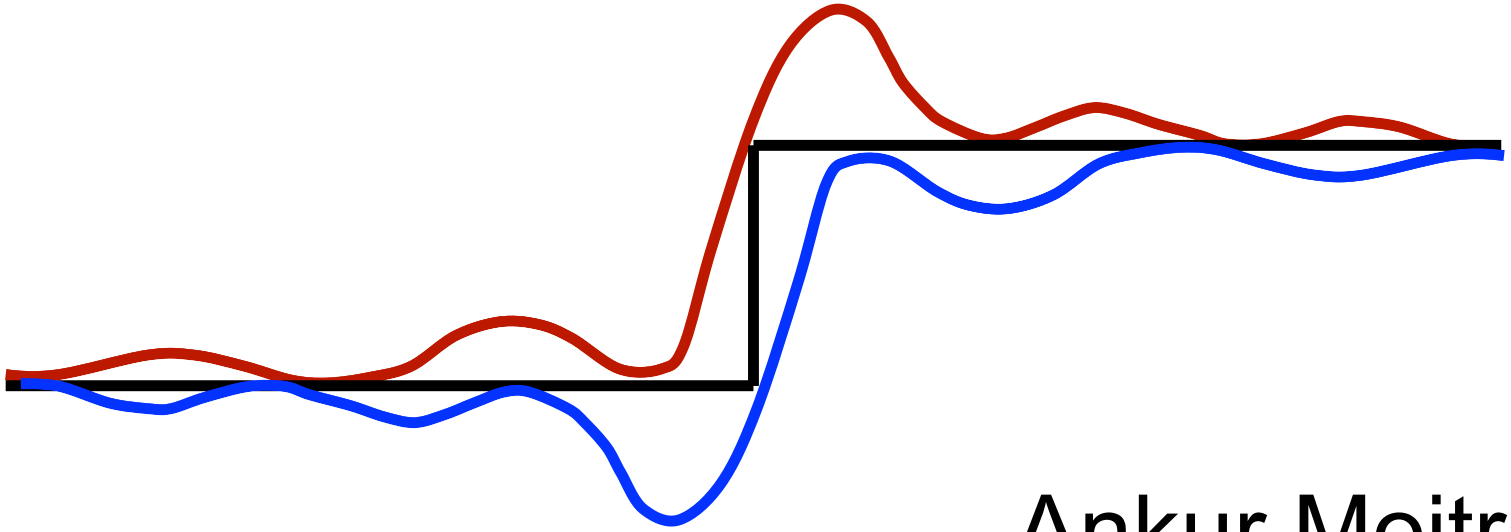
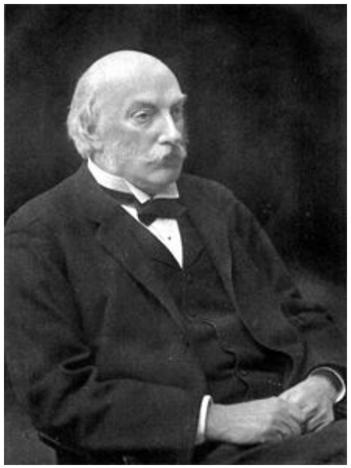
The Threshold for Super-resolution



Ankur Moitra

Massachusetts Institute of Technology

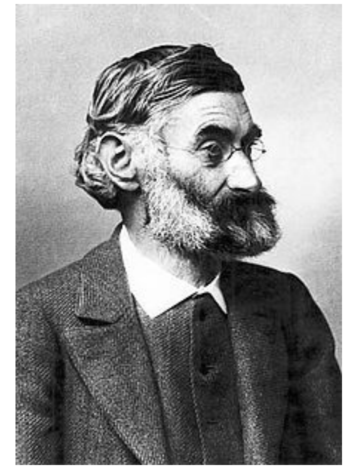
Limits to Resolution



Lord Rayleigh
(1842-1919)

$$d = \frac{\lambda}{2n\sin\theta}$$

numerical aperture



Ernst Abbe
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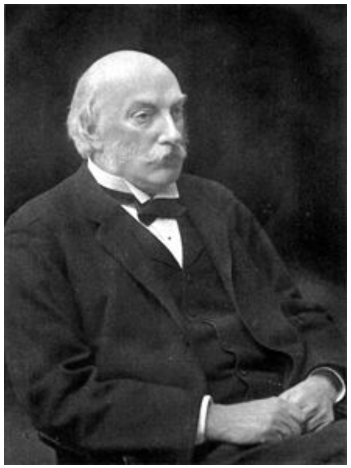

