Simple and Practical Algorithm for the Sparse Fourier Transform

Haitham Hassanieh   Piotr Indyk   Dina Katabi   **Eric Price**

MIT

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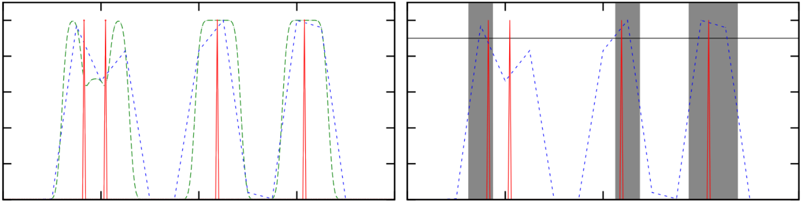


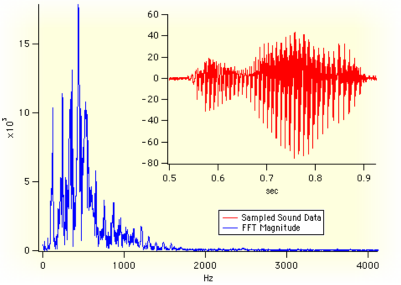
Outline

The Dicrete Fourier Transform

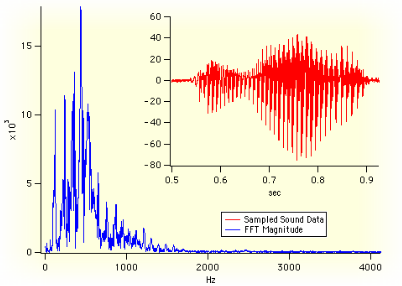
- Discrete Fourier transform: given $x \in \mathbb{C}^n$, find

$$\widehat{x}_i = \sum x_j \omega^{ij}$$

- Fundamental tool
  - Compression (audio, image, video)
  - Signal processing
  - Data analysis
  - ...
- FFT: $O(n \log n)$ time.



Sparse Fourier Transform



- Often the Fourier transform is dominated by a small number of "peaks"
  - Precisely the reason to use for compression.
- If most of mass in *k* locations, can we compute FFT faster?

Previous work

- Boolean cube: [KM92], [GL89]. What about $\mathbb{C}$?
- [Mansour-92]: $k^c \log^c n$.
- Long list of other work [GGIMS02, AGS03, Iwen10, Aka10]
- Fastest is [Gilbert-Muthukrishnan-Strauss-05]: $k \log^4 n$.
  - All have poor constants, many logs.
  - Need $n/k > 40{,}000$ or $\omega(\log^3 n)$ to beat FFTW.
  - Our goal: beat FFTW for smaller n/k in theory and practice.
  - Result: $n/k > 2,000$ or $\omega(\log n)$ to beat FFTW.

Our result

- Simple, practical algorithm with good constants.
- Compute the k-sparse Fourier transform in $O(\sqrt{kn}\log^{3/2} n)$ time.
- Get $\widehat{x'}$ with approximation error

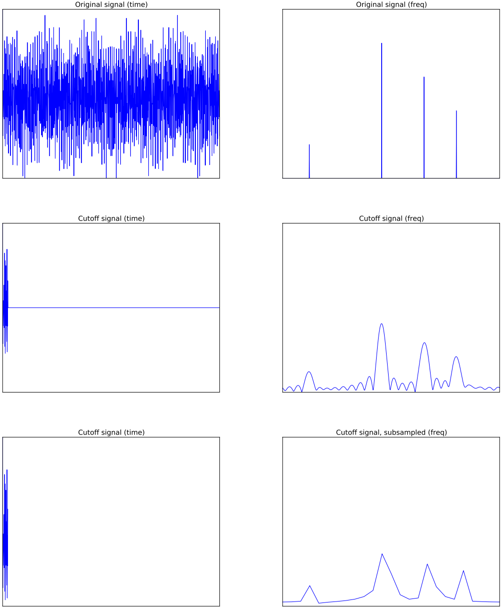
$$\|\widehat{x'} - \widehat{x}\|_\infty^2 \leq \frac{1}{k}\|\widehat{x} - \widehat{x}_k\|_2^2$$

- If $\widehat{x}$ is sparse, recover it exactly.
- Caveats:
  - Additional $\|x\|_2/n^{\Theta(1)}$ error.
  - n must be a power of 2.

