

# Convex Computation of the Reachable Set for Controlled Polynomial Hybrid Systems

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**Abstract**—This paper presents an approach to computing the time-limited backwards reachable set (BRS) of a semialgebraic target set for controlled polynomial hybrid systems with semialgebraic state and input constraints. By relying on the notion of occupation measures, the computation of the BRS of a target set that may be distributed across distinct subsystems of the hybrid system, is posed as an infinite dimensional linear program (LP). Computationally tractable approximations to this LP are constructed via a sequence of semidefinite programs each of which is proven to construct an outer approximation of the true BRS with asymptotically vanishing conservatism. In contrast to traditional Lyapunov based approaches, the presented approach is convex and does not require any form of initialization. The performance of the presented algorithm is illustrated on 2 nonlinear controlled hybrid systems.

## I. INTRODUCTION

Hybrid systems have been widely adopted as a modeling tool due to their expressive power while describing the dynamics of systems undergoing continuous and discrete transitions simultaneously. Consequently, the development of computationally tractable algorithms for reachability analysis is critical not only for verifying safe system behavior, but also due to its applicability during incremental control design [20]. This paper presents a numerical approach to construct the set of points that reach a given target set at a specified finite time for controlled polynomial hybrid systems.

Many algorithms have been proposed to compute this *backwards reachable set* (BRS) for a hybrid system. The most popular of such techniques rely either upon the linearity of the distinct subsystems of the hybrid system under investigation [10], [21], the Hamilton-Jacobi-Bellman Equation [22], or Lyapunov-type analysis [14], [17], [18]. Though the Hamilton-Jacobi-Bellman based methods work even in the presence of general nonlinear dynamics, they rely upon state-space discretization which can restrict their applicability to systems of low dimensionality. In contrast, Lyapunov based methods can be applied to higher dimensional systems. These approaches rely on checking Lyapunov’s criteria for stability using sums-of-squares (SOS) programming which are formulated as semi-algebraic constraints and casted as SOS constraints using the S-procedure. However, constructing these

Lyapunov functions requires solving a nonconvex bilinear program that is solved using some form of alternation, which is not guaranteed to converge to global or even local optima and requires feasible initialization.

In this paper, we address these issues by presenting a convex approach to computing the BRS of a semialgebraic target set for a controlled polynomial hybrid system in the presence of semialgebraic state and input constraints. Our approach is inspired by the method presented in [8], which describes a framework based on *occupation measures* for computing the BRS for classical polynomial dynamical systems. Our contributions are three-fold. First, in Section II, we formulate the determination of the BRS as an infinite dimensional linear program (LP) over the space of nonnegative measures. The target set in this formulation can in fact be divided amongst the distinct subsystems of the hybrid system. Second, in Section IV, we construct a sequence of finite dimensional relaxations to our infinite dimensional LP in terms of semidefinite programs (SDP)s. Finally, in Section IV-B, we prove that each solution to the sequence of SDPs is an *outer approximation* to the largest possible BRS with asymptotically vanishing conservatism. In Section V, we demonstrate the performance of our approach on 2 examples.

## II. PRELIMINARIES

In this section, we formalize our problem of interest, construct an infinite dimensional LP, and note that the solution of this LP is equivalent to solving our problem of interest. We make substantial use of measure theory, and the unfamiliar reader may wish to consult [6] for an introduction.

### A. Notation

Given an element  $y \in \mathbb{R}^n$ , let  $[y]_i$  denote the  $(i)$ -th component of  $y$ . We use the same convention for elements belonging to any multidimensional vector space. Let  $\mathbb{N}$  we denote the non-negative integers, and  $\mathbb{N}_k^n$  refer to those  $\alpha \in \mathbb{N}^n$  with  $|\alpha| = \sum_{i=1}^n [\alpha]_i \leq k$ . Let  $\mathbb{R}[y]$  denote the ring of real polynomials in the variable  $y$ . For a compact set  $K \subset \mathbb{R}^n$ , let  $\mathcal{M}(K)$  denote the space of signed Radon measures supported on  $K$ . The elements of  $\mathcal{M}(K)$  can be identified with linear functionals acting on the space of continuous functions  $C(K)$ , that is, as elements of the dual space  $C(K)'$  [6, Corollary 7.18]. The duality pairing of a measure  $\mu \in \mathcal{M}(K)$  on a test function  $v \in C(K)$  is:

$$\langle \mu, v \rangle = \int_K v(z) \mu(z). \quad (1)$$

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In addition, let  $\text{spt}(\mu)$  denote the support of a measure  $\mu$  and  $\lambda_n$  denote the Lebesgue measure on  $\mathbb{R}^n$ . Given  $n \in \mathbb{N}$  and  $D \subset \mathbb{R}^n$ ,  $\partial D$  denotes the boundary of  $D$ . Recall that given a collection of sets  $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ , the *disjoint union* of this collection is  $\coprod_{\alpha \in \mathcal{A}} S_\alpha = \bigcup_{\alpha \in \mathcal{A}} S_\alpha \times \{\alpha\}$ . Throughout the paper we abuse notation and say that given  $\bar{\alpha} \in \mathcal{A}$  and  $x \in S_{\bar{\alpha}}$ , then  $x \in \coprod_{\alpha \in \mathcal{A}} S_\alpha$ , even though we should write  $\iota_{\bar{\alpha}}(x) \in \coprod_{\alpha \in \mathcal{A}} S_\alpha$ , where  $\iota_{\bar{\alpha}}: S_{\bar{\alpha}} \rightarrow \coprod_{\alpha \in \mathcal{A}} S_\alpha$  is the *canonical identification*  $\iota_{\bar{\alpha}}(x) = (x, \bar{\alpha})$ .

### B. Controlled Polynomial Hybrid Systems

Motivated by the definition in [4], we define the class of hybrid systems of interest in this paper.

**Definition 1:** A controlled polynomial hybrid system is a tuple  $\mathcal{H} = (\mathcal{J}, \Gamma, \mathcal{D}, U, \mathcal{FG}, \mathcal{S}, \mathcal{R})$ , where:

- $\mathcal{J}$  is a finite set indexing the discrete states of  $\mathcal{H}$ ;
- $\Gamma \subset \mathcal{J} \times \mathcal{J}$  is a set of edges, forming a directed graph structure over  $\mathcal{J}$ ;
- $\mathcal{D} = \coprod_{j \in \mathcal{J}} X_j$  is a disjoint union of domains, where  $X_j = \{x \in \mathbb{R}^{n_j} \mid h_{X_j}(x) \geq 0, h_{X_j} \in \mathbb{R}[x], \forall i = \{1, \dots, n_{X_j}\}\}$  is a compact subset where  $n_j \in \mathbb{N}$ ;
- $U = \{u \in \mathbb{R}^m \mid h_U \geq 0, h_U \in \mathbb{R}[u], \forall i \in \{1, \dots, n_U\}\}$  is a compact, convex subset which describes the range space of the control inputs where  $m \in \mathbb{N}$ ;
- $\mathcal{FG} = \{(f_j, g_j)\}_{j \in \mathcal{J}}$  is a set of control affine systems, where  $f_j: \mathbb{R} \times X_j \rightarrow \mathbb{R}^{n_j}$ ,  $g_j: \mathbb{R} \times X_j \rightarrow \mathbb{R}^{n_j}$ , for  $t \in \mathbb{R}$ ,  $x \in X_j$ , and  $u \in U$ ,  $f(t, x) + g(t, x)u$  is a tangent vector on  $X_j$  at  $x$ , and  $f, g \in \mathbb{R}[t, x]$ ;
- $\mathcal{S} = \coprod_{e \in \Gamma} S_e$  is a disjoint union of the guards, where each  $S_{(j', j)} = \{x \in \partial X_j \mid h_{(j', j)}(x) = 0, h_{(j', j)} \in \mathbb{R}[x], \forall i = \{1, \dots, n_{(j', j)}\}\}$  is a compact, co-dimension 1 guard that defines a transition from domain  $j \in \mathcal{J}$  to domain  $j' \in \mathcal{J}$  and  $S_{(j', j)} \cap S_{(k', j)} = \emptyset$ ,  $\forall (j', j), (k', j) \in \Gamma$  when  $k' \neq j'$ ;
- $\mathcal{R} = \{R_e\}_{e \in \Gamma}$  is a set of reset maps, where each  $R_{(j', j)}: S_{(j', j)} \rightarrow X_{j'}$  defines the transition from guard  $S_{(j', j)}$  to  $X_{j'}$ ,  $R_{(j', j)} \in \mathbb{R}[x]$  where  $x \in X_j$ , and  $R_{(j', j)}$  is an injective, continuously differentiable function whose Jacobian is nonzero for every  $x \in S_{(j', j)}$ .

