail(i) event.
  
  – At most one init(*)_k,c,i event occurs in β.

And we define the consensus safety guarantees:

• Well-formedness: For any i ∈ members(c):
  
  – No decide(*)_k,c,i event is preceded by a fail(i) event.
  
  – At most one decide(*)_k,c,i event occurs in β.
  
  – If a decide(*)_k,c,i event occurs in β, then it is preceded by an init(*)_k,c,i event.

• Agreement: If decide(v)_k,c,i and decide(v')_k,c,i events occur in β, then v = v'.

• Validity: If a decide(v)_k,c,i event occurs in β, then it is preceded by an init(v)_k,c,i.
We assume that the $Cons(k, c)$ service is implemented using the Paxos algorithm [9], as described formally in [14]. This satisfies the safety guarantees described above, based on the safety assumptions:

**Theorem 6.3.1** If $\beta$ is a trace of Paxos that satisfies the safety assumptions of $Cons(k, c)$, then $\beta$ also satisfies the (well-formedness, agreement, and validity) safety guarantees of $Cons(k, c)$.

The Paxos algorithm also satisfies the following latency result:

**Theorem 6.3.2** Consider a timed execution $\alpha$ of the Paxos algorithm and a prefix $\alpha'$ of $\alpha$. Suppose that:

1. The underlying system “behaves well” after $\alpha'$, in the sense that timing is “normal” (what is called “regular” in [14])\(^1\) and no process failures or message losses occur.

2. For every $i$ that does not fail in $\alpha$, an init$(\ast)_i$ event occurs in $\alpha'$.

3. There exist $R \in$ read-quorums($c$) and $W \in$ write-quorums($c$) such that for all $i \in R \cup W$, no fail$_i$ event occurs in $\alpha$.

Then for every $i$ that does not fail in $\alpha$, a decide$(\ast)_i$ event occurs, no later than $9d + \varepsilon$ time after the end of $\alpha'$ ($\varepsilon > 0$).

### 6.4 Recon automata

A Recon$_i$ process is responsible for initiating consensus executions to help determine new configurations, for telling the local Reader-Writer$_i$ process about a newly-determined configuration, and for disseminating information about newly-determined configurations to the members of those configurations. The signature and state of Recon$_i$ appear in Figures 6-3, and the transitions in Figure 6-4.

\(^1\)In [14], regular timing implies that messages are delivered reliably within time $d$, that local processing time is 0, and that information is “gossiped” at intervals of $d$. 
**Signature:**

Input:
- `join(recon)_i`
- `recon(c, c')_i, c, c' ∈ C, i ∈ members(c)`
- `decide(c)_k,i, c ∈ C, k ∈ N^+`
- `recv((config, c, k))_j,i, c ∈ C, k ∈ N^+`, i ∈ members(c), j ∈ I - {i}
- `recv((init, c, c', k))_j,i, c, c' ∈ C, k ∈ N^+`, i, j ∈ members(c), j ≠ i
- `fail`

Output:
- `join-ack(recon)_i`
- `new-config(c, k)_i, c ∈ C, k ∈ N^+`
- `init(c, c')_k,i, c, c' ∈ C, k ∈ N^+`, i ∈ members(c)
- `recon-ack(b)_i, b ∈ {ok, nok}`
- `report(c)_i, c ∈ C`
- `send((config, c, k))_k,j, c ∈ C, k ∈ N^+`, j ∈ members(c) - {i}
- `send((init, c, c', k))_k,j,c, c' ∈ C, k ∈ N^+`, i, j ∈ members(c), j ≠ i

**State:**

- `status ∈ {idle, active}, initially idle`
- `rec-cmap ∈ CMap, initially rec-cmap(0) = c_0`
  - and `rec-cmap(k) = ⊥` for all `k ≠ 0`.
- `did-init ⊆ N^+`, initially ⊥
- `did-new-config ⊆ N^+`, initially ⊥
- `cons-data ∈ (N^+ → (C × C)); initially ⊥ everywhere`
- `rec-status ∈ {idle, active}, initially idle`
- `outcome ∈ {ok, nok, ⊥}, initially ⊥`
- `reported ⊆ C, initially ⊥`
- `failed`, a Boolean, initially `false`

Figure 6-3: Recon_i: Signature and state

Location `i` joins the Recon service when a `join(recon)` input occurs. Recon_i responds with a `join-ack`.

Recon_i includes a state variable `rec-cmap`, which holds a CMap: `rec-cmap(k) = c` indicates that `i` knows that `c` is the `k`th configuration identifier. If Recon_i has learned that `c` is the `k`th configuration identifier, it can convey this to its local Reader-Writer_i process using a `new-config(c, k)`, output action, and it can inform any other Recon_j process, `j ∈ members(c)`, by sending a `(config, c, k)` message. Recon_i learns about new configurations either by receiving a `decide` input from a Cons service, or by receiving a `config` or `init` message from another process.

Recon_i receives a reconfiguration request from its environment via a `recon(c, c')_i` event. Upon receiving such a request, Recon_i determines whether (a) `i` is a member of the known configuration `c` with the largest index `k - 1` and (b) it has not already prepared data for a consensus for the next larger index `k`. If both (a) and (b) hold, Recon_i prepares such data, consisting of the pair `<c, c'>`, where `c` is the `k - 1` st configuration identifier and `c'` is the proposed configuration identifier. Otherwise, Recon_i responds negatively to the new reconfiguration request.

Recon_i initiates participation in a `Cons(k, c)` algorithm when its consensus data are pre-
Input \text{join}(\text{recon})_i \\
\text{Effect:} \\
\quad \text{if } \neg \text{failed} \text{ then} \\
\quad \quad \text{if } \text{status} = \text{idle} \text{ then} \\
\quad \quad \quad \text{status} \leftarrow \text{active} \\

Output \text{join-ack}(\text{recon})_i \\
\text{Precondition:} \\
\quad \neg \text{failed} \\
\quad \text{status} = \text{active} \\
\text{Effect:} \\
\quad \text{none} \\

Output \text{new-config}(c, k)_i \\
\text{Precondition:} \\
\quad \neg \text{failed} \\
\quad \text{status} = \text{active} \\
\quad \text{rec-cmap}(k) = c \\
\quad k \notin \text{did-new-config} \\
\text{Effect:} \\
\quad \text{did-new-config} \leftarrow \text{did-new-config} \cup \{k\} \\

Output \text{send}((\text{config}, c, k))_{i,j} \\
\text{Precondition:} \\
\quad \neg \text{failed} \\
\quad \text{status} = \text{active} \\
\quad \text{rec-cmap}(k) = c \\
\text{Effect:} \\
\quad \text{none} \\

Input \text{recv}((\text{config}, c, k))_{j,i} \\
\text{Effect:} \\
\quad \text{if } \neg \text{failed} \text{ then} \\
\quad \quad \text{if } \text{status} = \text{active} \text{ then} \\
\quad \quad \quad \text{rec-cmap}(k) \leftarrow c \\

Output \text{report}(c)_i \\
\text{Precondition:} \\
\quad \neg \text{failed} \\
\quad \text{status} = \text{active} \\
\quad c \notin \text{reported} \\
\quad S = \{\ell : \text{rec-cmap}(\ell) \in C\} \\
\quad c = \text{rec-cmap}(\text{max}(S)) \\
\text{Effect:} \\
\quad \text{reported} \leftarrow \text{reported} \cup \{c\} \\

Input \text{recon}(c, c')_i \\
\text{Effect:} \\
\quad \text{if } \neg \text{failed} \text{ then} \\
\quad \quad \text{if } \text{status} = \text{active} \text{ then} \\
\quad \quad \quad \text{rec-status} \leftarrow \text{active} \\
\quad \quad \quad \text{let } S = \{\ell : \text{rec-cmap}(\ell) \in C\} \\
\quad \quad \quad \text{if } S \neq \emptyset \text{ and } c = \text{rec-cmap}(\text{max}(S)) \\
\quad \quad \quad \quad \text{and } \text{cons-data}(\text{max}(S) + 1) = \bot \text{ then} \\
\quad \quad \quad \quad \quad \text{cons-data}(\text{max}(S) + 1) \leftarrow \langle c, c' \rangle \\
\quad \quad \quad \quad \text{else outcome} \leftarrow \text{nok} \\

Output \text{init}(c')_{k, c, i} \\
\text{Precondition:} \\
\quad \neg \text{failed} \\
\quad \text{status} = \text{active} \\
\quad \text{cons-data}(k) = \langle c, c' \rangle \\
\quad k \geq 1 \text{ then } k \in \text{did-new-config} \\
\quad k \notin \text{did-init} \\
\text{Effect:} \\
\quad \text{did-init} \leftarrow \text{did-init} \cup \{k\} \\

Output \text{send}((\text{init}, c, c', k))_{i,j} \\
\text{Precondition:} \\
\quad \neg \text{failed} \\
\quad \text{status} = \text{active} \\
\quad \text{cons-data}(k) = \langle c, c' \rangle \\
\quad k \in \text{did-init} \\
\text{Effect:} \\
\quad \text{none} \\

Input \text{recv}((\text{init}, c, c', k))_{j,i} \\
\text{Effect:} \\
\quad \text{if } \neg \text{failed} \text{ then} \\
\quad \quad \text{if } \text{status} = \text{active} \text{ then} \\
\quad \quad \quad \text{rec-cmap}(k) \leftarrow c' \\
\quad \quad \quad \text{if cons-data}(k) = \bot \text{ then cons-data}(k) \leftarrow \langle c, c' \rangle \\

Input \text{decide}(c')_{k, c, i} \\
\text{Effect:} \\
\quad \text{if } \neg \text{failed} \text{ then} \\
\quad \quad \text{if } \text{status} = \text{active} \text{ then} \\
\quad \quad \quad \text{rec-cmap}(k) \leftarrow c' \\
\quad \quad \quad \text{if rec-status = active then} \\
\quad \quad \quad \quad \text{if cons-data}(k) = \langle c, c' \rangle \text{ then outcome} \leftarrow \text{ok} \\
\quad \quad \quad \quad \text{else outcome} \leftarrow \text{nok} \\

Output \text{recon-ack}(b)_i \\
\text{Precondition:} \\
\quad \neg \text{failed} \\
\quad \text{status} = \text{active} \\
\quad \text{rec-status} = \text{active} \\
\quad b = \text{outcome} \\
\text{Effect:} \\
\quad \text{rec-status} = \text{idle} \\
\quad \text{outcome} \leftarrow \bot \\

Input \text{fail}_i \\
\text{Effect:} \\
\quad \text{failed} \leftarrow \text{true} 

Figure 6-4: Recon\textsubscript{i}: Transitions.
pared. After initiating participation in a consensus algorithm, it sends \texttt{init} messages to inform the other members of \( c \) about its initiation of consensus. The other members use this information to prepare to participate in the same consensus algorithm (and also to update their \texttt{rec-cmap} if necessary). Thus, there are two ways in which \( \text{Recon}_i \) can initiate participation in consensus: as a result of a local \texttt{recon} event, or by receiving an \texttt{init} message from another \( \text{Recon}_j \) process.

When \( \text{Recon}_i \) receives a \texttt{decide}(\( c' \))\(_{k,i} \) directly from \( \text{Cons}(k, c) \), it records configuration \( c' \) in \texttt{rec-cmap} It also determines if a response to its local client is necessary (if a local reconfiguration operation is active), and determines the response based on whether the consensus decision is the same as the locally-proposed configuration identifier.

Each consensus service \( \text{Cons}(k, c) \) is responsible for conveying consensus decisions to \texttt{members}(\( c \)) . The \( \text{Recon}_i \) components are responsible for telling \texttt{members}(\( c' \)) about \( c' \) by sending \texttt{new-config} messages.

\textbf{Theorem 6.4.1} \textit{The Recon implementation guarantees well-formedness, agreement, and validity.}
Chapter 7

Conditional Performance Analysis

In this chapter we give a conditional latency analysis of the new algorithm, focusing on the improvements realized by the aggressive configuration-upgrade mechanism. We show that the new algorithm allows the system to recover rapidly after a period of unreliable network connectivity or bursty reconfiguration. In particular, we prove that if configurations do not fail too rapidly, then progress is guaranteed. First, in Section 7.1, we present a few general definitions. In Section 7.2 and 7.3, we define the executions being considered, and the environmental assumptions that these executions satisfy. Then in Sections 7.5, 7.6, and 7.7, we prove a series of lemmas that describe how long it takes configuration-upgrade operations to complete. Finally, in Section 7.8 we state the main stabilization theorem, and prove that operations will complete as long as the execution assumptions are met. Throughout this chapter, we compare the results with those proved in Section 9 of the RAMBO technical report [13].

7.1 Definitions

In this section, we present a few basic definitions. These definitions do not depend on timing, but are needed only for the conditional performance analysis. For these definitions, assume that \( \alpha \) is an execution.

First we define what it means for a configuration to be installed: configuration \( e \) is
installed when either of the following holds: (i) \(c = c_0\) or (ii) for some \(k > 0\), for all non-failed \(i \in \text{members}(c(k-1))\), a \text{decide}(c)_{k,i} \) event occurs in \(\alpha\). That is, configuration \(c = c(k)\) is installed when every non-failed member of configuration \(c(k-1)\) performs a \text{decide}(c(k)) \) event.

Next, we define an event that occurs when a configuration is guaranteed to be ready to be upgraded (though an upgrade operation may occur earlier than this event). We define the \text{upgrade-ready}(k) \) event, for \(k > 0\), to be the first event in \(\alpha\) after which, \(\forall \ell \leq k\), the following hold: (i) configuration \(c(\ell)\) is installed, and (ii) \(\forall i \in \text{members}(c(k-1))\) such that \(i\) has not failed at the time of the event, \(cmap(\ell)_i \neq \bot\).

### 7.2 Limiting Nondeterminism

The algorithm, as presented, is highly nondeterministic. Therefore for the purposes of analysis, we restrict our attention to a subset of executions in which automata follow certain timing-related rules. For the rest of this thesis we assume a fixed constant \(d > 0\). We assume that gossip occurs at fixed intervals of time \(d\), and also that in times of good behavior messages are delivered within time \(d\).

1. Each node, \(i \in I\), performs a \text{send}_{i,j} \) for all \(j \in \text{world}_i\) every time \(d\) as measured by the local clock of \(i\).

2. Each node, \(i \in I\), performs a \text{send}_{i,j} \) (an “important” send) whenever any of the following occurs:
   
   - Just after a \text{recv}(\text{join})_{j,i} \) event occurs, if \(status_i = \text{active}\).
   - (Responses for messages) Just after a \text{recv}(*, *, *, *), \text{pns}, *\)_{j,i} \) event occurs, if \(pns > pnum2(j)_i\) and \(status_i = \text{active}\).
   - Just after a \text{new-config}(c, k)_i \) event occurs if \(status_i = \text{active}\) and \(j \in \text{world}_i\).
   - Just after a \text{recv}(*, *, *, cm, *, *)_{j,i} \) event occurs, if \(op.phase_i \neq \text{idle}\) and for some \(k\), \(cm(k)_i \neq \bot\) and \(cmap(k)_i = \bot\).

\footnote{It seems, perhaps, that we should not be using \(d\) to represent both these quantities; however for consistency with the original RAMBO presentation, we continue to use this convention.}
• Just after a \texttt{read}_i, \texttt{write}_i, or \texttt{query-fix}_i event occurs, if \( j \in \text{members}(c) \), for some \( c \) in the range of \texttt{op.cmap}_i.

• Just after a \texttt{cfg-upgrade}(k)_i event occurs for configuration-upgrade \( \gamma \), if \( j \in \text{members(ckmap}(k'_i)) \) for any \( k'_i \in \text{removal-set}(\gamma) \).

• Just after a \texttt{cfg-upg-query-fix}(k)_i event occurs for configuration-upgrade \( \gamma \), if \( j \in \text{members(ckmap}(k'_i)) \) where \( k'_i = \text{target}(\gamma) \).

3. Locally controlled actions of any automaton in the system that have no effects, other than the important sends described just above, are performed only once.

4. If an action is enabled to occur at node \( i \), and has not yet been performed (and therefore is not restricted by the previous rule), then it occurs immediately, with zero time passing.

7.3 The Behavior of the Environment

Much of the analysis in the original \textsc{Rambo} algorithm makes guarantees about the latency of requests when “normal behavior” holds. In Section 9 of [13], Lynch and Shvartsman begin to examine how the system behaves in executions that achieve normal behavior \textit{after some point}. Here we adopt a similar model. We first define what it means for an execution to exhibit “normal behavior” from some point onward.

For the rest of the thesis, we use the following notation: given some time \( t \in \mathbb{R}^+ \), \( J(t, e, \alpha) \) represents the set of all nodes \( j \) such that \texttt{join-ack}_j occurs no later than time \( t - e - 2d \) in \( \alpha \). (Recall that \( d \) has been fixed, above.) In most cases, we will use the notation \( J(t) \), when \( e \) and \( \alpha \) are clear from the context.
7.3.1 Normal Timing Behavior from Some Point Onward

Let $\alpha$ be an admissible timed execution, and $\alpha'$ a finite prefix of $\alpha$. Arbitrary behavior is allowed in $\alpha'$: messages may be lost or delivered late, clocks may run at arbitrary rates, and in general any asynchronous behavior may occur. However we assume that after $\alpha'$, good behavior resumes. We say that $\alpha$ is an $\alpha'$-normal execution if the following assumptions hold:

1. **Initial time:** The join-ack$_{i_0}$ event occurs at time 0, completing the join protocol for node $i_0$, the node that created the data object.\(^2\)

2. **Regular timing:** The local clocks of all RAMBO II automata (i.e., Reader-Writer$_i$, Recon$_i$, Joiner$_i$) at all nodes progress at exactly the rate of real time, after $\alpha'$.

3. **Reliable message delivery:** No message sent in $\alpha$ after $\alpha'$ is lost.

