New Constructions of RIP Matrices with Fast Multiplication and Fewer Rows

aka, **sparse** recovery from **Fourier**-like measurements with applications to **fast** Johnson-Lindenstrauss **transforms**, **etc.**

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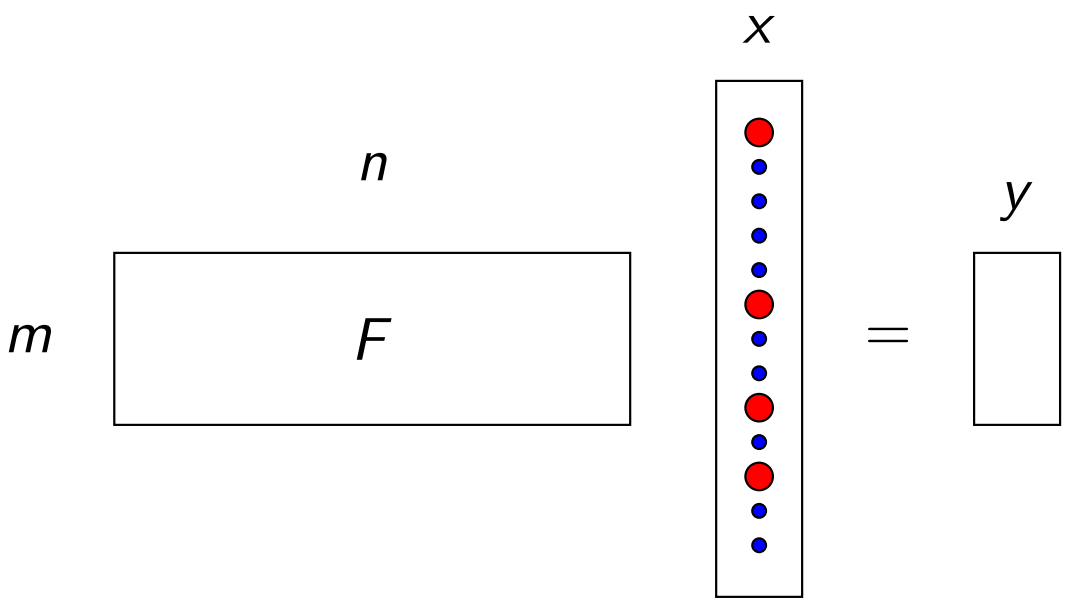
Compressed Sensing

Given: A few linear measurements of an (approximately) k-sparse vector $x \in \mathbb{R}^n$.
Goal: Recover x (approximately).



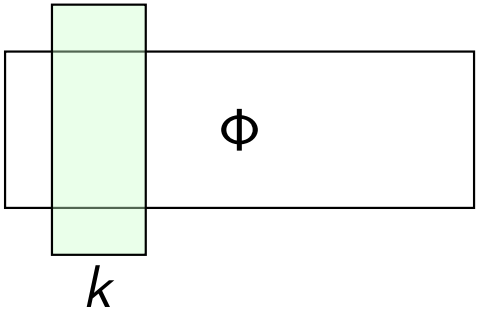
Algorithms for compressed sensing

- A lot of people use linear programming.
- Also Iterative Hard Thresholding, CoSaMP, OMP, StOMP, ROMP....
- For all of these:
  - the time it takes to multiply by Φ or Φ^* is the bottleneck.
  - the *Restricted Isometry Property* is a sufficient condition.

Restricted Isometry Property (RIP)

$$(1-\varepsilon)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1+\varepsilon)\|x\|_2^2$$

for all k-sparse $x \in \mathbb{R}^n$.



All of these submatrices are well conditioned.

Goal

Matrices Φ which have the RIP and support fast multiplication.

An open question

If the rows of Φ are random rows from a Fourier matrix, how many measurements do you need to ensure that Φ has the RIP?

- $m = O(k \log(n) \log^3(k))$ [CT06, RV08, CGV13].