For convenience, we sometimes refer to controlled polynomial hybrid systems as just hybrid systems, and we refer to the distinct vertices within the graph structure associated with a controlled polynomial hybrid system as modes.

Next, we define an *execution* of a hybrid system via construction in Algorithm 1. This definition agrees with the traditional intuition about executions of hybrid systems, which describes an execution as evolving as a dynamical system until a guard is reached, at which point a discrete transition occurs to a new domain using a reset map, and evolution continues again as a dynamical system.

Hybrid systems can suffer from Zeno executions, i.e. executions that undergo an infinite number of discrete transitions in a finite amount of time. Since the state of the trajectory after the Zeno occurs maybe undefined, we do not consider hybrid systems with Zeno executions:

**Assumption 1:**  $\mathcal{H}$  has no Zeno execution.

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### Algorithm 1 Execution of Hybrid System $\mathcal{H}$

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**Require:**  $t = 0$ ,  $j \in \mathcal{J}$ ,  $(x_0, j) \in \mathcal{D}$ , and  $u: \mathbb{R} \rightarrow U$  Lebesgue measurable.

- 1: Set  $x(0) = x_0$ .
  - 2: **loop**
  - 3: Let  $\gamma: I \rightarrow X_j$  be an absolutely continuous function<sup>a</sup> such that:
    - (i)  $\dot{\gamma}(s) = f(s, \gamma(s)) + g(s, \gamma(s))u(s)$  for almost every  $s \in I$  with respect to the Lebesgue measure on  $I \subset [0, \infty]$ ,
    - (ii)  $\gamma(t) = x(t)$ , and
    - (iii) for any other  $\tilde{\gamma}: \tilde{I} \rightarrow X_j$  satisfying (i) and (ii)  $\tilde{I} \subset I$ .
  - 4: Let  $t' = \sup I$  and  $x(s) = \gamma(s)$  for each  $s \in [t, t']^b$ .
  - 5: **if**  $t' = \infty$ , **or**  $\nexists (j', j) \in \Gamma$  such that  $\gamma(t') \in S_{(j', j)}$  **then**
  - 6: Stop.
  - 7: **end if**
  - 8: Let  $(j', j) \in \Gamma$  be such that  $\gamma(t') \in S_{(j', j)}$ .
  - 9: Set  $x(t') = R_{(j', j)}(\gamma(t'))$ ,  $t = t'$ , and  $j = j'$ .
  - 10: **end loop**
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<sup>a</sup>Note that the existence of a curve satisfying conditions (i),(ii), and (iii) follows from [3, Theorem 10.1.4]

<sup>b</sup>Note if  $t' < \infty$ , then  $\gamma(t') \in \partial X_j$ . [3, Theorem 10.1.12]

### C. Problem Statement

Next, we describe the target set whose BRS we are interested in computing. First, we define the projection of the target set in each mode  $j \in \mathcal{J}$ :

$$X_{T_j} = \{x \in X_j \mid h_{T_j}(x) \geq 0, h_{T_j} \in \mathbb{R}[x], \forall i = \{1, \dots, n_{T_j}\}\}. \quad (2)$$

The *target set* is then defined as:

$$X_T = \prod_{j \in \mathcal{J}} X_{T_j}. \quad (3)$$

Given a finite final time  $T > 0$ , our goal is to compute the time-limited BRS of  $X_T$  which is defined as:

$$\mathcal{X} = \left\{ (x_0, j) \in \mathcal{D} \mid \exists u: [0, T] \rightarrow U \text{ Lebesgue measurable} \right. \\ \left. \text{s.t. } x: [0, T] \rightarrow \mathcal{D} \text{ defined via Algorithm 1,} \right. \\ \left. x(0) = x_0 \in X_j, x(T) \in X_T \right\}. \quad (4)$$

We denote the projection of  $\mathcal{X}$  in each mode  $j$  as:

$$\mathcal{X}_j = \left\{ x_0 \in X_j \mid (x_0, j) \in \mathcal{X} \right\} \quad (5)$$

We make the following assumption to solve this problem:

**Assumption 2:**  $X_{T_j}$  is compact for all  $j \in \mathcal{J}$ .

### D. Liouville's Equation

We compute  $\mathcal{X}$  by defining measures over  $[0, T] \times X_j$  for each  $j \in \mathcal{J}$  whose supports' model the evolution of *families of trajectories* in each mode. An initial condition and its relationship with respect to the terminal set can be understood via Algorithm 1, but the relationship between a

family of trajectories and the terminal set is best understood differently. To appreciate this distinct perspective, first define the linear operator  $\mathcal{L}_j : C^1([0, T] \times X_j) \rightarrow C([0, T] \times X_j \times U)$  on a test function  $v$  as:

$$\mathcal{L}_j v = \frac{\partial v}{\partial t} + \sum_{i=1}^{n_j} \frac{\partial v}{\partial x_i} \left( [f(t, x)]_i + [g(t, x)u(t)]_i \right), \quad (6)$$

and its adjoint operator  $\mathcal{L}'_j : C([0, T] \times X_j \times U)' \rightarrow C^1([0, T] \times X_j)'$  by the adjoint relation:

$$\langle \mathcal{L}'_j \mu, v \rangle = \langle \mu, \mathcal{L}_j v \rangle = \int_{[0, T] \times X_j \times U} \mathcal{L}_j v(t, x, u) d\mu(t, x, u) \quad (7)$$

for all  $\mu \in \mathcal{M}([0, T] \times X_j \times U)$  and  $v \in C^1([0, T] \times X_j)$ .

Using this operator, we can understand the evolution of any test function  $v \in C^1([0, T] \times X_j)$  in  $X_j$ . To make this explicit, consider the evolution of a point  $x_{\tau_i} \in X_j$  beginning at time  $\tau_i \in [0, T]$  under the control input  $u(\cdot | \tau_i, x_{\tau_i})$  according to Algorithm 1, which we denote by  $x(\cdot | \tau_i, x_{\tau_i})$ . Define the first hitting time of a guard in  $X_j$  as:

$$\tau_f(\tau_i, x_{\tau_i}) = \min \left\{ T, \inf \{ \tau \geq \tau_i \mid \exists k \in \mathcal{J} \text{ s.t. } x(\tau | \tau_i, x_{\tau_i}) \in G_{k,j} \} \right\}. \quad (8)$$

It follows from Equation (6) that:

$$v(\tau_f, x(\tau_f | \tau_i, x_{\tau_i})) - v(\tau_i, x_{\tau_i}) = \int_{\tau_i}^{\tau_f} \frac{d}{dt} v(t, x(t | \tau_i, x_{\tau_i})) dt \quad (9)$$

$$= \int_{\tau_i}^{\tau_f} \mathcal{L}_j v(t, x(t | \tau_i, x_{\tau_i}), u(t | \tau_i, x_{\tau_i})) dt, \quad (10)$$

where we have suppressed the dependence on  $\tau_i$  and  $x_{\tau_i}$  in  $\tau_f$ . A standard approach to determining the BRS of a system imposes Lyapunov conditions on the test functions and their derivatives. However, this results in nonconvex bilinear matrix inequalities. Instead we examine conditions on the space of measures—the dual to the space of continuous functions—in order to arrive at a convex formulation.