Structure of this section

- If $\widehat{x}$ is k-sparse with known support S, find $\widehat{x}_S$ exactly in $O(k \log^2 n)$ time.
- In general, estimate $\widehat{x}$ approximately in $\widetilde{O}(\sqrt{nk})$ time.

Intuition

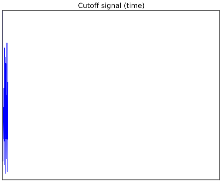


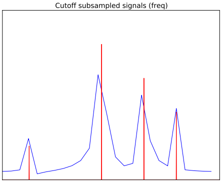
n-dimensional DFT

n-dimensional DFT
of first *B* terms.

B-dimensional DFT
of first *B* terms.

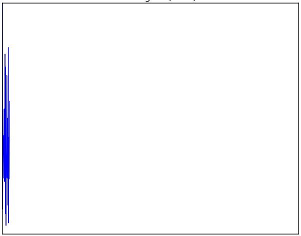
Framework

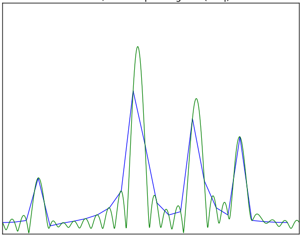




- "Hashes" into B buckets in $B \log B$ time.
- Issues:
  - "Hashing" needs a random hash function
    - ★ Access $x'_t = \omega^{-at} x_{\sigma t}$, so $\widehat{x'}_t = \widehat{x}_{\sigma^{-1} t + a}$ [GMS-05]
  - Collisions
    - ★ Have $B > 4k$, repeat $O(\log n)$ times and take median. [Count-Sketch, CCF02]
  - Leakage
  - Finding the support. [Porat-Strauss-12], talk at 9:45am.

Leakage



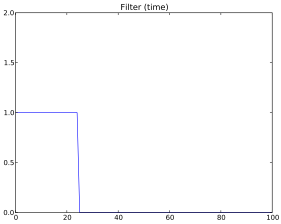


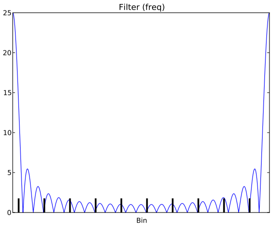
- Let $F_i = \begin{cases} 1 & i < B \\ 0 & \text{otherwise} \end{cases}$ be the "boxcar" filter. (Used in [GGIMS02,GMS05])
- Observe

$$\mathrm{DFT}(F \cdot x, B) = \mathrm{subsample}(\mathrm{DFT}(F \cdot x, n), B) = \mathrm{subsample}(\widehat{F} * \widehat{x}, B).$$

- DFT $\widehat{F}$ of boxcar filter is sinc, decays as $1/i$.
- Need a better filter F!

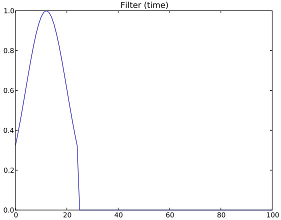
Filters

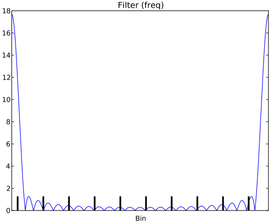




- Observe subsample($\widehat{F} * \widehat{x}, B$) in $O(B \log B)$ time.
- Needs for F:
  - $\text{supp}(F) \in [0, B]$
  - $|\widehat{F}| < \delta = 1/n^{\Theta(1)}$ except “near” 0.
  - $\widehat{F} \approx 1$ over $[-n/2B, n/2B]$.