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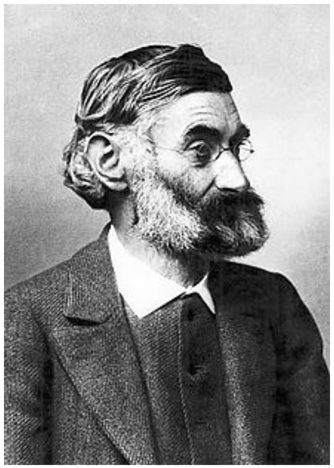
numerical aperture (= $n\sin\theta$)



Ernst Abbe
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In microscopy, it is difficult to observe sub-wavelength structures (**Rayleigh Criterion**, **Abbe Limit**, …)

Many devices are inherently **low-pass**:

Super-resolution: Can we recover **fine**-grained structure from **coarse**-grained measurements?

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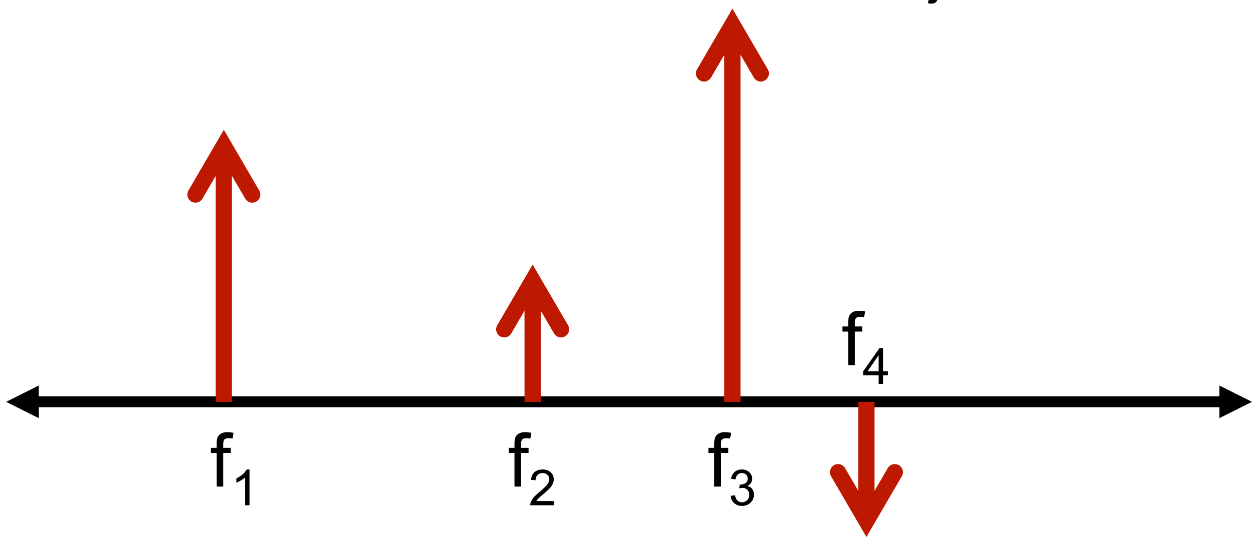
2014 Nobel Prize in Chemistry!

Super-resolution Cameras

Eric Betzig, Stefan Hell, William Moerner

A Mathematical Framework [Donoho, '91]:

Super-position of k spikes, each f_j in [0,1):