To do this, we begin by defining an *occupation measure*:

$$\mu(A \times B \times C | \tau_i, x_{\tau_i}) = \int_0^T I_{A \times B \times C}(t, x(t | \tau_i, x_{\tau_i}), u(t | \tau_i, x_{\tau_i})) dt, \quad (11)$$

for all subsets  $A \times B \times C$  in the Borel  $\sigma$ -algebra of  $[0, T] \times X_j \times U$ , where  $I_{A \times B \times C}(\cdot)$  denotes the indicator function on a set  $A \times B \times C$ . As a result of its definition, the occupation measure of a set  $A \times B \times C$  quantifies the amount of time the graph of a solution and its associated control,  $(t, x(t | \tau_i, x_{\tau_i}), u(t | \tau_i, x_{\tau_i}))$ , spends in  $A \times B \times C$ . For any measurable function  $h : [0, T] \times X_j \times U \rightarrow \mathbb{R}$  an occupation measure by construction satisfies the following property:

$$\int_0^T h(t, x(t | \tau_i, x_{\tau_i}), u(t | \tau_i, x_{\tau_i})) dt = \int_{[0, T] \times X_j \times U} h(t, x, u) d\mu(t, x, u | \tau_i, x_{\tau_i}) \quad (12)$$

As a result, Equation (10) then becomes:

$$v(\tau_f, x(\tau_f | \tau_i, x_{\tau_i})) - v(\tau_i, x_{\tau_i}) = \int_{[0, T] \times X_j \times U} \mathcal{L}_j v(t, x, u) d\mu(t, x | \tau_i, x_{\tau_i}). \quad (13)$$

If the initial state whose evolution was of interest was not just a single point at a specific time, but was a family of points each beginning at distinct times, then we could define an *initial measure*,  $\mu_i \in \mathcal{M}([0, T] \times X_j)$ , whose support coincided with this family of points and their initialization times. We could then define an *average occupation measure*,  $\mu \in \mathcal{M}([0, T] \times X_j \times U)$  by:

$$\mu(A \times B \times C) = \int_X \mu(A \times B \times C | \tau_i, x_{\tau_i}) d\mu_i(\tau_i, x_{\tau_i}), \quad (14)$$

and a *final measure*,  $\mu_f \in \mathcal{M}([0, T] \times X_j)$  by:

$$\mu_f(A \times B) = \int_{[0, T] \times X_j} I_{A \times B}(\tau_f(\tau_i, x_{\tau_i}), x(\tau_f | \tau_i, x_{\tau_i})) d\mu_i(\tau_i, x_{\tau_i}). \quad (15)$$

Integrating with respect to  $\mu_i$ , introducing the average occupation measure and final measure, and using the property defined in Equation (12), Equation (13) becomes:

$$\langle \mu_f, v \rangle - \langle \mu_i, v \rangle = \langle \mu, \mathcal{L}_j v \rangle, \quad \forall v \in C^1([0, T] \times X_j). \quad (16)$$

The support of  $\mu$  models the flow of trajectories beginning in the support of  $\mu_i$ , and the support of  $\mu_f$  represents the distribution of states at some final time after being transported along system trajectories from the initial measure.

Notice that Equation (16) is linear in its measure components. Since Equation (16) must hold for all test functions, we obtain a linear operator equation:

$$\mu_f - \mu_i = \mathcal{L}_j \mu, \quad (17)$$

called Liouville's Equation, which is a classical result in statistical physics that describes the evolution of a density of particles within a fluid [2]. The occupation measures  $\mu_i$ ,  $\mu$  and  $\mu_f$ , along with Liouville's equation allow us to reason about *families* of trajectories of a classical dynamical system. This equation is satisfied by families of trajectories generated according to Algorithm 1 starting from the initial distribution  $\mu_i$ . The converse statement is true for control affine systems with a convex admissible control set, as we have assumed. We refer the reader to [8, Appendix A] for an extended discussion of Liouville's Equation. Extending the applicability of this result to hybrid systems requires careful modification and selection of the initial and final measures.

### III. INFINITE DIMENSIONAL LINEAR PROGRAM

In this section, we derive an infinite-dimensional LP characterization of the BRS of  $X_T$ . The basic idea is to introduce and then maximize the mass of initial occupation measures defined just at  $t = 0$  in each of the hybrid modes, denoted  $\mu_0^j$ , under the constraint that it is dominated by the Lebesgue measure, i.e.,  $\lambda \geq \mu_0^j$ . System dynamics in each mode are captured by Liouville's Equation (17) on each mode and state and terminal constraints are handled

by suitable constraints on the support of the measures. To describe all trajectories of a hybrid system, two modifications are required.

First, trajectories arriving at a guard in each mode must be detected. By splitting the final measure in each mode into two types of measures this detection can be accomplished:

$$\mu_f^j = \mu_T^j + \sum_{(j',j) \in \Gamma} \mu^{S(j',j)} \quad (18)$$

with  $\mu_T^j \in \mathcal{M}(\{T\} \times X_{T_j})$  and  $\mu^{S(j',j)} \in \mathcal{M}([0, T] \times S_{(j',j)})$ . Second, once trajectories arriving at a guard,  $S_{(j',j)}$ , are detected they must be re-initialized after the application of the reset map,  $R_{(j',j)}$ , in mode  $j' \in \mathcal{J}$ . To accomplish this task, the mass of the occupation measure at each guard at each time between  $[0, T]$  must be transferred exactly to the new domain after resetting:

**Lemma 1:** *Let  $\mathcal{H}$  be a controlled polynomial hybrid system as in Definition 1 and  $\mu^{S(j',j)} \in \mathcal{M}([0, T] \times S_{(j',j)})$ . Let  $\sigma \in \mathcal{M}([0, T] \times R_{(j',j)}(S_{(j',j)}))$  be such that:*

$$\langle \sigma, v \rangle = \langle R_{(j',j)}^* \mu^{S(j',j)}, v \rangle \quad \forall v \in C([0, T] \times X_{j'}), \quad (19)$$

where  $\langle R_{(j',j)}^* \mu^{S(j',j)}, v \rangle$  is defined as:

$$\int_{[0, T] \times S_{(j',j)}} v(t, R_{(j',j)}(x)) |\det(DR_{(j',j)}(x))| d\mu^{S(j',j)}(t, x), \quad (20)$$

and  $DR_{(j',j)}$  is the Jacobian of  $R_{(j',j)}$ , then  $x \in \text{spt}(\mu^{S(j',j)})$  ( $x \notin \text{spt}(\mu^{S(j',j)})$ ) if and only if  $R_{(j',j)}(x) \in \text{spt}(\sigma)$  ( $x \notin \text{spt}(\sigma)$ ).