Filters





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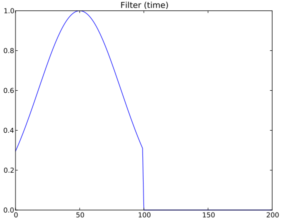
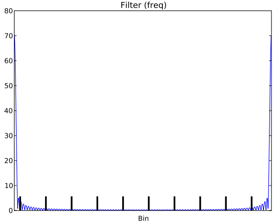
- Gaussians:
  - Standard deviation $\sigma = B/\sqrt{\log n}$
  - DFT has $\widehat{\sigma} = (n/B)\sqrt{\log n}$
  - Nontrivial leakage into $O(\log n)$ buckets.
  - But likely trivial contribution to correct bucket.

Filters





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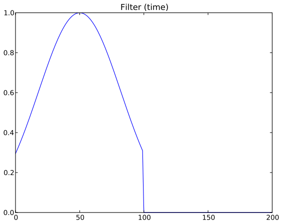
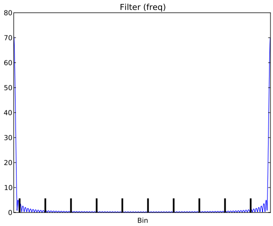
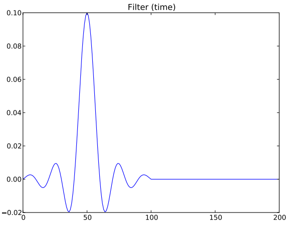
Filters

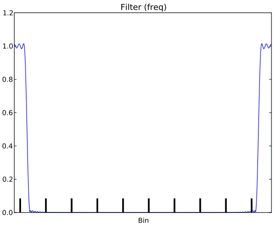




- Let G be Gaussian with $\sigma = B\sqrt{\log n}$
- H be box-car filter of length n/B.

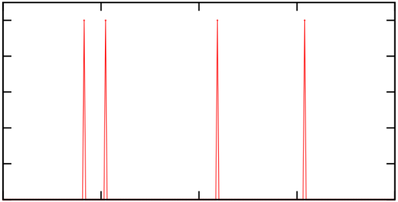
Filters





- Let G be Gaussian with $\sigma = B\sqrt{\log n}$
- H be box-car filter of length n/B.
- Use $\widehat{F} = \widehat{G} * H$.
  - $F = G \cdot \widehat{H}$, so $\text{supp}(F) \subset [0, B \log n]$.
  - $|\widehat{F}| < 1/n^{\Theta(1)}$ outside $-n/B, n/B$.
  - $|\widehat{F}| = 1 \pm 1/n^{\Theta(1)}$ within $n/2B, n/B$.
- Hashes correctly to one bucket, leaks to at most 1 bucket.
- Replace Gaussians with "Dolph-Chebyshev window functions": factor 2 improvement.

Algorithm to estimate $\widehat{x}_S$



- For $O(\log n)$ different permutations of $\widehat{x}$, compute subsample($\widehat{F} * \widehat{x}, B$).
- Estimate each x_i as median of values it maps to.

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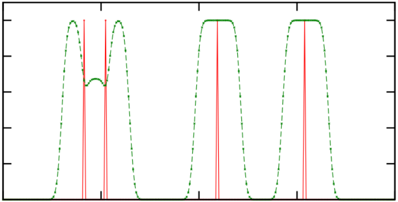

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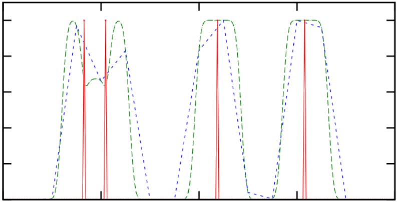