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Super-position of k spikes, each f_j in [0,1):

$$x(t) = \sum_{j=1}^{k} u_j \, \delta_{f_j}(t)$$

coefficient — delta function at f_j

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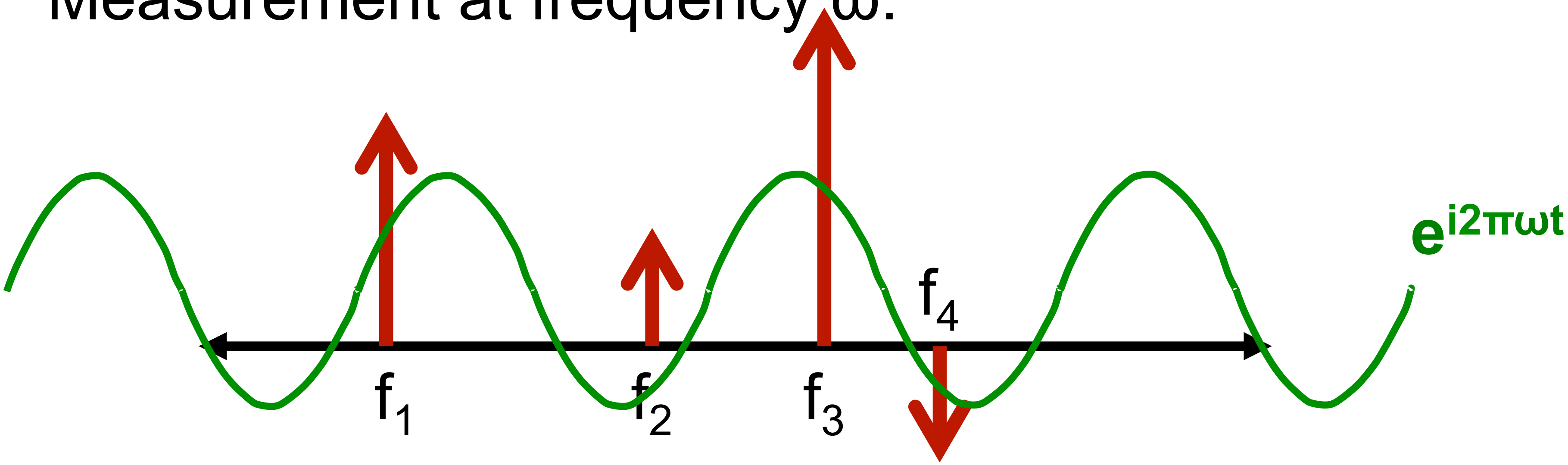

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noise

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cut-off frequency

Measurement at frequency ω, |ω| ≤ m:

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[Prony (1795), Pisarenko (1973), Matrix Pencil (1990), …]

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And is there an algorithm?

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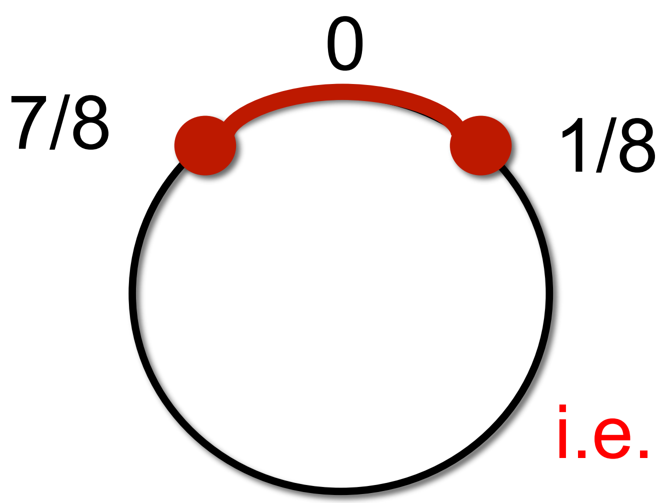

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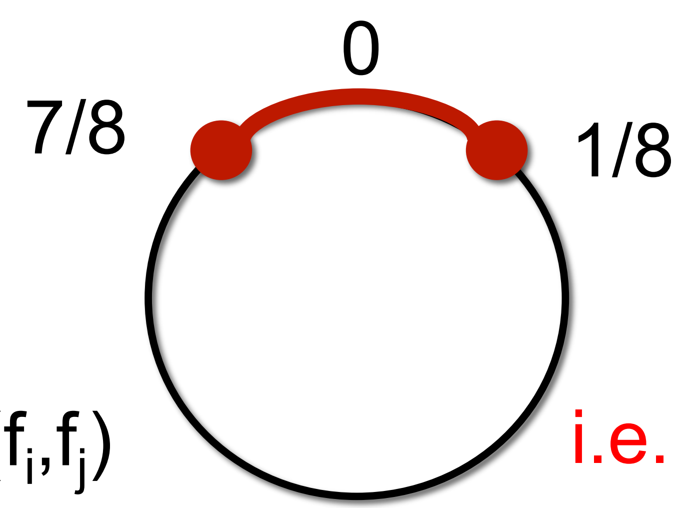

