The Threshold for Super-resolution

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Limits to Resolution

\[ d = \frac{\lambda}{2n \sin \theta} \]

Lord Rayleigh (1842-1919)

numerical aperture

Ernst Abbe (1840-1905)
Limits to Resolution

In microscopy, it is difficult to observe sub-wavelength structures (Rayleigh Criterion, Abbe Limit, …)

\[ d = \frac{\lambda}{2n\sin \theta} \]

Lord Rayleigh (1842-1919)

Ernst Abbe (1840-1905)
Many devices are inherently low-pass:

**Super-resolution:** Can we recover fine-grained structure from coarse-grained measurements?
Many devices are inherently **low-pass**:

**Super-resolution**: Can we recover **fine**-grained structure from **coarse**-grained measurements?

Applications in medical imaging, microscopy, astronomy, radar detection, geophysics, ...
Many devices are inherently low-pass:

**Super-resolution:** Can we recover fine-grained structure from coarse-grained measurements?

Applications in medical imaging, microscopy, astronomy, radar detection, geophysics, …

2014 Nobel Prize in Chemistry!

Super-resolution Cameras

Eric Betzig, Stefan Hell, William Moerner
A Mathematical Framework [Donoho, ‘91]:

Super-position of k spikes, each $f_j$ in [0,1):

\[
\begin{align*}
&f_1 \\
&f_2 \\
&f_3 \\
&f_4
\end{align*}
\]
A Mathematical Framework [Donoho, ‘91]:

Super-position of k spikes, each $f_j$ in [0,1):

$$x(t) = \sum_{j=1}^{k} u_j \delta_{f_j}(t)$$

coefficient delta function at $f_j$
A Mathematical Framework [Donoho, ‘91]:

Super-position of $k$ spikes, each $f_j$ in $[0,1)$:

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Measurement at frequency $\omega$:

$$e^{i2\pi \omega t}$$
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$$x(t) = \sum_{j=1}^{k} u_j \delta_{f_j}(t)$$

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Measurement at frequency $\omega$:

$$v_\omega = \int_{0}^{1} e^{i2\pi \omega t} x(t) \, dt + \eta_\omega$$
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Measurement at frequency $\omega$:

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A Mathematical Framework [Donoho, ‘91]:

Super-position of $k$ spikes, each $f_j$ in $[0,1)$:

$$x(t) = \sum_{j=1}^{k} u_j \delta_{f_j}(t)$$

Measurement at frequency $\omega$, $|\omega| \leq m$:

$$v_\omega = \sum_{j=1}^{k} u_j e^{i2\pi f_j \omega} + \eta_\omega$$

cut-off frequency
Are there **algorithms** for enhancing resolution?
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When can we recover the coefficients \((u_j)'s\) and locations \((f_j)'s\) from low freq measurements?
Are there **algorithms** for enhancing resolution?

**Proposition 1:** When there is no noise ($\eta_\omega = 0$), there is a polynomial time algorithm to recover the $u_j$’s and $f_j$’s exactly with $m = k$ – i.e. measurements at $\omega = -m, -m+1, \ldots, m-1, m$
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[Proeny (1795), Pisarenko (1973), Matrix Pencil (1990), ...]
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**When can we recover the coefficients (u_j’s) and locations (f_j’s) from low freq measurements?**

What if there is noise?
Are there **algorithms** for enhancing resolution?

What if there is noise? Under what conditions is there an estimator $f_j$ and $u_j$ which converges at an inverse poly-rate (in $1/|\eta_\omega|$)?

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What if there is noise? Under what conditions is there an estimator

\[
\hat{f}_j \rightarrow f_j \quad \text{and} \quad \hat{u}_j \rightarrow u_j
\]

which converges at an inverse poly-rate (in $1/|\eta_\omega|$)?