*Proof.* This follows directly from [7, Theorem 263D].  $\square$   
As a result of this lemma, for any  $t \in [0, T]$  the support of  $R_{(j',j)}^* \mu^{S(j',j)}(t, \cdot)$  characterizes exactly the reinitialization in mode  $j'$  after the application of the reset map  $R_{(j',j)}$  of points arriving at the guard  $S_{j',j}$  at time  $t$ . By splitting the initial measure in each mode into two types of measures this reinitialization can be accommodated:

$$\mu_i^j = \mu_0^j + \sum_{(j,j') \in \Gamma} R_{(j,j')}^* \mu^{S(j,j')} \quad (21)$$

with  $\mu_0^j \in \mathcal{M}(\{0\} \times X_j)$  and  $\mu^{S(j,j')} \in \mathcal{M}([0, T] \times S_{(j,j')})$ .

With these two modifications, we can define an infinite dimensional LP,  $P$ , that maximizes the size of the BRS, modeled by  $\sum_{j \in \mathcal{J}} \text{spt}(\mu_0^j)$ , for a given target set, modeled by  $\text{spt}(\mu_T^j)$  for each  $j \in \mathcal{J}$ . That is, define  $P$  as:

$$\begin{aligned} \sup \quad & \sum_{j \in \mathcal{J}} \mu_0^j(X_j) & (P) \\ \text{s.t.} \quad & \mathcal{L}_j^* \mu^j = \mu_T^j + \sum_{(j',j) \in \Gamma} \mu^{S(j',j)} - \mu_0^j - \sum_{(j,j') \in \Gamma} R_{(j,j')}^* \mu^{S(j,j')} \quad \forall j \in \mathcal{J}, \\ & \mu_0^j + \hat{\mu}_0^j = \lambda_{n_j} \quad \forall j \in \mathcal{J}, \\ & \sum_{j \in \mathcal{J}} \mu_T^j(X_{T_j}) = \sum_{j \in \mathcal{J}} \mu_0^j(X_j), \\ & \mu^j, \mu_0^j, \hat{\mu}_0^j, \mu_T^j \geq 0 \quad \forall j \in \mathcal{J}, \\ & \mu^{S_e} \geq 0 \quad \forall e \in \Gamma, \end{aligned}$$

<sup>1</sup>Refer to Theorem 2 to relate maximizing  $\text{spt}(\mu_0^j)$  to  $\mu_0^j(X_j)$

where the given data are  $\mathcal{H}$  and  $X_T$  and the supremum is taken over a tuple of measures  $(\mu, \mu_0, \hat{\mu}_0, \mu_T, \mu_S) \in (\mathcal{M}([0, T] \times \mathcal{D}) \times \mathcal{M}(\{0\} \times \mathcal{D}) \times \mathcal{M}(\{0\} \times \mathcal{D}) \times \mathcal{M}(\{T\} \times X_T) \times \mathcal{M}([0, T] \times \mathcal{S}))$ . For notational convenience, we denote the  $j \in \mathcal{J}$  slice of  $\mu$  using the super index  $j$  (i.e. for any  $(t, x) \in [0, T] \times X_j$  set  $\mu^j(t, x) = \mu(t, x, j)$ ) and applied a similar convention to  $\mu_0, \hat{\mu}$ , and  $\mu_T$ . Similarly, we denote the  $(j, j') \in \Gamma$  slice of  $\mu_S$  using the notation  $\mu^{S(j,j')}$ .

The constraint  $\sum_{j \in \mathcal{J}} \mu_T^j(X_{T_j}) = \sum_{j \in \mathcal{J}} \mu_0^j(X_j)$  ensures that the BRS computes only points arriving at the target set at time  $T$  rather than at any of the guards at time  $T$ . The slack measures (denoted with “hats”) are introduced to impose the constraints  $\lambda \geq \mu_0^j$  which ensures that the optimal value of  $P$  is the Lebesgue measure of the largest achievable BRS. These observations are summarized next:

**Theorem 2:** *The optimal value of  $P$  is equal to  $\sum_{j \in \mathcal{J}} \lambda_{n_j}(\mathcal{X}_j)$ , i.e. the sum of the Lebesgue measures of the BRS restricted to each domain of the hybrid system.*

*Proof.* Notice that for any initial condition in  $x_0 \in \mathcal{X}$  there is a hybrid trajectory constructed via Algorithm 1 with  $x(T) \in X_T$ . As a result for any initial measure  $\mu_0$  with  $\text{spt}(\mu_0) \subset \mathcal{X}$  there exists occupation measures  $\hat{\mu}_0, \mu, \mu^T$ , and  $\mu_S$  such that the constraints of  $P$  are satisfied. Thus  $p^* \geq \sum_{j \in \mathcal{J}} \lambda_{n_j}(\mathcal{X}_j)$ .

To prove the other direction, notice that as a result of [8, Theorem 1]  $p^* = \sum_{j \in \mathcal{J}} \lambda_{n_j}(\text{spt}(\mu_0^j))$ . Suppose then for contradiction that  $\exists j \in \mathcal{J}$  such that  $\lambda_{n_j}(\text{spt}(\mu_0^j) \setminus \mathcal{X}_j) > 0$ . Using [8, Lemma 3] there exists a family of trajectories generated via Algorithm 1 starting from  $\mu_0^j$  generating the occupation measures  $\mu^j$  and  $\mu_T^j + \sum_{j',j} \mu^{S(j',j)}$ . If the trajectories arrive at  $X_T$ , then we have a contradiction, so suppose there exists some  $j' \in \mathcal{J}$  such that the trajectories arrive at  $S_{(j',j)}$ . Using Lemma 1, we can reset these trajectories and reapply the same argument to extend the family of trajectories through modes of the system. Notice that due to Assumption 1 the times of transitions for this family of trajectories are strictly monotonic. If at the final time  $T$  the family of trajectories generated according to Algorithm 1 are in  $\text{spt}(\mu_T)$ , then we have a contradiction, so suppose that they are at a guard. In that instance  $\sum_{j \in \mathcal{J}} \mu_T^j(X_{T_j}) < \sum_{j \in \mathcal{J}} \mu_0^j(X_j)$  which would violate a constraint in  $P$ , so we have a contradiction and  $p^* = \sum_{j \in \mathcal{J}} \lambda_{n_j}(\text{spt}(\mu_0^j)) \leq \sum_{j \in \mathcal{J}} \lambda_{n_j}(\mathcal{X}_j)$ .  $\square$

Next, lets define the dual program to  $P$  denoted  $D$  as:

$$\begin{aligned} \inf \quad & \sum_{j \in \mathcal{J}} \int_{X_j} w_j(x) d\lambda_{n_j}(x) & (D) \\ \text{s.t.} \quad & \mathcal{L}_j v_j(t, x, u) \leq 0 \quad \forall (t, x, j, u) \in [0, T] \times \mathcal{D} \times U \\ & w_j(x) \geq v_j(0, x) + p + 1 \quad \forall (x, j) \in \mathcal{D}, \\ & w_j(x) \geq 0 \quad \forall (x, j) \in \mathcal{D} \\ & v_j(T, x) \geq -p \quad \forall (x, j) \in X_T \\ & v_j(t, x) \geq v_{j'}(t, R_{(j',j)}(x)) \quad \forall (t, (x, (j', j))) \in [0, T] \times \mathcal{S} \end{aligned}$$

where the given data are  $\mathcal{H}$  and  $X_T$  and the infimum is taken over the tuple  $(v, w, p) \in (C^1([0, T] \times \mathcal{D}) \times C(\mathcal{D}) \times \mathbb{R})$ . As before, for notational convenience, we denote the  $j \in \mathcal{J}$  slice of  $v$  using the subscript  $j$  (i.e. for every  $(t, x) \in [0, T] \times X_j$

set  $v_j(t, x) = v(t, x, j)$ ) and apply a similar convention to  $w$ . The next result verifies that there is no duality gap between the two programs:

**Theorem 3:** *There is no duality gap between  $P$  and  $D$ .*

*Proof.* Due to space limitations, we omit the proof, which follows from [1, Theorem 3.10].  $\square$