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Proposition 2 [M '14]: There is a polynomial time algorithm to recover estimates where

$$\min_{\text{matchings } \sigma} \max_{j} \left| \widehat{f}_{\sigma(j)} - f_j \right| + \left| \widehat{u}_{\sigma(j)} - u_j \right| \leq \varepsilon$$

provided $|\eta_\omega| \leq \text{poly}(\varepsilon, 1/m, 1/k)$, and $m > 1/\Delta + 1$

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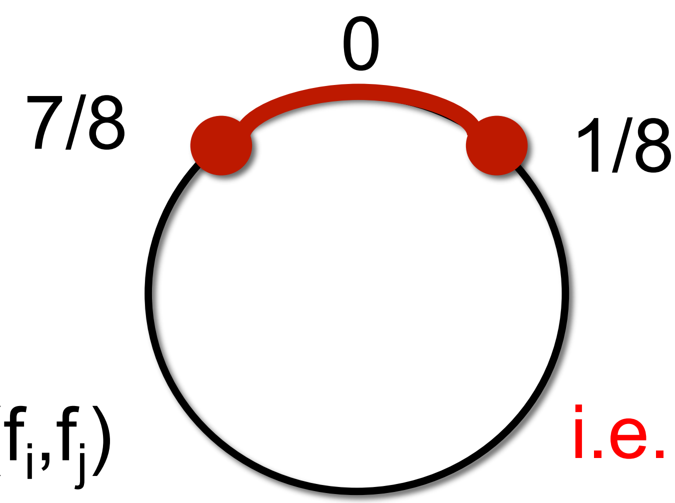

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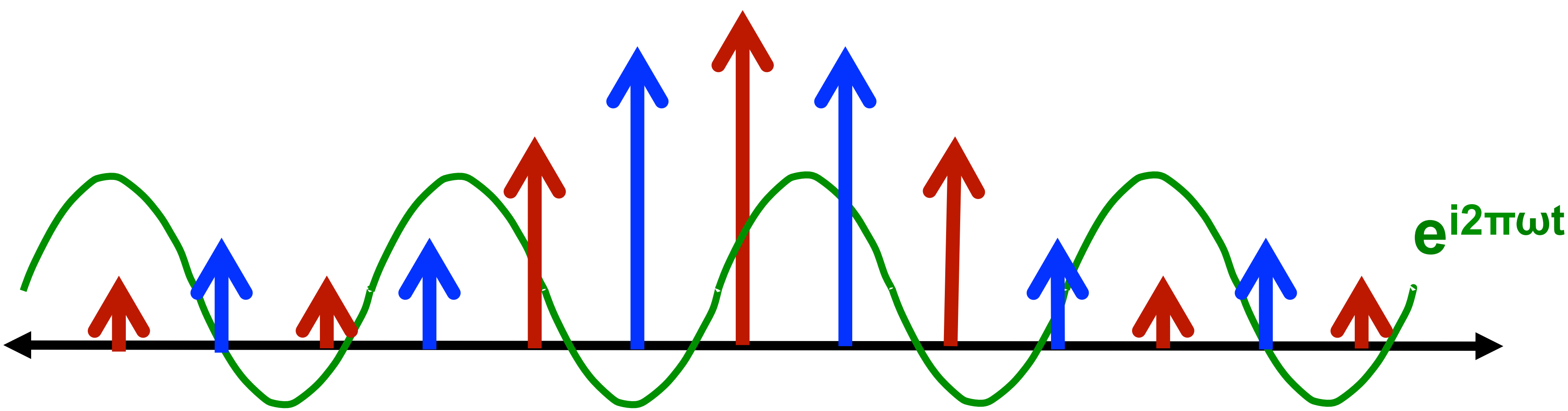

