L_2-Gain Optimization for Robust Bipedal Walking on Unknown Terrain

Hongkai Dai and Russ Tedrake

***Abstract*— In this paper we seek to quantify and explicitly optimize the robustness of a control system for a robot walking on terrain with uncertain geometry. Geometric perturbations to the terrain enter the equations of motion through a relocation of the hybrid event "guards" which trigger an impact event; these perturbations can have a large effect on the stability of the robot and do not fit into the traditional robust control analysis and design methodologies without additional machinery. We attempt to provide that machinery here. In particular, we quantify the robustness of the system to terrain perturbations by defining an L_2 gain from terrain perturbations to deviations from the nominal limit cycle. We show that the solution to a periodic dissipation inequality provides a sufficient upper bound on this gain for a linear approximation of the dynamics around the limit cycle, and we formulate a semidefinite programming problem to compute the L_2 gain for the system with a fixed linear controller. We then use either binary search or an iterative optimization method to construct a linear robust controller and to minimize the L_2 gain. The simulation results on canonical robots suggest that the L_2 gain is closely correlated to the actual number of steps traversed on the rough terrain, and our controller can improve the robot's robustness to terrain disturbances.**

I. INTRODUCTION

Bipedal robots are subject to many sources of uncertainty during walking; these could include a push from human, an unexpected gust of wind, or parametric uncertainties of unmodeled friction forces. Among all of these uncertainties, we focus in this paper on geometric perturbations to the terrain height. Unlike uncertainties which affect the continuous dynamics of the system, which can be accommodated with traditional approaches to robust control analysis and synthesis, terrain uncertainty manifests itself directly in the hybrid dynamical systems nature of a walking robot. A perturbation in terrain height appears as changes in the timing and dynamics of a ground contact event. Changes in the ground contact events can have a major stabilizing or de-stabilizing effect on legged robots. Although it is natural to apply robust control analysis and/or synthesis to a (typically numerical) approximation of walking dynamics on the Poincaré map, or even to apply time-domain methods from robust control to the continuous phases of the dynamics, applying robust control to the hybrid systems uncertainty requires additional care.

In this paper, we define an L_2 gain to quantify the robustness of bipedal robots to terrain disturbances. Moreover, we present a semidefinite programming formulation for computing an upper bound of the L_2 gain based on dissipation inequality. We further demonstrate that through binary search or iterative optimization, that upper bound can be optimized by constructing robust linear controllers. We validate our paradigm on canonical robots.

This work is supported by NSF contract CNS-0960061 and DARPA Maximum Mobility and Manipulation program, BAA-10-65-M3-FP-024.

H. Dai and R. Tedrake are with the Computer Science and Artificial Intelligence Lab, Massachusetts Institute of Technology, Cambridge, MA, 02139, USA. {daih,russt}@csail.mit.edu

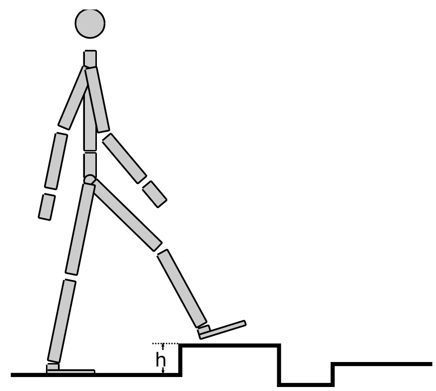

Fig. 1: A simple humanoid walking over uneven terrain; h is the terrain height at the incoming ground impact.

II. RELATED WORK

Extensive research has been performed on dealing with uncertainties for a continuous linear system [24], [16]. A common approach is to quantify the robustness of the system by its L_2 gain. By searching over the storage function that satisfies the dissipation inequality, an upper bound of the L_2 gain can be determined for a closed-loop system, and an H_∞ controller can be constructed for a given L_2 gain upper bound [12]. In this paper, we extend this approach to hybrid dynamical systems, with uncertainty existing in the guard function.

For limit cycle walkers, the robotics community has realized that careful local linearization of the dynamics around a limit cycle can provide powerful tools for orbital stability analysis[7] and control design[17]. This analysis can also lead to regional stability analysis[10] and can be lead to receding-horizon control strategies for dealing with terrain sensed at runtime[9]. These results suggest that linear controllers in the transversal coordinate can be used for nonlinear hybrid systems like bipedal robots. In this paper, we will show that robust linear controller even in the original (non-reduced) coordinates can improve the stability of a robot walking over unknown terrain.

Morimoto employs DDP method to optimize an H_∞ cost function, so as to improve the robustness of a bipedal walker[13]. His simulation results demonstrate that this controller enables the robot traverse longer distance under joint disturbances. This result indicates that H_∞ norm, and thus the L_2 gain is a good robust measure of the nonlinear system like bipedal robots.

One common approach in analyzing periodic legged locomotion is to construct a discrete step-to-step function, namely the Poincaré map, and analyze the properties of this discrete map [11], [14], [21]. Based on the Poincaré map, Hobbelen defines the gait sensitivity norm to measure the robustness of limit cycle walkers [8], Byl uses mean first-passage time to measure the robustness to unknown terrain given that the terrain height is drawn from a known distribution [4]. Moreover, Park designs an H_∞ controller for the discrete Poincaré map [15] for a bipedal walker. However, there some important limitations to Poincaré map analysis and control. In Poincaré analysis, it is difficult to include continuous dynamics uncertainty, and in Poincaré synthesis, control decisions can be made only once per one cycle, so opportunities for mid-step corrections are missed. For most systems, the Poincaré map does not have a closed form representation; it can only be numerically approximated instead of being exactly computed. In our approach, instead of relying on the Poincaré map, we study the continuous formulation of the hybrid dynamical system directly.

This paper is organized as follows: In Section III we present the definition of the L_2 gain for a limit cycle walker (III-A); a semidefinite programming formulation to compute an upper bound of the L_2 gain (III-B); control synthesis for a given L_2 gain upper-bound (III-C); and a paradigm to optimize such an upper bound (III-D). In section IV, we validate our approach on canonical robots. We then conclude our work in the last section.