4. **Message delay bound:** If a message is sent at time $t$ in $\alpha$ and it is delivered, then it is delivered by time $\max(t, \elltime(\alpha')) + d$.

7.3.2 Configuration–Viability

Next we will define configuration-viability, which is the key assumption needed to guarantee that read and write operations complete. As in all quorum-based algorithms, liveness depends on all the nodes in some quorums remaining alive. In RAMBO II, a node can make progress only if it is able to communicate with the read and write quorums of all extant configurations. We say that a configuration has failed when either: (i) some node in every read-quorum of the configuration has failed, or (ii) some node in every write-quorum of the configuration has failed. If a configuration fails before a new configuration is installed and the old configuration removed, then the system will be effectively crashed: no future read or write request will ever complete. In order to guarantee that operations complete, then, it is necessary for the client using the RAMBO II system to initiate appropriate reconfigurations.

\(^2\)This assumption was assumed implicitly in the initial RAMBO papers, and was missing from the list of assumptions.
to ensure that quorums remain accessible. The configuration viability assumption is a complex property, depending on the behavior of the algorithm, the client initiating appropriate reconfigurations, and on the patterns of node failure and message loss.

We define what it means for an execution to be \((\alpha', e, \tau)-configuration-viable\): Let \(\alpha\) be an admissible timed execution, and let \(\alpha'\) be a finite prefix of \(\alpha\). Let \(e, \tau \in \mathbb{R}^{\geq 0}\). Then \(\alpha\) is \((\alpha', e, \tau)-configuration-viable\) if the following holds:

For all \(i, c, k\) such that \(cmap(k)_i = c\) in some state in \(\alpha\), there exist \(R \in \text{read-quorums}(c)\) and \(W \in \text{write-quorums}(c)\) such that at least one of the following holds:

1. No process in \(R \cup W\) fails in \(\alpha\).

2. There exists a finite prefix \(\alpha_{\text{install}}\) of \(\alpha\) such that for all \(\ell \leq k + 1\), configuration \(c(\ell)\) is installed in \(\alpha_{\text{install}}\) and no process in \(R \cup W\) fails in \(\alpha\) by time \(\max(\ell \text{time}(\alpha') + e, \ell \text{time}(\alpha_{\text{install}})) + \tau\).

By assuming that an execution is \((\alpha', e, \tau)-configuration-viable\), we ensure that the algorithm has at least time \(\tau\) after a new configuration is installed to clean up obsolete configurations. Also, since all configurations are viable until at least time \(e + \tau\) after \(\alpha'\), the algorithm has at least time \(e + \tau\) after the system stabilizes to clean up obsolete configurations.

### 7.3.3 Recon-Spacing

While reconfigurations cannot impede a read/write operation, too frequent reconfigurations can slow down a read/write operation by introducing new quorums that must be contacted. In order to bound the time required for a read/write operation, we need to bound the frequency of reconfigurations.

There are two components to Recon-Spacing. Let \(\alpha\) be an \(\alpha'-normal\) execution, and \(e \in \mathbb{R}^{\geq 0}\). Then \(\alpha\) satisfies:

1. \((\alpha', e)-recon-spacing-1\): if for any \(\text{recon}(c, \ast)\) event in \(\alpha\) after \(\alpha'\) the preceding \(\text{report}(c)\) event occurs at least time \(e\) earlier.
2. \((\alpha', e)\)-recon-spacing-2: if for any \(\text{recon}(c, \ast)_i\) event in \(\alpha\) after \(\alpha'\) there exists a write-quorum \(W \in \text{write-quorums}(c)\) such that for all \(j \in W\), \(\text{report}(c)_j\) precedes the \(\text{recon}(c, \ast)_i\) event in \(\alpha\).

We say that \(\alpha\) satisfies \((\alpha', e)\)-recon-spacing if it satisfies both \((\alpha', e)\)-recon-spacing-1 and \((\alpha', e)\)-recon-spacing-2.

Notice that, instead of assuming the second part of this requirement, we could instead modify the \text{recon} automaton to enforce this ordering: the automaton could collect gossip messages indicating which nodes had performed \(\text{report}(c)\), and delay or abort the next \text{recon} if it preceded an appropriate set of \text{report} events. We choose to instantiate this as an assumption, rather than as a modification to the automaton for two reasons. First, we prefer to retain compatibility with the original RAMBO analysis. Second, by stating this as an assumption, it is possible that the client using the RAMBO II algorithm might choose to violate the second part of the assumption. As a result, those guarantees that depend on this assumption will not hold; however reconfigurations may be more performed more frequently. Even if the second part of this assumption is violated, safety is still guaranteed: atomicity is maintained, and read and write operations are guaranteed to terminate. However, operations might not terminate rapidly in \(8d\), as in Section 7.8.

### 7.3.4 Join-Connectivity

The hypothesis of \textit{join-connectivity} is designed to ensure that all non-failing joining processes are able to learn about each other. Otherwise, it is possible for the processes to join and fail in such a way that the world-views of the nodes are partitioned into multiple components, with different nodes aware of different, disconnected pieces of the world. It is also important for the latency analysis to bound how long this process takes. If two nodes both complete the join protocol and do not fail, then they should learn about each other within a bounded time. For this reason, we define the notion of \textit{join-connectivity} as follows:

Let \(\alpha\) be an \(\alpha'\)-normal execution, \(e \in \mathbb{R}^{\geq 0}\). We say that \(\alpha\) satisfies \((\alpha', e)\)-join-connectivity provided that: for any time \(t\) and nodes \(i, j \in J(t, e, \alpha)\), if neither \(i\) nor \(j\) fails until after \(\max(t - 2d, \elltime(\alpha') + e)\), then by time \(\max(t - 2d, \elltime(\alpha') + e)\), \(i \in \text{world}_j\).
This indicates, then, that if two nodes both complete joining by some time \( t \) after \( \alpha' \), then within time \( e \) the two nodes are aware of each other. If two nodes both complete joining by some time \( t \) during \( \alpha' \), then within time \( e \) after \( \alpha' \) the two nodes are aware of each other.

Prior results on joining from [13] suggest that in some cases it can be shown that the current simple join protocol in the RAMBO II algorithm provides \((\alpha', d + d\lceil \log(|J|) \rceil)\)-join-connectivity. However we will not prove - or depend on - this earlier result. Instead we will assume that the system provides \((\alpha', e)\)-join-connectivity for some \( e \), and prove our results in this context. We leave it as an open problem to determine the exact value of \( e \); a more complicated and interactive join protocol might well provide better results.

### 7.3.5 Recon-Readiness

The next assumption we make is related to the problem of reconfiguration by a node that has recently joined. We will assume that every node that is proposed to be a member of a configuration has been a member of the RAMBO II system for a reasonable period of time. This allows us to conclude that everyone is aware of nodes that are part of active configurations.

An \( \alpha' \)-normal execution \( \alpha \) satisfies \((\alpha', e)\)-recon-readiness if the following property holds: if for some node \( i \) and some configurations \( c \) and \( c' \), a \( \text{recon}(c, c')_i \) event occurs in \( \alpha \) at time \( t \), then:

- If \( j \in \text{members}(c') \), then \( j \) performs a join-ack prior to the \( \text{recon} \) event.
- If the \( \text{recon} \) event occurs after \( \alpha' \), and if \( j \in \text{members}(c') \), then \( j \in J(t, e, \alpha) \).

This prohibits nodes that have just joined the system, but are not yet in anyone’s world view from forming new configurations. As long as \( e \) is not too large, this seems a reasonable requirement.

### 7.3.6 Upgrade-Readiness

The last assumption we make ensures that a node initiates an upgrade operation only if it has joined sufficiently long ago. This ensures that when a node performs an upgrade, it has
relatively up-to-date information.

We say that an $\alpha'$-normal execution $\alpha$ satisfies $(\alpha', e)$-upgrade-readiness if the following property holds: if for some $i$ a $\text{cfg-upgrade}(\ast)_i$ event occurs in $\alpha$ after $\alpha'$ at time $t$, then $i \in J(t)$.

In particular, we suggest that in an implementation of this algorithm, only members of configuration $c(k)$ initiate operations to upgrade configuration $c(k)$. In this case, recon-readiness guarantees upgrade-readiness.

### 7.3.7 Fixed Parameters

We have already fixed $d$ such that gossip occurs at fixed intervals of time $d$, and in times of good behaviour messages are delivered with time $d$. We now fix $e$ as well. Additionally, for the rest of the thesis, we fix $\alpha$ and $\alpha'$, and assume that $\alpha$ is an $\alpha'$-normal execution. We therefore sometimes suppress these parameters, as they are clear from context. For example, we will use the notation $J(t)$ to represent $J(t, e, \alpha)$. When we refer to join-connectivity, we mean $(\alpha', e)$-join-connectivity; recon-readiness is used to mean $(\alpha', e)$-recon-readiness; upgrade-readiness is used to mean $(\alpha', e)$-upgrade-readiness; $\tau$-recon-spacing is used to mean $(\alpha', \tau)$-recon-spacing; $\tau$-configuration-viability is used to mean $(\alpha', e, \tau)$-configuration viability.

### 7.4 Basic Lemmas

In this section, we prove a few basic lemmas that will be useful in the rest of the thesis.

The following two lemmas demonstrate some basic facts about the sets $J(\ast)$:

**Lemma 7.4.1**

1. If $t \leq t'$, then $J(t) \subseteq J(t')$.

2. For all $t, t'$, $J(t) \subseteq J(\max(t, t'))$.

**Proof.** By definition of $J(\cdot)$. □

The following lemma uses the recon-readiness assumption to say something stronger about the joining time of members of a configuration:
Lemma 7.4.2 Assume that $\alpha$ is an $\alpha'$-normal execution satisfying $(\alpha', \epsilon)$-recon-readiness. If $h$ is a configuration proposed at time $t'$ by a recon($\ast, h$) event, $t \geq t'$, and $t \geq \elltime(\alpha') + \epsilon + 2d$, then \text{members}(h) \subseteq J(t).

Proof. First, assume that $t' \geq \elltime(\alpha')$. Then the result follows immediately by recon-readiness and Lemma 7.4.1. Assume then, that $t' < \elltime(\alpha')$. By recon-readiness, every member of configuration $h$ performs a join-ack by $\elltime(\alpha')$. Therefore, by definition of $J$, \text{members}(h) \subseteq J(\elltime(\alpha') + \epsilon + 2d). Since $t \geq \elltime(\alpha') + \epsilon + 2d$, Lemma 7.4.1 implies that $J(\elltime(\alpha') + \epsilon + 2d) \subseteq J(t)$.

The next lemma shows a similar result about upgrade-readiness:

Lemma 7.4.3 Assume that $\alpha$ is an $\alpha'$-normal execution satisfying $(\alpha', \epsilon)$-upgrade-readiness. If a \text{cfg-upgrade}($\ast$)$_i$ event occurs in $\alpha$ at time $t$, for some node $i$, then $i \in J(\max(t, \elltime(\alpha') + \epsilon + 2d))$.

Proof. First, assume that the \text{cfg-upgrade} event occurs after $\alpha'$. Then the lemma follows immediately by the definition of upgrade-readiness and Lemma 7.4.1. Assume then, that the \text{cfg-upgrade} event occurs in $\alpha'$. By the precondition of \text{cfg-upgrade}, $i$ must perform a join-ack prior to the \text{cfg-upgrade} event: otherwise $\text{status}_i \neq \text{active}$ when the \text{cfg-upgrade} occurs, which contradicts the precondition of the \text{cfg-upgrade}. Therefore $i$ performs a join-ack$_i$ at latest at time $\elltime(\alpha')$, and therefore $i \in J(\elltime(\alpha') + \epsilon + 2d)$, and the lemma again follows by Lemma 7.4.1.
7.5 Propagation of Information

In this section, we introduce the notion of information being in the “mainstream”. Once a sufficient set of nodes know a particular fact, then, under appropriate assumptions, this fact will never be forgotten by the system as a whole. In particular, we show that this is true about information in the cmap: updates to the cmap are propagated. Once every non-failed node in $J(t)$ updates its cmap, then at any time in the future, at time $t' ≥ t + 2d$, every non-failed node in $J(t')$ will be aware of this update.

If $cm$ is a CMap and $β$ is a finite prefix of $α$ with $ëtime(β) = t ≥ e + 2d$, then we say that $cm$ is mainstream after $β$ provided that the following holds: For every $i ∈ J(t)$ such that $\text{fail}_i$ does not occur in $β$, $cm ≤ ëstate(β). cmap_i$.

Further, we define the following notation: given an execution $α$ and a time $t ∈ \mathbb{R}^{≥0}$, we define $β(t, α)$ to be the finite prefix of $α$ such that $ëtime(β(t, α)) = t$ and every event that occurs at time $t$ occurs in $β(t, α)$. As we have already fixed $α$, for the rest of this paper we use the simpler notation of $β(t)$. We then say that a CMap $cm$ is mainstream after $t$ if it is mainstream after $β(t)$.

The first lemma shows a basic property of mainstream cmaps:

**Lemma 7.5.1** Assume that $α$ is an execution, $t$ is a time, and $cm$, $cm2$ are CMaps. If $cm ≤ cm2$, and $cm2$ is mainstream after $t$, then $cm$ is mainstream after $t$.

**Proof.** Immediate from the definition of mainstream.  

The following lemma shows that a node’s cmap is monotone:

**Lemma 7.5.2** Assume that $α''$ is a finite prefix of execution $α$, and that $α'''$ is a prefix of $α''$. Assume that $i$ is a node. Then $ëstate(α'''). cmap_i ≤ ëstate(α''). cmap_i$.

**Proof.** In the algorithm, $cmap_i$ is only modified by the update function, and the update function is monotone; that is, for all CMaps new-cmap, $cmap ≤ update(cmap, new-cmap)$.

**Lemma 7.5.3** Assume that $α$ is an execution, and $t$ and $t'$ are times, and that $t ≤ t'$. Assume that $i$ is a node, and $cm$ is a CMap.
Lemma 7.5.4

1. If \( cm \leq \ellstate(\beta(t)).cmap_i \), then \( cm \leq \ellstate(\beta(t')).cmap_i \).

2. \( \ellstate(\beta(t)).cmap_i \leq \ellstate(\beta(t')).cmap_i \).

Proof. This follows by Lemma 7.5.2, where \( \alpha'' = \beta(t) \) and \( \alpha'' = \beta(t') \).

Next, we demonstrate a particular case when a \( cmap \) becomes mainstream.

Lemma 7.5.4 Let \( \alpha \) be an \( \alpha' \)-normal execution satisfying \((\alpha',e)\)-join-connectivity. Let \( t \) be a time such that \( t \geq \elltime(\alpha') + e \). If \( i \in J(t+2d) \), and \( i \) does not fail in \( \beta(t+d) \), then \( \ellstate(\beta(t)).cmap_i \) is mainstream after \( t + 2d \).

Proof. Let \( cm = \ellstate(\beta(t)).cmap_i \). To show that \( cm \) is mainstream after \( t + 2d \), we need to show that for all \( j \in J(t + 2d) \) such that \( j \) does not fail in \( \beta(t + 2d) \), \( cm \leq \ellstate(\beta(t + 2d)).cmap_j \). Fix any such \( j \). By join-connectivity, \( j \in world_i \) by time \( \max(t, \elltime(\alpha') + e) \leq t \).

By time \( t + d \), \( i \) sends a gossip message, \( msg \), to node \( j \) such that \( cm \leq msg.cmap_i \). By time \( t + 2d \), \( j \) receives the gossip message and updates \( cmap_j \) with \( msg.cmap \). By the monotonicity of the \( update \) function, \( msg.cmap \leq update(cmap_j, msg.cmap) \). Therefore \( cm \leq \ellstate(\beta(t + 2d)).cmap_j \), as required.

The following lemma shows that if two nodes are both in the set \( J(t + 2d) \), then information is propagated from one to the other.

Lemma 7.5.5 Let \( \alpha \) be an \( \alpha' \)-normal execution satisfying \((\alpha',e)\)-join-connectivity. Assume that \( t \) and \( t' \) are times, and \( t' - 2d \geq t \geq \elltime(\alpha') + e \). Assume that \( i \) and \( j \) are nodes, and \( i, j \in J(t + 2d) \). Also, assume that \( i \) does not fail in \( \beta(t + 2d) \), and \( j \) does not fail in \( \beta(t') \).

If \( cm \leq \ellstate(\beta(t)).cmap_j \), then \( cm \leq \ellstate(\beta(t')).cmap_j \).
\[ \begin{array}{cccc}
\alpha' & \alpha' + e + 2d & t & t' + 2d \\
\end{array} \]

\( \text{cm mainstream after } t \implies \text{cm mainstream after } t' + 2d \)

**Figure 7-5: Lemma 7.5.6**

**Proof.** By Lemma 7.5.4, \( \ell\text{state}(\beta(t)).\text{cm}_i \) is mainstream after \( t + 2d \). Notice that \( j \in J(t + 2d) \), and therefore, by the definition of mainstream, \( \ell\text{state}(\beta(t)).\text{cm}_i \leq \ell\text{state}(\beta(t + 2d)).\text{cm}_j \). Since \( t + 2d \leq t' \), by Lemma 7.5.3, \( \ell\text{state}(\beta(t + 2d)).\text{cm}_j \leq \ell\text{state}(\beta(t')).\text{cm}_j \).

Putting the inequalities together, \( \text{cm} \leq \ell\text{state}(\beta(t')).\text{cm}_j \).

We now show that once a \( \text{cm} \) is in the mainstream, after \( 2d \) it will always be in the mainstream. First, Lemma 7.5.6 considers a special case: it considers only times \( t' \) after the system has stabilized, when a \( \text{recon}(h, h') \) event occurs. Second, Lemma 7.5.7 handles the case where the \( \text{cm} \) is in the mainstream at a time in \( \alpha' \). Third, Lemma 7.5.8 proves the existence of a configuration with some necessary special properties to prove the main theorem. Finally, Lemmas 7.5.9 and 7.5.10 prove the general result, as summarized in Lemma 7.5.11.