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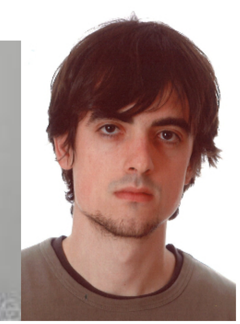

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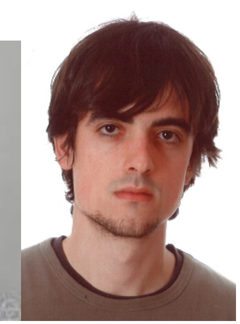

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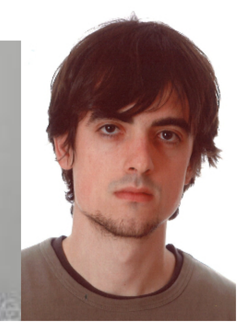

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Vandermonde Matrices

$$V_m^k = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & & \alpha_k \\ \alpha_1^2 & \alpha_2^2 & & \alpha_k^2 \\ \vdots & \vdots & & \vdots \\ \alpha_1^{m-1} & \alpha_2^{m-1} & \cdots & \alpha_k^{m-1} \end{bmatrix}$$

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This matrix plays a key role in many *exact* **inverse problems** (poly interpolation, sparse recovery, …)

Matrix Pencil Method

Notation: $D_u = \text{diag}(\{u_j\})$ and $D_\alpha = \text{diag}(\{\alpha_j\})$

$A = V_m^k D_u (V_m^k)^H$ and $B = V_m^k D_\alpha D_u (V_m^k)^H$

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Claim 2: If α_j's are distinct and $m \geq k$ and u_j's are non-zero, the unique solns to

$$Ax = \lambda Bx$$

are $\lambda = 1/\alpha_j$

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super-resolution

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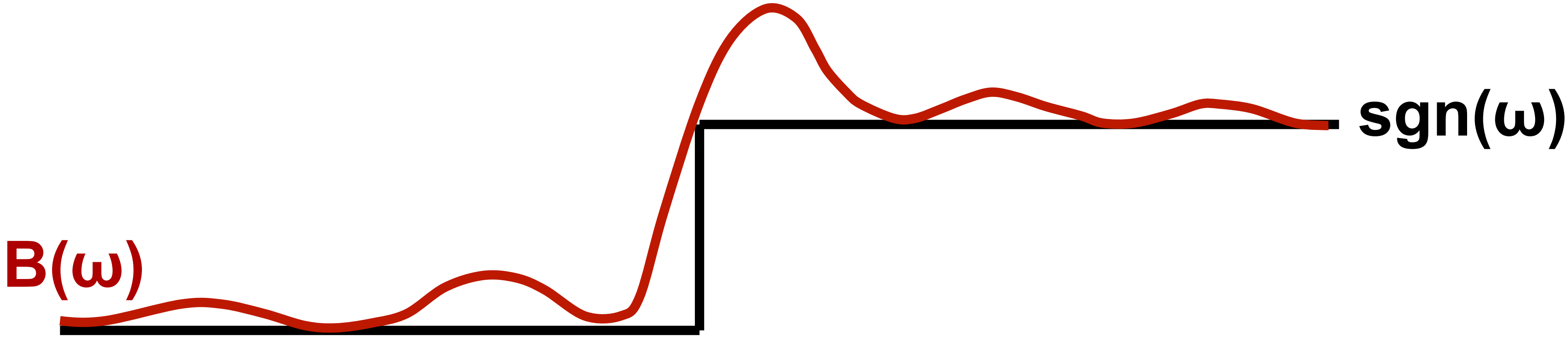
robust recovery ⟺ V_m^k is well-conditioned

We show a **phase transition** for its condition number

An Interlude

The **Beurling-Selberg majorant**:



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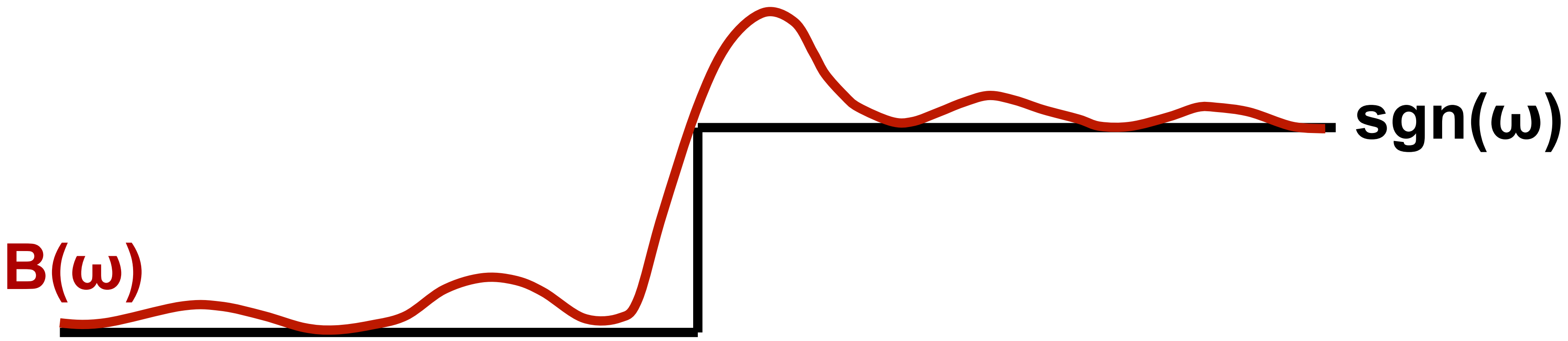

Properties: (1) $\mathrm{sgn}(\omega) \leq B(\omega)$

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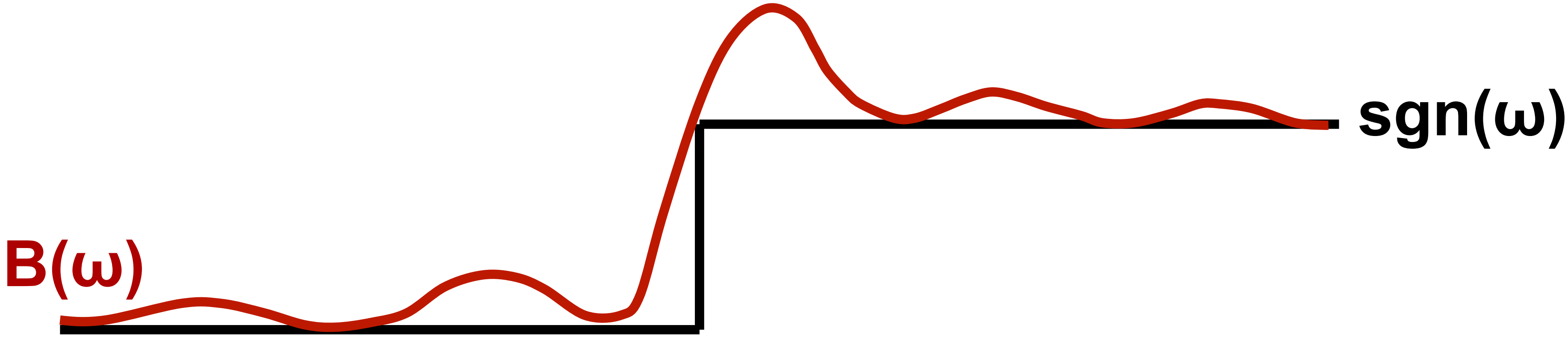

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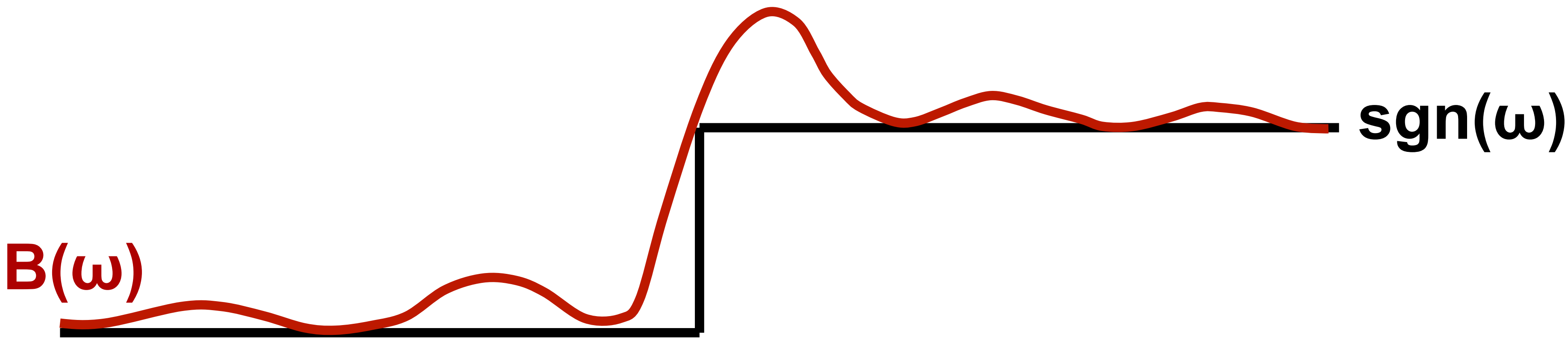