III. APPROACH

Bipedal walking robots with pin feet are commonly modeled as hybrid systems with continuous modes interconnected by transition functions [22], [5]. Suppose the robot foot hits the ground with inelastic impact. The ground transition is then modeled as an impulsive mapping. For simplicity, we consider the system with only one continuous mode and one transition function

$$\dot{x} = f(x,u) \quad \text{if} \quad \phi(x,h) > 0 \tag{1}$$

$$x_h^+ = \Delta(x_h^-) \quad \text{if} \quad \phi(x_h^-,h) = 0 \tag{2}$$

Where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, h is the height of the terrain at the incoming ground impact. ϕ represents the distance between the foot and the terrain. When the distance decreases to zero, the ground impact occurs and the transition function maps the pre-impact state x_h^- to the post-impact state x_h^+. We suppose that we have access to perfect state information. Notice that unlike disturbances in the continuous dynamical system, in the rough terrain walking case, the terrain disturbance exists only in the guard function, the hybrid part of the system.

Terrain uncertainty can arise from many situations, like terrain perception error. Also, when we have planned a nominal gait, and do not want to re-plan for every small difference in the perceived height, that small difference can also be regarded as terrain uncertainty. Suppose the nominal terrain is h^*. On such terrain h^*, we have planned a limit cycle (x^*,u^*) as the desired walking pattern. Denote the terrain disturbance as $\bar{h} = h - h^*$. The goal is to design a phase tracking controller such that the error between the actual trajectory and the desired one (x^*,u^*) is small, while the unknown terrain disturbance is present. The magnitude of the error signal for an infinite time horizon is defined as

$$\int_0^\infty |e(t)|^2 dt = \int_0^\infty |x(t) - x^*(\tau)|_Q^2 + |u(t) - u^*(\tau)|_R^2 \tag{3}$$

where Q,R are given positive definite matrices for state error and control error respectively; τ is the phase being tracked on the nominal trajectory. For simplicity we suppose the clock of the tracked phase elapses at the same speed as the world clock in the continuous mode, (more advanced phase tracking, like the transversal coordinate, is also possible.) For the ground impact, since it does not make sense for the perturbed trajectory after the impact to track the nominal trajectory prior to the impact, the phase is reset by a function Π to the post-impact phase τ_h^+ every time the impact takes place. Namely

$$\dot{\tau} = 1 \quad \text{if} \quad \phi(x(t),h) > 0 \tag{4}$$

$$\Pi(x_h^+, x^*(\tau_h^+)) = 0 \quad \text{if} \quad \phi(x_h^-,h) = 0 \tag{5}$$

Notice that the projection function Π can be an implicit function of τ_h^+. For example, τ_h^+ may be chosen such that the stance leg angle of the post-impact state x_h^+ is the same as the stance leg angle of the nominal state $x^*(\tau_h^+)$ [1].

We measure the influence of terrain disturbance $\bar{h}$ on the error signal e by its L_2 gain, which is defined as the infimum of γ such that the equation below is lower bounded.

$$-\int_0^\infty |e(t)|^2 dt + \gamma^2 \sum_{n=1}^{\infty} \bar{h}[n]^2 > -\infty \tag{6}$$

where $\bar{h}[n]$ is the terrain height disturbance at the n^{th} ground impact.

We can not use Equation 6 directly to compute the L_2 gain of the system. The challenge lies in the following two aspects:

1) The time horizon stretches to infinity, while numerically determining the boundedness of the summation of an infinite time signal is not straightforward. The effects of terrain disturbances accumulate over time, and such effect is prohibitively difficult to explicitly compute.
2) The disturbance is a discrete signal, while the error is continuous. We lack the tools to tackle such a "hybrid" condition.

To overcome the above difficulties, rather than relying on Poincaré map, which is the traditional approach, we analyze the hybrid dynamical system (Equation 1, 2) directly in the following sections. We rely on the fact that the nominal trajectory (x^*,u^*) is periodic.

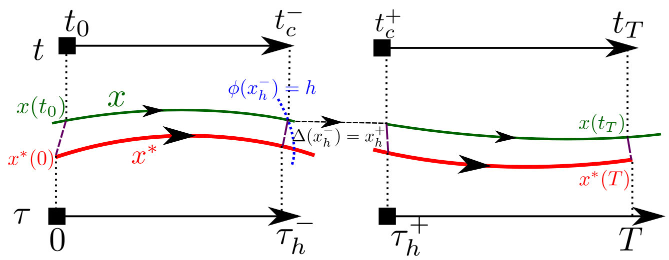

Fig. 2: The perturbed state trajectory x and the nominal state trajectory x^* within one step. Suppose the step starts at time t_0. The perturbed trajectory x hits the impact surface $\phi(x,h)=0$ with the pre-impact state x_h^- at time t_c^-. τ_h^- is the value of the tracked phase when the ground impact happens, $\tau_h^- = t_c^- - t_0$. The post-impact state x_h^+ is mapped by the state-transition function Δ from x_h^-. The post-impact phase τ_h^+ is determined by the phase reset function Π. The world clock does not change at the impact, namely $t_c^- = t_c^+$.

A. L_2 gain for a periodic phase tracking system

To get rid of the infinite time horizon problem, our solution is to break it into small steps, and only analyze the continuous state propagation within each step independently. We define a step based on the nominal trajectory being tracked. If we fix a certain event $\mathbb{E}$ on the nominal trajectory (such an event should be distinct from the ground impact, for example, the robot reaches its apex with that nominal state configuration), a step starts at $\tau = 0$ right after event $\mathbb{E}$ happens on the nominal trajectory, and ends when the the same event $\mathbb{E}$ takes place again on the nominal trajectory $(x^*(\tau), u^*(\tau))$. Since the nominal trajectory is periodic, the state/control on the nominal trajectory that triggers event $\mathbb{E}$ at the end of the step exactly equals the state/control at the beginning of the step. We denote the time length in between the two events as T, and we reset the tracked phase from T to 0 at the end of each step. The propagation of the perturbed and nominal trajectories within one step is illustrated in Fig.2. The step interval is $\tau \in [0,\ T]$, with only one ground impact taking place in the middle of the step. Note that when the perturbed trajectory hits the impact surface $\phi = 0$ at time t_c^- with state x_h^-, the nominal trajectory does not necessarily hit the same impact surface. To summarize, when the robot walks on rough terrain, there are two clocks running simultaneously, the world clock and the tracked phase clock. The infinite horizon is split every time the tracked phase clock jumps from T to 0.