First, we define a successful \( \text{recon} \) event as follows: a \( \text{recon}(s, c) \) event is successful if at some time afterwards a \( \text{decide}(c)_k \) event occurs for some \( k \) and \( i \).

**Lemma 7.5.6** Let \( \alpha \) be an \( \alpha' \)-normal execution satisfying: (i) \((\alpha', e)\)-join-connectivity, (ii) \((\alpha', e)\)-recon-readiness, (iii) \((\alpha', 2d)\)-recon-spacing-1, and (iv) \((\alpha', e, 2d)\)-configuration-viability.

Assume that \( t \) and \( t' \) are times, and that \( t \geq \ell\text{time}(\alpha') + e + 2d \) and \( t' \geq t \). Let \( h \) and \( h' \) be two configurations, and assume that \( \text{recon}(h, h') \) occurs at time \( t' \), and that this is a successful \( \text{recon} \) event.

If \( \text{cm} \) is mainstream after \( t \), then \( \text{cm} \) is mainstream after \( t' + 2d \).

**Proof.** Fix \( t \) and \( \text{cm} \) such that \( \text{cm} \) is mainstream after \( t \). We prove the result by induction on the number of successful \( \text{recon} \) events that occur at or after time \( t \).
As the base case, consider the first successful $\text{recon}(h, h')$ event that occurs in $\alpha$ at a time $t' \geq t$. We need to show that $cm$ is mainstream after $t' + 2d$. Therefore fix some $j' \in J(t' + 2d)$ such that $\text{fail}_{j'}$ does not occur in $\beta(t' + 2d)$. We will show that $cm \leq \ell state(\beta(t' + 2d)).\text{cmap}_{j'}$.

Choose some node $j \in \text{members}(h)$ such that $j$ does not fail in $\beta(t' + 2d)$; that is, $j$ does not fail until after $t' + 2d$. Configuration-viability ensures that such a node exists. Notice that $j \in J(t)$, by Lemma 7.4.2. Since $cm$ is mainstream after $t$, then $cm \leq \ell state(\beta).\text{cmap}_{j'}$, as needed.

Next we show the inductive step. Inductively assume the following: if $\text{recon}(\ast, \ast)$ is one of the first $n$ successful $\text{recon}$ events in $\alpha$ that occur at time $t' \geq t$, then $cm$ is mainstream after $t'$.

Consider the $(n+1)^{\text{st}}$ successful $\text{recon}(h, h')$ event in $\alpha$ that occurs at or after $t$. Assume this event occurs at time $t'$. We need to show that $cm$ is mainstream after $t' + 2d$. Therefore fix some $j' \in J(t' + 2d)$ such that $\text{fail}_{j'}$ does not occur in $\beta(t' + 2d)$. We will show that $cm \leq \ell state(\beta(t' + 2d)).\text{cmap}_{j'}$.

Let $\rho$ be the $n^{\text{th}}$ successful $\text{recon}(\ast, h)$ event, and assume that $\rho$ occurs at time $t_1$. Note that the first argument of the $(n+1)^{\text{st}}$ successful $\text{recon}$ event must be the configuration proposed by the $n^{\text{th}}$ successful $\text{recon}$ event.

$2d$-recon-spacing-1 guarantees that $t' \geq t_1 + 2d$. The inductive hypothesis shows that $cm$ is mainstream after $t_1 + 2d$.

Choose some node $j \in \text{members}(h)$ such that no $\text{fail}_j$ occurs in $\beta(t' + 2d)$. Configuration-viability ensures that such a node exists. By recon-readiness and Lemma 7.4.1, $j \in J(t' + 2d)$. By assumption $j' \in J(t' + 2d)$ and $j'$ does not fail in $\beta(t' + 2d)$. By Lemma 7.5.5, $cm \leq \ell state(\beta(t' + 2d)).\text{cmap}_{j'}$, as needed.

The next lemma considers the case where a $\text{cmap}$ is mainstream in $\alpha'$ or soon after, and shows that it is mainstream after $\ell time(\alpha') + e + 4d$. 

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Lemma 7.5.7 Let $\alpha$ be an $\alpha'$-normal execution satisfying (i) $(\alpha', e)$-join-connectivity, (ii) $(\alpha', e)$-recon-readiness, (iii) $(\alpha', 2d)$-recon-spacing-1, and (iv) $(\alpha', e, 4d)$-configuration-viability.

Assume that $t$ is a time and that $e + 2d \leq t \leq \ell\text{time}(\alpha') + e + 2d$. If $cm$ is mainstream after $t$, then $cm$ is mainstream after $\ell\text{time}(\alpha') + e + 4d$.

Proof. Consider configuration $c_0$. By configuration-viability, there exists a read-quorum, $R \in \text{read-quorums}(c_0)$, and a write-quorum, $W \in \text{write-quorums}(c_0)$ such that no node in $R \cup W$ fails by $\ell\text{time}(\alpha') + e + 4d$.

Let $t_1 = \ell\text{time}(\alpha') + e + 2d$. Consider $i_0 \in R \cup W$; $i_0$ does not fail by $\ell\text{time}(\alpha') + e + 4d$. Since $i_0$ performs a join-ack at time 0, by the assumption that $\alpha$ is an $\alpha'$-normal execution, and since $t \geq e + 2d$, $i_0 \in J(t)$. Also note that by Lemma 7.5.3, $i_0 \in J(t_1)$.

Since $cm$ is mainstream after $t$, $cm \leq \ell\text{state}(\beta(t)).cmap_{i_0}$. Therefore, we know by Lemma 7.5.3 that $cm \leq \ell\text{state}(\beta(t_1)).cmap_{i_0}$. By Lemma 7.5.4, we know that $\ell\text{state}(\beta(t_1)).cmap_{i_0}$ is mainstream after $t_1 + 2d$. Therefore by Lemma 7.5.1, $cm$ is mainstream after $t_1 + 2d$; that is, $cm$ is mainstream after $\ell\text{time}(\alpha') + e + 4d$. \qed

The next lemma shows the existence of a certain configuration, $h'$, with some particular properties. This will be useful in proving Lemma 7.5.11.

Lemma 7.5.8 Let $\alpha$ be an $\alpha'$-normal execution satisfying: (i) $(\alpha', e)$-join-connectivity, (ii) $(\alpha', e)$-recon-readiness, (iii) $(\alpha', 2d)$-recon-spacing-1, and (iv) $(\alpha', e, 4d)$-configuration-viability.

Assume that $t$ and $t'$ are times. Assume that $\ell\text{time}(\alpha') + e + 2d \leq t \leq t' - 2d$ and $\ell\text{time}(\alpha') + e + 6d \leq t'$. Assume that $cm$ is mainstream after $t$. Then there exists a configuration $h$, with index $k$, with the following properties:
1. $\text{members}(h) \subseteq J(t')$.

2. For all members $i$ of configuration $h$ that do not fail in $\beta(t')$, $cm \leq \ell\text{state}(\beta(t' - 2d)).cmap_i$.

3. No successful $\text{recon}(h, *)$ event occurs in $\beta(t' - 4d)$.

Proof. There are three different sub-cases to consider.

1. No successful $\text{recon}$ event occurs in $\beta(t' - 4d)$:

   Let $h = c_0$. Notice that $\text{members}(h) \subseteq J(t)$, since $i_0$ (the only member of $c_0$) completes a join-ack at time 0 (by assumption on $\alpha$), and $t > \ell\text{time}(\alpha') + e + 2d$. This, then, implies Property 1 by Lemma 7.4.1. Since $i_0 \in J(t)$ and $cm$ is mainstream after $t$, $cm \leq \ell\text{state}(\beta(t)).cmap_{i_0}$. Therefore, since $t \leq t' - 2d$, by Lemma 7.5.3, $cm \leq \ell\text{state}(\beta(t' - 2d)).cmap_{i_0}$, as required for Property 2. Property 3 holds trivially.

2. A successful $\text{recon}$ event occurs in $\beta(t' - 4d)$ after time $t$:

   Consider the last successful $\text{recon}$ event in $\alpha$ that occurs in $\beta(t' - 4d)$; let $h$ be the configuration identifier appearing as the second argument in this $\text{recon}$ event. Assume that this $\text{recon}$ event occurs at time $t_{rec}$. Note that $t < t_{rec} \leq t' - 4d$. Therefore (since $t' \geq \ell\text{time}(\alpha') + e + 6d$ and $t' \geq t_{rec}$) by Lemma 7.4.2, $\text{members}(h) \subseteq J(t')$, as required for Property 1. Since $t_{rec} > t$, Lemma 7.5.6 shows that $cm$ is mainstream after $t_{rec} + 2d$. Recall that $t_{rec} + 2d \leq t' - 2d$. By the mainstream property, for every member, $i$, of configuration $h$ that does not fail in $\beta(t' - 2d)$, $cm \leq \ell\text{state}(\beta(t_{rec} + 2d)).cmap_i$; therefore, for each of these members, $i$, by Lemma 7.5.3, $cm \leq \ell\text{state}(\beta(t' - 2d)).cmap_i$, as required for Property 2. Property 3 holds by the selection of the last successful $\text{recon}$ event in $\beta(t' - 4d)$.

3. Neither Case 1 nor Case 2 holds, that is, a successful $\text{recon}$ event occurs in $\beta(t' - 4d)$, but no such $\text{recon}$ event occurs after time $t$:

   Consider the last successful $\text{recon}$ event in $\alpha$ that occurs in $\beta(t' - 4d)$; let $h$ be the configuration identifier appearing as the second argument in this $\text{recon}$ event. Assume that this $\text{recon}$ event occurs at time $t_{rec}$. Notice, then, that $t_{rec} \leq t$. (Otherwise, Case
2 would hold.) Since \( t \geq \ell \text{time}(\alpha') + e + 2d \), then by Lemma 7.4.2, \( \text{members}(h) \subseteq J(t) \).

By Lemma 7.5.3, then, \( \text{members}(h) \subseteq J(t') \), which implies Property 1. Since \( cm \) is mainstream after \( t \) (and \( \text{members}(h) \subseteq J(t) \)), for all \( j \in \text{members}(h) \) such that no \( \text{fail}_j \) event occurs in \( \beta(t) \), \( cm \leq \ell \text{state}(\beta(t)).cmap_j \). Since \( t \leq t' - 2d \), by Lemma 7.5.3, for all \( j \) such that no \( \text{fail}_i \) event occurs by time \( t' - 2d \), \( cm \leq \ell \text{state}(\beta(t' - 2d)).cmap_j \), as required for Property 2. Property 3 holds by the selection of the last successful \( \text{recon} \) event that occurs in \( \beta(t' - 4d) \).

\[ \square \]

Finally we prove the main lemma of this section, showing that if a \( cmap \) is mainstream at time \( t \), then the \( cmap \) is also mainstream at times \( t' \geq t + 2d \). There are two cases to consider: (i) \( t \geq \ell \text{time}(\alpha') + e + 2d \), and (ii) \( t < \ell \text{time}(\alpha') + e + 2d \). Lemma 7.5.9 shows the first case, Lemma 7.5.10 shows the second case, and Lemma 7.5.11 presents the overall conclusion.

**Lemma 7.5.9** Let \( \alpha \) be an \( \alpha' \)-normal execution satisfying (i) \((\alpha', e)\)-join-connectivity, (ii) \((\alpha', e)\)-recon-readiness, (iii) \((\alpha', 2d)\)-recon-spacing-1, and (iv) \((\alpha', e, 4d)\)-configuration-viability.

Assume that \( t \) and \( t' \) are times. Assume that \( e + 2d \leq t \leq t' - 2d \) and \( \ell \text{time}(\alpha') + e + 6d \leq t' \). Additionally assume that \( t \geq \ell \text{time}(\alpha') + e + 2d \). If \( cm \) is a mainstream CMap after \( t \), then \( cm \) is mainstream after \( t' \).

**Proof.** By assumption, \( t \geq \ell \text{time}(\alpha') + e + 2d \). Lemma 7.5.8 shows that there exists a configuration, \( h \), with index \( k \) with the following three properties:

1. \( \text{members}(h) \subseteq J(t') \).
2. For all members $i$ of configuration $h$ that do not fail in $\beta(t')$, $cm \leq \ell\text{state}(\beta(t' - 2d)).cmap_j$.

3. No successful $\text{recon}(h, \ast)$ event occurs in $\beta(t' - 4d)$.

Configuration-viability guarantees that some node of configuration $h$ does not fail until after the next configuration is installed. No successful $\text{recon}(h, \ast)$ event occurs in $\beta(t' - 4d)$, by Property 3. Therefore some node, $j \in \text{members}(h)$ does not fail in $\beta(t')$ (and therefore does not fail in $\beta(t' - d)$), by $4d$-configuration-viability. By Property 1 of $h$, node $j \in J(t')$. Therefore, by Lemma 7.5.4, $\ell\text{state}(\beta(t' - 2d)).cmap_j$ is mainstream after $t'$.

Further, we know by Property 2 that $cm \leq \ell\text{state}(\beta(t' - 2d)).cmap_j$. Therefore by Lemma 7.5.1, $cm$ is mainstream after $t'$.

The following lemma considers the case where $t < \ell\text{time}(\alpha') + e + 2d$:

**Lemma 7.5.10** Let $\alpha$ be an $\alpha'$-normal execution satisfying (i) $(\alpha', e)$-join-connectivity, (ii) $(\alpha', e)$-recon-readiness, (iii) $(\alpha', 2d)$-recon-spacing-1, and (iv) $(\alpha', e, 4d)$-configuration-viability.

Assume that $t$ and $t'$ are times. Assume that $e + 2d \leq t \leq t' - 2d$ and $\ell\text{time}(\alpha') + e + 6d \leq t'$. Additionally, assume that $t < \ell\text{time}(\alpha') + e + 2d$. If $cm$ is a mainstream $\text{CMap}$ after $t$, then $cm$ is mainstream after $t'$.

**Proof.** By assumption, $t < \ell\text{time}(\alpha') + e + 2d$. Let $t_1 = \ell\text{time}(\alpha') + e + 2d$. By Lemma 7.5.7, $cm$ is mainstream after $t_1 + 2d$. By assumption, $t_1 + 2d \leq t' - 2d$, and $\ell\text{time}(\alpha') + e + 2d \leq t_1 + 2d$. By Lemma 7.5.9, however, we know that since $cm$ is mainstream after $t_1 + 2d$, then $cm$ is mainstream after $t'$.
The following lemma combines the previous two lemmas into a single conclusion. This lemma is the main result of this section, and is used throughout the rest of the proof.

Lemma 7.5.11 Let \( \alpha \) be an \( \alpha' \)-normal execution satisfying (i) \((\alpha', e)\)-join-connectivity, (ii) \((\alpha', e)\)-recon-readiness, (iii) \((\alpha', 2d)\)-recon-spacing-1, and (iv) \((\alpha', e, 4d)\)-configuration-viability.

Assume that \( t \) and \( t' \) are times. Assume that \( e+2d \leq t \leq t'-2d \) and \( \elltime(\alpha')+e+6d \leq t' \). If \( cm \) is a mainstream CMap after \( t \), then \( cm \) is mainstream after \( t' \).

Proof. By Lemmas 7.5.9 and 7.5.10. \( \square \)

7.6 Upgrade-Ready Viability

In this section, we show the relationship between a configuration being upgrade-ready, and a configuration being viable. In particular, we prove that if an execution \( \alpha \) is \((\alpha', e, 22d)\)-configuration-viable, then configuration \( c(k) \) is viable until at least \( 15d \) after the upgrade-ready \((c(k+1))\) event.

The first lemma shows that soon after a configuration is installed, every node that joined a while ago learns about the new configuration.

Lemma 7.6.1 Let \( \alpha \) be an \( \alpha' \)-normal execution satisfying: (i) \((\alpha', e)\)-join-connectivity, (ii) \((\alpha', e)\)-recon-readiness, (iii) \((\alpha', e, 4d)\)-configuration-viability.

Assume that \( t \in \mathbb{R}_{\geq 0} \) is a time, and configuration \( c(k) \) is installed at time \( t \). Then there exists a CMap, \( cm \), such that \( cm(k) \neq \bot \), and \( cm \) is mainstream after \( \max(t, \elltime(\alpha')+e)+2d \).

Proof. We first find a node \( j \in \text{members}(c(k-1)) \) such that \( j \in J(\max(t, \elltime(\alpha')+e)+2d) \) and \( j \) does not fail in \( \beta(\max(t, \elltime(\alpha')+e)+d) \). Configuration-viability guarantees that there exists a read-quorum \( R \in \text{read-quorums}(c(k-1)) \) and a prefix \( \alpha'' \) of \( \alpha \) such that \( c(k) \) is installed in \( \alpha \) and no node in \( R \) fails by \( \max(\elltime(\alpha''), \elltime(\alpha')+e)+4d \). Since configuration \( c(k) \) is installed at time \( t \), we know that \( t \leq \elltime(\alpha'') \), and therefore no node in \( R \) fails
by \( \max(t, \ell\text{time}(\alpha')+e) + 4d \). Therefore no node in \( R \) fails in \( \beta(\max(t, \ell\text{time}(\alpha')+e) + d) \). Choose some node \( j \in R \).

Assume that configuration \( c(k-1) \) is proposed at time \( t_{rec} \). We next apply Lemma 7.4.2 where \( h = c(k-1), t' = t_{rec} \), and \( t = \max(t, \ell\text{time}(\alpha')+e) + 2d \):

- \( \max(t, \ell\text{time}(\alpha')+e) + 2d \geq t_{rec} \): \( c(k-1) \) is proposed at \( t_{rec} \leq t \), since \( c(k-1) \) must be proposed prior to configuration \( c(k-1) \) being installed, which must occur prior to configuration \( c(k) \) being installed; \( t \leq \max(t, \ell\text{time}(\alpha')+e) + 2d \).