Importantly, the dual allows us to obtain outer approximations of the BRS:

**Theorem 4:** *For every  $j \in \mathcal{J}$ ,  $\mathcal{X}_j \subset \{x_0 \in X_j | w_j(x) \geq 1\}$  for any feasible  $w_j$  of  $D$ . Furthermore, there is a sequence of feasible solutions to  $D$  such that for each  $j \in \mathcal{J}$ , the  $w_j$ -component converges from above to  $I_{\mathcal{X}_j}$  in the  $L^1$  norm and almost uniformly.*

*Proof.* To prove the first result, consider a feasible  $(v, w)$  in  $D$ . Given any  $(x_0, j_0) \in \mathcal{X}$ , there exists a  $u$  such that  $u(t) \in U$  for all  $t \in [0, T]$  and  $x(T) \in X_T$  where  $x : [0, T] \rightarrow \mathcal{D}$  is generated via Algorithm 1. Let  $\{\tau_i\}_{i=0}^n \subset [0, T]$  be the strictly monotonic and finite sequence of transition times of the trajectory (which exists by Assumption 1) with  $\tau_0 = 0$  and  $\tau_n = T$  and  $\{j_i\}_{i=0}^n \subset \mathcal{J}$  be the sequence of visited modes. Due to  $D$ 's constraints:

$$\begin{aligned} -p \leq v_{j_n}(T, x(T)) &= v_{j_{n-1}}(\tau_{n-1}, R_{(j_n, j_{n-1})}(x(\tau_{n-1}))) + \\ &+ \int_{\tau_{n-1}}^{\tau_n} \mathcal{L}_{j_n} v_{j_n}(t, x(t), u(t)) dt \\ &\leq v_{j_{n-1}}(\tau_{n-1}^-, x(\tau_{n-1}^-)) \\ &= v_{j_{n-2}}(\tau_{n-2}, R_{(j_{n-1}, j_{n-2})}(x(\tau_{n-2}))) + \\ &+ \int_{\tau_{n-2}}^{\tau_{n-1}} \mathcal{L}_{j_{n-1}} v_{j_{n-1}}(t, x(t), u(t)) dt \\ &\dots \\ &\leq v_{j_0}(0, x_0) \leq w_{j_0}(x_0) - p - 1, \end{aligned}$$

which proves the first statement since  $(x_0, j_0)$  was arbitrary. The second result follows from a straightforward extension to [8, Theorem 3]  $\square$

#### IV. NUMERICAL IMPLEMENTATION

The infinite-dimensional problems  $P$  and  $D$  are not directly amenable to computation. However, a sequence of finite-dimensional approximations in terms of SDPs can be obtained by characterizing measures in  $P$  by their *moments*, and restricting the space of functions in  $D$  to polynomials. The solutions to each of the SDPs in this sequence can be used to construct outer approximations that converge to the solution of the infinite-dimensional LP. A comprehensive introduction to such *moment relaxations* can be found in [11].

For each  $j \in \mathcal{J}$ , measures on the set  $[0, T] \times X_j$  are completely determined by their action (via integration) on a dense subset of the space  $C^1([0, T] \times X_j)$  [6]. Since  $[0, T] \times X_j$  is compact by assumption, the Stone–Weierstrass Theorem [6, Theorem 4.45] allows us to choose the set of polynomials as this dense subset. Every polynomial on  $\mathbb{R}^n$ , say  $p \in \mathbb{R}[x]$ , can be expanded in the monomial basis via:

$$p(x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha, \quad (22)$$

where  $\alpha \in \mathbb{N}^n$  ranges over vectors of non-negative integers,  $x^\alpha = \prod_{i=1}^n [x]_i^{\alpha_i}$ , and  $\text{vec}(p) = (p_\alpha)_{\alpha \in \mathbb{N}^n}$  is the vector of coefficients of  $p$ . By definition, the  $p_\alpha$  are real and only finitely many are non-zero. We define  $\mathbb{R}_k[x]$  to be those polynomials such that  $p_\alpha$  is non-zero only for  $\alpha \in \mathbb{N}_k^n$ . The degree of a polynomial,  $\text{deg}(p)$ , is the smallest  $k$  such that  $p \in \mathbb{R}_k[x]$ .

The moments of a measure  $\mu \in \mathcal{M}(K)$  for  $K \subset \mathbb{R}^n$  are:

$$y_\mu^\alpha = \int x^\alpha d\mu(x). \quad (23)$$

Integration of a polynomial with respect to a measure  $\mu$  can be expressed as a linear functional of its coefficients:

$$\langle \mu, p \rangle = \int p(x) d\mu(x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha y_\mu^\alpha = \text{vec}(p)^T y_\mu. \quad (24)$$

Integrating the square of a polynomial  $p \in \mathbb{R}_k[x]$ , we obtain:

$$\int p(x)^2 d\mu(x) = \text{vec}(p)^T M_k(y_\mu) \text{vec}(p), \quad (25)$$

where  $M_k(y_\mu)$  is the *truncated moment matrix* defined by

$$[M_k(y_\mu)]_{(\alpha, \beta)} = y_\mu^{\alpha + \beta} \quad (26)$$

for  $\alpha, \beta \in \mathbb{N}_k^n$ . Note that for any positive measure  $\mu$ , the matrix  $M_k(y_\mu)$  must be positive semidefinite. Similarly, given  $h \in \mathbb{R}[x]$  one has:

$$\int p(x)^2 h(x) d\mu(x) = \text{vec}(p)^T M_k(h, y_\mu) \text{vec}(p), \quad (27)$$

where  $M_k(h, y)$  is a *localizing matrix* defined by

$$[M_k(h, y_\mu)]_{(\alpha, \beta)} = \sum_{\gamma \in \mathbb{N}^n} h_\gamma y_\mu^{\alpha + \beta + \gamma} \quad (28)$$

for all  $\alpha, \beta \in \mathbb{N}_k^n$ . Note that the positive semidefiniteness of a localizing matrix for a moment sequence guarantees the existence of a Borel measure on the semialgebraic set defined by  $h$  [11, Theorem 3.8]. The localizing and moment matrices are symmetric and linear in the moments.

##### A. Approximating Problems

Finite dimensional SDPs approximating  $P$  can be obtained by replacing constraints on measures with constraints on moments. All of the equality constraints of  $P$  can be expressed as an infinite-dimensional linear system of equations which the moments of the measures appearing in  $P$  must satisfy. This linear system is obtained by restricting to polynomial test functions:  $v(t, x) = t^\alpha x^\beta$  and  $w(x) = x^\beta$ ,  $\forall \alpha \in \mathbb{N}$  and  $\forall \beta \in \mathbb{N}^n$ . For example, the Liouville equation in  $P$  is obtained via:

$$\begin{aligned} &\int_{[0, T] \times X_j \times U} \mathcal{L}_j v_j(t, x, u) d\mu^j(t, x, u) = \int_{X_{T_j}} v_j(T, x) d\mu_T(x) + \\ &+ \sum_{(j', j) \in \Gamma} \int_{[0, T] \times S_{(j', j)}} v_j(t, x) d\mu^{S_{(j', j)}}(t, x) - \int_{X_j} v_j(0, x) d\mu_0(x) - \\ &+ \sum_{(j, j') \in \Gamma} \int_{[0, T] \times S_{(j, j')}} v(t, R_{(j, j')}(x)) |\det(DR_{(j, j')}(x))| d\mu^{S_{(j, j')}}(t, x) \end{aligned}$$

Notice in particular that  $|\det(DR_{(j,j')}(x))|$  is a polynomial function by Definition 1.