Suppose that at the start of the n^{th} step, the world clock of the perturbed trajectory is at time $t_{\mathbb{E}}[n]$, and without loss of generality, we can assume $t_{\mathbb{E}}[1] = 0$. Equation 6 can be reformulated as the summation of disturbance and error signal within each step in Equation 7.

$$\sum_{n=1}^{\infty}\left(\gamma^2 \bar{h}[n]^2 - \int_{t_{\mathbb{E}}[n]}^{t_{\mathbb{E}}[n+1]} |x(t)-x^*(\tau)|_Q^2 + |u(t)-u^*(\tau)|_R^2 dt\right) > -\infty \tag{7}$$

For the given step n, we define the state error within that step as $\bar{x}^{(n)}(\tau) = x(t) - x^*(\tau)$, the control error as $\bar{u}^{(n)}(\tau) = u(t) - u^*(\tau)$, with the initial condition $\bar{x}^{(n)}(0) = x(t_{\mathbb{E}}[n]) - x^*(0)$ and $\bar{u}^{(n)}(0) = u(t_{\mathbb{E}}[n]) - u^*(0)$. We get the following lemma on the L_2 gain of the system with an infinite time horizon:

Lemma 3.1: For the hybrid dynamical system described by Equation 1 and 2, a sufficient condition for the L_2 gain no larger than a constant γ, is that there exists a storage function $V : [0,\ T] \times \mathbb{R}^n \to \mathbb{R}$, such that the following conditions 8-10 hold

$$\begin{aligned}&\gamma^2\bar{h}^2 - \int_0^{\tau_h^-} |\bar{x}^{(n)}(\tau)|_Q^2 + |\bar{u}^{(n)}(\tau)|_R^2 d\tau \\ &- \int_{\tau_h^+}^{T} |\bar{x}^{(n)}(\tau)|_Q^2 + |\bar{u}^{(n)}(\tau)|_R^2 d\tau \\ &\geq V(T, \bar{x}^{(n)}(T)) - V(0, \bar{x}^{(n)}(0)) \quad \forall\, \bar{h} \in \mathbb{R},\ \bar{x}^{(n)}(0) \in \mathbb{R}^n\end{aligned} \tag{8}$$

And the constraints on the two ends of the step

$$V(T,z) \geq V(0,z) \quad \forall\, z \in \mathbb{R}^n \tag{9}$$

And

$$V(T,z) \geq 0 \quad \forall\, z \in \mathbb{R}^n \tag{10}$$

Proof:

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\gamma^2\bar{h}[n]^2 - \int_{t_{\mathbb{E}}[n]}^{t_{\mathbb{E}}[n+1]} |x(t)-x^*(\tau)|_Q^2 + |u(t)-u^*(\tau)|_R^2 dt\right) \\ &= \sum_{n=1}^{\infty}\left(\gamma^2\bar{h}[n]^2 - \int_0^{\tau_h^-} |x(t)-x^*(\tau)|_Q^2 + |u(t)-u^*(\tau)|_R^2 d\tau \right. \\ &\quad \left. - \int_{\tau_h^+}^{T} |x(t)-x^*(\tau)|_Q^2 + |u(t)-u^*(\tau)|_R^2 d\tau\right) \\ &\geq \sum_{n=1}^{\infty} V(T, x(t_{\mathbb{E}}[n+1]) - x^*(T)) - V(0, x(t_{\mathbb{E}}[n]) - x^*(0)) \\ &= \sum_{n=2}^{\infty} \left(V(T, x(t_{\mathbb{E}}[n+1]) - x^*(T)) - V(0, x(t_{\mathbb{E}}[n+1]) - x^*(0))\right) \\ &\quad + \lim_{m\to\infty} V(T, x(t_{\mathbb{E}}[m]) - x^*(T)) - V(0, x(t_{\mathbb{E}}[1]) - x^*(0)) \\ &> -\infty\end{aligned}$$

Note that due to the periodicity of the nominal trajectory, $x^*(T) = x^*(0)$. ■

Lemma 3.1 enables us to get rid of the infinite time horizon problem and only analyze conditions 8-10 within one step. It is conservative as it does not capture the recovery motions that take one bad step before taking a different good step. In the sequel, we will drop the superscript "n" in $\bar{x}$ and $\bar{u}$, as each step is analyzed independent of the other steps.

Still, within one step, we have the continuous error signal e and the discrete terrain disturbance $\bar{h}$ in condition (8). To separate them, we consider the following sufficient conditions for equation (8)

$$\dot{V}(\tau, \bar{x}(\tau)) \leq -|\bar{x}(\tau)|_Q^2 - |\bar{u}(\tau)|_R^2 \quad \text{if } \phi(x^*(\tau)+\bar{x}(\tau), h^*+\bar{h}) > 0 \tag{11}$$

$$V(\tau_h^+, \bar{x}^+) - V(\tau_h^-, \bar{x}^-) \leq \gamma^2\bar{h}^2 \quad \text{if } \phi(x^*(\tau_h^-)+\bar{x}^-, h^*+\bar{h}) = 0 \tag{12}$$

where $\bar{x}^+ = \Delta(x^*(\tau_h^-)+\bar{x}^-) - x^*(\tau_h^+)$ is the post-impact state error. Notice that we isolate the mode transition out and treat both pre-impact state error $\bar{x}^-$ and post-impact state error $\bar{x}^+$ as time-independent variables.

For simplicity, we restrict the storage function V to a quadratic form,

$$V(\tau,\bar{x}) = \bar{x}' S(\tau)\bar{x} \tag{13}$$

where $S \in \mathbb{R}^{n\times n}$ satisfies the following conditions

$$\dot{V} = \bar{x}'\dot{S}(\tau)\bar{x} + 2\dot{\bar{x}}' S(\tau)\bar{x} \leq -|\bar{x}|_Q^2 - |\bar{u}|_R^2 \quad \text{if } \phi(x^*(\tau)+\bar{x}(\tau), h^*+\bar{h}) > 0 \tag{14a}$$

$$\bar{x}^{+\prime} S(\tau_h^+)\bar{x}^+ - \bar{x}^{-\prime} S(\tau_h^-)\bar{x}^- \leq \gamma^2 \bar{h}^2 \quad \text{if } \phi(x^*(\tau_h^-)+\bar{x}^-, h^*+\bar{h}) = 0 \tag{14b}$$

$$S(T) \succeq S(0) \tag{14c}$$

$$S(T) \succeq 0 \tag{14d}$$

Equations 14c and 14d are obtained by substituting Equation 13 into Equation 9 and 10 respectively.