- \( \max(t, \ell\text{time}(\alpha')+e) + 2d \geq \ell\text{time}(\alpha')+e + 2d \): Immediate.

We therefore conclude that \( \text{members}(c(k-1)) \subseteq J(\max(t, \ell\text{time}(\alpha')+e) + 2d) \). Therefore we have shown that \( j \in \text{members}(c(k-1)), j \in J(\max(t, \ell\text{time}(\alpha')+e) + 2d), \) and \( j \) does not fail in \( \beta(\max(t, \ell\text{time}(\alpha')+e) + d) \).

Since configuration \( c(k) \) is installed at time \( t \) and \( j \in \text{members}(c(k-1)) \), \( \ell\text{state}(\beta(t)), cmap(k)_j \neq \bot \), by the definition of a configuration being installed, and therefore (by Lemma 7.5.3) \( \ell\text{state}(\beta(\max(t, \ell\text{time}(\alpha')+e))), cmap(k)_j \neq \bot \). We let \( cm = \ell\text{state}(\beta(\max(t, \ell\text{time}(\alpha')+e))) \). \( cmap(k)_j; cm(k) \neq \bot \), as required.

We next apply Lemma 7.5.4, where \( t = \max(t, \ell\text{time}(\alpha')+e) \) and \( i = j \):

- \( \max(t, \ell\text{time}(\alpha')+e) \geq \ell\text{time}(\alpha')+e \): Immediate.

- \( j \in J(\max(t, \ell\text{time}(\alpha')+e) + 2d) \): Shown above.

- \( j \) does not fail in \( \beta(\max(t, \ell\text{time}(\alpha')+e) + d) \): Shown above.

We therefore conclude that \( \ell\text{state}(\beta(\max(t, \ell\text{time}(\alpha')+e))), cmap_i \) is mainstream after \( \max(t, \ell\text{time}(\alpha')+e) + 2d \), that is, \( cm \) is mainstream after \( t, \ell\text{time}(\alpha')+e + 2d \). \( \square \)

The next lemma shows that soon after smaller configurations are installed, a configuration is upgrade-ready.

**Lemma 7.6.2** Let \( \alpha \) be an \( \alpha' \)-normal execution satisfying: (i) \( (\alpha', e) \)-join-connectivity, (ii) \( (\alpha', e) \)-recon-readiness, (iii) \( (\alpha', 2d) \)-recon-spacing-1, and (iv) \( (\alpha', e, 4d) \)-configuration-viability.
Let $c$ be a configuration with index $k$, and assume that for all $\ell \leq k$, configuration $c(\ell)$ is installed in $\alpha$ by time $t$.

Then upgrade-ready($k$) occurs in $\beta(\max(t, \elltime(\alpha') + e) + 6d)$.

**Proof.** For every configuration $c(\ell)$ with index $\ell \leq k$, let $t_\ell$ be the time at which configuration $c(\ell)$ is installed. Therefore $t \geq \max(t_\ell)$.

We first show that for all $\ell \leq k$, there exists a CMap, $c_{m_\ell}$ such that $c_{m_\ell}(\ell) \neq \bot$ and $c_{m_\ell}$ is mainstream after $\max(t, \elltime(\alpha') + e) + 6d$. Fix some $\ell \leq k$.

Lemma 7.6.1, where $t = t_\ell$ and $k = \ell$, shows that there exists a CMap, $c_{m_\ell}$ such that $c_{m_\ell}(\ell) \neq \bot$ and $c_{m_\ell}$ is mainstream after time $\max(t_\ell, \elltime(\alpha') + e) + 2d$.

We next apply Lemma 7.5.11, where $t = \max(t_\ell, \elltime(\alpha') + e) + 2d$ and $t' = \max(t, \elltime(\alpha') + e) + 6d$:

- $\max(t_\ell, \elltime(\alpha') + e) + 2d \geq e + 2d$: Immediate.
- $\max(t_\ell, \elltime(\alpha') + e) + 2d \leq \max(t, \elltime(\alpha') + e) + 6d - 2d$: We know that $t_\ell \leq t$, and $\elltime(\alpha') + e + 2d \leq \elltime(\alpha') + e + 4d$.
- $\max(t, \elltime(\alpha') + e) + 6d \geq \elltime(\alpha') + e + 6d$: Immediate.
- $c_{m_\ell}$ is mainstream after $\max(t_\ell, \elltime(\alpha') + e) + 2d$: Shown above.

We therefore conclude that $c_{m_\ell}$ is mainstream after $\max(t, \elltime(\alpha') + e) + 6d$. We have thus shown that for all $\ell \leq k$, there exists a CMap, $c_{m_\ell}$ such that $c_{m_\ell}(\ell) \neq \bot$ and $c_{m_\ell}$ is mainstream after $\max(t, \elltime(\alpha') + e) + 6d$.

Recall that upgrade-ready($k$) is designated as the first event after which (i) all configurations with index $\leq k$ have been installed, and (ii) for all $\ell < k$, for all members of configuration $c(\ell - 1)$ that do not fail prior to the upgrade event, $cmap(\ell) \neq \bot$. The first component occurs by time $t$, and therefore by time $\max(t, \elltime(\alpha') + e) + 6d$, by assumption.

We therefore need to show the second part. Fix some node $j \in members(c(k - 1))$ such that $j$ does not fail in $\beta(\max(t, \elltime(\alpha') + e) + 6d)$. Fix some $\ell < k$. We apply Lemma 7.4.2, where $h = c(k - 1)$, $t = \max(t, \elltime(\alpha') + e) + 6d$, and $t'$ is the time at which configuration $c(k - 1)$ is proposed:
- \( \max(t, \ell\text{time}(\alpha') + e) + 6d \) is \( \geq \) the time at which configuration \( c(k - 1) \) is proposed: \( c(k - 1) \) is proposed prior to time \( t_{k-1} \) (the time at which configuration \( c(k - 1) \) is installed), which is \( \leq t = \max(t, \ell\text{time}(\alpha') + e) + 6d \).

- \( \max(t, \ell\text{time}(\alpha') + e) + 6d \geq \ell\text{time}(\alpha') + e + 2d \): Immediate.

We therefore conclude that \( \text{members}(c(k-1)) \subseteq J(\max(t, \ell\text{time}(\alpha') + e) + 6d) \), and therefore \( j \in J(\max(t, \ell\text{time}(\alpha') + e) + 6d) \).

We know from above that \( cm_\ell \) is mainstream after \( \max(t, \ell\text{time}(\alpha') + e) + 6d \), which implies, by the definition of being mainstream, that \( cm_\ell \leq \ell\text{state}(\beta(\max(t, \ell\text{time}(\alpha') + e) + 6d)) \cdot cmap(\ell)_j \). This in turn implies that \( \ell\text{state}(\beta(\max(t, \ell\text{time}(\alpha') + e) + 6d)) \cdot cmap(\ell)_j \neq \perp \), as required. Therefore \( \text{upgrade-ready}(k) \) occurs in \( \beta(\max(t, \ell\text{time}(\alpha') + e) + 6d) \). \( \square \)

The next lemma directly relates the time when all quorums of configuration \( c(k - 1) \) fail to the time at which \( \text{upgrade-ready}(k) \) occurs.

**Lemma 7.6.3** Let \( \alpha \) be an \( \alpha' \)-normal execution satisfying: (i) (\( \alpha', e \))-join-connectivity, (ii) (\( \alpha', e \))-recon-readiness, (iii) (\( \alpha', 2d \))-recon-spacing-1, and (iv) (\( \alpha', e, 22d \))-configuration-viability.

Let \( c \) be a configuration with index \( k \), and assume that the \( \text{upgrade-ready}(k) \) event occurs at time \( t \). Then there exists a read-quorum, \( R \), and a write-quorum, \( W \), of configuration \( c(k - 1) \) such that no node in \( R \cup W \) fails in \( \beta(\max(t, \ell\text{time}(\alpha') + e) + 16d) \).

**Proof.** Let \( \alpha'' \) be the shortest prefix of \( \alpha \) such that every configuration with index \( \leq k \) is installed in \( \alpha \). Let \( t' = \ell\text{time}(\alpha'') \). Notice that for all \( \ell \leq k \), configuration \( c(\ell) \) is installed in \( \beta(t') \).

Lemma 7.6.2, where \( t = t' \) and \( c \) and \( k \) are as defined above, shows that the \( \text{upgrade-ready}(k) \) event occurs in \( \beta(\max(t', \ell\text{time}(\alpha') + e) + 6d) \), that is, \( t \leq \max(t', \ell\text{time}(\alpha') + e) + 6d \).

Configuration-viability guarantees that there exists a read-quorum, \( R \), and a write-quorum, \( W \), of configuration \( c(k - 1) \) such that either (1) no process in \( R \cup W \) fails in \( \alpha \), or (2) there exists a finite prefix, \( \alpha_{\text{install}} \) of \( \alpha \) such that for all \( \ell \leq k \), configuration \( c(\ell) \) is installed in \( \alpha_{\text{install}} \) and no process in \( R \cup W \) fails in \( \alpha \) by time \( \max(\ell\text{time}(\alpha_{\text{install}}), \ell\text{time}(\alpha') + e) + 22d \).
In the former case, we are done. We now consider the second case. Since \( \alpha'' \) is the shortest prefix of \( \alpha \) such that every configuration with index \( k \) is installed, we know that \( \alpha'' \) is a prefix of \( \alpha_{\text{install}} \), and therefore \( t' = \ell_{\text{time}}(\alpha'') \leq \ell_{\text{time}}(\alpha_{\text{install}}) \). Therefore we know that there exists a read-quorum, \( R \in \text{read-quorums}(c(k - 1)) \), and a write-quorum, \( W \in \text{write-quorums}(c(k - 1)) \), such that no node in \( R \cup W \) fails by time \( \max(t', \ell_{\text{time}}(\alpha') + e) + 22d \).

Then, \( \max(t, \ell_{\text{time}}(\alpha') + e) + 16d \leq \max(t', \ell_{\text{time}}(\alpha') + e) + 22d \), and as a result, no node in \( R \cup W \) fails by time \( \max(t, \ell_{\text{time}}(\alpha') + e) + 16d \). That is, no node in \( R \cup W \) fails in \( \beta(\max(t, \ell_{\text{time}}(\alpha') + e) + 16d) \).

The final lemma shows that if no \text{upgrade-ready}(k) occurs in \( \alpha \), then configuration \( c(k - 1) \) is always viable.

**Lemma 7.6.4** Let \( \alpha \) be an \( \alpha' \)-normal execution satisfying: (i) \((\alpha', e)\)-join-connectivity, (ii) \((\alpha', e)\)-recon-readiness, (iii) \((\alpha', 2d)\)-recon-spacing-1, and (iv) \((\alpha', e, 4d)\)-configuration-viability.

Let \( c \) be a configuration with index \( k \), and assume that no \text{upgrade-ready}(k + 1) event occurs in \( \alpha \). Then there exists a read-quorum, \( R \in \text{read-quorums}(c) \), and a write-quorum, \( W \in \text{write-quorums}(c) \), such that no node in \( R \cup W \) fails in \( \alpha \).

**Proof.** Assume that for some \( \ell \leq k + 1 \), configuration \( c(\ell) \) is not installed in \( \alpha \). By the definition of configuration-viability, then, there exists a read-quorum, \( R \in \text{read-quorums}(c) \), and a write-quorum, \( W \in \text{write-quorums}(c) \), such that no node in \( R \cup W \) fails in \( \alpha \).

Assume, instead, that for every \( \ell \leq k + 1 \), configuration \( c(\ell) \) is installed in \( \alpha \). Then by Lemma 7.6.2, an \text{upgrade-ready}(k + 1) event occurs in \( \alpha \), contradicting the hypothesis. \( \square \)

### 7.7 Configuration-Upgrade Latency Results

In this section we show that configuration-upgrade operations terminate rapidly, and that any obsolete configuration is rapidly removed. In particular, these results hold in executions that include periods of bad behavior. The configuration-upgrade mechanism in RAMBO does not make these guarantees. The original RAMBO latency analysis required the assumption

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of \((\alpha', \infty)\)-configuration-viability\(^3\) for the entire execution. This is an unrealistic assumption in a long-lived dynamic system. As a result of the new configuration-upgrade mechanism, we need to assume only bounded configuration-viability to ensure liveness.

First we state a lemma about configuration-upgrade after the system stabilizes and good behavior resumes.

**Lemma 7.7.1** Let \(\alpha\) be an \(\alpha'\)-normal execution. Let \(t \in R^{\geq 0}\) be a time. Let \(i\) be a node that does not fail until after \(\max(t, \ell time(\alpha') + d) + 4d\).

Assume a \(\text{cfg-upgrade}(k)_i\) event occurs in \(\alpha\) at time \(t\). Additionally, assume that for every configuration \(c(\ell)\) such that \(upg.cmap(\ell)_i \in C\), there exists a read-quorum, \(R_\ell\), and a write-quorum, \(W_\ell\), of configuration \(c(\ell)\) such that no node in \(R_\ell \cup W_\ell\) fails by time \(t + 3d\).

Then a \(\text{cfg-upgrade-ack}(k)_i\) event occurs no later than \(t + 4d\).

**Proof.** There are two cases to consider.

**Case 1:** \(t > \ell time(\alpha')\). At time \(t\), node \(i\) begins the configuration-upgrade, with phase-number \(p_1 = upg.pnum_i\). By triggered gossip, node \(i\) immediately sends out messages to every node in \(world_i\). Therefore for every configuration \(c(\ell)\) such that \(upg.cmap(\ell)_i \in C\), every node \(j \in R_\ell \cup W_\ell\) receives a message by time \(t + d\).

By triggered gossip, then, each of these nodes sends a response with phase-number \(p_1\). Each response is received by time \(t + 2d\), at which point a \(\text{cfg-upg-query-fix}(k)_i\) event occurs. Node \(i\) then chooses a new phase-number, \(p_2\), and sets \(upg.pnum_i = p_2\).

Immediately, by triggered gossip node \(i\) sends out messages to every process in \(world_i\), including every node in \(R_\ell \cup W_\ell\), for every configuration \(c(\ell)\) such that \(upg.cmap(\ell)_i \in C\). Again, a response is sent by time \(t + 3d\), and node \(i\) receives a response from each with phase-number \(p_2\) by time \(t + 4d\). Immediately, then, a \(\text{cfg-upg-query-fix}(k)\) event occurs. This is followed by a \(\text{cfg-upgrade-ack}(k)_i\), proving our claim.

**Case 2:** \(t \leq \ell time(\alpha')\). At time \(t\), node \(i\) begins the configuration-upgrade, with phase-number \(p_1 = upg.pnum_i\). By occasional gossip, \(i\) sends out messages to every node in

\(^3\)Although we have not formally defined \((\alpha', \infty)\)-configuration-viability here, one can understand it to mean \((\alpha', e)\)-configuration-viability for arbitrarily large \(e\).
world. Therefore for every configuration $c(\ell)$ such that $upg.cmap(\ell)_i \in C$, every node $j \in R_\ell \cup W_\ell$ receives a message by time $\max(t, \ell time(\alpha') + d) + d$.

By triggered gossip, then, each of these nodes sends a response with phase-number $p_1$. Each response is received by time $\max(t, \ell time(\alpha') + d) + 2d$, at which point a $\text{cfg-upg-query-fix}(k)_i$ event occurs. Node $i$ then chooses a new phase-number, $p_2$, and sets $upg.pnum_i = p_2$.

Immediately, by triggered gossip node $i$ sends out messages to every process in $\text{world}_i$, including every node in $R_\ell \cup W_\ell$, for every configuration $c(\ell)$ such that $upg.cmap(\ell)_i \in C$. Again, a response is sent by time $\max(t, \ell time(\alpha') + d) + 3d$, and node $i$ receives a response from each with phase-number $p_2$ by time $\max(t, \ell time(\alpha')) + 4d$. Immediately, then, a $\text{cfg-upg-query-fix}(k)$ event occurs. This is followed by a $\text{cfg-upgrade-ack}(k)$, proving our claim.

Next, we provide a conditional guarantee that a configuration is viable: if for some time $t$ every earlier $\text{cfg-upgrade}$ operation completes rapidly within $4d$, then every configuration that is extant at time $t$ will remain viable until $t + 3d$.

We do this in four steps. First, Lemma 7.7.2 demonstrates that a node with certain good properties exists. Second, Lemma 7.7.3 shows that this certain node with good properties will begin an upgrade operation, in certain situations. Third, Lemma 7.7.4 shows that soon after a configuration is $\text{upgrade-ready}(k)$, some node completes an upgrade operation on configuration $c(k)$. Finally, Lemma 7.7.5 uses these preliminary lemmas to show that under certain conditions, configurations remain viable sufficiently long.

**Lemma 7.7.2** Let $\alpha$ be an $\alpha'$-normal execution satisfying (i) (\(\alpha', e\))-join-connectivity, (ii)($\alpha'$, $e$)-recon-readiness, (iii) ($\alpha'$, $e$)-upgrade-readiness, (iv) ($\alpha'$, $2d$)-recon-spacing-1, (v) ($\alpha'$, $e$, $22d$)-configuration-viability.

Assume that an $\text{upgrade-ready}(k_2)$ event occurs at time $t$ for some configuration $c_2$ and assume that $k_2 \geq 1$. Let $k_1 = k_2 - 1$, and $c_1 = c(k_1)$. Then there exists a node $i$ such that the following hold:
1. \( i \) is a member of configuration \( c_1 \).