A *finite-dimensional* linear system is obtained by truncating the degree of the polynomial test functions to  $2k$ . Let  $\Xi_{\mathcal{J}} = \prod_{j \in \mathcal{J}} \mu^j$ ,  $\Xi_0 = \prod_{j \in \mathcal{J}} \{\mu_0^j, \hat{\mu}_0^j\}$ ,  $\Xi_{\Gamma} = \prod_{e \in \Gamma} \mu^{S_e}$ , and  $\Xi_T = \prod_{j \in \mathcal{J}} \mu_T^j$ . Let  $\mathbf{y}_k^{\eta} = (y_{k,\xi}) \subset \mathbb{R}$  be a vector of sequences of moments truncated to degree  $2k$  for each  $(\xi, j) \in \Xi_{\eta}$  and for each  $\eta \in \{\mathcal{J}, 0, \Gamma, T\}$ . The finite-dimensional linear system is then represented by the linear system:

$$A_k(\mathbf{y}_k^{\mathcal{J}}, \mathbf{y}_k^0, \mathbf{y}_k^{\Gamma}, \mathbf{y}_k^T) = b_k. \quad (29)$$

Constraints on the support of the measures also need to be imposed (see [11] for details). Let the  $k$ -th relaxed SDP representation of  $P$ , denoted  $P_k$ , be defined as:

$$\begin{aligned} \sup \quad & \sum_{j \in \mathcal{J}} y_{k,\mu_0^j}^0 & (P_k) \\ \text{s.t.} \quad & A_k(\mathbf{y}_k^{\mathcal{J}}, \mathbf{y}_k^0, \mathbf{y}_k^{\Gamma}, \mathbf{y}_k^T) = b_k, \\ & M_k(y_{k,\xi}) \succeq 0 \quad \forall (\xi, j) \in \{\Xi_{\mathcal{J}}, \Xi_0, \Xi_{\Gamma}, \Xi_T\}, \\ & M_{k_{X_{j_i}}}(h_{X_{j_i}}, y_{k,\xi}) \succeq 0 \quad \forall (i, \xi, j) \in \{1, \dots, n_{X_j}\} \times \Xi_{\mathcal{J}}, \\ & M_{k_{X_{j_i}}}(h_{X_{j_i}}, y_{k,\xi}) \succeq 0 \quad \forall (i, \xi, j) \in \{1, \dots, n_{X_j}\} \times \Xi_0, \\ & M_{k_{S_{e_i}}}(h_{e_i}, y_{k,\xi}) \succeq 0 \quad \forall (i, \xi, e) \in \{1, \dots, n_e\} \times \Xi_{\Gamma}, \\ & M_{k_{T_{j_i}}}(h_{T_{j_i}}, y_{k,\xi}) \succeq 0 \quad \forall (i, \xi, j) \in \{1, \dots, n_{T_j}\} \times \Xi_T, \\ & M_{k-1}(h_{\tau}, y_{k,\xi}) \succeq 0 \quad \forall (\xi, j) \in \{\Xi_{\mathcal{J}}, \Xi_{\Gamma}\}, \end{aligned}$$

where the given data are  $\mathcal{H}$  and  $X_T$  and the supremum is taken over the sequence of moments,  $\mathbf{y}_k^{\mathcal{J}}, \mathbf{y}_k^0, \mathbf{y}_k^{\Gamma}, \mathbf{y}_k^T$ ,  $h_{\tau} = t(T - t)$ ,  $k_{X_{j_i}} = k - \lceil \deg(h_{X_{j_i}})/2 \rceil$ ,  $k_{S_{e_i}} = k - \lceil \deg(h_{e_i})/2 \rceil$ ,  $k_{T_{j_i}} = k - \lceil \deg(h_{T_{j_i}})/2 \rceil$ , and  $\succeq 0$  denotes positive semi-definiteness. For each  $k \in \mathbb{N}$ ,  $p_k^*$  denote the supremum of  $P_k$ .

The dual of  $P_k$  can be constructed as a sums-of-squares (SOS) program denoted  $D_k$  for each  $k \in \mathbb{N}$ . It is obtained by restricting the optimization space in the  $D$  to the polynomial functions with degree truncated to  $2k$  and replacing the non-negativity constraint  $D$  with an SOS constraint [16]. To make this explicit, for each  $j \in \mathcal{J}$ , let  $Q_k(h_{X_{j_1}}, \dots, h_{X_{j_n_{X_j}}}) \subset \mathbb{R}_{2k}[x]$  be the set of polynomials  $q \in \mathbb{R}_{2k}[x]$  (i.e. of total degree less than  $2k$ ) expressible as:

$$q = s_0 + \sum_{i=1}^{n_{X_j}} s_i h_{X_{j_i}}, \quad (30)$$

for some SOS polynomials  $\{s_i\}_{i=0}^{n_{X_j}} \subset \mathbb{R}_{2k}[x]$ . Every such polynomial is clearly non-negative on  $X_j$ . Similarly, for each  $j \in \mathcal{J}$  and  $e \in \Gamma$ , define  $Q_{2k}(h_{\tau}, h_{X_{j_1}}, \dots, h_{X_{j_n_{X_j}}}) \subset \mathbb{R}_{2k}[t, x]$ ,  $Q_{2k}(h_{T_{j_1}}, \dots, h_{T_{j_n_{T_j}}}) \subset \mathbb{R}_{2k}[x]$ , and  $Q_{2k}(h_{\tau}, h_{T_{e_1}}, \dots, h_{T_{e_{n_e}}}) \subset \mathbb{R}_{2k}[t, x]$ . Employing this notation, the  $k$ -th relaxed SDP representation of  $D$ , denoted

$D_k$ , is defined as:

$$\begin{aligned} \inf \quad & \sum_{j \in \mathcal{J}} l^T \text{vec}(w_j) & (D_k) \\ \text{s.t.} \quad & -\mathcal{L}_j v_j \in Q_{2k}(h_{\tau}, h_{X_{j_1}}, \dots, h_{X_{j_n_{X_j}}}) \quad \forall j \in \mathcal{J}, \\ & w_j - v_j(0, \cdot) - p - 1 \in Q_{2k}(h_{X_{j_1}}, \dots, h_{X_{j_n_{X_j}}}) \quad \forall j \in \mathcal{J}, \\ & w_j \in Q_{2k}(h_{X_{j_1}}, \dots, h_{X_{j_n_{X_j}}}) \quad \forall j \in \mathcal{J}, \\ & v_j(T, \cdot) + p \in Q_{2k}(h_{T_{j_1}}, \dots, h_{T_{j_n_{T_j}}}) \quad \forall j \in \mathcal{J}, \\ & v_j - v_{j'} \circ (1, R_e) \in Q_{2k}(h_{\tau}, h_{T_{e_1}}, \dots, h_{T_{e_{n_e}}}) \quad \forall e := (j', j) \in \Gamma, \end{aligned}$$

where the given data are  $\mathcal{H}$  and  $X_T$ , the infimum is taken over the vector of polynomials  $(v, w, p) \in \prod_{j \in \mathcal{J}} \mathbb{R}_{2k}[t, x] \times \prod_{j \in \mathcal{J}} \mathbb{R}_{2k}[x] \times \mathbb{R}$ , and  $l$  is a vector of moments associated with the Lebesgue measure (i.e.  $\int_X w_j d\lambda = l^T \text{vec}(w_j)$  for all  $w_j \in \mathbb{R}_{2k}[x]$  and  $j \in \mathcal{J}$ ). For notational convenience in the description of  $D_k$  we denote the  $j \in \mathcal{J}$  slice of  $v$  using the subscript  $j$  (i.e. for every  $(t, x) \in [0, T] \times X_j$  we let  $v_j(t, x) = v(t, x, j)$ ) and apply a similar convention to  $w$ . For each  $k \in \mathbb{N}$ , let  $d_k^*$  denote the infimum of  $D_k$ . In fact, the following result holds:

**Theorem 5:** For each  $k \in \mathbb{N}$ , there is no duality gap between  $P_k$  and  $D_k$ .

*Proof.* Due to space limitations, we omit the proof, which follows by establishing that  $P_k$  is bounded due to the constraint  $\mu_0 + \hat{\mu}_0 = \lambda$  and then arguing that the feasible set of the SDP,  $D_k$ , has an interior point which is sufficient to establish zero duality gap [23, Theorem 5].  $\square$

### B. Convergence of Approximating Problems

Next, we prove the convergence properties of  $P_k$  and  $D_k$ . We begin by proving that the polynomial  $w_j$  approximates the indicator function on  $\mathcal{X}_j$ . As we increase  $k$ , this approximation gets tighter. The following theorem, whose proof we omit since it is a straightforward extension of Theorem 5 in [8], makes this statement precise.