We aim to design a linear time-varying controller $\bar{u}(\tau) = K(\tau)\bar{x}(\tau)$ and search for the storage matrix S satisfying Conditions 14a-14d, which guarantee that the closed-loop system has an L_2 gain no larger than γ. condition 14b for the mode transition is tricky, as the post-impact phase τ_h^+ is determined by the phase reset function Π, thus we cannot compute $\bar{x}^+$ from the state transition function Δ only, it has to be computed with the phase reset and guard function together; moreover, there exists a constraint that the guard function should be zero, we also wish to get rid of this constraint. For simplicity, we focus on the linearized case, and consider a small perturbation of the terrain height $\bar{h}$, and a small drift of the pre-impact state $\bar{x}^-$. Note that when there is no terrain disturbance and pre-impact state error is zero, the ground impact happens at the nominal impact time $\tau_{h^*}^-$, which is the phase that the nominal trajectory hits the nominal terrain guard function. Moreover, the post impact state error $\bar{x}^+$ will also be zero when $\bar{h} = 0$ and $\bar{x}^- = 0$. To analyze the situation when both $\bar{x}^-$ and $\bar{h}$ are small, by taking full derivative of the guard function, the following condition should hold.

$$\phi_x f(x^*(\tau_{h^*}^-), u^*(\tau_{h^*}^-))d\tau_h^- + \phi_x dx^- + \phi_h dh = 0 \tag{15}$$

where we denote dq as the infinitesimal variation of some variable q. We suppose that the robot does not graze on the ground when impact happens, thus $\phi_x f(x^*(\tau_{h^*}^-), u^*(\tau_{h^*}^-)) \neq 0$, and the variation of the pre-impact time $d\tau_h^-$ can be uniquely determined by the pre-impact state variation dx^- and terrain variation dh.

$$d\tau_h^- = \frac{-\phi_h dh - \phi_x dx^-}{\phi_x f(x^*(\tau_{h^*}^-), u^*(\tau_{h^*}^-))} \tag{16}$$

Likewise, we take full differentiation on the state transition function Δ and phase reset function Π also. Combining these two full differentiations with Equation 15, we get the following matrix equation.

$$L \begin{bmatrix} d\tau_h^- \\ d\tau_h^+ \\ dx^+ \end{bmatrix} = \begin{bmatrix} -\phi_h & -\phi_x \\ 0 & \Delta_x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dh \\ dx^- \end{bmatrix} \tag{17}$$

Where L is a matrix defined in Equation 18. We suppose the phase reset function uniquely determines the post-impact phase, hence the term $\Pi_{x^*} f(x^*(\tau_{h^*}^+), u^*(\tau_{h^*}^+)) \neq 0$. So the matrix L is nonsingular, and we can solve $[d\tau_h^-,\ d\tau_h^+,\ dx^+]$ from $[dh,\ dx^-]$ as a linear mapping. Suppose that

$$dx^+ = T_1 dh + T_2 dx^- \tag{19}$$

And we do the first order expansion of the term $\bar{x}^+, S(\tau_h^+)$ and $S(\tau_h^-)$ as follows:

$$\bar{x}^+ = T_1\bar{h} + T_2\bar{x}^- + o(\bar{h}) + o(\bar{x}^-) \tag{20a}$$

$$S(\tau_h^+) = S(\tau_{h^*}^+) + \dot{S}(\tau_{h^*}^+)\bar{\tau}_h^+ + o(\bar{\tau}_h^+) \tag{20b}$$

$$S(\tau_h^-) = S(\tau_{h^*}^-) + \dot{S}(\tau_{h^*}^-)\bar{\tau}_h^- + o(\bar{\tau}_h^-) \tag{20c}$$

where $\bar{\tau}_h^+, \bar{\tau}_h^-$ are variations of pre- and post impact phases respectively. We use a second order approximation of Condition 14b around the nominal terrain height $\bar{h}^* = 0$ and nominal trajectory $\bar{x}^{-*} = 0$. By substituting 20a-20c into 14b, the second order approximation is

$$(T_1\bar{h} + T_2\bar{x}^-)' S(\tau_{h^*}^+)(T_1\bar{h} + T_2\bar{x}^-) - \bar{x}^{-\prime} S(\tau_{h^*}^-)\bar{x}^- \leq \gamma^2\bar{h}^2 \tag{21}$$

Notice that we will then only need to verify the storage matrix S at the nominal pre-impact time $\tau_{h^*}^-$ and post-impact time $\tau_{h^*}^+$. Equation 21 is equivalent to the linear matrix inequality (LMI) below

$$\begin{bmatrix} T_2' \\ T_1' \end{bmatrix} S(\tau_{h^*}^+) \begin{bmatrix} T_2 & T_1 \end{bmatrix} - \begin{bmatrix} S(\tau_{h^*}^-) & 0 \\ 0 & \gamma^2 \end{bmatrix} \preceq 0 \tag{22}$$

For the continuous mode condition 14a, we linearize the state dynamics Equation 1. Denote $A = \frac{\partial f}{\partial x}, B = \frac{\partial f}{\partial u}$, and we require the following condition to hold

$$-\dot{S} \succeq (A+BK)'S + S(A+BK) + Q + K'RK \tag{23}$$

Notice that by dropping the constraint $\phi(x^*(\tau)+\bar{x}(\tau), h^*+\bar{h}) > 0$, equation 23 is more conservative than equation 14a, as we require the inequality holds for any state error $\bar{x}$. To summarize the discussions above, we have the following theorem

Theorem 3.2: For a hybrid system with dynamics defined as Equation 1 and 2, a sufficient condition for the controller $\bar{u} = K\bar{x}$ making the feedback system with L_2 gain no larger than γ, is that there exists a matrix $S : [0,\ T] \to \mathbb{R}^{n\times n}$ and satisfying the following conditions

$$-\dot{S} \succeq (A+BK)'S + S(A+BK) + Q + K'RK \tag{24a}$$

$$\begin{bmatrix} T_2' \\ T_1' \end{bmatrix} S(\tau_{h^*}^+) \begin{bmatrix} T_2 & T_1 \end{bmatrix} - \begin{bmatrix} S(\tau_{h^*}^-) & 0 \\ 0 & \gamma^2 \end{bmatrix} \preceq 0 \tag{24b}$$

$$S(T) \succeq S(0) \tag{24c}$$

$$S(T) \succeq 0 \tag{24d}$$

B. Computing L_2 gain

For a given linear controller $\bar{u}(\tau) = K(\tau)\bar{x}(\tau)$, our goal is determine an upper bound of its L_2 gain based on Theorem

$$L = \begin{bmatrix} \phi_x f(x^*(\tau_{h^*}^-), u^*(\tau_{h^*}^-)) & 0 & 0 \\ -\Delta_x f(x^*(\tau_{h^*}^-), u^*(\tau_{h^*}^-)) & f(x^*(\tau_{h^*}^+), u^*(\tau_{h^*}^+)) & I \\ 0 & (\Pi_{x^+} + \Pi_{x^*}) f(x^*(\tau_{h^*}^+), u^*(\tau_{h^*}^+)) & \Pi_{x^+} \end{bmatrix} \tag{18}$$