2. \( i \) does not fail in \( \beta(\max(t, \ell\text{time}(\alpha') + e + d) + 10d) \),

3. \( i \in J(\max(t, \ell\text{time}(\alpha') + e + d) + 8d) \),

4. \( i \in J(\max(t, \ell\text{time}(\alpha') + e + 2d)) \),

5. \( i \) performs a join-ack prior to the upgrade-ready(\( k_2 \)) event in \( \alpha \).

**Proof.** Lemma 7.6.3, applied with \( c = c_2 \), \( k = k_2 \), and \( t \) as defined above, implies that there exists a read-quorum, \( R \), of configuration \( c_1 \) such that no member of \( R \) fails in \( \beta(\max(t, \ell\text{time}(\alpha') + e + d) + 16d) \). Then we know that no member of \( R \) fails in \( \beta(\max(t, \ell\text{time}(\alpha') + e + d) + 14d) \). We therefore choose a node \( i \in R \subseteq \text{members}(c_1) \). We know that \( i \) does not fail in \( \beta(\max(t, \ell\text{time}(\alpha') + e + d) + 10d) \). This \( i \) satisfies Parts 1 and 2.

Let \( t_{c_1} \) be the time at which configuration \( c_1 \) is proposed. Notice that \( \max(t, \ell\text{time}(\alpha') + e + 2d) \geq t_{c_1} \), because \( t \), the time of the upgrade-ready(\( k_2 \)) event, cannot be smaller than \( t_{c_1} \), the time at which configuration \( c_1 \) is proposed (since an upgrade-ready(\( k_2 \)) event cannot occur until after a recon(\( c_1, c_2 \)) event, which cannot occur until after a recon(\( \ast, c_1 \)) event). Therefore, Lemma 7.4.2, applied where \( h = c_1 \), \( t' = t_{c_1} \), and \( t = \max(t, \ell\text{time}(\alpha') + e + 2d) \), guarantees that \( \text{members}(c_1) \subseteq J(\max(t, \ell\text{time}(\alpha') + e + 2d)) \). Since \( i \in \text{members}(c_1) \), we know that \( i \in J(\max(t, \ell\text{time}(\alpha') + e + 2d)) \), satisfying Part 4.

Since \( \max(t, \ell\text{time}(\alpha') + e + 2d) \leq \max(t, \ell\text{time}(\alpha') + e + d) + 10d \) (since \( \ell\text{time}(\alpha') + e + 2d \leq \ell\text{time}(\alpha') + e + 10d \)), Lemma 7.4.1, applied where \( t = \max(t, \ell\text{time}(\alpha') + e + 2d) \) and \( t' = \max(t, \ell\text{time}(\alpha') + e + d) + 10d \), implies that \( J(\max(t, \ell\text{time}(\alpha') + e + 2d)) \subseteq J(\max(t, \ell\text{time}(\alpha') + e + d) + 10d) \), and thus \( i \in J(\max(t, \ell\text{time}(\alpha') + e + d) + 10d) \), satisfying Part 3.

Finally, notice that recon-readiness requires that \( i \) performs a join-ack prior to the recon(\( \ast, c_1 \)) event, and therefore prior to the cfg-upgrade(\( k_2 \)) event. This satisfies Part 5. \( \square \)

The next lemma claims that when a configuration is upgrade-ready, and a node with certain properties (as in Lemma 7.7.2) exists, then either the configuration is removed or an upgrade operation begins.
Lemma 7.7.3 Let $\alpha$ be an $\alpha'$-normal execution satisfying (i) $(\alpha', e)$-join-connectivity, (ii) $(\alpha', e)$-recon-readiness, (iii) $(\alpha', e)$-upgrade-readiness, (iv) $(\alpha', 2d)$-recon-spacing-1, (v) $(\alpha', e, 22d)$-configuration-viability.

Assume upgrade-ready$(k_2)$ occurs at time $t$ and $k_2 \geq 1$. Let $k_1 = k_2 - 1$ and $c_1 = c(k - 1)$. Further, assume that node $i$ has the following properties:

1. $i$ is a member of configuration $c_1$.
2. $i$ does not fail in $\beta(\max(t, \ell time(\alpha') + e + d) + 10d)$,
3. $i \in J(\max(t, \ell time(\alpha') + e + d) + 8d)$,
4. $i \in J(\max(t, \ell time(\alpha') + e + 2d))$,
5. $i$ performs a join-ack prior to the upgrade-ready$(k_2)$ event.

Let $t'$ be a time such that $t \leq t' < \max(t, \ell time(\alpha') + e + d) + 13d$. Let $\alpha''$ be a prefix of $\alpha$ such that:

1. $t' = \ell time(\alpha'')$,
2. an upgrade-ready$(k_2)$ event is in $\alpha''$,
3. $\ell state(\alpha''). upg. phase_i = idle$.

Then either:

1. $\ell state(\beta(t')). cmap(k_1)_i = \pm$, or
2. $i$ performs a cfg-upgrade$(k')_i$ at time $t'$, for some $k' \geq k_2$.

Proof. If $\ell state(\alpha''). cmap(k_1)_i = \pm$, then the conclusion holds, since $\alpha''$ is a prefix of $\beta(t')$: by Lemma 7.5.3, $\ell state(\beta(t')). cmap(k_1)_i = \pm$. Assume, then, that $\ell state(\alpha''). cmap(k_1)_i \neq \pm$.

We examine in turn the preconditions for cfg-upgrade$(k')_i$ just after $\alpha''$ (from Figure 3-1):

1. $\neg \ell state(\alpha''). failed_i$: By Part 2 of the assumption on $i$, we know that $i$ does not fail in $\beta(\max(t, \ell time(\alpha') + e + d) + 10d)$. However, $t' < \max(t, \ell time(\alpha') + e + d) + 10d$, and thus $i$ does not fail in $\beta(t')$. Since $\alpha''$ is a prefix of $\beta(t')$, $i$ does not fail in $\alpha''$. 92
2. \( \ellstate(\alpha'').status_i = \text{active} \): By Part 5 of the assumption on \( i \) we know that \( i \) performs a join-ack prior to the upgrade-ready(\( k_2 \)) event.

3. \( \ellstate(\alpha'').upg-phase_i = \text{idle} \): By assumption, this holds.

4. \( \forall \ell \in \mathbb{N}, \ell \leq k_2 : \ellstate(\alpha'').cmap(\ell)_i \neq \bot \): It suffices to show that by the point in the execution at which the upgrade-ready(\( k_2 \)) event occurs, node \( i \) has already learned of configuration \( c_2 \) and all configurations with smaller indices. Let \( \alpha''' \) be the prefix of \( \alpha \) ending in the upgrade-ready(\( k_2 \)) event. Part (ii) of the definition of the upgrade-ready(\( k_2 \)) event guarantees that: for all \( \ell \leq k_2 \), for all \( j \in \text{members}(c_1) \) that do not fail in \( \alpha''' \), \( \ellstate(\alpha''').cmap(\ell)_j \neq \bot \). Notice that by Part 1 of the assumption about \( i \), \( i \in \text{members}(c_1) \) and that by Part 2 of the assumption about \( i \), \( i \) does not fail in \( \alpha'' \), since \( \elltime(\alpha''') = t \leq \max(t, \elltime(\alpha') + e + d) \). Therefore we can conclude by part (ii) that for all \( \ell \leq k_2, \ellstate(\alpha''').cmap(\ell)_i \neq \bot \). Since \( \alpha''' \) is a prefix of \( \alpha'' \) (by assumption that upgrade-ready(\( k_2 \)) is included in \( \alpha'' \)), by Lemma 7.5.2 we know that for all \( \ell \leq k_2, \ellstate(\alpha'').cmap(\ell)_i \neq \bot \), as desired.

5. \( \ellstate(\alpha'').cmap(k_2)_i \in C \): By assumption, \( \ellstate(\alpha'').cmap(k_1)_i \neq \pm \). Invariant 4.3.3 then implies that \( \ellstate(\alpha'').cmap(k_2)_i \neq \pm \), since \( k_1 < k_2 \). Part 4, above, shows that \( \ellstate(\alpha'').cmap(k_2)_i \neq \bot \), thus implying the desired result.

6. \( \ellstate(\alpha'').cmap(k_1)_i \in C \): By assumption, \( \ellstate(\alpha'').cmap(k_1)_i \neq \pm \). Part 4, above, shows that \( \ellstate(\alpha'').cmap(k_1)_i \neq \bot \), since \( k_1 \leq k_2 \), thus implying the desired result.

Since enabled events occur in zero time (by assumption), either the event becomes disabled, in which case \( \ellstate(\beta(t')).cmap(k_1)_i = \pm \), satisfying Part 1 of the conclusion, or at time \( t' = \elltime(\alpha'') \) a \text{cfg-upgrade} event for some configuration \( c \) with index \( k' \geq k_2 \) occurs, satisfying Part 2 of the conclusion. \( \square \)

The next lemma conditionally guarantees that soon after a new configuration is upgrade-ready, the old configuration is removed.

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Lemma 7.7.4 Let $\alpha$ be an $\alpha'$-normal execution satisfying (i) $(\alpha', e)$-join-connectivity, (ii) $(\alpha', e)$-recon-readiness, (iii) $(\alpha', e)$-upgrade-readiness, (iv) $(\alpha', 2d)$-recon-spacing-1, (v) $(\alpha', e, 22d)$-configuration-viability.

Assume that $t \in \mathbb{R}^{\geq 0}$ is a time such that $t > \elltime(\alpha') + e + 14d$. Assume that $c_1$ is a configuration, and for some finite prefix $\alpha''$ of $\alpha$, where $t = \elltime(\alpha'')$, for some node $i \in J(t)$ that does not fail in $\alpha''$, for some index $k_1$, $\ellstate(\alpha'').cmap(k_1)_i = c_1$.

Also, we assume the Upgrades-Complete Hypothesis: for every $\text{cfg-upgrade}(\star)_j$ event that occurs in $\alpha$ at some time $t_{\text{upg}} < t$ at some node $j \in J(\max(t_{\text{upg}}, \elltime(\alpha') + e + 2d))$ where $j$ does not fail in $\beta(\max(t_{\text{upg}}, \elltime(\alpha') + e + d) + 4d)$, a matching $\text{cfg-upg-ack}(\star)_j$ occurs by time $\max(t_{\text{upg}}, \elltime(\alpha') + e + d) + 4d$.

Assume that an $\text{upgrade-ready}(k_1 + 1)$ event occurs at time $t' < t - 13d$. Let $k_2 = k_1 + 1$ and $c_2 = c(k_2)$. Then for some node $i' \in J(\max(t', \elltime(\alpha') + e + d) + 8d)$ that does not fail in $\beta(\max(t', \elltime(\alpha') + e + d) + 10d)$, $\ellstate(\beta(\max(t', \elltime(\alpha') + e + d) + 8d)).cmap(k_1)_v = \pm$.

Proof. We first identify a node, $i'$, that is suitable. Then we show that $i'$ completes an upgrade operation in the allotted time.

We apply Lemma 7.7.2, where $t = t'$, and therefore conclude that there exists a node $i'$ with the following five properties:

1. $i'$ is a member of configuration $c_1$,

2. $i'$ does not fail in $\beta(\max(t', \elltime(\alpha') + e + d) + 10d)$,

3. $i' \in J(\max(t', \elltime(\alpha') + e + d) + 8d)$,

4. $i' \in J(\max(t', \elltime(\alpha') + e + 2d))$,

5. $i'$ performs a join-ack prior to the $\text{upgrade-ready}(k_2)$ event.

Notice that Part 2 and Part 3 satisfy the first two requirements for $i'$ in the conclusion of this lemma. It remains to show that $i'$ marks configuration $c_1$ as $\pm$ at the appropriate point.

We consider what happens at time $\max(t', \elltime(\alpha') + e + d)$. Let $\alpha'''$ be the prefix of $\alpha$ that is the longer of the following two prefixes: (i) $\beta(\elltime(\alpha') + e + d)$, or (ii) the shortest prefix of
α that includes the \texttt{cfg-upgrade}(k_2) event. Notice that \( \elltime(\alpha'') = \max(t', \elltime(\alpha') + e + d) \), and that the \texttt{cfg-upgrade}(k_2) event is in \( \alpha'' \).

If \( \ellstate(\alpha'').cmap(k_1) \neq \pm \), then the claim is immediate: Lemma 7.5.2 implies that \( \ellstate(\alpha'').cmap \leq \ellstate(\beta(\max(t', \elltime(\alpha') + e + d) + 8d)).cmap \), since \( \elltime(\alpha'') = \max(t', \elltime(\alpha') + e + d) < \max(t', \elltime(\alpha') + e + d) + 8d. \) Therefore, if \( \ellstate(\alpha'').cmap(k_1) = \pm \), then \( \ellstate(\beta(\max(t', \elltime(\alpha') + e + d) + 8d)).cmap(k_1) = \pm \).

We thus assume that \( \ellstate(\alpha'').cmap(k_1) \neq \pm \), and consider what happens at time \( \max(t', \elltime(\alpha') + e + d) \). There are now two cases to consider:

1. \( \ellstate(\alpha'').upg.phase = \text{idle} \) or
2. \( \ellstate(\alpha'').upg.phase \neq \text{idle} \).

**Case 1:** Assume that \( \ellstate(\alpha'').upg.phase = \text{idle} \). We apply Lemma 7.7.3, where \( t = t' \), \( t' = \max(t', \elltime(\alpha') + e + d) \), \( \alpha'' = \alpha'' \), and \( i' \) is as chosen above:

- \( t' \leq \max(t', \elltime(\alpha') + e + d) < \max(t', \elltime(\alpha') + e + d) + 13d \): immediate,
- \( i' \) satisfies the criteria, by the properties of \( i' \) above,
- \( \elltime(\alpha'') = \max(t', \elltime(\alpha') + e + d) \) and \texttt{upgrade-ready}(k_2) occurs in \( \alpha'' \): by the way in which \( \alpha'' \) was chosen,
- \( \ellstate(\alpha'').upg.phase = \text{idle} \): by the case assumption.

From this lemma, we conclude that either:

1. \( \ellstate(\beta(\max(t', \elltime(\alpha') + e + d))).cmap(k_1) = \pm \), or
2. \( i' \) performs a \texttt{cfg-upgrade}(k') at time \( \max(t', \elltime(\alpha') + e + d) \), for some \( k' \geq k_2 \).

In the first case, where \( \ellstate(\beta(\max(t', \elltime(\alpha') + e + d))).cmap(k_1) = \pm \), we are done: Lemma 7.5.3 implies that \( \ellstate(\beta(\max(t', \elltime(\alpha') + e + d) + 8d)).cmap(k_1) = \pm \). Consider the second case, that is, \( i' \) performs a \texttt{cfg-upgrade}(k') at time \( \max(t', \elltime(\alpha') + e + d) \), for some \( k' \geq k_2 \).

We then apply the Upgrades-Complete Hypothesis, where \( j = i' \) and \( t_{upg} = t' \); notice that:
\[ i' \in J(max(t', \elltime(\alpha') + e + 2d)) : \text{by } 4^ih \text{ property of } i', \]

\[ i' \text{ does not fail in } \beta(max(t', \elltime(\alpha') + e + d) + 4d) : \text{by Part 2 of the way in which } i' \text{ was chosen, and} \]

\[ \text{max}(t', \elltime(\alpha') + e + d) < t : t' + 13d < t, \text{by assumption, and } \elltime(\alpha') + e + 14d < t, \text{by assumption, and therefore max}(t', \elltime(\alpha') + e + d) + 13d < t. \]

Therefore, by the Upgrades-Complete Hypothesis we conclude that a \( \text{cfg-upg-ack}(k')_\nu \) occurs by time \( \text{max}(t', \elltime(\alpha') + e + d) + 4d. \) Since \( k' \geq k_2 \), then by the precondition of a \( \text{cfg-upg-ack} \) operation we know that \( \ellstate(\beta(max(t', \elltime(\alpha') + e + d) + 4d).\text{cmap}(k_1)_\nu) = \pm. \) Lemma 7.5.3 implies that \( \ellstate(\beta(max(t', \elltime(\alpha') + e + d) + 8d).\text{cmap}(k_1)_\nu) = \pm, \text{as desired.} \)

**Case 2:** Assume that \( \ellstate(\alpha'').\text{upg-phase}_\nu \neq \text{idle}. \) For this to occur, a \( \text{cfg-upgrade}(k')_\nu \) event must occur prior to the \( \text{upgrade-ready}(k_2) \) event in \( \alpha \) with no matching \( \text{cfg-upg-ack}(k')_\nu \) event prior to the \( \text{upgrade-ready}(k_2) \) event, where \( k' = \ellstate(\alpha'').\text{upg-target}_\nu. \) Otherwise, if there were no ongoing upgrade operation, \( i' \) would be idle. Let \( t_1 \) be the time at which this earlier \( \text{cfg-upgrade}(k')_\nu \) operation occurs.

We can then apply the Upgrades-Complete Hypothesis, where \( j = i' \) and \( t_{\text{upg}} = t_1; \) notice that:

\[ i' \in J(max(t_1, \elltime(\alpha') + e + 2d)) : \text{Lemma 7.4.3, applied where } t = t_1 \text{ and } i = i', \text{ shows that } i' \in J(max(t_1, \elltime(\alpha') + e + 2d)). \]

\[ i' \text{ does not fail in } \beta(max(t_1, \elltime(\alpha') + e + d) + 4d) : \text{By Part 2 of the way in which } i' \text{ was chosen, } i' \text{ does not fail in } \beta(max(t', \elltime(\alpha') + e + d) + 10d). \text{ Notice that } t_1 \leq \max(t', \elltime(\alpha') + e + d), \text{ since the earlier upgrade event occurs in } \alpha'' \text{ prior to the } \text{upgrade-ready}(k_2) \text{ event. Therefore } i' \text{ does not fail in } \beta(max(t_1, \elltime(\alpha') + e + d) + 4d). \]

\[ \text{max}(t_1, \elltime(\alpha') + e + d) < t : \text{Again, notice that } \max(t_1, \elltime(\alpha') + e + d) \leq \max(t', \elltime(\alpha') + e + d), \text{ since } t_1 \leq t'. \text{ Also, } t' + 13d < t, \text{ by assumption, and } \elltime(\alpha') + e + 14d < t, \text{ by assumption. Therefore, } \max(t', \elltime(\alpha') + e + d) < t, \text{ implying that } \max(t_1, \elltime(\alpha') + e + d) < t. \]
We can then conclude that a \texttt{cfg-upgrade-ack}(k'_{
u}) occurs in \( \alpha \) by time \( \max(t_1, \elltime(\alpha') + e + d) + 4d \leq \max(t', \elltime(\alpha') + e + d) + 4d \). If \( k' \geq k_2 \), then by the precondition of the \texttt{cfg-upgrade-ack}(k'_{
u}) action, \( t' \) marks \( cmap(k_1) = \pm \), and we are done.