And is there an algorithm?
Proposition 2 [M ‘14]: There is a polynomial time algorithm for noisy super-resolution if $m > 1/\Delta + 1$
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...where $d_w$ is the “wrap-around” distance:

$$d_w(1/8, 7/8) = 1/4$$

i.e. $d_w(1/8, 7/8) = 1/4$
Proposition 2 [M ‘14]: There is a polynomial time algorithm for noisy super-resolution if \( m > \frac{1}{\Delta} + 1 \)

...where \( d_w \) is the “wrap-around” distance:

\[
\Delta = \min_{i \neq j} d_w(f_i, f_j)
\]

i.e. \( d_w(1/8, 7/8) = 1/4 \)
Proposition 2 [M ‘14]: There is a polynomial time algorithm to recover estimates where

$$\min_{\text{matchings } \sigma} \max_j \left| \hat{f}_{\sigma(j)} - f_j \right| + \left| \hat{u}_{\sigma(j)} - u_j \right| \leq \varepsilon$$

provided $|\eta_\omega| \leq \text{poly}(\varepsilon, 1/m, 1/k)$, and $m > 1/\Delta + 1$

...where $d_w$ is the “wrap-around” distance:

$$\Delta = \min_{i \neq j} d_w(f_i, f_j)$$

i.e. $d_w(1/8, 7/8) = 1/4$
Proposition 3 [M ‘14]: For any $m \leq (1-\varepsilon)/\Delta$ and $k$, there is a pair of $\Delta$-separated signals $x$ and $\hat{x}$ where

$$\left| \sum_{j=1}^{k} u_j e^{i2\pi f_j \omega} - \sum_{j=1}^{k} \hat{u}_j e^{i2\pi f_j \omega} \right| \leq e^{-\varepsilon k}$$

for any $|\omega| \leq m$
Proposition 3 [M ‘14]: For any $m \leq (1-\varepsilon)/\Delta$ and $k$, there is a pair of $\Delta$-separated signals $x$ and $\hat{x}$ where
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for any $|\omega| \leq m$.
[Donoho, ’91]:
Asymptotic bounds for $m = 1/\Delta$, on a grid
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compressed sensing off-the-grid
[Donoho, ’91]:
Asymptotic bounds for $m = 1/\Delta$, on a grid
(Beurling’s balyage)

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compressed sensing
off-the-grid
Vandermonde Matrices

$$V_m^k = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \alpha_k \\
\alpha_1^2 & \alpha_2^2 & \alpha_k^2 \\
\vdots & \vdots & \vdots \\
\alpha_1^{m-1} & \alpha_2^{m-1} & \alpha_k^{m-1}
\end{bmatrix}$$

$$\alpha_j = e^{i2\pi f_j}$$
Vandermonde Matrices

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\end{bmatrix} \]

\[ \alpha_j \overset{\text{def}}{=} e^{i2\pi f_j} \]

This matrix plays a key role in many exact inverse problems (poly interpolation, sparse recovery, \ldots)
Matrix Pencil Method

Notation: \(D_u = \text{diag}\{u_j\}\) and \(D_\alpha = \text{diag}\{\alpha_j\}\)

\[
A = V_m^k D_u (V_m^k)^H \quad \text{and} \quad B = V_m^k D_\alpha D_u (V_m^k)^H
\]
Matrix Pencil Method

Claim 1: The entries of $A$ and $B$ correspond to $v_{\omega}$ with $-m+1 \leq \omega \leq m$

Notation: $D_u = \text{diag} \{u_j\}$ and $D_\alpha = \text{diag} \{\alpha_j\}$

$A = V_m^k D_u (V_m^k)^H$ and $B = V_m^k D_\alpha D_u (V_m^k)^H$
Matrix Pencil Method

Claim 1: The entries of A and B correspond to $v_\omega$ with $-m+1 \leq \omega \leq m$

Claim 2: If $\alpha_j$’s are distinct and $m \geq k$ and $u_j$’s are non-zero, the unique solns to

$$Ax = \lambda Bx$$

are $\lambda = 1/\alpha_j$

Notation: $D_u = \text{diag}(\{u_j\})$ and $D_\alpha = \text{diag}(\{\alpha_j\})$

$$A = V_m^k D_u (V_m^k)^H \text{ and } B = V_m^k D_\alpha D_u (V_m^k)^H$$
Vandermonde Matrices

\[ V^k_m = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \alpha_k \\
\alpha_1^2 & \alpha_2^2 & \alpha_k^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{m-1} & \alpha_2^{m-1} & \cdots & \alpha_k^{m-1}
\end{bmatrix} \]