3.2, the problem is formulated as

$$\min_{S(.),\gamma} \quad \gamma \tag{25a}$$
$$\text{s.t } -\dot{S} \succeq (A+BK)'S + S(A+BK) + Q + K'RK \tag{25b}$$
$$\begin{bmatrix} T_2' \\ T_1' \end{bmatrix} S(\tau_{h^*}^+) \begin{bmatrix} T_2 & T_1 \end{bmatrix} - \begin{bmatrix} S(\tau_{h^*}^-) & 0 \\ 0 & \gamma^2 \end{bmatrix} \preceq 0 \tag{25c}$$
$$S(T) \succeq S(0) \tag{25d}$$
$$S(T) \succeq 0 \tag{25e}$$

Thus we need to search for the matrix S parameterized by time. Unfortunately we cannot use the condition 25b directly, for two reasons

1) By parameterizing S over time, there are infinitely many S to be determined, it cannot be solved through optimization program which only handles finitely many decision variables.
2) The differential term $\dot{S}$ cannot be computed in a closed form representation.

However, we know that the following celebrated equation for a general Lyapunov function

Lemma 3.3: For a general differential Lyapunov function

$$-\dot{P} = F'P + PF + M \tag{26}$$

The solution is given by

$$P(t) = \Phi_F(t_f,t)'P(t_f)\Phi_F(t_f,t) + \int_t^{t_f} \Phi_F(\sigma,t)'M(\sigma)\Phi_F(\sigma,t)d\sigma \tag{27}$$

Where $\Phi_F(\sigma,t)$ is the state transition matrix associated with F from time t to time σ.

For the differential Lyapunov inequality, the following corollary holds

Corollary 3.4: The matrix P satisfying the differential Lyapunov inequality

$$-\dot{P} \succeq F'P + PF + M \tag{28}$$

is equivalent to the satisfaction of the the following condition

$$P(t) \succeq \Phi_F(t_f,t)'P(t_f)\Phi_F(t_f,t) + \int_t^{t_f} \Phi_F(\sigma,t)'M(\sigma)\Phi_F(\sigma,t)d\sigma \quad \forall\, t < t_f \tag{29}$$

Moreover, given two matrices $P_1, P_2 \in \mathbb{R}^{n\times n}$, if the following condition holds

$$P_1 \succeq \Phi_F(t_f,t_0)'P_2\Phi_F(t_f,t_0) + \tag{30}$$
$$\int_t^{t_f} \Phi_F(\sigma,t_0)'M(\sigma)\Phi_F(\sigma,t_0)d\sigma \tag{31}$$

Then there exists a continuous time matrix function $P(t), t \in [t_0,\ t_f]$, such that $P(.)$ is the solution to the differential Lyapunov inequality (28), and satisfies $P(t_0) = P_1, P(t_f) = P_2$.

Based on Corollary 3.4, if we denote $\bar{F} = A + BK, \bar{M} = Q + K'RK$, condition 25b can be reformulated as

$$S(\tau_{h^*}^+) \succeq \Phi_{\bar{F}}(T,\tau_{h^*}^+)'S(T)\Phi_{\bar{F}}(T,\tau_{h^*}^+) + \int_{\tau_{h^*}^+}^{T} \Phi_{\bar{F}}(\sigma,\tau_{h^*}^+)'(Q+K'RK)\Phi_{\bar{F}}(\sigma,\tau_{h^*}^+)d\sigma \tag{32a}$$
$$S(0) \succeq \Phi_{\bar{F}}(\tau_{h^*}^-,0)'S(\tau_{h^*}^-)\Phi_{\bar{F}}(\tau_{h^*}^-,0) + \int_0^{\tau_{h^*}^-} \Phi_{\bar{F}}(\sigma,0)'(Q+K'RK)\Phi_{\bar{F}}(\sigma,0)d\sigma \tag{32b}$$

The state transition matrix and the integral term can be numerically computed. The decision variables are reduced to the storage matrix S at time $0, \tau_{h^*}^-, \tau_{h^*}^+, T$ and a scalar γ. We can further reduce the number of decision variables by checking the special properties of the optimal solution. Notice that if the tuple (γ, S_1) satisfies Condition 25b-25e, then we can construct a new tuple (γ, S_2) satisfying

$$S_2(0) = S_2(T) \tag{33a}$$
$$S_2(\tau_{h^*}^-) = S_1(\tau_{h^*}^-) + \frac{\lambda_{min}(S_2(0) - S_1(0))}{\lambda_{max}(\Phi_{\bar{F}}(\tau_{h^*}^-,0)'\Phi_{\bar{F}}(\tau_{h^*}^-,0))} I \tag{33b}$$
$$S_2(\tau_{h^*}^+) = S_1(\tau_{h^*}^+) \tag{33c}$$
$$S_2(T) = S_1(T) \tag{33d}$$

It can be easily verified that S_2 satisfies constraints 32a, 32b, 25d and 25e. Moreover, since $S_2(\tau_{h^*}^-) \succeq S_1(\tau_{h^*}^-), S_2(\tau_{h^*}^+) = S_1(\tau_{h^*}^+)$, (γ, S_2) also satisfies constraint 25c, hence (γ, S_2) is a feasible solution to the program 25a-25e. Namely, in the optimization program 25a-25e, by replacing the constraint 25d with strict equality 33a, the optimal value does not change. Thus the decision variable $S(0)$ can be dropped, as it can be replaced by $S(T)$.

We further show that the optimization program can be simplified by dropping $S(T)$. Condition 25e is the only constraint involving $S(T)$. But if $S(T) \succeq 0$, then inequality 32a implies $S(\tau_{h^*}^+) \succeq 0$. On the other hand, if $S(\tau_{h^*}^+) \succeq 0$, then inequality 25c implies $S(\tau_{h^*}^-) \succeq T_2'S(\tau_{h^*}^-)T_2 \succeq 0$, and by inequality 32b, we have $S(0) \succeq \Phi_{\bar{F}}(T,\tau_{h^*}^+)'S(\tau_{h^*}^-)\Phi_{\bar{F}}(T,\tau_{h^*}^+) \succeq 0$, thus $S(T) = S(0) \succeq 0$. So $S(T) \succeq 0$ iff $S(\tau_{h^*}^+) \succeq 0$. Constraint 25e can be replaced by $S(\tau_{h^*}^+) \succeq 0$, and the decision variable $S(T)$ can be dropped.