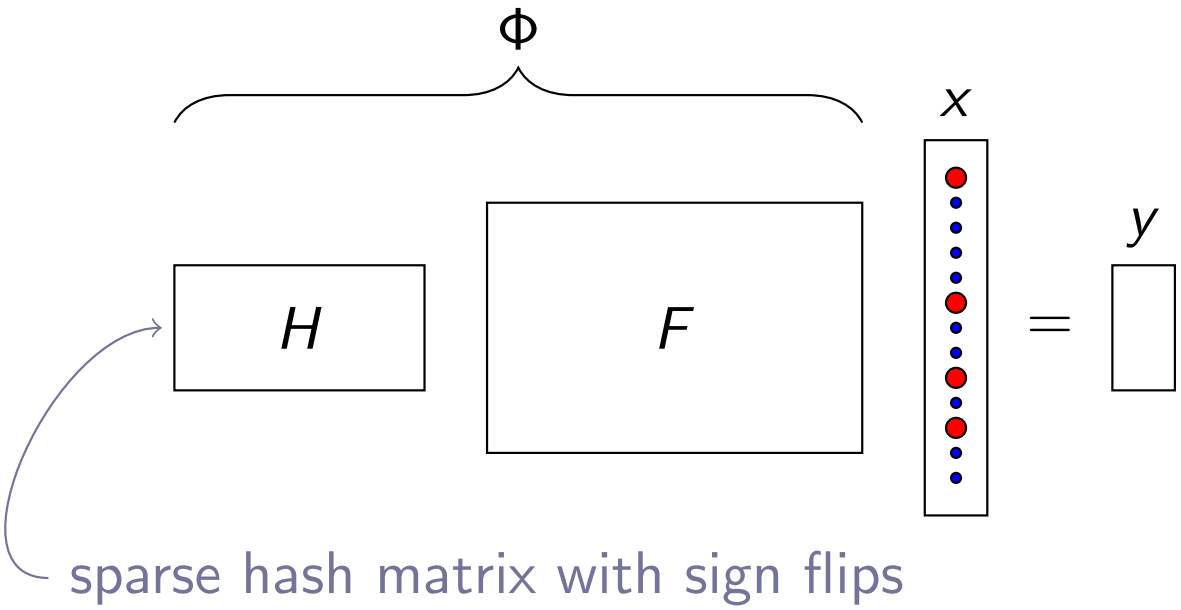
Ideal:

- $m = O(k \log(n/k))$.

(Related: how about partial circulant matrices?)

- $m = O(k \log^2(n) \log^2(k))$ [RRT12, KMR13].

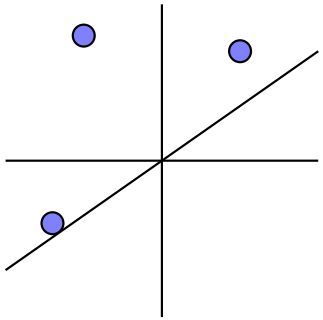
In this work



- Can still multiply by Φ quickly.
- Our result: has the RIP with

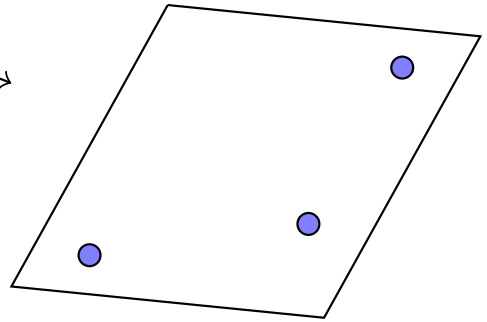
$$m = O(k \log(n) \log^2(k)).$$

Another motivation:
Johnson Lindenstrauss (JL) Transforms



High dimensional data
$S \subset \mathbb{R}^n$

Linear map, Φ



Φ preserves
the geometry of S

Low dimensional sketch
$\Phi(S) \in \mathbb{R}^m$

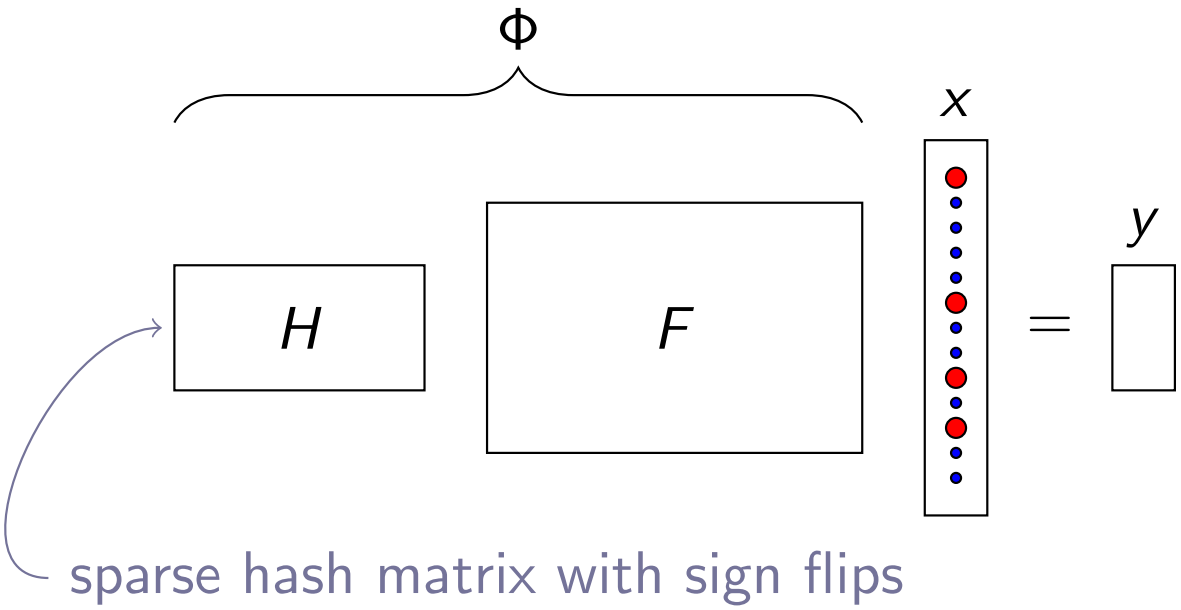
What do we want in a JL matrix?

