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Cambridge, Massachusetts

Fourier Methods in Array Processing

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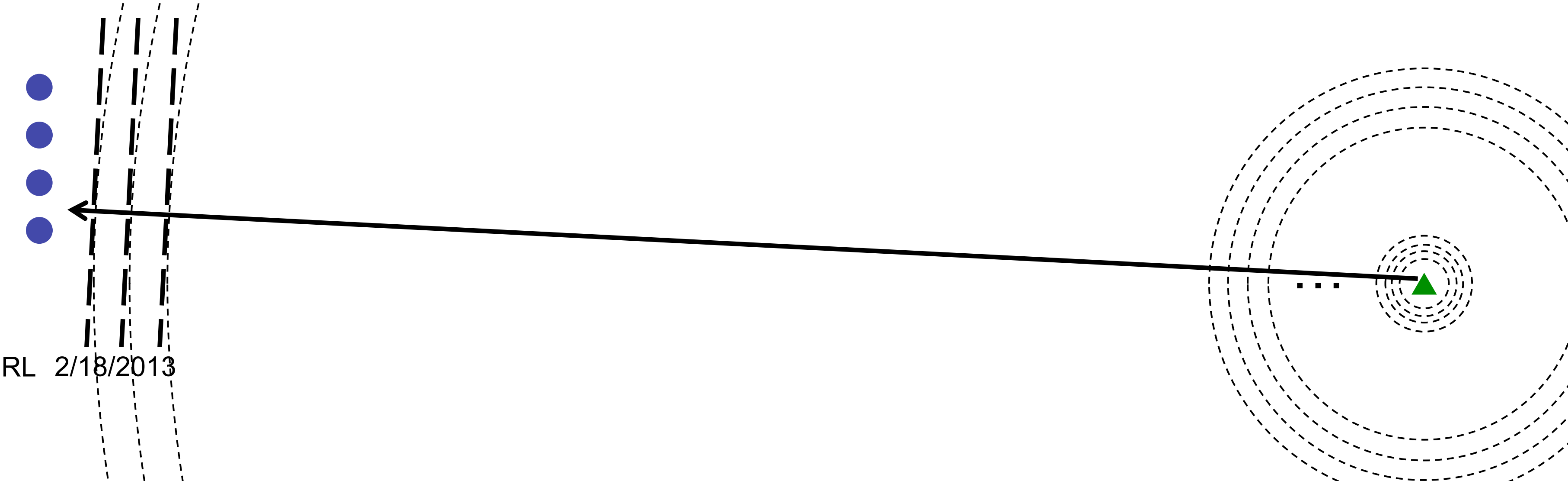
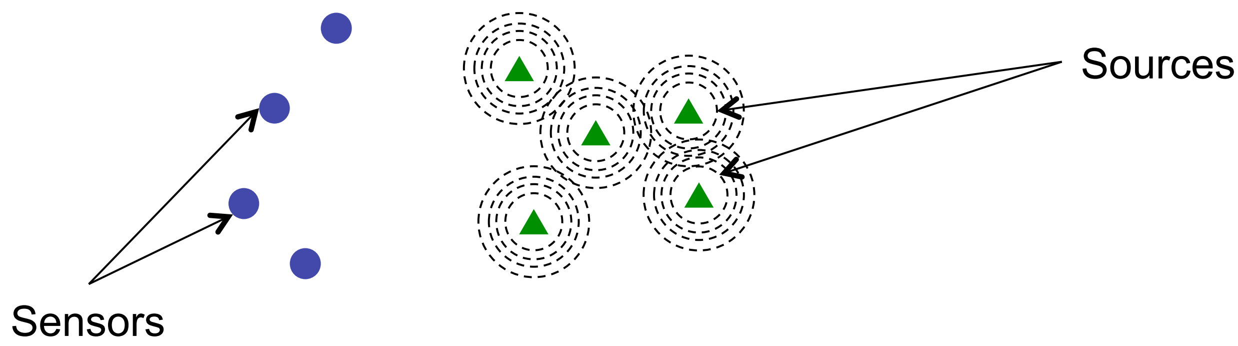


Array Processing Problem



A number of sensors are sensing a scene.

A number of sources transmit in a scene.

Can we localize and reconstruct sources in the scene?