In all, with powerful conic programming solver like *SeDuMi* [18], we can solve the following semidefinite programming problem (SDP) to determine an upper bound of the L_2 gain, given a fixed linear controller $\bar{u} = K\bar{x}$.

$$\min_{\zeta, S^+, S^-} \quad \zeta \tag{34a}$$
$$\text{s.t } S^+ \succeq \Psi'S^-\Psi + \bar{N} \tag{34b}$$
$$\begin{bmatrix} T_2' \\ T_1' \end{bmatrix}' S^+ \begin{bmatrix} T_2 & T_1 \end{bmatrix} - \begin{bmatrix} S^- & 0 \\ 0 & \zeta \end{bmatrix} \preceq 0 \tag{34c}$$
$$S^+ \succeq 0 \tag{34d}$$

Where

$$S^{+}=S(\tau_{h^*}^{+}), S^{-}=S(\tau_{h^*}^{-}), \zeta=\gamma^2$$
$$\Psi=\Phi_{\bar{F}}(\tau_{h^*}^{-},0)\Phi_{\bar{F}}(T,\tau_{h^*}^{-})$$
$$\bar{N}=\int_{\tau_{h^*}^{+}}^{T}\Phi_{\bar{F}}(\sigma,\tau_{h^*}^{+})'(Q+K'RK)\Phi_{\bar{F}}(\sigma,\tau_{h^*}^{+})d\sigma+\Phi_{\bar{F}}(T,\tau_{h^*}^{+})'\left(\int_{0}^{\tau_{h^*}^{-}}\Phi_{\bar{F}}(\sigma,0)'(Q+K'RK)\Phi_{\bar{F}}(\sigma,0)d\sigma\right)\Phi_{\bar{F}}(T,\tau_{h^*}^{+})$$

where $\Psi,\bar{N}$ can be numerically computed.

C. Robust control synthesis

Given a constant γ and a hybrid system defined in Equation 1, 2, we want to construct a linear controller $\bar{u}(\tau)=K(\tau)\bar{x}(\tau)$, such that the closed loop system has an L_2 gain no larger than γ. Based on Theorem 3.2, it is equivalent to computing the storage matrix S and the control gain K satisfying constraints 24a-24d. When $\gamma^2>T_1'S(\tau_{h^*}^{+})T_1$, by using *Schur Complement*, the LMI constraint 24b is equivalent to

$$S(\tau_{h^*}^{-})\succeq T_2'S(\tau_{h^*}^{+})T_1(\gamma^2-T_1'S(\tau_{h^*}^{+})T_1)^{-1}T_1'S(\tau_{h^*}^{+})T_2 \quad (35)$$
$$+T_2'S(\tau_{h^*}^{+})T_2 \quad (36)$$

Condition 24a, 24c, 25e and 35 implies that finding the control gain K for the original hybrid system becomes equivalent to the robust control synthesis for the following periodic linear system $\mathscr{S}$

$$\dot{v}(\tau)=A(\tau)v(\tau)+B(\tau)f(\tau) \quad \tau\neq,\tau_{h^*}^{-}+nT,\tau\neq kT,n,k\in\mathbb{Z} \quad (37)$$
$$v(\tau^{+})=T_2v(\tau^{-})+T_1w \quad \tau=\tau_{h^*}+nT,n\in\mathbb{Z} \quad (38)$$
$$v(\tau^{+})=v(\tau^{-}) \quad \tau=nT,n\in\mathbb{Z} \quad (39)$$

where v is the state, f is the control input, w is the disturbance. A,B are periodic matrices satisfying $A(t+T)=A(t),B(t+T)=B(t)\ \forall\, t$. The robust control synthesis for this periodic linear system is well established [12], [23], [25], [2]. If the system $\mathscr{S}$ has an L_2 gain less than γ, then there exists a periodic gain matrix K, satisfying the following conditions

$$K(\tau)=-R^{-1}B(\tau)'S(\tau) \quad (40a)$$
$$-\dot{S}(\tau)=A(\tau)'S(\tau)+S(\tau)A(\tau)-S(\tau)B(\tau)R^{-1}B(\tau)'S(\tau)+Q \quad \tau\neq\tau_{h^*},T \quad (40b)$$
$$S(\tau_{h^*}^{-})=T_2'S(\tau_{h^*}^{+})T_1(\gamma^2-T_1'S(\tau_{h^*}^{+})T_1)^{-1}T_1'S(\tau_{h^*}^{+})T_2+T_2'S(\tau_{h^*}^{+})T_2 \quad (40c)$$
$$S(T)=S(0) \quad (40d)$$

Namely, there exists a periodic solution to the differential Riccati equation 40b and the jump Riccati equation 40c. The usual approach to solving the periodic Riccati equation is to numerically integrate it backward until the solution converges [3], given that matrix (A,B) is controllable. So for a fixed γ, if we know that there exists a linear controller for the hybrid system (Equation 1, 2) and a storage matrix S verifying that γ is an upper bound of the closed-loop system L_2 gain, based on Theorem 3.2, then by solving the periodic Riccati equation 40a-40d, we obtain such a robust controller.

D. Optimizing L_2 gain

If our goal is to design the linear controller for the hybrid system (Equation 1, 2) to minimize the L_2 gain of the closed-loop system, we can use binary search on γ. Namely, for a given γ, if the periodic Riccati equations 40b-40d have a convergent solution, then this γ is a valid upper bound of the L_2 gain for the closed-loop system; otherwise we cannot verify the existence of a linear controller making the closed-loop system with L_2 gain smaller than γ. Alternatively, we can also use the iterative optimization Algorithm 1. It is obvious that $\{\gamma_i\}$ is a non-decreasing sequence in Algorithm 1, as in each iteration, the tuple (γ_i,P_i) which solves the Riccati equation is a feasible solution to the semidefinite programming problem in the next iteration.