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V^k_m = \begin{bmatrix}
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\]

This matrix plays a key role in many exact inverse problems (poly interpolation, sparse recovery, ...) and super-resolution.
Vandermonde Matrices

$$V^k_m = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_k \\
\alpha_1^2 & \alpha_2^2 & \cdots & \alpha_k^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{m-1} & \alpha_2^{m-1} & \cdots & \alpha_k^{m-1}
\end{bmatrix}$$

$$\alpha_j \overset{\text{def}}{=} e^{i2\pi f_j}$$

exact recovery $\leftrightarrow$ $V^k_m$ is full rank
Vandermonde Matrices

\[ V_{m}^{k} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{m-1} & \alpha_2^{m-1} & \cdots & \alpha_k^{m-1} \end{bmatrix} \]

\[ \alpha_j = e^{i2\pi f_j} \]

robust recovery \[\leftrightarrow\] \(V_{m}^{k}\) is well-conditioned
Vandermonde Matrices

\[ V_m^k = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_k \\
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\end{bmatrix} \]

\[ \alpha_j = e^{i2\pi f_j} \]

robust recovery \[\iff\] \(V_m^k\) is well-conditioned

We show a **phase transition** for its condition number
An Interlude

The Beurling-Selberg majorant:

$B(\omega)$

$\text{sgn}(\omega)$
An Interlude

The Beurling-Selberg majorant:

Properties: \( \text{sgn}(\omega) \leq B(\omega) \)
An Interlude

The Beurling-Selberg majorant:

Properties:

1. \( \text{sgn}(\omega) \leq B(\omega) \)
2. \( \hat{B}(x) \) supported in \([-1,1]\)
An Interlude

The **Beurling-Selberg majorant**: 

Properties:

1. $\text{sgn}(\omega) \leq B(\omega)$
2. $\hat{B}(x)$ supported in $[-1,1]$
3. $\int_{-\infty}^{\infty} B(\omega) - \text{sgn}(\omega) \, d\omega = 1$
An Interlude

The **Beurling-Selberg majorant**:  

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3. \( \int_{-\infty}^{\infty} B(\omega) - \text{sgn}(\omega) \, d\omega = 1 \)
An Interlude

The **Beurling-Selberg majorant**:

\[
\left(\frac{\text{sign}(\pi \omega)}{\pi}\right)^2 \left(\sum_{j = 1}^{\infty} (\omega - j)^{-2} - \sum_{j = -\infty}^{-1} (\omega - j)^{-2} + \frac{2}{\omega}\right)
\]

**Properties:**

1. \(\text{sgn}(\omega) \leq B(\omega)\)
2. \(\hat{B}(x)\) supported in \([-1,1]\)
3. \(\int_{-\infty}^{\infty} B(\omega) - \text{sgn}(\omega) \, d\omega = 1\)
An Interlude

The **Beurling-Selberg minorant**: 

\[ \text{sgn}(\omega) \]

\[ b(\omega) \]
An Interlude

The **Beurling-Selberg minorant**:

- **Properties:**
  1. $b(\omega) \leq \text{sgn}(\omega)$
  2. $b(x)$ supported in $[-1, 1]$
  3. $\int_{-\infty}^{\infty} \text{sgn}(\omega) - b(\omega) \, d\omega = 1$
Many applications in **analytic number theory**
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We will use them to bound $\kappa(V_m^k)$ ...
Many applications in **analytic number theory**

**Theorem 1:** There are functions $C_E(\omega)$ and $c_E(\omega)$ for $E = [0, m-1]$ that satisfy:

1. $c_E(\omega) \leq I_E(\omega) \leq C_E(\omega)$
2. $\hat{c}_E(x)$ and $\hat{C}_E(x)$ supported in $[-\Delta, \Delta]$
3. $\int_{-\infty}^{\infty} C_E(\omega) - I_E(\omega) \, d\omega = \int_{-\infty}^{\infty} I_E(\omega) - c_E(\omega) \, d\omega = 1/\Delta$
Many applications in **analytic number theory**