Otherwise, we apply Lemma 7.7.3 to show that another \texttt{cfg-upgrade} operation begins: let \( t_2 \) be the time at which the \texttt{cfg-upgrade-ack}(k'_{
u}) occurs and \( \alpha_2 \) be the prefix of \( \alpha \) ending in the \texttt{cfg-upgrade-ack}(k'_{
u}) event. Notice that:

\begin{itemize}
  \item \( t' \leq \max(t_2, \elltime(\alpha') + e + d) \): By the way in which the \texttt{cfg-upgrade}(k') was chosen, it has to complete no earlier than \( t' \).
  \item \( \max(t_2, \elltime(\alpha') + e + d) < \max(t', \elltime(\alpha') + e + d) + 13d \): Above, we showed that that \texttt{cfg-upgrade-ack}(k'_{
u}) occurs by \( \max(t', \elltime(\alpha') + e + d) + 4d \), that is, \( t_2 \leq \max(t_1, \elltime(\alpha') + e + d) + 4d \leq \max(t', \elltime(\alpha') + e + d) + 4d \), since \( t_1 \leq t' \). Therefore, \( t_2 < \max(t', \elltime(\alpha') + e + d) + 13d \). Also, \( \elltime(\alpha') + e + d < \elltime(\alpha') + e + 14d \).
\end{itemize}

Then we apply Lemma 7.7.3 with \( t = t' \), \( t' = \max(t_2, \elltime(\alpha') + e + d) \), \( \alpha'' = \alpha_2 \), and \( \nu' \) as chosen above:

\begin{itemize}
  \item \( t' \leq \max(t_2, \elltime(\alpha') + e + d) < \max(t', \elltime(\alpha') + e + d) + 13d \): as shown above,
  \item \( \nu' \) satisfies the criteria, by the properties of \( \nu' \) above,
  \item \( \elltime(\alpha_2) = \max(t_2, \elltime(\alpha') + e + d) \) and \texttt{upgrade-ready}(k_2) occurs in \( \alpha'' \): by the way in which \( \alpha_2 \) was chosen and the fact that the \texttt{cfg-upgrade-ack}(k'_{
u}) must come after the \texttt{upgrade-ready}(k_2) event,
  \item \( \ellstate(\alpha_2)_{.upg.phase_{\nu}} = \text{idle} \): by the effect of the \texttt{cfg-upg-ack}(k'_{
u}) event that is the last event in \( \alpha'' \).
\end{itemize}

We then conclude that either:

1. \( \ellstate(\beta(\max(t_2, \elltime(\alpha') + e + d))), cmap(k_1)_{\nu} = \pm \), or
2. \( \nu' \) performs a \texttt{cfg-upgrade}(k''_{
u}) at time \( \max(t_2, \elltime(\alpha') + e + d) \), for some \( k'' \geq k_2 \).
Again, if the first case holds, we are done: since \( t_2 \leq \max(t', \ell t_{time}(\alpha') + e + d) + 8d \), Lemma 7.5.3 implies that \( \ell s \text{state}(\beta(\max(t', \ell t_{time}(\alpha') + e + d) + 8d)), \ell \text{map}(k_1)_v = \pm \). Therefore, we can assume that the second part holds, and \( i' \) performs a \text{cfg-upgrade}(k''\alpha') at time \( \max(t_2, \ell t_{time}(\alpha') + e + d) \), for some \( k'' \geq k_2 \).

Once more, we apply the Upgrades-Complete Hypothesis, where \( j = i' \) and \( t_{upg} = t_2 \); notice that:

- \( i' \in J(\max(t_2, \ell t_{time}(\alpha') + e + 2d)) \): Recall that \( i' \in J(\max(t_1, \ell t_{time}(\alpha') + e + 2d)) \), above. Since \( \max(t_1, \ell t_{time}(\alpha') + e + 2d) \leq \max(t_2, \ell t_{time}(\alpha') + e + 2d) \) (i.e., the upgrade begins before it completes), by Lemma 7.4.1, where \( t = \max(t_1, \ell t_{time}(\alpha') + e + 2d) \) and \( t' = \max(t_2, \ell t_{time}(\alpha') + e + 2d) \), \( J(\max(t_1, \ell t_{time}(\alpha') + e + 2d)) \subseteq J(\max(t_2, \ell t_{time}(\alpha') + e + 2d)) \), implying that \( i' \in J(\max(t_2, \ell t_{time}(\alpha') + e + 2d)) \).

- \( i' \) does not fail in \( \beta(\max(t_2, \ell t_{time}(\alpha') + e + d) + 4d) \): By Part 2 of the way in which \( i' \) was chosen, \( i' \) does not fail in \( \beta(\max(t', \ell t_{time}(\alpha') + e + d) + 10d) \). Notice that \( t_2 \leq \max(t', \ell t_{time}(\alpha') + e + d) + 4d \), as shown above. Therefore \( \max(t_2, \ell t_{time}(\alpha') + e + d) + 4d \leq \max(t', \ell t_{time}(\alpha') + e + d) + 8d \), and as a result \( i' \) does not fail in \( \beta(\max(t_2, \ell t_{time}(\alpha') + e + d) + 4d) \).

- \( \max(t_2, \ell t_{time}(\alpha') + e + d) < t \): Again, notice that \( \max(t_2, \ell t_{time}(\alpha') + e + d) \leq \max(t', \ell t_{time}(\alpha') + e + d) + 4d \). Also, \( t' + 13d < t \), by assumption, and \( \ell t_{time}(\alpha') + e + d + 13d < t \), by assumption. Therefore, \( \max(t', \ell t_{time}(\alpha') + e + d) + 13d < t \). Therefore, \( \max(t_2, \ell t_{time}(\alpha') + e + d) \leq \max(t', \ell t_{time}(\alpha') + e + d) + 4d < t - 9d \), as desired.

We can then conclude that a \text{cfg-upgrade-ack}(k'\alpha') occurs in \( \alpha' \) by time \( \max(t_2, \ell t_{time}(\alpha') + e + d) + 4d \leq \max(t', \ell t_{time}(\alpha') + e + d) + 8d \). Since \( k'' \geq k_2 \), then by the precondition of the \text{cfg-upgrade-ack}(k\alpha') action, \( i' \) marks \( \ell \text{map}(k_1) = \pm \), and Lemma 7.5.3 implies that \( \ell \text{state}(\beta(\max(t', \ell t_{time}(\alpha') + e + d) + 8d)), \ell \text{map}(k_1)_v = \pm \).
Lemma 7.7.5 Let α be an α'-normal execution satisfying (i) (α', e)-join-connectivity, (ii) (α', e)-recon-readiness, (iii) (α', e)-upgrade-readiness, (iv) (α', 2d)-recon-spacing-1, (v) (α', e, 22d)-configuration-viability.

Assume that $t \in R^{\geq 0}$ is a time such that $t > \ell \text{time}(\alpha') + e + 14d$. Assume that $c_1$ is a configuration, and for some finite prefix $\alpha''$ of $\alpha$, where $t = \ell \text{time}(\alpha'')$, for some node $i \in J(\max(t, \ell \text{time}(\alpha') + e + 2d))$ that does not fail in $\alpha''$, for some index $k_1$, $\ell \text{state}(\alpha''). \text{cmap}(k_1)_i = c_1$.

Also we assume the Upgrades-Complete Hypothesis: for all $\text{cfg-upgrade}(*)_j$ events that occur in $\alpha$ at some time $t_{upp} < t$ at some node $j \in J(\max(t_{upp}, \ell \text{time}(\alpha') + e + 2d))$ where $j$ does not fail in $\beta(\max(t_{upp}, \ell \text{time}(\alpha') + e + d) + 4d$, a matching $\text{cfg-upg-ack}(*)_j$ occurs by time $\max(t_{upp}, \ell \text{time}(\alpha') + e + d) + 4d$.

Then there exists a read-quorum, $R \in \text{read-quorums}(c_1)$, and a write-quorum, $W \in \text{write-quorums}(c_1)$, such that no node in $R \cup W$ fails in $\beta(t + 3d)$.

Proof. Let $k_2 = k_1 + 1$, and let $c_2 = c(k_2)$. First, consider the case where no $\text{upgrade-ready}(k_2)$ event occurs in $\alpha$. We apply Lemma 7.6.4, where $c = c_1$ and $k = k_1$: this implies, then, that there exists a read-quorum, $R \in \text{read-quorums}(c_1)$, and a write-quorum, $W \in \text{write-quorums}(c_1)$, such that no node in $R \cup W$ fails in $\alpha$.

Next, consider the case where an $\text{upgrade-ready}(k_2)$ event occurs in $\alpha$. Let $t'$ be the time at which the $\text{upgrade-ready}(k_2)$ event occurs. We claim that $\text{upgrade-ready}(k_2)$ occurs no earlier than $t - 13d$. That is, $t' + 13d \geq t$.

Assume, in contradiction, that $t' + 13d < t$. We now apply Lemma 7.7.4 to conclude that there exists a node $i' \in J(\max(t', \ell \text{time}(\alpha') + e + d) + 8d)$ that does not fail in $\beta(\max(t', \ell \text{time}(\alpha') + e + d + 10d)$ such that $\ell \text{state}(\beta(\max(t', \ell \text{time}(\alpha') + e + d + 8d)). \text{cmap}(k_1)_{i'} = \pm$.

We now show that the information about configuration $c_1$'s removal is propagated from node $i'$ to node $i$. That is, we show the following:

Claim: $\ell \text{state}(\alpha''). \text{cmap}(k_1)_i = \pm$.

Proof of claim: We do this in three steps. First, we show that $\ell \text{state}(\beta(\max(t', \ell \text{time}(\alpha') + e + d) + 10d). \text{cmap}_{i'}$ is mainstream after $\max(t', \ell \text{time}(\alpha') + e + d) + 10d$. Second, we show
that $\ell \text{state}(\beta(\max(t', \ell \text{time}(\alpha') + e + d) + 8d)).cmap_{i}$ is mainstream after $t - d$. Third, we conclude that $\ell \text{state}(\alpha'').cmap_{i}(k_{1})\ = \pm$.

Step 1: We already know that $i' \in J(\max(t', \ell \text{time}(\alpha') + e + d) + 8d)$, and does not fail in $\beta(\max(t', \ell \text{time}(\alpha') + e + d) + 10d)$. We then apply Lemma 7.5.4, where $t = \max(t', \ell \text{time}(\alpha') + e + d) + 8d$, and $i = i'$:

- $\max(t', \ell \text{time}(\alpha') + e + d) + 8d \geq \ell \text{time}(\alpha') + e$: Immediate.

- $i' \in J(\max(t', \ell \text{time}(\alpha') + e + d) + 8d + 2d)$: $i' \in J(\max(t', \ell \text{time}(\alpha') + e + d) + 8d)$, as shown above, therefore this follow from Lemma 7.4.1, where $t = \max(t', \ell \text{time}(\alpha') + e + d) + 8d$ and $t' = \max(t', \ell \text{time}(\alpha') + e + d) + 10d$.

- $i'$ does not fail in $\beta(\max(t', \ell \text{time}(\alpha') + e + d) + 8d + d)$, since $i'$ does not fail in $\beta(\max(t', \ell \text{time}(\alpha') + e + d) + 8d + 2d)$ as shown above.

Therefore we can conclude that $\ell \text{state}(\beta(\max(t', \ell \text{time}(\alpha') + e + d) + 8d)).cmap_{i}$ is mainstream after $\max(t, \ell \text{time}(\alpha') + e + d) + 10d$.

Step 2: We have assumed above that $t' < t - 13d$, that is, $t' + 10d < t - d - 2d$. Also, we have assumed that $\ell \text{time}(\alpha') + e + 14d < t$, that is, $\ell \text{time}(\alpha') + e + d + 10d < t - d - 2d$. Therefore, $\max(t', \ell \text{time}(\alpha') + e + d) + 10d < t - 3d$. We now apply Lemma 7.5.11, where $t = \max(t', \ell \text{time}(\alpha') + e + d) + 10d$, $t' = t - d$, and $cm = \ell \text{state}(\beta(\max(t', \ell \text{time}(\alpha') + e + d) + 8d)).cmap_{i}$:

- $e + 2d \leq \max(t', \ell \text{time}(\alpha') + e + d) + 10d$,

- $\max(t', \ell \text{time}(\alpha') + e + d) + 10d \leq t - 3d$,

- $\ell \text{state}(\beta(\max(t', \ell \text{time}(\alpha') + e + d) + 8d)).cmap_{i}$ is mainstream after $\max(t, \ell \text{time}(\alpha') + e + d) + 10d$.

We therefore conclude that $\ell \text{state}(\beta(\max(t', \ell \text{time}(\alpha') + e + d) + 8d)).cmap_{i}$ is mainstream after $t - d$.

Step 3: Notice, then, that by assumption $i \in J(t)$ and $i$ does not fail in $\beta(t - d)$. Therefore by the definition of mainstream, $\ell \text{state}(\beta(\max(t', \ell \text{time}(\alpha') + e + d) + 8d)).cmap_{i} \leq$
Lemma 7.5.3 then implies that \( \ellstate(\beta(t-d)).cmap_i \leq \ellstate(\alpha^n).cmap_i \), since \( \beta(t-d) \) is a prefix of \( \alpha^n \). Therefore, \( \ellstate(\beta(\max(t', \elltime(\alpha') + e + d) + 8d)).cmap_i \leq \ellstate(\alpha^n).cmap_i \). Since \( \ellstate(\beta(\max(t', \elltime(\alpha') + e + d) + 8d)).cmap(k_1)_i = \pm \) (as shown above), this means that \( \ellstate(\alpha^n).cmap(k_1)_i = \pm \), as claimed above, concluding Step 3.

This claim that \( \ellstate(\alpha^n).cmap(k_1)_i = \pm \), though, leads to a contradiction: by assumption of this lemma, \( \ellstate(\alpha^n).cmap(k_1)_i = c_1 \). Therefore, we conclude that our assumption that \( t' < t - 13d \) is incorrect: that is, we must have \( t' \geq t - 13d \). That is, we have shown that the upgrade-ready(\( k_2 \)) event occurs at most \( 13d \) prior to time \( t \).

We now apply Lemma 7.6.3, where \( c = c_2 \), \( k = k_2 \), and \( t = t' \), to conclude that there exists a read-quorum, \( R \), and a write-quorum, \( W \), of configuration \( c_1 \) such that no node in \( R \cup W \) fails in \( \beta(\max(t', \elltime(\alpha') + e) + 16d) \). Above we showed that \( t' + 13d \geq t \), therefore \( t' + 16d \geq t + 3d \), which implies that \( \max(t', \elltime(\alpha') + e) + 16d \geq t + 3d \). Therefore, we can conclude that there exists a read-quorum, \( R \), and a write-quorum, \( W \), of configuration \( c_1 \) such that no node in \( R \cup W \) fails in \( \beta(t + 3d) \). \( \square \)

The next two lemmas claim that every configuration-upgrade operation completes soon after it begins, or soon after the network stabilizes. The first lemma handles the case where the upgrade begins before the network stabilizes, or during stabilization. The second lemma handles the general case, for all \( t \).

**Lemma 7.7.6** Let \( \alpha \) be an \( \alpha' \)-normal execution satisfying: (i) \( \alpha' \)-join-connectivity, (ii) \( \alpha' \)-recon-readiness, (iii) \( \alpha', 2d \)-recon-spacing-1, (iv) \( \alpha', e, 22d \)-configuration-viability.

Assume that \( t \in \mathbb{R}_{\geq 0} \) is a time such that \( t \leq \elltime(\alpha') + e + 14d \), and that a \( \text{cfg-upgrade}(k)_i \) occurs at time \( t \) at node \( i \). Assume that node \( i \in J(t) \) and that \( i \) does not fail in \( \beta(\max(t, \elltime(\alpha') + d) + 4d) \).

Then a \( \text{cfg-upg-ack}(k)_i \) occurs no later than time \( \max(t, \elltime(\alpha') + d) + 4d \).