Algorithm 1 Iterative Optimization

1: At iteration i, given a linear controller gain K_i, solve the following semidefinite programming problem to determine γ_i as an upper bound of the L_2 gain of the closed-loop system

$$\min_{S_i^{+},S_i^{-},\zeta_i} \quad \zeta_i$$
$$\text{s.t} \quad \begin{bmatrix}T_2'\\T_1'\end{bmatrix}S_i^{+}\begin{bmatrix}T_2 & T_1\end{bmatrix}-\begin{bmatrix}S_i^{-} & 0\\0 & \zeta_i\end{bmatrix}\preceq 0$$
$$S_i^{+}\succeq\Psi_{F_i}'S_i^{-}\Psi_{F_i}+\bar{N}_i$$
$$S_i^{+}\succeq 0$$
$$\zeta_i>T_1'S_i^{+}T_1$$

Where

$$F_i=A+BK_i$$
$$\bar{N}_i=\int_{\tau_{h^*}^{+}}^{T}\Phi_{F_i}(\sigma,\tau_{h^*}^{+})'(Q+K'RK)\Phi_{F_i}(\sigma,\tau_{h^*}^{+})d\sigma+\Phi_{F_i}(T,\tau_{h^*}^{+})'\left(\int_{0}^{\tau_{h^*}^{-}}\Phi_{F_i}(\sigma,0)'(Q+K'RK)\Phi_{F_i}(\sigma,0)d\sigma\right)\Phi_{F_i}(T,\tau_{h^*}^{+})$$
$$\gamma_i=\sqrt{\zeta_i}$$

2: For the upper bound γ_i, construct a γ_i sub-optimal controller by computing a periodic solution to the following Riccati equation

$$P_i(\tau_{h^*}^{-})=T_2'P(\tau_{h^*}^{+})T_1(\gamma^2-T_1'P(\tau_{h^*}^{+})T_1)^{-1}T_1'P(\tau_{h^*}^{+})T_2+T_2'S(\tau_{h^*}^{+})T_2$$
$$-\dot{P}_i=A'P_i+P_iA+Q-P_iBR^{-1}B'P_i \quad \forall\,\tau\neq\tau_{h^*}$$
$$P(T)=P(0)$$

Find the γ_i-suboptimal controller K_i

$$K_i=-R^{-1}B'P_i$$

3: $i\leftarrow i+1$, with $K_{i+1}=K_i$

The binary search approach scales better for systems with high *degrees of freedom* (DOF). But the initial guess of

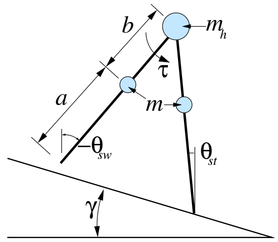

Fig. 3: A compass gait walking down a slope [19]

γ is unclear, and sometimes suffers the numeric tolerance issue of determining convergence. We can combine these two approaches, given an initial linear controller (not necessarily robust), we can compute the L_2 gain of the closed-loop system associated with that controller, and such L_2 gain can be an initial guess of the binary search. When convergence is hard to be decided by numeric integrating Riccati equation, we can switch to the iterative optimization to get a good estimate of the optimal γ.

IV. Results

A. Compass Gait

A compass gait robot is the simplest dynamic walking model (Fig 3). It is well known that the compass gait robot has a very narrow region of attraction and can easily fall down over rough terrain. We compare two limit cycles, the passive one and the robust one [6]. We run simulations of the robot walking over unknown terrain with virtual slope drawn from $[2^\circ, 8^\circ]$. The comparison is summarized in Table I. γ is computed through the semidefinite programming formulation in Section III-D. The limit cycle that has smaller γ can traverse much longer distance than the one with large γ. This good agreement between the big gap of γ and the distinction of their actual performance on the rough terrain suggests that the L_2 gain is a good indicator for the capability of traversing unknown terrain.

TABLE I: Comparison between the passive limit cycle and the robust one with the same LQR controller on a rugged terrain.

Limit Cycle	Passive	Robust
γ	292.6646	21.2043
Average number of steps before falling down	< 10	∞

For the passive limit cycle, we construct two robust controllers based on the algorithm in Section III-D, and compute their L_2 gain by solving the the semidefinite programming problem in Section III-B. We run 100 simulations of the compass gait robot walking on a rough terrain with the LQR controller and the two robust controllers. The virtual slope of the terrain is drawn uniformly within $[4^\circ, 6^\circ]$. Their performance is compared in Table II. The controller with the least γ, Robust1, can take more than twice as many steps as the LQR controller, the one with the largest γ. It is also note that the control gain of the Robust1 and Robust2 are quite different, the gain at $\tau_{h^*}^-$ of Robust1 is more than 2.3 times larger than that of Robust2, but since their γ values are close, their average number of steps on the rough terrain become also close. The comparison between those three controllers are shown in Fig 4 and 5, the straight line is $y = x$. Each dot represents a simulation of the compass gait on the same terrain for three controllers. In Fig 4, most points are above the $y = x$ line, indicating that Robust1 controller enables the robot to traverse more steps than LQR does. In Fig.5, most points are along the $y = x$ line, indicating the performance of the two robust controllers is close.

TABLE II: Comparison between three controllers for the passive limit cycle on a mild terrain.

controller	LQR	Robust2	Robust1
γ	292.6646	274.2038	269.9908
Average number of steps before falling down	40.61	89.27	96.21

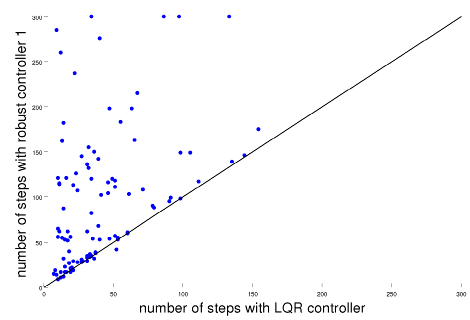

Fig. 4: LQR vs Robust1

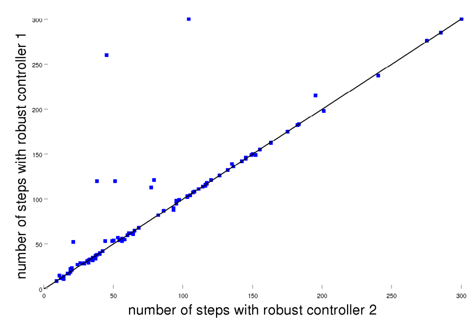

Fig. 5: Robust2 vs Robust1

B. RABBIT

RABBIT is a five-link planar bipedal walker constructed jointly by several French laboratories. Grizzle's group in University of Michigan have done extensive research into modeling and control of this robot [20], [5]. We apply our approach to the rigid-body model of RABBIT created by Westervelt (Fig.6).

We consider two different walking gaits for RABBIT. For gait 1 we construct an LQR controller, the SDP program computes that for the closed-loop system of RABBIT following gait 1, its $\gamma = 522.0822$; for gait 2 we use binary search to find a controller, which makes $\gamma \in [5000, 6000]$. We then take 40 simulations of the robot model walking on a rough terrain. The virtual slope of the terrain changes from step to step, and the angle is drawn uniformly from $[-2^\circ, 2^\circ]$. The comparison of their average number of steps traversed

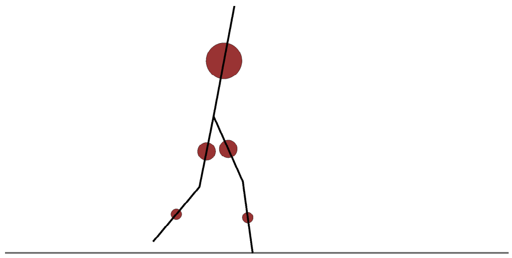

Fig. 6: A model of RABBIT walking on flat ground

is shown in Table III. This again shows that the upper bound of L_2 gain is a good indicator of the capability of traversing rough terrain. Moreover, it suggests that our method scales well to a system of RABBIT's complexity.