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**Theorem 1:** There are functions $C_E(\omega)$ and $c_E(\omega)$ for $E = [0,m-1]$ that satisfy:

1. $c_E(\omega) \leq I_E(\omega) \leq C_E(\omega)$

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3. $\int_{-\infty}^{\infty} C_E(\omega) - I_E(\omega) \, d\omega = \int_{-\infty}^{\infty} I_E(\omega) - c_E(\omega) \, d\omega = 1/\Delta$
Theorem 2: \( |V_m^k u|^2 = (m-1 \pm 1/\Delta) |u|^2 \)

Proof:
\[
|V_m^k u|^2 = \sum_{\omega = 0}^{m-1} |v_\omega|^2
\]
Theorem 2: $|V_m^k u|^2 = (m-1 \pm 1/\Delta) |u|^2$

Proof:

$$|V_m^k u|^2 = \sum_{\omega = 0}^{m-1} |v_\omega|^2$$

\(h(\omega)\) ... (Dirac comb)
Theorem 2: \(|V_m^k u|^2 = (m-1 \pm 1/\Delta) |u|^2\)

Proof:

\[
|V_m^k u|^2 = \sum_{\omega = 0}^{m-1} |v_\omega|^2
\]

Let \(h(\omega) = \sum_{t = -\infty}^{\infty} \delta_t(\omega)\) (Dirac comb)
Theorem 2: \[ |V_m^k u|^2 = (m-1 \pm 1/\Delta) |u|^2 \]

Proof:

\[ |V_m^k u|^2 = \sum_{\omega = 0}^{m-1} |v_\omega|^2 = \int_{-\infty}^{\infty} h(\omega) I_E(\omega) |v_\omega|^2 \, d\omega \]

Let \( h(\omega) = \sum_{t = -\infty}^{\infty} \delta_t(\omega) \)  

(Dirac comb)
Theorem 2: \(|V_m^k u|^2 = (m-1 \pm 1/\Delta) |u|^2\)

Proof:

\[ |V_m^k u|^2 = \sum_{\omega = 0}^{m-1} |v_\omega|^2 = \int_{-\infty}^{\infty} h(\omega) I_E(\omega) |v_\omega|^2 d\omega \]

\[ \leq \int_{-\infty}^{\infty} h(\omega) C_E(\omega) |v_\omega|^2 d\omega \]

Let \( h(\omega) = \sum_{t = -\infty}^{\infty} \delta_t(\omega) \) (Dirac comb)
Theorem 2: \(|V_m^k u|^2 = (m-1 \pm 1/\Delta) |u|^2\)

Proof:

\[
\int_{-\infty}^{\infty} h(\omega) C_E(\omega) |v_\omega|^2 \, d\omega =
\]

Let \(h(\omega) = \sum_{t = -\infty}^{\infty} \delta_t(\omega)\) (Dirac comb)
Theorem 2: \[|V_m^ku|^2 = (m-1 \pm 1/\Delta) |u|^2\]

Proof:

\[
\int_{-\infty}^{\infty} h(\omega) C_E(\omega) |v_\omega|^2 d\omega =
\sum_{k \geq 1} \sum_{k' \geq 1} u_j \overline{u}_{j'} \int_{-\infty}^{\infty} h(\omega) C_E(\omega) e^{i2\pi(f_j-f_{j'})}\omega d\omega
\]

Let \[h(\omega) = \sum_{t = -\infty}^{\infty} \delta_t(\omega)\] (Dirac comb)
Theorem 2: $|V_m^k u|^2 = (m-1 \pm 1/\Delta) |u|^2$

Proof:

Let $h(\omega) = \sum_{t=-\infty}^{\infty} \delta_t(\omega) = \sum_{t=-\infty}^{\infty} e^{i2\pi t \omega}$ (Dirac comb)

$$\int_{-\infty}^{\infty} h(\omega) C_E(\omega) |v_\omega|^2 \, d\omega = \sum_{j=1}^{k} \sum_{j'=1}^{k} u_j \overline{u}_{j'} \int_{-\infty}^{\infty} h(\omega) C_E(\omega) e^{i2\pi(f_j-f_{j'})\omega} \, d\omega$$
Theorem 2: \( |V_m^k u|^2 = (m-1 \pm 1/\Delta) |u|^2 \)