- Target dimension should be small (like $\log(|S|)$).
- Fast multiplication.
  - Approximate numerical algebra problems (e.g., linear regression, low-rank approximation)
  - k-means clustering

How do we get a JL matrix?

- Gaussians will do.
- Best way known for *fast JL*: By [KW11], RIP $\Rightarrow$ JL.*
- So our result also gives fast JL transforms with the fewest rows known.

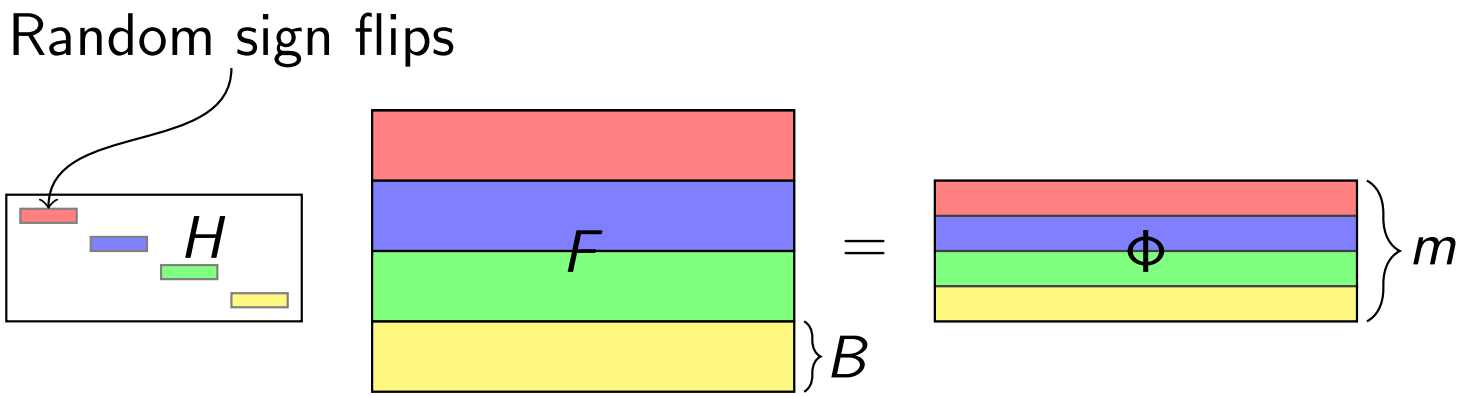
Our results



- Can still multiply by Φ quickly.
- Our result: has the RIP with

$$m = O(k \log(n) \log^2(k)).$$

More precisely



- If A has mB rows, then Φ has m rows.
- The "buckets" of H have size B.

Theorem

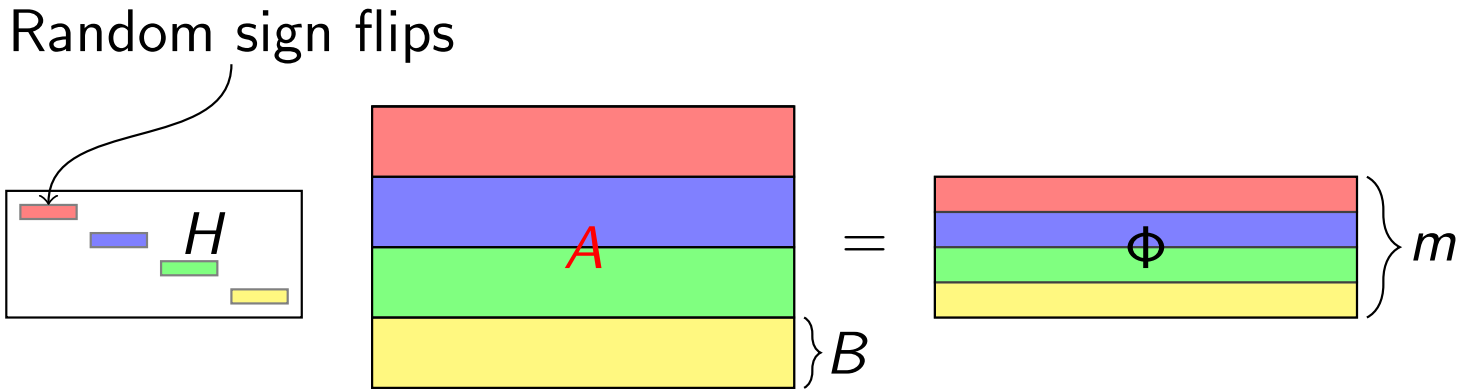
If $B \simeq \log^{2.5}(n)$, $m \simeq k \log(n) \log^2(k)$, and F is a random partial Fourier matrix, then Φ has the RIP with probability at least $2/3$.