**Theorem 6:** For each  $k \in \mathbb{N}$  and  $j \in \mathcal{J}$ , let  $w_{j_k} \in \mathbb{R}_{2k}[x]$  denote the  $j$ -slice of the  $w$ -component of the solution to  $D_k$ , and let  $\bar{w}_{j_k}(x) = \min_{i \leq k} w_{j_i}(x)$ . Then,  $w_{j_k}$  converges from above to  $I_{\mathcal{X}_j}$  in the  $L^1$  norm, and  $\bar{w}_{j_k}(x)$  converges from above to  $I_{\mathcal{X}_j}$  in the  $L^1$  norm and almost uniformly.

As a result of Theorems 1 and 6 we have:

**Corollary 1:**  $\{d_k^*\}_{k=1}^{\infty}$  and  $\{p_k^*\}_{k=1}^{\infty}$  converge monotonically from above to the optimal value of  $D$  and  $P$ .

Next, we prove that for each  $j \in \mathcal{J}$  the 1-superlevel set of  $w_j$  converges in Lebesgue measure to  $\mathcal{X}_j$ .

**Theorem 7:** For each  $k \in \mathbb{N}$  and  $j \in \mathcal{J}$ , let  $w_{j_k} \in \mathbb{R}_{2k}[x]$  denote the  $j$ -slice of the  $w$ -component of the solution to  $D_k$ , and let  $\mathcal{X}_{j_k} := \{x \in \mathbb{R}^n | w_{j_k}(x) \geq 1\}$ . Then,  $\lim_{k \rightarrow \infty} \lambda_{n_j}(\mathcal{X}_{j_k} \setminus \mathcal{X}_j) = 0$ .

*Proof.* Via Theorem 4 we see  $w_{j_k} \geq I_{\mathcal{X}_{j_k}} \geq I_{\mathcal{X}_j}$ . From Theorem 6,  $w_{j_k} \rightarrow I_{\mathcal{X}_j}$  in the  $L^1$  norm on  $X_j$ . Hence:

$$\lambda_{n_j}(\mathcal{X}_j) = \lim_{k \rightarrow \infty} \int_{X_j} w_{j_k} d\lambda_{n_j} \geq \lim_{k \rightarrow \infty} \int_{X_j} I_{\mathcal{X}_{j_k}} d\lambda_{n_j} = \lim_{k \rightarrow \infty} \lambda_{n_j}(\mathcal{X}_{j_k}).$$

Since  $\mathcal{X}_j \subset \mathcal{X}_{j_k}$  for all  $k$ ,  $\lim_{k \rightarrow \infty} \lambda_{n_j}(\mathcal{X}_{j_k}) = \lambda_{n_j}(\mathcal{X}_j)$ .  $\square$

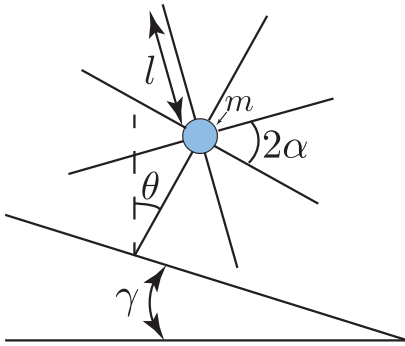


Fig. 1. Rimless wheel system

## V. IMPLEMENTATION AND EXAMPLES

In this section, we describe the performance of our approach on two examples. The relaxed problems were prepared using YALMIP and solved using MOSEK on a machine with an Intel i7-4820k 3.70GHz processor with 16GB RAM [12]. We briefly describe an extension of our algorithm to the case when our goal is to determine whether an initial condition is able to reach a target set within a pre-specified time  $T$  rather than exactly at  $T$ , which we refer to as the time-free backwards reachable set problem. This is done by allowing the support of  $\mu_T \in \mathcal{M}([0, T] \times \mathcal{D})$  in  $P$ . Consequently, the only modification on the dual program  $D$  is that the non-negativity constraint on  $v$  is imposed for all  $t \in [0, T]$ . Each of the aforementioned corollaries, lemmas, and theorems extend with nearly identical proof and the numerical implementation extends in a straightforward manner. The measures are supported on variables corresponding to time, states and control inputs, totaling  $(1+n+m)$  variables where  $n$  is the number of states and  $m$  is the number of control inputs. The total number of moments in the primal problem scales as  $O((1+n+m)^k)$  for a fixed relaxation  $k$  and  $O(k^{1+n+m})$  for a fixed  $n, m$ . Assuming a linear reset map, the number of variables in the dual problem scale as  $O(\max(n_v, n_{f_j}, n_v, n_{g_j}, n_u))$  where  $n_v, n_{f_j}, n_{g_j}, n_u$  are the degrees of  $v, f_j, g_j,$  and  $u$ , respectively.

### A. Rimless Wheel

The rimless wheel is a simple planar walking model illustrated in Figure 1. It consists of a single point mass with spokes radiating outward with dynamics given by an inverted pendulum  $f(\theta, \dot{\theta}) = [\dot{\theta}; \sin(\theta)]$ , where  $\theta$  is the angle between the vertical and pinned spoke. The front spoke hits the ground when  $\theta = \alpha + \gamma$ , upon which the system undergoes a reset where  $[\theta^+, \dot{\theta}^+] := [2\gamma - \theta^-, \cos(2\alpha)\dot{\theta}^-]$ .

The limit cycle and basin of attraction of this system have been studied analytically[5]. By choosing  $\alpha = 0.4, \gamma = 0.2$ , the rimless wheel has a stable limit cycle where the energy lost during ground impact is equal to the change in potential energy through the cycle. Figure 2 illustrates this limit cycle along with the phase portrait, with the guard shown in dotted red and the image of the reset map shown in solid red. We considered the task of determining the BRS of the limit cycle with  $T = 10$  and for implementation we considered a third-order Taylor expansion of the dynamics and reset map. The

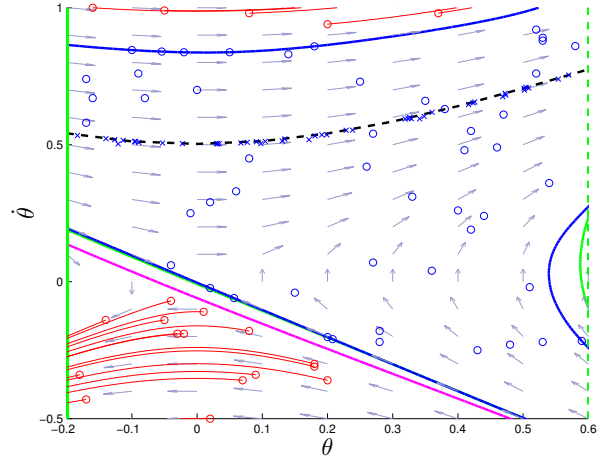


Fig. 2. BRS of the Rimless Wheel. The limit cycle of the system is drawn in as a dotted black line. The magenta, green, and blue lines denote outer-approximations of the BRS degree 6, 8, and 10, respectively. 50 points were sampled drawn in blue circles within the degree 10 BRS, used as initial conditions, and evolved using the dynamics of the Rimless Wheel (not the Taylor approximation). Their final time state is plotted with the blue X. 20 points were sampled outside the BRS starting at the red circles and their evolutions are plotted.

result of our computation is illustrated in Figure 2. The time to compute the BRS for degrees 6, 8, and 10 were 1.5s, 4.8s, and 18s, respectively.