TABLE III: Comparison between two control strategies of RABBIT

γ	522.0822	[5000, 6000]
Average number of steps	20.325	1.675

V. Conclusions and Future Works

In this paper, we study the robustness of a bipedal robot to unknown terrain elevations. Given a desired walking pattern, we define a continuous error signal for the hybrid dynamical system. We quantify the robustness to terrain disturbance by the L_2 gain of the closed-loop system. Given a fixed linear controller, we present a semidefinite programming approach to compute an upper bound of the L_2 gain, and a control synthesis scheme to design a robust controller so as to bring down the L_2 gain. The simulation results validate that the L_2 gain is a good indicator of the capability to traverse unknown terrain. And our robust controller can improve such capability.

It will be easy to extend our scheme to include disturbances in the continuous time, since the analysis is performed on the continuous signal in our paper.

VI. Acknowledgement

The work is supported by National Science Foundation under grant number CNS-0960061 and DARPA Maximum Mobility and Manipulation program, BAA-10-65-M3-FP-024. We wish to thank the colleagues in Robot Locomotion Group, for the intriguing discussions on this paper. We also appreciate Prof. Megretski for his tutorial on robust control, and Prof. Grizzle and Westervelt for sharing their code on RABBIT model.

References

[1] Yeuhi Abe and Jovan Popovic. Simulating 2d gaits with a phase-indexed tracking controller. *IEEE Computer Graphics and Applications*, 31(4):22–33, 2011.

[2] Pierre Basar, Tamer;Bernhard. *H-Infinity Optimal Control and Related Minimax Design Problems*. Birkhäuser, 1995.

[3] S. Bittanti, P. Colaneri, and G. De Nicolao. The periodic riccati equation. In S. Bittanti, A. J. Laub, and J. C. Willems, editors, *The Riccati Equation*, chapter 6. Springer-Verlag, 1991.

[4] Katie Byl and Russ Tedrake. Metastable walking machines. *International Journal of Robotics Research*, 28(8):1040–1064, August 1 2009.

[5] C. Chevallereau, G. Abba, Y. Aoustin, F. Plestan, E. R. Westervelt, C. Canudas-De-Wit, and J. W. Grizzle. Rabbit: a testbed for advanced control theory. *IEEE Control Systems Magazine*, 23(5):57–79, Oct. 2003.

[6] Hongkai Dai and Russ Tedrake. Optimizing robust limit cycles for legged locomotion on unknown terrain. In *Proceedings of the IEEE Conference on Decision and Control*, 2012.

[7] J. Hauser and A. Banaszuk. Approximate feedback linearization around a trajectory: application to trajectory planning. *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, 1:7–11 vol.1, Dec 1997.

[8] Hobbelen, D.G.E., Wisse, and M. A disturbance rejection measure for limit cycle walkers: The gait sensitivity norm. *Robotics, IEEE Transactions on*, 23(6):1213 –1224, dec. 2007.

[9] Ian R. Manchester, Uwe Mettin, Fumiya Iida, and Russ Tedrake. Stable dynamic walking over rough terrain: Theory and experiment. In *Proceedings of the International Symposium on Robotics Research (ISRR)*, 2009.

[10] Ian R. Manchester, Mark M. Tobenkin, Michael Levashov, and Russ Tedrake. Regions of attraction for hybrid limit cycles of walking robots. *Proceedings of the 18th IFAC World Congress, extended version available online: arXiv:1010.2247 [math.OC]*, 2011.

[11] Tad McGeer. Dynamics and control of bipedal locomotion. *Journal of Theoretical Biology*, 163:277–314, 1993.

[12] Alexandre Megretski. Lecture Notes for 6.245, 2008.

[13] Jun Morimoto and Christopher Atkeson. Minimax differential dynamic programming: An application to robust biped walking. *Advances in Neural Information Processing Systems*, 2002.

[14] Jun Morimoto, Jun Nakanishi, Gen Endo, Gordon Cheng, Christopher G Atkeson, and Garth Zeglin. Poincare-map-based reinforcement learning for biped walking. *ICRA*, 2005.

[15] Hae-Won Park. *Control of a Bipedal Robot Walker on Rough Terrain*. PhD thesis, University of Michigan, 2012.

[16] Petersen, I.R., Ugrinovskii, V.A., Savkin, and A.V. *Robust control design using H-8 methods*. Communications and control engineering series. Springer, 2000.

[17] Anton S. Shiriaev, Leonid B. Freidovich, and Ian R. Manchester. Can we make a robot ballerina perform a pirouette? orbital stabilization of periodic motions of underactuated mechanical systems. *Annual Reviews in Control*, 32(2):200–211, Dec 2008.

[18] Jos F. Sturm. Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1-4):625 – 653, 1999.

[19] Russ Tedrake. *Underactuated Robotics: Learning, Planning, and Control for Efficient and Agile Machines: Course Notes for MIT 6.832*. Working draft edition, 2012.

[20] Westervelt, E. R., Buche, G., Grizzle, and J. W. Experimental validation of a framework for the design of controllers that induce stable walking in planar bipeds. *The International Journal of Robotics Research*, 23(6):559–582, 2004.

[21] E. R. Westervelt, J. W. Grizzle, and D. E. Koditschek. Hybrid zero dynamics of planar biped walkers. *IEEE Transactions on Automatic Control*, 48(1):42–56, Jan 2003.

[22] Eric R. Westervelt, Jessy W. Grizzle, Christine Chevallereau, Jun Ho Choi, and Benjamin Morris. *Feedback Control of Dynamic Bipedal Robot Locomotion*. CRC Press, Boca Raton, FL, 2007.

[23] Lihua Xie, Carlos E de Souza, and M.D Fragoso. h_∞ filtering for linear periodic systems with parameter uncertainty. *Systems & Control Letters*, 17(5):343 – 350, 1991.

[24] Zhou, Kemin, Doyle, John C., Glover, and Keith. *Robust and optimal control*. Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1996.

[25] Kemin Zhou and John C. Doyle. *Essentials of Robust Control*. Prentice Hall, 1997.