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$$\left(\frac{\operatorname{sign}(\pi\omega)}{\pi}\right)^2 \left(\sum_{j=1}^{\infty} (\omega - j)^{-2} - \sum_{j=-\infty}^{-1} (\omega - j)^{-2} + \frac{2}{\omega} \right)$$

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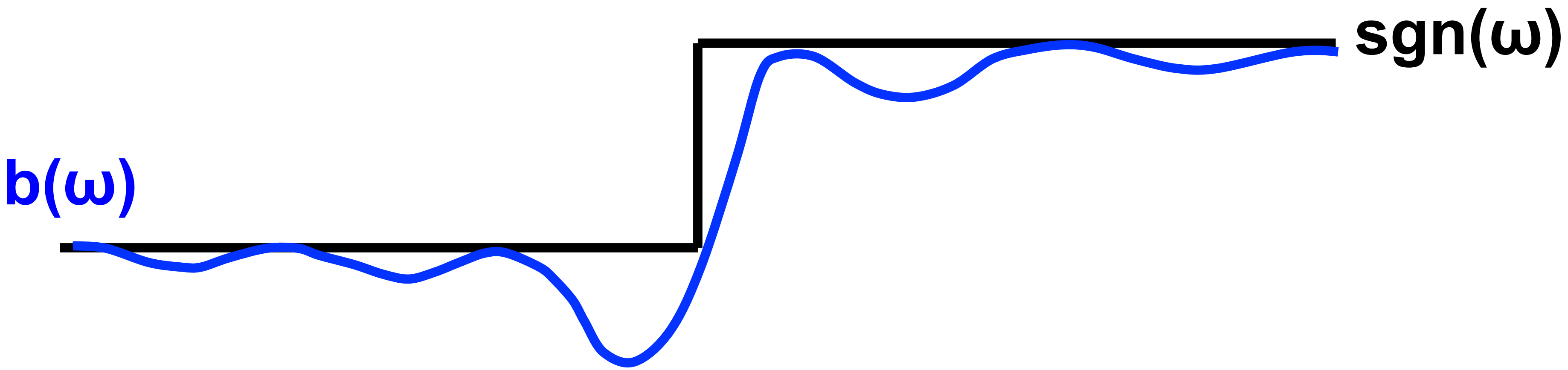

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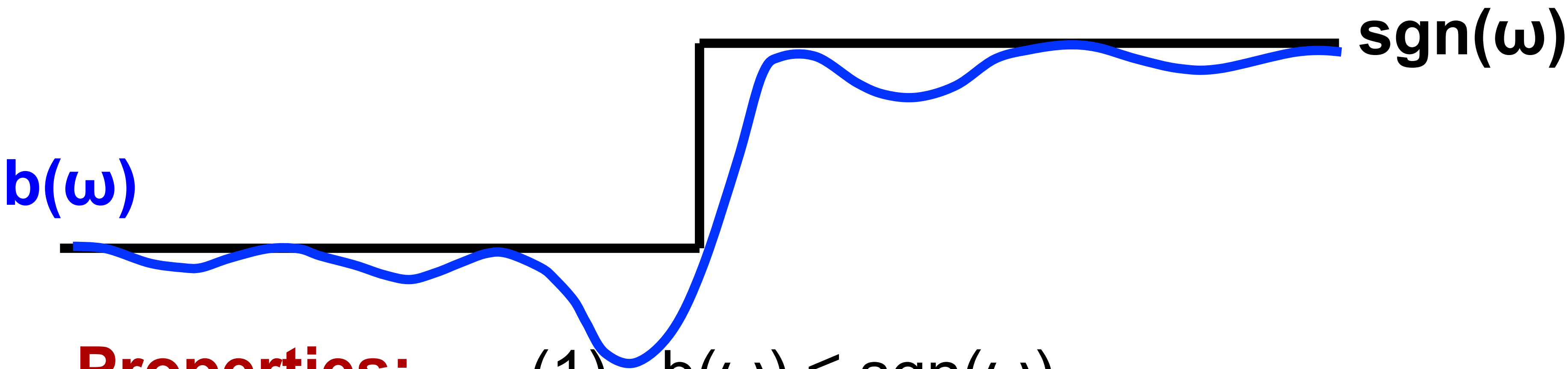