Proof:

\[
\int h(\omega) C_E(\omega) |v_\omega|^2 \, d\omega = \sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} \sum_{j'=1}^{\infty} u_j \overline{u}_{j'} \int_{-\infty}^{\infty} e^{i2\pi t \omega} C_E(\omega) e^{i2\pi(f_j-f_{j'}) \omega} d\omega
\]

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\sum_{j=1}^{k} \sum_{j'=1}^{k} \sum_{t=-\infty}^{\infty} u_j \overline{u}_{j'} \hat{C}_E(f_j-f_{j'}+t)
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Let \( h(\omega) = \sum_{t=-\infty}^{\infty} \delta_t(\omega) = \sum_{t=-\infty}^{\infty} e^{i2\pi t \omega} \) (Dirac comb)
Theorem 2: $|V_m^k u|^2 = (m-1 \pm 1/\Delta) |u|^2$

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Let $h(\omega) = \sum_{t = -\infty}^{\infty} \delta_t(\omega) = \sum_{t = -\infty}^{\infty} e^{i2\pi t \omega}$ (Dirac comb)

\[
\int_{-\infty}^{\infty} h(\omega) C_E(\omega) |v_\omega|^2 \, d\omega = \sum_{j=1}^{k} \sum_{j'=1}^{k} \sum_{t=-\infty}^{\infty} u_j \overline{u}_{j'} \hat{C}_E(f_j-f_{j'}+t)
\]

zero for cross-terms
Theorem 2: $|V_m^ku|^2 = (m-1 \pm 1/\Delta) |u|^2$

Proof:

$$\int_{-\infty}^{\infty} h(\omega) C_E(\omega) |v_\omega|^2 d\omega = \sum_{j=1}^{k} |u_j|^2 \hat{C}_E(0)$$

Let $h(\omega) = \sum_{t=-\infty}^{\infty} \delta_t(\omega) = \sum_{t=-\infty}^{\infty} e^{i2\pi t \omega}$ (Dirac comb)
Theorem 2: $|V_m^k u|^2 = (m-1 \pm 1/\Delta) |u|^2$

Proof:

Let $h(\omega) = \sum_{t=-\infty}^{\infty} \delta_t(\omega) = \sum_{t=-\infty}^{\infty} e^{i2\pi t \omega}$ (Dirac comb)

\[
\int_{-\infty}^{\infty} h(\omega) C_E(\omega) |v_\omega|^2 d\omega = \sum_{j=1}^{k} |u_j|^2 \hat{C}_E(0) = \sum_{j=1}^{k} (|E| + 1/\Delta) |u_j|^2
\]
Theorem 2: $|V_m^ku|^2 = (m-1 \pm 1/\Delta) |u|^2$

Proof:

Let $h(\omega) = \sum_{j=1}^{k} |u_j|^2 \hat{C}_E(0) = \sum_{j=1}^{k} (|E| + 1/\Delta) |u_j|^2$

Let $h(\omega) = \sum_{t=-\infty}^{\infty} \delta_t(\omega) = \sum_{t=-\infty}^{\infty} e^{i2\pi t\omega}$ (Dirac comb)
The threshold for super-resolution is $1/\Delta$
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And the condition number of the Vandermonde matrix has an identical phase transition
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And the condition number of the Vandermonde matrix has an identical phase transition

**Theme:** Test functions are used in harmonic analysis to prove various inequalities
The threshold for super-resolution is $1/\Delta$

And the condition number of the Vandermonde matrix has an identical phase transition

Theme: Test functions are used in harmonic analysis to prove various inequalities

These functions can be interpreted as preconditioners for $V_m^k$, and can yield faster, new algorithms…
Summary:

- Noisy super-resolution needs separation, and there is a sharp phase transition for when it is possible.
- Applications of Beurling-Selberg extremal functions in the analysis of algorithms.
- A new interpretation of test functions in harmonic analysis as preconditioners for the Vandermonde matrix.
- Can these tools be applied to compressed sensing off-the-grid? Other inverse problems?