This talk: overview of basic models and methods

(Linearized) Wave Propagation

Distance: $d_{k,m}$

Sensor (array element) m

Propagation delay: $\tau_{k,m} = d_{k,m}/c$

Source k

Signal delayed according to distance and speed of wave propagation in medium

$$y_m(t) = x_k(t - \tau_{k,m})$$

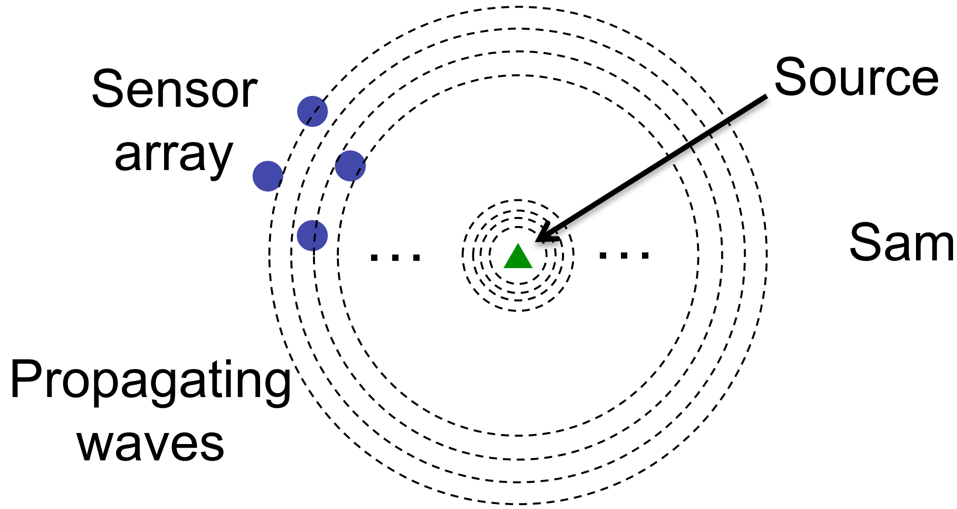
$$\Longleftrightarrow Y_m(\omega) = e^{-i\omega\tau_{k,m}} X_k(\omega)$$

Superposition: Signal at receiver sum of all transmitted signals

Narrowband approximation: Phase delay same for all ω

Free space assumption: No secondary reflections

Wave Propagation, Far-field Approximation



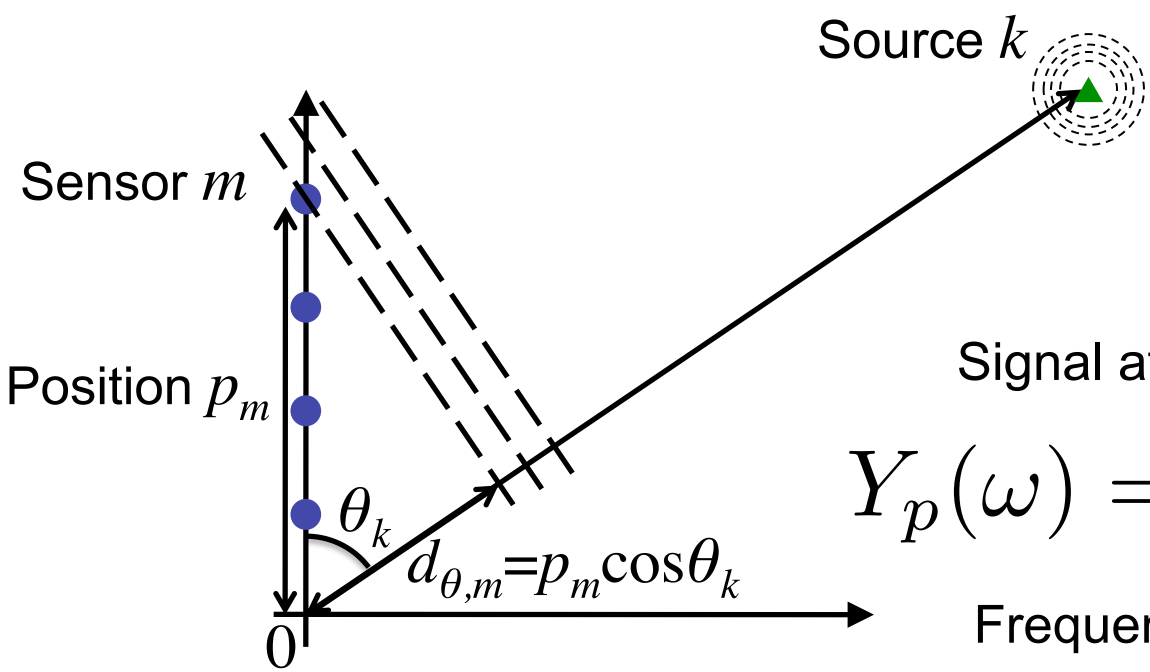
Propagating waves are circular:
Same delay for same distance from source