**Proof.** Let \( \gamma \) be the configuration-upgrade operation associated with the \( \text{cfg-upgrade}(k) \) action. Lemma 7.7.1 shows that proving the following is sufficient to prove the lemma: for every configuration in \( \text{removal-set}(\gamma) \) there exists a read-quorum, \( R \) and a write-quorum, \( W \), such that no node in \( R \cup W \) fail by time \( \max(t, \elltime(\alpha') + d) + 3d \).
Consider any configuration, $c_1$ with index $k_1$ in $\text{removal-set}(\gamma)$. If $t_1$ is the time at which configuration $c(k_1 + 1)$ is installed, configuration-viability ensures that configuration $c_1$ does not fail until $\max(t_1, \ell\text{time}(\alpha') + e) + 22d$. Notice that $\ell\text{time}(\alpha') + e + 22d > t + 3d$, since $t \leq \ell\text{time}(\alpha') + e + 14d$. Therefore, this guarantees that there exists a read-quorum, $R$, and a write-quorum, $W$ for configuration $c_1$ such that no node in $R \cup W$ fails untilafter $\ell\text{time}(\alpha') + e + 22d > \max(t, \ell\text{time}(\alpha') + d) + 3d$. \hfill \Box

Lemma 7.7.7 \ Let $\alpha$ be an $\alpha'$-normal execution satisfying: (i) $(\alpha', e)$-join-connectivity, (ii) $(\alpha', e)$-recon-readiness, (iii) $(\alpha', 2d)$-recon-spacing-1, (iv) $(\alpha', e, 22d)$-configuration-viability.
Assume that $t \in \mathbb{R}^{\geq 0}$ is a time, and that a $\text{cfg-upgrade}(k)_i$ occurs in $\alpha$ at time $t$ at node $i$. Assume that node $i \in J(t)$ and that $i$ does not fail in $\beta(\max(t, \ell\text{time}(\alpha') + e + d) + 4d)$.

Then a $\text{cfg-upg-ack}(k)_i$ occurs no later than time $\max(t, \ell\text{time}(\alpha') + e + d) + 4d$.

Proof. We prove this lemma by proving a stronger statement by strong induction on the number of $\text{cfg-upgrade}$ events in $\alpha$: if a $\text{cfg-upgrade}^{(*)}_j$ event occurs in $\alpha$ at some time $t_{upg} \leq t$ at some node $j \in J(t_{upg})$, and $j$ does not fail in $\beta(\max(t_{upg}, \ell\text{time}(\alpha') + e + d) + 4d)$, then a matching $\text{cfg-upg-ack}^{(*)}_j$ occurs no later than time $\max(t_{upg}, \ell\text{time}(\alpha') + e + d) + 4d$.

As this is strong induction, we now examine the inductive step. Consider configuration-upgrade $\gamma$, the $k + 1$st upgrade operation in $\alpha$ that occurs at time $t_{upg} \leq t$ at node $j \in J(t)$ that does not fail in $\beta(\max(t_{upg}, \ell\text{time}(\alpha') + e + d) + 4d)$. Assume, inductively, that if $\gamma'$ is one of the first $k$ upgrade operations that occurs at time $t' \leq t$ at some node $j' \in J(t')$ that does not fail in $\beta(\max(t', \ell\text{time}(\alpha') + e + d) + 4d)$, then a matching $\text{cfg-upg-ack}^{(*)}$ occurs no later than time $\max(t', \ell\text{time}(\alpha') + e + d) + 4d$. There are two cases to consider.

Case 1: $t_{upg} \leq \ell\text{time}(\alpha') + e + 14d$.

Recall that the $\text{cfg-upgrade}$ event occurs at node $j \in J(t_{upg})$ where $j$ does not fail in $\beta(\max(t_{upg}, \ell\text{time}(\alpha') + e + d) + 4d)$. Lemma 7.7.6 shows that a $\text{cfg-upg-ack}(k)_j$ occurs by time $\max(t_{upg}, \ell\text{time}(\alpha') + d) + 4d \leq \max(t_{upg}, \ell\text{time}(\alpha') + e + d) + 4d$.

Case 2: $t_{upg} > \ell\text{time}(\alpha') + e + 14d$.

Lemma 7.7.1 shows that proving the following is sufficient to prove the lemma: for every configuration in $\text{removal-set}(\gamma)$ there exists a read-quorum, $R$ and a write-quorum, $W$,
such that no node in $R \cup W$ fails in $\beta(\max(t_{upg}, \ell\text{time}(\alpha') + d) + 3d)$. Let $\alpha''$ be the prefix of $\alpha$ ending with the $\text{cfg-upgrade}$ event $\gamma$. Fix some configuration $c \in \text{removal-set}(\gamma)$ with index $k$; that is, $\ell\text{state}(\alpha'').cmap(k)_j = c$. We now apply Lemma 7.7.5, where $c_1 = c$, $k_1 = k$, $\alpha''$ is as just defined, and $t = t_{upg}$:

- $t_{upg} > \ell\text{time}(\alpha'') + e + 14d$.
- $t_{upg} = \ell\text{time}(\alpha'')$.
- $\ell\text{state}(\alpha'').cmap(k)_j = c$, since $c \in \text{removal-set}(\gamma)$ and $\alpha''$ is the execution ending with the event $\gamma$.
- $j \in J(\max(t_{upg}, \ell\text{time}(\alpha') + e + 2d))$, since $j \in J(t_{upg})$ and $t_{upg} > \ell\text{time}(\alpha') + e + 14d$.
- Upgrades-Complete Hypothesis: for every $\text{cfg-upgrade}(\ast)_j$ event that occurs in $\alpha$ at some time $t' < t_{upg}$ at some node $j' \in J(\max(t_{upg}, \ell\text{time}(\alpha') + e + 2d))$ where $j'$ does not fail in $\beta(\max(t_{upg}, \ell\text{time}(\alpha') + e + d) + 4d)$, a matching $\text{cfg-upgrade}_{j'}$ occurs by time $\max(t_{upg}, \ell\text{time}(\alpha') + e + d) + 4d$: this is the inductive hypothesis, since any $\text{cfg-upgrade}$ occurring at time $t' < t_{upg}$ must be one of the first $k$ upgrade events.

Therefore, we conclude that there exists a read-quorum, $R \in \text{read-quorums}(c)$, and a write-quorum, $W \in \text{write-quorums}(c)$, such that no node in $R \cup W$ fails in $\beta(t + 3d)$. Since this is true for all $c \in \text{removal-set}(\gamma)$, this then shows the desired result.

\[\square\]

We next present two corollaries that follow from these lemmas. First, we present the unconditional version of Lemma 7.7.5:

**Corollary 7.7.8** Let $\alpha$ be an $\alpha'$-normal execution satisfying (i) $(\alpha', e)$-join-connectivity, (ii)$(\alpha', e)$-recon-readiness, (iii) $(\alpha', 2d)$-recon-spacing-1, (iv) $(\alpha', e, 22d)$-configuration-viability.
Assume that $t \in R^{\geq 0}$ is a time. Assume that $c$ is a configuration, and for some finite prefix $\alpha''$ of $\alpha$ where $t = \ell \text{time}(\alpha'')$, some node $i \in J(t)$ that does not fail in $\alpha''$, for some index $k$, $\ell \text{state}(\alpha'').\text{cmap}(k)_i = c$.

Then there exists a read-quorum, $R$, and a write-quorum, $W$, such that no node in $R \cup W$ fails in $\beta(\max(t, \ell \text{time}(\alpha') + e + d) + 3d)$.

**Proof.** If $t \geq \ell \text{time}(\alpha') + e + 14d$, then we show that the result follows from Lemma 7.7.7 and Lemma 7.7.5. We apply Lemma 7.7.7 where $c_1 = c$, $k_1 = k$: notice that Lemma 7.7.5 assumes that:

- $t > \ell \text{time}(\alpha') + e + 14d$: By assumption.
- $t = \ell \text{time}(\alpha'')$: By assumption.
- $\ell \text{state}(\alpha'').\text{cmap}(k)_i = c$: By assumption.
- $i \in J(\max(t, \ell \text{time}(\alpha') + e + 2d))$: $t > \ell \text{time}(\alpha') + e + 14d$.
- $i$ does not fail in $\alpha''$: By assumption.

- Upgrade-Completes Hypothesis: Fix some $\text{cfg-upgrade}(\ast)_j$ event that occurs at time $t_{upg} < t$ at node $j \in J(\max(t_{upg}, \ell \text{time}(\alpha') + e + 2d)$ where $j$ does not fail in $\beta(\max(t_{upg}, \ell \text{time}(\alpha') + e + d) + 4d)$. We apply Lemma 7.7.7, where $t = t_{upg}$ and $i = j$. (Notice that $j \in J(t_{upg})$ by Lemma 7.4.1.) We therefore conclude that a $\text{cfg-upgrade}(\ast)_j$ occurs no later than $\max(t_{upg}, \ell \text{time}(\alpha') + e + d) + 4d$, as required by the conclusion of the Upgrade-Completes Hypothesis.

We thus conclude that there exists a read-quorum, $R \in \text{read-quorums}(c)$ and a write-quorum, $W \in \text{write-quorums}(c)$ such that no node in $R \cup W$ fails in $\beta(t + 3d)$. Since $t > \ell \text{time}(\alpha') + e + 14d$, this implies that no node in $R \cup W$ fails in $\beta(\max(t, \ell \text{time}(\alpha') + e + d) + 3d)$.

Alternatively, if $t \leq \ell \text{time}(\alpha') + e + 14d$, configuration-viability guarantees that there exists a read-quorum, $R$, and a write-quorum, $W$, such that no node in $R \cup W$ fails in $\beta(\ell \text{time}(\alpha') + e + 22d)$, and again the result follows. \qed
The second corollary guarantees the liveness of the system; that is, the following corollary shows that read and write operations always terminate eventually:

**Corollary 7.7.9** Let \( \alpha \) be an \( \alpha' \)-normal execution satisfying (i) \((\alpha', e)\)-join-connectivity, (ii)\((\alpha', e)\)-recon-readiness, (iii) \((\alpha', 2d)\)-recon-spacing-1, (iv) \((\alpha', e, 22d)\)-configuration-viability.

Assume that \( t \in R^{\geq 0} \). Assume that at time \( t \), for some \( i \in J(t) \) that does not fail in \( \alpha^4 \), a read\(_i\) or write\(_i\) occurs in \( \alpha \). Then the operation eventually completes.

**Proof.** The read or write operation completes if each phase of the operation completes. Let \( \psi \) be the read\(_i\), write\(_i\), query-fix\(_i\), or recv\(_i\) action that sets op.cmap to cmap, beginning the phase. Each phase completes when for all \( \ell : \) op.cmap\((\ell)\) \( \in C \), \( i \) has sent a gossip message to an appropriate quorum of nodes in \( c(\ell) \), and received a response. The only way an operation can fail to terminate, then, is if there does not exist a non-failed read-quorum or a write-quorum of some configuration in op.cmap.

Assume that \( c \) is a configuration with index \( k \) such that \( \text{op.cmap}(k) \); is set to \( c \) at some time \( t' \) after \( \psi \), and before the phase completes. Then for some \( \alpha'' \) where \( t' = \ell\text{time}(\alpha'') \), \( \ell\text{state}(\alpha'').\text{cmap}(k) ; = c \), since \( \text{op.cmap} \) is set by copying a truncated version of \( \text{cmap}_r \). By Corollary 7.7.8, there exists a read-quorum, \( R \), and a write-quorum, \( W \), such that no node in \( R \cup W \) fails in \( \beta(\max(t, \ell\text{time}(\alpha')+e+d)+3d) \). No later than time \( \max(t, \ell\text{time}(\alpha')+e+d)+d \), node \( i \) sends a gossip message to every node in \( R \cup W \). By time \( \max(t, \ell\text{time}(\alpha')+e+d)+2d \) the message is received by every node in \( R \cup W \), and each node sends a response to \( i \). By time \( \max(t, \ell\text{time}(\alpha')+e+d)+3d \), node \( i \) receives the response, and \( R \cup W \subseteq \text{acc} \). Therefore, for all configurations the read and write quorums survive long enough, and so the phase completes.

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\(^4\)More specifically, we are assuming that \( i \) does not fail until after the operation terminates; since we do not here bound how long the operation may take, we instead assume that \( i \) does not fail in \( \alpha \). Obviously \( i \) failing after the operation completes has no effect on the operation completing.
7.8 Read-Write Latency Results

In this section we state and prove the main result of the latency analysis: if an execution contains a period of time of good behavior, and if this section of the executions is 22d-configuration-viable, then all read and write operations terminate, and terminate within 8d. Notice that in the original RAMBO paper, a similar result required the stronger assumption of \(\infty\)-configuration-viability, an arbitrarily unbounded failure assumption, depending on events earlier in the execution. Here there is no dependency on earlier events: the algorithm is guaranteed to stabilize rapidly, as soon as the network stabilizes.

We need one more lemma. This lemma shows that once a \texttt{report}(c) action occurs for some configuration with index \(k\), then soon every node has set \(cmap(\ell) \neq \bot\), for all \(\ell \leq k\). This will allow us to show that if a read or write operation begins long enough after a certain \texttt{report}(c) operation, then it cannot be interrupted by learning about new configurations with smaller indices.

**Lemma 7.8.1** Let \(\alpha\) be an \(\alpha'\)-normal execution satisfying: (i) \((\alpha, e)\)-join-connectivity, (ii) \((\alpha', e)\)-recon-readiness, (iii) 6d-recon-spacing, (iv) \((\alpha', e, 4d)\)-configuration-viability.

Assume that \(\alpha\) contains \texttt{decide} events for infinitely many configurations. Let \(\ell\) be a configuration index. Let \(c_1\) be the configuration with index \(\ell\), and \(c_2\) be the configuration with index \(\ell + 1\).

Let \(i\) be the node at which the first \texttt{recon}(\(c_1, c_2\)) event, \(\pi\), occurs. Let \(t\) be the time at which the \texttt{report}(\(c_1\)) event, \(\phi\), occurs.

Then there exists a CMap, \(cm\), such that:

1. \(cm(\ell) \neq \bot\), and

2. \(cm\) is mainstream after \(\max(t, \elltime(\alpha') + e + d) + 6d\).

**Proof.** There are two cases to consider. In each case, we first demonstrate an appropriate \(cm\): we identify a node that performs a \texttt{report}(\(c_1\)) and does not fail too soon. We then show that the cmap of that node is mainstream after \(\max(t, \elltime(\alpha') + e + d) + 6d\).
Case 1: \( \text{recon}(c_1, c_2)_i \) occurs at some time \( \ell \text{time}(\alpha') + e + 2d \).

In this case, we use the Recon-Spacing-2 assumption to identify a node with an appropriate cmap, and then use configuration-viability to show that this node survives long enough for its cmap to become mainstream after \( \ell \text{time}(\alpha') + e + 4d \), which then allows us to show that its cmap is mainstream after \( \max(t, \ell \text{time}(\alpha') + e + d) + 6d \).

By the Recon-Spacing-2 assumption, there exists a write-quorum, \( W \in \text{write-quorums}(c_1) \), such that for every node \( j \in W \), a \( \text{report}(c_1)_j \) occurs in \( \alpha \) prior to \( \pi \), the \text{recon} event that proposes configuration \( c_2 \). By configuration-viability, there exists some node \( j \in W \) that does not fail by time \( \ell \text{time}(\alpha') + e + 4d \), since there exists some read-quorum, \( R \), that does not fail by time \( \ell \text{time}(\alpha') + e + 4d \), and by assumption \( R \cap W \neq \emptyset \).

We now show that \( \text{cmmap}_j \) satisfies Property 1 after \( \ell \text{time}(\alpha') + e + 2d \). Notice that:

\[
\ell \text{state}(\beta(\text{time}(\pi))). \text{cmmap}(\ell)_j \neq \bot,
\]

since the \text{report} action notifies \( j \) of the configuration \( c_1 \) prior to \( \pi \). By assumption we know that \( \text{time}(\pi) \leq \ell \text{time}(\alpha') + e + 2d \). Therefore we know that \( \ell \text{state}(\beta(\ell \text{time}(\alpha') + e + 2d)). \text{cmmap}_j \neq \bot \).

Let \( cm = \ell \text{state}(\beta(\ell \text{time}(\alpha') + e + 2d)). \text{cmmap}_j \). We know, then, that \( cm(\ell) \neq \bot \), as desired.

Next we show that \( cm \) is mainstream after \( \ell \text{time}(\alpha') + e + 4d \). We apply Lemma 7.5.4, where \( i = j \), \( t = \ell \text{time}(\alpha') + e + 2d \):

- \( j \in J(\ell \text{time}(\alpha') + e + 4d) \): If \( \ell = 0 \), then \( j = i_0 \) and we have, by assumption, that \( i_0 \) performs a \text{join-ack}_{i_0} \) at time 0, immediately implying the statement by the definition of \( J \).

Otherwise, we apply Lemma 7.4.2, where \( h = c_1 \), \( t' = \text{time(\text{recon}(c(\ell - 1), c_1)))} \), and \( t = \ell \text{time}(\alpha') + e + 2d \). Notice that \( \ell \text{time}(\alpha') + e + 2d \geq \text{time(\text{recon}(c(\ell - 1), c_1)))} \) since \( \ell \text{time}(\alpha') + e + 2d \geq \text{time}(\pi) \), and \( \text{time}(\pi) \geq \text{time(\text{recon}(c(\ell - 1), c_1)))} \). We therefore conclude that \( \text{members}(c_1) \subseteq J(\ell \text{time}(\alpha') + e + 2d) \). In particular, this
means that \( j \in J(\ell \text{time}(\alpha') + e + 2d) \). Next we apply Lemma 7.4.1, where \( t = \ell \text{time}(\alpha') + e + 2d \) and \( t' = \ell \text{time}(\alpha') + e + 4d \) to see that \( j \in J(\ell \text{time}(\alpha') + e + 4d) \).

- \( \ell \text{time}(\alpha') + e + 2d \geq \ell \text{time}(\alpha') + e \): Immediate.

- \( j \) does not fail in \( \beta(\ell \text{time}(\alpha') + e + 3d) \): as shown above \( j \) does not fail in \( \beta(\ell \text{time}(\alpha') + e + 4d) \), by choice of \( j \) and configuration-viability.

We then conclude, since \( cm = \ell \text{state}(\beta(\ell \text{time}(\alpha') + e + 2d)).cmap_j \), that \( cm \) is mainstream after \( \ell \text{time}(\alpha') + e + 4d \).

We next apply Lemma 7.5.11, where \( t = \ell \text{time}(\alpha') + e + 4d \), \( t' = \max(t, \ell \text{time}(\alpha') + e + d) + 6d \), and \( cm \) is as defined above:

- \( e + 2d \leq \ell \text{time}(\alpha') + e + 4d \): Immediate.