### B. Vehicle Dynamics

Next, we compute the forward reachable set of the vehicle illustrated in Figure 3a. Motivated by [19] we model its dynamics as follows:

$$\begin{aligned} m\ddot{x}_1 &= m\dot{x}_2\dot{\psi} + 2F_{x_1f} + 2F_{x_1r} \\ m\ddot{x}_2 &= -m\dot{x}_1\dot{\psi} + 2F_{x_2f} + 2F_{x_2r} \\ I_z\ddot{\psi} &= 2aF_{x_2f} - 2bF_{x_2r} \end{aligned} \quad (31)$$

where  $m = 2050$  and  $I_z = 3344$  denote the vehicle mass and inertia, respectively,  $a = 1.43$  and  $b = 1.47$  denote the distances from the vehicle's center of gravity to the front and rear axles, respectively. The states  $\dot{x}_1$  and  $\dot{x}_2$  denote the vehicle's longitudinal and lateral velocities, respectively, and  $\dot{\psi}$  denotes the vehicle's yaw rate about the center of gravity.  $F_{x_1f}$  and  $F_{x_2f}$  are the forces of the front tire in the longitudinal and lateral axis, respectively, and  $F_{x_1r}$  and  $F_{x_2r}$  are the forces of the rear tire in the longitudinal and lateral axis, respectively.

The longitudinal and lateral tire force components in the vehicle body frame are modeled as:

$$\begin{aligned} F_{x_1\star} &= F_{l\star} \cos(u_\star) - F_{c\star} \sin(u_\star), \\ F_{x_2\star} &= F_{l\star} \sin(u_\star) + F_{c\star} \cos(u_\star), \end{aligned} \quad (32)$$

where  $\star$  denotes  $f$  or  $r$  for the front and rear tire and  $u_\star$  denotes the steering angle at the wheel. For this example, we assume only the front tire can be controlled, thus  $u_f = u$  and  $u_r = 0$ . The longitudinal force in the tire frame is  $F_{l\star} = 4269.125$ . As described in [15], we model the dynamics as a hybrid system by splitting  $F_{c\star}$ , the force due to tire friction,

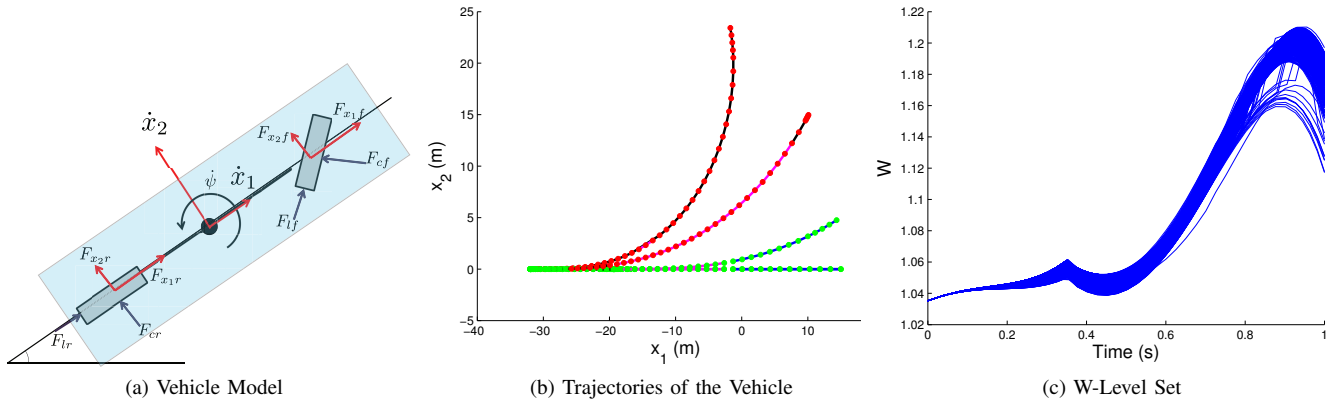


Fig. 3. (a) An illustration depicting the forces modeled in the vehicle body-fixed frame ( $F_{x1\star}$  and  $F_{x2\star}$ ), the forces in the tire-fixed frame ( $F_{l\star}$  and  $F_{c\star}$ ), and the rotational and translational velocities used in the vehicle model described in Equation (31). (b) An illustration of four trajectories of the vehicle beginning at  $x_0$  with constant input  $-0.3, -0.2, -0.1$  and  $0$ . The green dots on the trajectory denote states that when evaluated with respect to the computed  $w$  belonged to the 1-superlevel set. The red dots along the trajectory denote corresponding states that did not belong to the 1-superlevel set of  $w$ . Note that the trajectory with constant input  $0.2$  and  $0.3$ , in contrast with the other two trajectories, has portions of its trajectory that are red. Portions of the trajectory drawn in black, magenta, and blue illustrate when the system was in mode 1, 2, and 3, respectively. (c) An illustration of the value of the computed  $w$  for different trajectories starting at  $x_0$  with 200 distinct inputs all constrained to  $[-0.1, 0.1]$ . Notice that for all time, the value of  $w$  is greater than 1 indicating the trajectories lie in the forward reachable set.

into different zones depending upon the longitudinal velocity:

$$F_{c\star} = \begin{cases} -\frac{\dot{x}_2 + \alpha\dot{\psi}}{18}, & \text{if } \dot{x}_1 \in [15, 21) \text{ - Mode 1} \\ -\frac{\dot{x}_2 + \alpha\dot{\psi}}{24}, & \text{if } \dot{x}_1 \in [21, 27) \text{ - Mode 2} \\ -\frac{\dot{x}_2 + \alpha\dot{\psi}}{30}, & \text{if } \dot{x}_1 \in [27, 33) \text{ - Mode 3} \end{cases}$$

For this example, we look at the forward reachable set of a vehicle beginning at  $x_0$  which is equal to  $x_1 = -32, x_2 = 0, \psi = 0, \dot{x}_1 = 15, \dot{x}_2 = 0$ , and  $\dot{\psi} = 0$  with the steering input  $u$  constrained to  $[-0.1, 0.1]$ . We use small angle approximations for the trigonometric terms in Equation (32). We compute the full forward reachable set, as illustrated in Figures 3b and 3c, by running our degree 6 relaxation of our algorithm on the time-reversed dynamics and by solving the time-free backwards reachable set problem. It took 40m to compute the BRS.

## VI. CONCLUSION

We presented an approach for computing the BRS of a hybrid system using an infinite dimensional LP over the space of non-negative measures. Finite dimensional approximations to this LP in terms of SDPs were then constructed to obtain outer approximations of the BRS. In contrast to previous approaches relying on Lyapunov's stability criteria, our method is convex and does not require feasible initialization. We are currently pursuing methods to extend our approach to feedback control synthesis, region of attraction computation, and inner approximations for hybrid systems as in [9], [13].

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