Far field approximation

Sources located **far relative to array size**

Propagating waves become **flat (planar)**

Linear Array, Far Field Approximation



Delay to origin unknown:
$T=D/c$

Signal at origin: $X_k(\omega)$

Signal at position p due to source k:

$$Y_p(\omega) = e^{-i\omega\frac{p\cos\theta_k}{c}} X_k(\omega)$$

Frequency: $\omega=2\pi f$, Wavelength: $\lambda=c/f$

Total signal at position p

$$Y_p(\omega) = \sum_k e^{i2\pi\frac{p}{\lambda}\cos\theta_k} X_k(\omega)$$

Linear Array, Far Field Approximation

$$Y_p(\omega) = \sum_k e^{i2\pi \frac{p}{\lambda} \cos\theta_k} X_k(\omega)$$

Drop ω from notation

Substitute variable $u_k = \cos\theta_k$

We get a (spatial, inverse) Fourier transform!

$$Y_p = \sum_k e^{i2\pi \frac{p}{\lambda} u_k} X_k$$

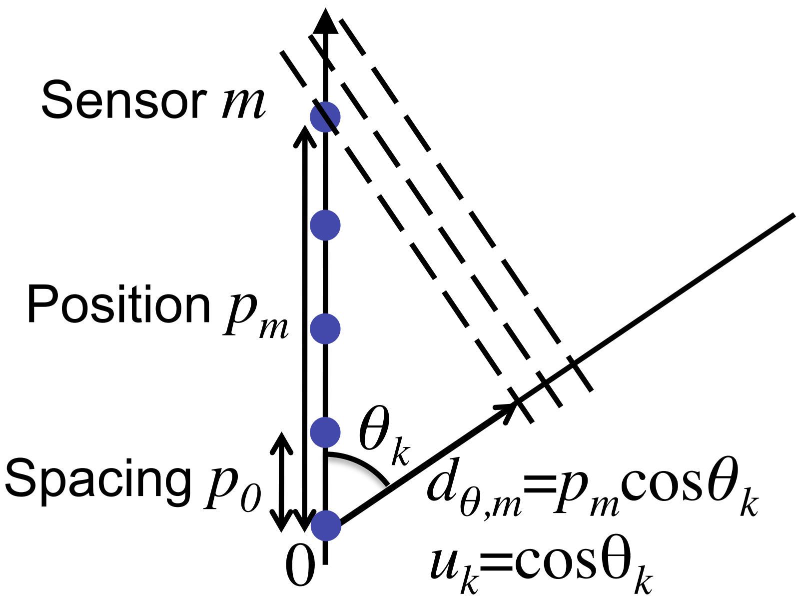
Received Signal

Sampling point In wavelengths

"Source Frequency" (cosine of angle)

Source Signal

Discretizing the (inverse) Fourier Transform



$$Y_p = \sum_k e^{i2\pi \frac{p}{\lambda} u_k} X_k$$

Uniform linear array:

$p_m = m p_0,\ m=1,\ldots,M$

Set $u \in [-1,1]$ on a grid

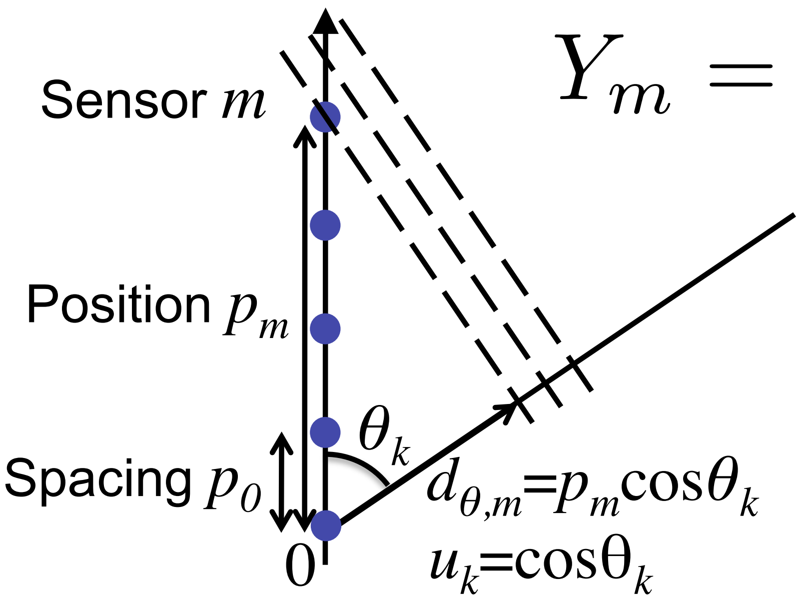
$u_n = -1 + 2n/N$

$$Y_m = \sum_n e^{i2\pi \frac{m p_0}{\lambda}\left(-1+\frac{2n}{N}\right)} X_n$$

K-Sparse

$$= e^{-i2\pi \frac{m p_0}{\lambda}} \sum_n e^{i\frac{2\pi m n}{N}\frac{2p_0}{\lambda}} X_n$$

Discretizing the (inverse) Fourier Transform



$$Y_m = e^{-i2\pi\frac{mp_0}{\lambda}} \sum_n e^{i\frac{2\pi mn}{N}\frac{2p_0}{\lambda}} X_n$$

Set this to one.
We get the DFT!

Half wavelength spacing: $P_m=\lambda/2$
$M=N$ array elements.

For other spacing p_0, use DFT manipulations:
Zero padding and aliasing (folding)

$$\mathbf{Y=FX}$$

Inversion Problem

$$\mathbf{Y{=}FX}$$

Inversion Problem: What $\mathbf{X}$ generated $\mathbf{Y}$?

Classical approach:

$$\widehat{\mathbf{X}} = \mathbf{F}^{\dagger}\mathbf{Y} = \mathbf{F}^{H}\mathbf{Y}$$

Common names: Beamforming, Backprojection, Matched Filter

Main design issue:

Given target at certain angle,
what does the inversion look like?

Inversion Problem

$$\mathbf{Y}=\mathbf{F}\mathbf{X}$$

Inversion Problem: What $\mathbf{X}$ generated $\mathbf{Y}$?

(possible) Sparse approach:

$$\widehat{\mathbf{X}} = \arg\min_{\mathbf{X}} \|\mathbf{Y} - \mathbf{F}\mathbf{X}\|_2 \text{ s.t. } \|\mathbf{X}\|_0 \leq K$$

Also uses $\mathbf{F}^H\mathbf{Y}$ for most algorithms: coherence is important

Main design issue:

Given target at certain angle,
what does the coherence look like?

Beampattern/coherence

Given target at certain angle, what does inversion look like?

$$\mathbf{F} = \begin{bmatrix} e^{j\frac{2\pi}{N}\frac{2p_0}{\lambda}} & \cdots & e^{j\frac{2\pi n}{N}\frac{2p_0}{\lambda}} & \cdots & e^{j\frac{2\pi N}{N}\frac{2p_0}{\lambda}} \\ \vdots & \ddots & \vdots & & \\ e^{j\frac{2\pi m}{N}\frac{2p_0}{\lambda}} & \cdots & e^{j\frac{2\pi mn}{N}\frac{2p_0}{\lambda}} & \cdots & e^{j\frac{2\pi mN}{N}\frac{2p_0}{\lambda}} \\ \vdots & \ddots & \vdots & & \\ e^{j\frac{2\pi M}{N}\frac{2p_0}{\lambda}} & \cdots & e^{j\frac{2\pi Mn}{N}\frac{2p_0}{\lambda}} & \cdots & e^{j\frac{2\pi MN}{N}\frac{2p_0}{\lambda}} \end{bmatrix}$$

$$\widehat{\mathbf{X}} = \mathbf{F}^{\dagger}\mathbf{Y} = \mathbf{F}^{H}\mathbf{Y}$$