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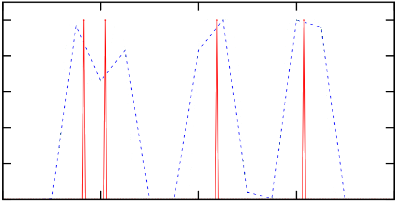

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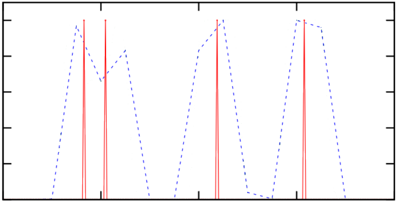
Algorithm in general



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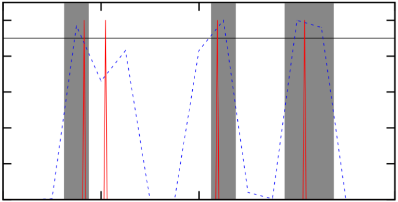

- For $O(\log n)$ different permutations of $\widehat{x}$, compute subsample$(\widehat{F} * \widehat{x}, B)$.
- Estimate each x_i as median of values it maps to.
- To find S: choose all that map to the top $2k$ values.

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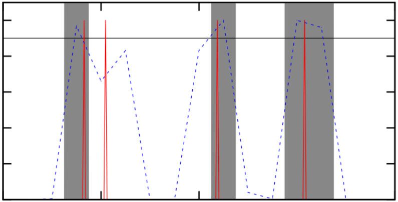

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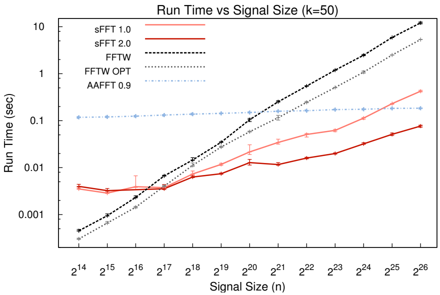
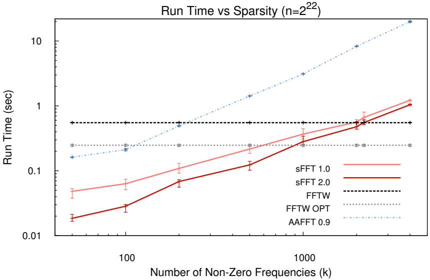
- Estimate each x_i as median of values it maps to.
- To find S: choose all that map to the top $2k$ values.
- nk/B candidates to update at each iteration: total

$$(\frac{nk}{B} + B \log n) \log n = \sqrt{nk} \log^{3/2} n$$

time.

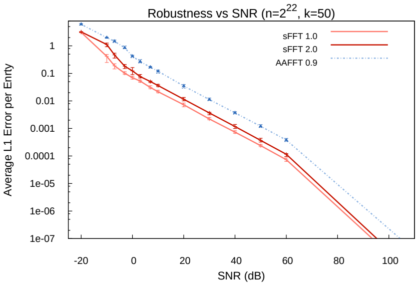
Empirical Performance: runtime





- Compare to FFTW, previous best sublinear algorithm (AAFFT).
- Offer a heuristic that improves time to $\widetilde{O}(n^{1/3}k^{2/3})$.
  - Filter from [Mansour '92].
  - Can't rerandomize, might miss elements.
- Faster than FFTW for $n/k > 2{,}000$.
- Faster than AAFFT for $n/k < 1{,}000{,}000$.

Empirical Performance: noise



- Just like in Count-Sketch, algorithm is noise tolerant.

Conclusions

- Roughly: fastest algorithm for $n/k \in [2 \times 10^3, 10^6]$.
- Recent improvements [HIKP12b?]
  - $O(k \log n)$ for exactly sparse $\widehat{x}$
  - $O(k \log \frac{n}{k} \log n)$ for approximation.
  - Beats FFTW for $n/k > 400$ (in the exact case).