Properties:

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(1) $c_E(\omega) \leq I_E(\omega) \leq C_E(\omega)$

(2) $\hat{c}_E(x)$ and $\hat{C}_E(x)$ supported in $[-\Delta,\Delta]$

(3) $\int_{-\infty}^{\infty} C_E(\omega) - I_E(\omega)\, d\omega = \int_{-\infty}^{\infty} I_E(\omega) - c_E(\omega)\, d\omega = 1/\Delta$

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Proof:

$$|V_m^k u|^2 = \sum_{\omega=0}^{m-1} |v_\omega|^2$$

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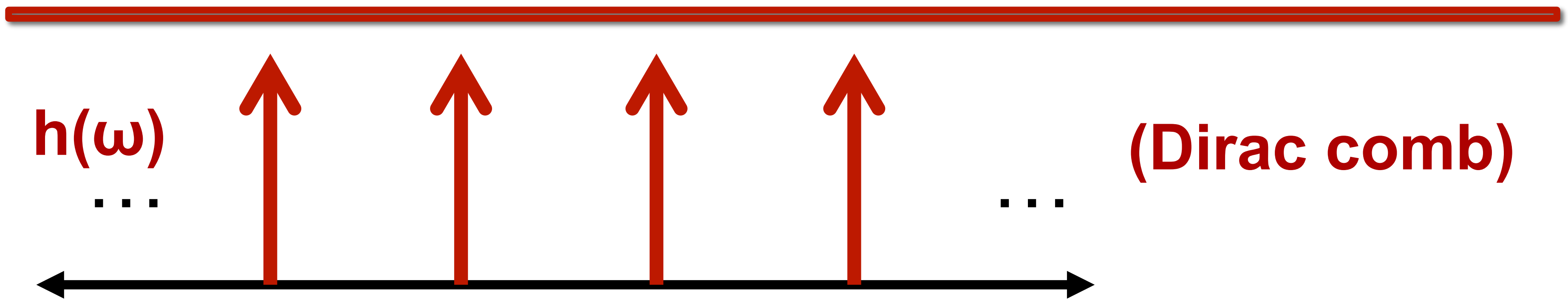

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$$\sum_{j=1}^{k} \sum_{j'=1}^{k} u_j\ \overline{u}_{j'} \int_{-\infty}^{\infty} h(\omega)\ C_E(\omega)\ e^{i2\pi(f_j - f_{j'})\omega} d\omega$$

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Proof:

$$\int_{-\infty}^{\infty} h(\omega)\ C_E(\omega)\ |v_\omega|^2\ d\omega\ =$$

$$\sum_{j=1}^{k} \sum_{j'=1}^{k} u_j\ \overline{u}_{j'} \int_{-\infty}^{\infty} h(\omega)\ C_E(\omega)\ e^{i2\pi(f_j - f_{j'})\omega} d\omega$$

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zero for cross-terms

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These functions can be interpreted as preconditioners for V_m^k, and can yield **faster**, new algorithms…

Thanks!

Summary:

- Noisy super-resolution needs separation, and there is a sharp phase transition for when it is possible
- Applications of **Beurling-Selberg extremal functions** in the analysis of algorithms
- A new interpretation of test functions in harmonic analysis as **preconditioners** for the Vandermonde matrix
- Can these tools be applied to compressed sensing off-the-grid? Other inverse problems?