- \( \ell \text{time}(\alpha') + e + 4d \leq \max(t, \ell \text{time}(\alpha') + e + d) + 6d - 2d \): Immediate.

- \( cm \) is mainstream after \( \ell \text{time}(\alpha') + e + 4d \): As shown above.

Therefore, we conclude that \( cm \) is mainstream after \( \max(t, \ell \text{time}(\alpha') + e + d) + 6d \), as desired.

**Case 2: recon(\( c_1, c_2 \))_i \) occurs at some time \( > \ell \text{time}(\alpha') + e + 2d \).

We first notice that \( i \) has been notified of configuration \( c_1 \) and then show that the cmap of \( i \) is mainstream after \( \max(t, \ell \text{time}(\alpha') + e + d) + 6d \).

Notice that \( \ell \text{state}(\beta(t)).cmap(\ell)_i \neq \bot \), since the report(\( c_1 \))_i \ event notifies \( i \) of configuration \( c_1 \).

We now apply Lemma 7.5.4, where \( i \) is as defined above and \( t = \max(t, \ell \text{time}(\alpha') + e + d) \), to show that \( cm \) is mainstream after \( \max(t, \ell \text{time}(\alpha') + e + d) + 2d \):

- \( \max(t, \ell \text{time}(\alpha') + e + d) + 2d \geq \ell \text{time}(\alpha') + e \): Immediate.

- \( i \in J(\max(t, \ell \text{time}(\alpha') + e + d) + 2d) \): We apply Lemma 7.4.2, where \( h = c_1 \), \( t' \) is the time at which \( c_1 \) is proposed, and \( t = \max(t, \ell \text{time}(\alpha') + e + d) + 2d \). Notice that \( \max(t, \ell \text{time}(\alpha') + e + d) + 2d \) is no earlier than the time at which \( c_1 \) is proposed, since a report(\( c_1 \)) occurs prior to \( \max(t, \ell \text{time}(\alpha') + e + d) + 2d \).
Also, \( \max(t, \ell \text{time}(\alpha') + e + d) + 2d \geq \ell \text{time}(\alpha') + e + 2d \). Therefore we conclude that \( \text{members}(c_1) \subseteq J(\max(t, \ell \text{time}(\alpha') + e + d) + 2d) \). This implies that \( i \in J(\max(t, \ell \text{time}(\alpha') + e + d) + 2d) \).

- \( i \) does not fail in \( \beta(\max(t, \ell \text{time}(\alpha') + e + d) + d) \): We know that \( i \) does not fail prior to event \( \pi \), that is, the \( \text{recon}(c_1, c_2)_i \) event. By Recon-Spacing-1, we know that \( \text{time}(\pi) \geq t + 6d \). By assumption of this case, we know that \( \text{time}(\pi) > \ell \text{time}(\alpha') + e + 2d \). Therefore \( i \) does not fail in \( \beta(\max(t, \ell \text{time}(\alpha') + e + d) + d) \).

We therefore conclude that \( cm \) is mainstream after \( \max(t, \ell \text{time}(\alpha') + e + d) + 2d \).

We next apply Lemma 7.5.11, where \( t = \max(t, \ell \text{time}(\alpha') + e + d) + 2d, t' = \max(t, \ell \text{time}(\alpha') + e + d) + 6d \), and \( cm \) is as defined above:

- \( e + 2d \leq \max(t, \ell \text{time}(\alpha') + e + d) + 2d \): Immediate.
- \( \max(t, \ell \text{time}(\alpha') + e + d) + 2d \leq \max(t, \ell \text{time}(\alpha') + e + d) + 6d - 2d \): Immediate.
- \( cm \) is mainstream after \( \text{time}(\pi_t) \): As shown above.

Therefore, we conclude that \( cm \) is mainstream after \( \max(t, \ell \text{time}(\alpha') + e + d) + 6d \), as desired.

\[ \square \]

We finally prove the main theorem, showing that read and write operations terminate rapidly. This result requires \( 12d + \epsilon \text{-recon-spacing} \), and is similar to Theorem 8.17 from [13]. This earlier theorem states that in a normal, steady-state case, with good environmental behavior, read and write operations terminate within time \( 8d \). Although the following theorem allows for more complicated behavior, deviating from the assumption of good environmental assumptions, read and write operations still complete rapidly.

**Theorem 7.8.2** Let \( \alpha \) be an \( \alpha' \)-normal execution satisfying: (i) \( (\alpha, \epsilon) \)-join-connectivity, (ii) \( (\alpha', \epsilon) \)-recon-readiness, (iii) \( 12d + \epsilon \text{-recon-spacing} \), (iv) \( (\alpha', \epsilon, 22d) \)-configuration-viability.

Let \( t > \ell \text{time}(\alpha') + e + 17d \), and assume a read or write operation starts at time \( t \) at some node \( i \). Assume \( i \in J(t + 8d) \), and does not fail until the read or write operation completes.
Also, assume that \( \alpha \) contains decide events for infinitely many configurations. Then node \( i \) completes the read or write operation by time \( t + 8d \).

**Proof.** Let \( c_0, c_1, c_2, \ldots \) denote the infinite sequence of successive configurations decided upon in \( \alpha \); by infinite reconfiguration, this sequence exists. For each \( k \geq 0 \), let \( \pi_k \) be the first \( \text{recon}(c_k, c_{k+1}) \) event in \( \alpha \), let \( i_k \) be the location at which this occurs, and let \( \phi_k \) be the corresponding, preceding \( \text{report}(c_k)_{i_k} \) event. (The special case of this notation for \( k = 0 \) is consistent with our usage elsewhere.)

We show that the time for each phase of the read or write operation is \( \leq 4d \) – this will yield the bound we need. Consider one of the two phases, and let \( \psi \) be the read, write, or query-fix event that begins the phase.

We claim that \( \text{time}(\psi) > \text{time}(\phi_0) + 8d \), that is, that \( \psi \) occurs more than \( 8d \) time after the \( \text{report}(0)_{i_0} \) event: We have that \( \text{time}(\psi) \geq t \), and \( t > \text{time}(\text{join-ack}_i) + 8d \) by assumption that \( i \in J(t + 8d) \). Also, \( \text{time}(\text{join-ack}_i) \geq \text{time}(\text{join-ack}_{i_0}) \). Furthermore, \( \text{time}(\text{join-ack}_{i_0}) \geq \text{time}(\phi_0) \), that is, when \( \text{join-ack}_{i_0} \) occurs, \( \text{report}(0)_{i_0} \) occurs with no time passage. Putting these inequalities together we see that \( \text{time}(\psi) > \text{time}(\phi_0) + 8d \).

Fix \( k \) to be the largest number such that \( \text{time}(\psi) > \text{time}(\phi_k) + 8d \). The claim in the preceding paragraph shows that such \( k \) exists.

Next, we show that within zero time of \( \psi \) occurring, \( \text{cmap}(\ell)_i \neq \perp \) for all \( \ell \leq k \). It is at this point that the proof diverges from that of Lemma 8.17 from [12].

For the purposes of the next two lemmas, fix any \( \ell \leq k \). We apply Lemma 7.8.1, where \( \ell \) is as fixed above, \( t = \text{time}(\phi_\ell) \), \( \phi = \phi_\ell \), \( \pi = \pi_\ell \), \( c_1 = c_\ell \) and \( i = i_\ell \). We therefore conclude that there exists a CMap \( cm \) such that:

1. \( cm(\ell) \neq \perp \), and

2. \( cm \) is mainstream after \( \max(\text{time}(\phi_\ell), \ell \text{time}(\alpha') + e + d) + 6d \).

We next apply Lemma 7.5.11, where \( t = \max(\text{time}(\phi_\ell), \ell \text{time}(\alpha') + e + d) + 6d \), \( t' = \text{time}(\psi) \), and \( cm \) is as above, to show that \( cm \) is mainstream after \( \text{time}(\psi) \):

- \( e + 2d \leq \max(\text{time}(\phi_\ell), \ell \text{time}(\alpha') + e + d) + 6d \): Immediate.
\[ \text{max}(time(\phi_k), \ell time(\alpha') + e + d) + 6d \leq time(\psi) - 2d: \] By the way in which \( k \) is chosen we know that \( time(\phi_k) + 8d < time(\psi) \). Also, \( time(\phi_\ell) \leq time(\phi_k) \): either \( \ell = k \), or \( \phi_\ell \) precedes \( \pi_\ell \) which precedes \( \phi_k \). By assumption we know that \( \ell time(\alpha') + e + 8d < t \), and \( t \leq time(\psi) \).

- \( cm \) is mainstream after \( \text{max}(time(\phi_\ell), \ell time(\alpha') + e) + 6d \): As shown above.

Therefore, we conclude that \( cm \) is mainstream after \( time(\psi) \). We know that \( i \in J(t) \), and \( t \leq time(\psi) \), so by Lemma 7.4.1, \( i \in J(time(\psi)) \). Also, \( i \) does not fail until the read or write operation completes, and therefore either the read or write operation completes at \( time(\psi) \) (in which case we have proved the desired bound) or \( i \) does not fail in \( \beta(time(\psi)) \).

Therefore by definition of a CMap being mainstream, if \( cm \) is mainstream after \( time(\psi) \), then \( cm \leq \ell state(\beta(time(\psi))).cmap_i \).

Having shown this for fixed \( \ell \leq k \), we now know that for all \( \ell \leq k \) there exists some CMap, \( cm \), such that \( cm(\ell) \neq \perp \) and \( cm \) is mainstream after \( time(\psi) \), this implies that for all \( \ell \leq k \), \( \ell state(\beta(time(\psi))).cmap_\ell_i \neq \perp \). Therefore we have shown that within zero time of \( \psi \) occurring, \( cmap(\ell)_i \neq \perp \) for all \( \ell \leq k \).

Now, by choice of \( k \), we know that \( time(\psi) \leq time(\phi_{k+1}) + 8d \). The Recon-Spacing condition implies that \( time(\pi_{k+1}) \) (the first recon event that requests the creation of the \((k + 2)^{nd} \) configuration) is \( > time(\phi_{k+1}) + 12d \). Therefore, for an interval of time of length \( > 4d \) after \( \psi \), the largest index of any configuration that appears anywhere in the system is \( k + 1 \). This implies that the phase of the read or write operation that starts with \( \psi \) completes with at most one additional delay (of \( 2d \)) for learning about a new configuration. This yields a total time of at most \( 4d \) for the phase, as claimed.

Finally, by Corollary 7.7.9, the operation eventually terminates, which guarantees that ever configuration in \( op.cmap \) remains viable for long enough. \( \square \)

This shows that assuming \((\alpha', e, 22d)-configuration-viability\) is sufficient to guarantee that read and write operations terminate quickly. As long as the reconfiguration algorithm can guarantee this level of viability, the RAMBO II algorithm will continue to make progress, regardless of any bad behavior the network may experience. Further, while \( 22d \) may seem
a long period of time to ensure viability, it must be remembered that $d$ is typically a small interval: we have been assuming that $d$ is a single message delay in the network. Note that simply deciding on a new configuration to install might take many intervals of $d$ (in [12], it is bounded by $11d$). Also, this $22d$ bound is fairly conservative: by making stronger assumptions as to who begins configuration-upgrade operations, and how gossip messages propagate information about completed configuration-upgrade operations, it is probably possible to improve this bound. In this thesis we are primarily interested in the fact that it is a constant time bound.
Chapter 8

Implementation and Preliminary Evaluation

Musial and Shvartsman [16] have developed a prototype distributed implementation that incorporates both the original RAMBO configuration management algorithm [12] and the new RAMBO II algorithm presented in this thesis. The system was developed by manually translating the Input/Output Automata specification to Java code. To mitigate the introduction of errors during translation, the implementers followed a set of precise rules, similar to [2], that guided the derivation of Java code from Input/Output Automata notation. The system is undergoing refinement and tuning, however an initial evaluation of the performance of the two algorithms has been performed in a local-area setting.

The platform consists of a Beowulf cluster with 13 machines running Linux (Red Hat 7.1). The machines are Pentium processors in the range from 90 MHz to 900 MHz, interconnected via a 100 Mbps Ethernet switch. The implementation of the two algorithms shares most of the code and all low-level routines. Any difference in performance is traceable to the distinct configuration management discipline used by each algorithm.

The machines vary significantly in speed. Given several very slow machines, Musial and Shvartsman do not evaluate absolute performance and instead focus initially on comparing the two algorithms.

The preliminary results in Figure 8-1 show the average latency of read/write operations
Figure 8-1: Preliminary empirical evaluation of the average operation latency (measured as the number of gossip intervals), as a function of reconfiguration frequency, measured as number of reconfigurations per one reconfiguration period.

as the frequency of reconfigurations grows from about two to twenty reconfigurations per one gossip period. In order to handle such frequent reconfigurations, a large gossip interval (8 seconds) is used. This interval is much larger than the round-trip message delay, thus reducing the effects of network congestion encountered when reconfiguring very frequently. The results show that the overall latency of read/write operations for the new algorithm progressively improve, as the frequency of reconfiguration increases. As expected, the decrease in latency becomes substantial for bursty reconfigurations (at 20 reconfigurations per gossip interval). For less frequent reconfigurations the latency is similar, at about 4 gossip intervals depending on the settings (not shown). This is expected and consistent with our analysis, since the two algorithms are essentially identical when $cmaps$ contain one or two configurations. Figure 8-2 shows the average number of configurations in $cmaps$ as a function of reconfiguration frequency. This further explains the difference in performance, since the average number of configurations in $cmaps$ is lower in the new algorithm as the frequency of reconfigurations increases.
Figure 8-2: Preliminary empirical evaluation of the average number of configurations in cmap's, as a function of reconfiguration frequency, measured as number of reconfigurations per one reconfiguration period.

Finally notice that the modest number of machines used in this study favored the original algorithm. This is because the machines are often members of multiple configurations, thus the number of messages needed to reach fixed-points by the read/write operations of the original algorithm is much lower than is expected when each processor is a member of a few configurations.

Also, notice that this evaluation does not examine the effects of message loss and lack of network connectivity. We hypothesize that, as in the case of frequent bursty reconfiguration, when there are intervals of time in which the network is disconnected, the new algorithm should recover more rapidly. This testing has not yet been performed.

Full performance evaluation is currently in progress. Shvartsman and Musial are investigating how the performance depends on the number of machines and various timing parameters.
Chapter 9

Conclusion and Open Problems

In this thesis we have presented a new algorithm, improving on the original RAMBO algorithm by Lynch and Shvartsman [12, 13]. While the original RAMBO algorithm is analyzed primarily in the context of good network behavior, we are able to show that our new algorithm functions well even when the network experiences transient periods of bad behavior, including message loss, clock skews, and arbitrary asynchrony, and when reconfiguration is bursty and uneven.

The key to this improvement is a new rapid configuration-upgrade mechanism, which allows the system to stabilize rapidly after a period of bad network behavior. In the previous RAMBO algorithm, it might take arbitrarily long to recover from a period of bad behavior. In this new algorithm, however, within a constant time, the system returns to a steady-state condition. This allows the algorithm to function more reliably in a long-running, dynamic system: when a system is expected to function for months and years without failure, it is necessary to rapidly recover from the inevitable transient network failures.

This improvement also makes practical the design of algorithms to choose new configurations. In the earlier version of RAMBO, it is unclear what properties a reconfiguration algorithm must support in order for it to be useful. This thesis shows that a reconfiguration automaton must provide exactly \((\alpha', 22d)\)-configuration-viability.

To design such a reconfiguration algorithm, then, is one of the major open problems posed by this thesis. In particular, it seems important to show that if the rate of failure
is bounded, then the algorithm continues to make progress. This is similar to the ideas introduced by Karger and Liben-Nowell in [10], in which they assume that the system has a bounded half-life: the time in which either half the processes fail or the number of active processes doubles. Under this assumption, they show that their algorithm operates correctly.

By similarly assuming a bounded rate of failures, it should be possible in certain cases to design a reconfiguration algorithm that guarantees liveness by initiating reconfiguration with some minimum frequency. By choosing appropriate quorums and appropriate numbers of reconfigurations, \((\alpha', 2d)\)-configuration-viability should be possible.

Other open problems include improving the join protocol, and designing a leave protocol to allow good detection of nodes that have exited the system. Currently, the join protocol is quite simple and it would seem beneficial to require more communication before allowing a node to initiate operations. And when nodes fail or leave, in the algorithm as stated, they are just ignored. By introducing a formal protocol to leave the system, and a method for detecting failed nodes, it might be possible to improve the long-run performance of the system.

Another open problem is to determine how to recover when viability fails (and data is inevitably lost). More generally, is a self-stabilizing version of RAMBO feasible? It would also be interesting to determine whether a version of RAMBO could be adapted to tolerate Byzantine faults.

RAMBO may also allow the construction of other data types, such as weakly consistent memory and sets. It may also be possible to optimize RAMBO to return read values more rapidly, in one phase, in certain cases. An important question would be to determine the most powerful data object that can be implemented using the RAMBO technique; one suspects that it is impossible to implement consensus in this manner.

Finally, it would be interesting to examine how the RAMBO algorithm could be adapted to specific platforms. The algorithm is presented in a fairly abstract fashion. In real implementations, it would be optimized depending on the target platform. In particular, we suspect that RAMBO should work well in sensor networks, mobile-networks, and peer-to-peer networks.

In conclusion, this thesis has presented a new algorithm for atomic memory in a highly
dynamic environment, proved that is always correct, and presented a set of conditions that guarantee liveness. This provides significant improvements over existing algorithms, rapidly recovering from transient network problems and bursty reconfiguration.
Bibliography


[16] Peter M. Musial and Alex A. Shvartsman. Implementing a reconfigurable atomic memory service for dynamic networks. submitted for publication.


# Index of Definitions

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