Previous results

Construction	Measurements m	Multiplication Time	Notes
[AL09, AR13]	$\frac{k\log(n)}{\varepsilon^2}$	$n\log(n)$	as long as $k \leq n^{1/2-\delta}$
Sparse JL matrices [KN12]	$\frac{k\log(n)}{\varepsilon^2}$	εmn	
Partial Fourier [RV08, CGV13]	$\frac{k\log(n)\log^3(k)}{\varepsilon^2}$	$n\log(n)$	
Partial Circulant [KMR13]	$\frac{k\log^2(n)\log^2(k)}{\varepsilon^2}$	$n\log(m)$	
Hash / partial Fourier [NPW12]	$\frac{k\log(n)\log^2(k)}{\varepsilon^2}$	$n\log(n)$ + $m\text{polylog}(n)$	
Hash / partial circulant [NPW12]	$\frac{k\log(n)\log^2(k)}{\varepsilon^2}$	$n\log(m)$ + $m\text{polylog}(n)$	

Approach

Our approach is actually more general:



General result

If A is a "decent" RIP matrix:

- A has too many (mB) rows, but *does* have the RIP (whp).
- RIP-ness degrades gracefully as number of rows decreases.

Then Φ is a better RIP matrix:

- Φ has the RIP (whp) with fewer (m) rows.
- Time to multiply by Φ = time to multiply by $A + mB$.

Proof overview

We want

$$\mathbb{E} \sup_{x \in \Sigma_k} \left| \|\Phi x\|_2^2 - \|x\|_2^2 \right| < \varepsilon,$$

where Σ_k is unit-norm k-sparse vectors.

Proof overview I: triangle inequality

$$\begin{aligned}
&\mathbb{E} \sup_{x \in \Sigma_k} \left| \|\Phi x\|_2^2 - \|x\|^2 \right| \\
&\quad \leq \mathbb{E} \sup_{x \in \Sigma_k} \left| \|\Phi x\|_2^2 - \|Ax\|_2^2 \right| + \mathbb{E} \sup_{x \in \Sigma_k} \left| \|Ax\|_2^2 - \|x\|_2^2 \right| \\
&\quad \ldots \\
&\quad \leq \mathbb{E} \sup_{x \in \Sigma_k} \left| \|X_x \xi\|_2^2 - \mathbb{E}_\xi \|X_x \xi\|_2^2 \right| + (\text{RIP constant of } A),
\end{aligned}$$

where X_x is some matrix depending x and A, and ξ is the vector of random sign flips used in H.

Proof overview I: triangle inequality

$$\mathbb{E}\sup_{x\in\Sigma_k}\left|\|X_x\xi\|_2^2-\mathbb{E}_\xi\|X_A(x)\xi\|_2^2\right|+(\text{RIP constant of } A)$$

By assumption, this is small.
(Recall A has too many rows)

This is a *Rademacher Chaos Process.*
We have to do some work to show that it is small.

Proof overview II: probability and geometry

By [KMR13], it suffices to bound

$$\gamma_2(\Sigma_k, \|\cdot\|_A)$$

Some norm induced by A

Captures how "clustered" Σ_k is with respect to $\|\cdot\|_A$

We estimate this by bounding the *covering number* of Σ_k with respect to $\|\cdot\|_A$. $\square$

Open Questions

(1) How many random fourier measurements do you need for the RIP?

(2) Can you remove the other two log factors from our construction?

- It seems like doing this would remove two log factors from (1) as well.

(3) Can you come up with any ensemble of RIP matrices with $k \log(N/k)$ rows and fast multiplication?

(4) Can you come up with any ensemble JL matrices with $\log(|S|)$ rows supporting fast multiplication?

Thanks!

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