Narrow beampattern
Grating lobes

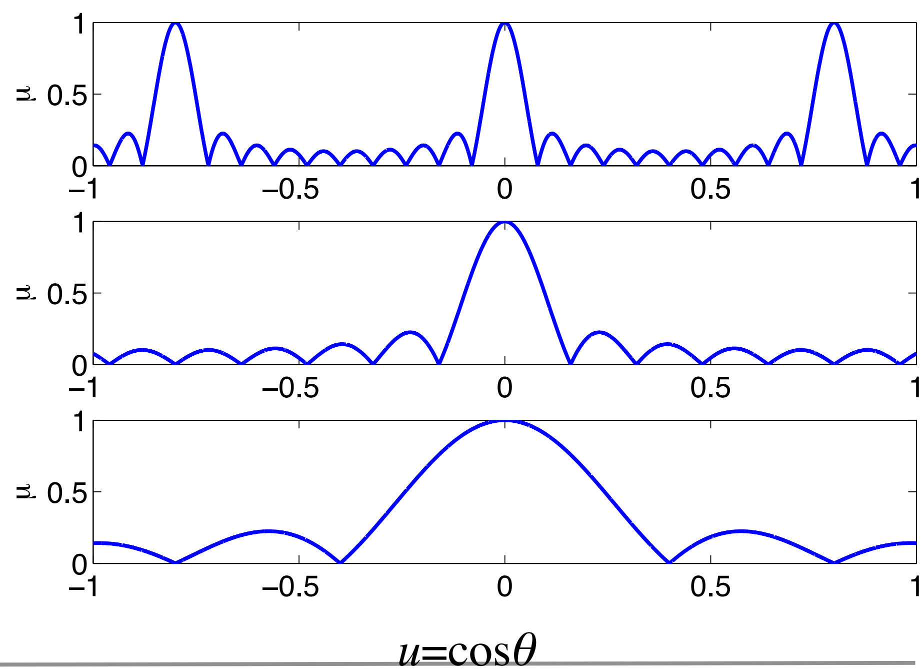
$p_0 > \lambda/2$

Wider beampattern
No grating lobes

$p_0 = \lambda/2$

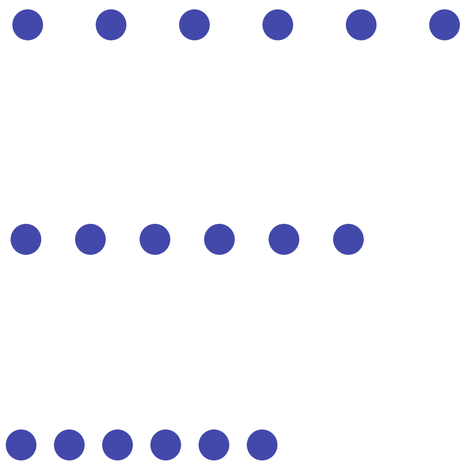
Too wide beampattern
No grating lobes

$p_0 < \lambda/2$



Beampattern/coherence

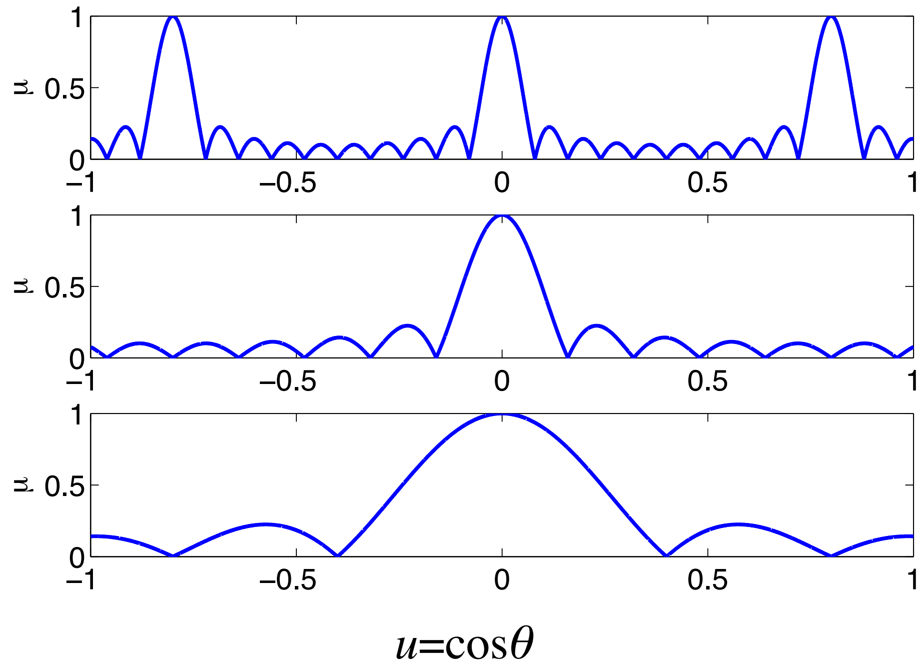
Given target at certain angle, what does inversion look like?



$p_0 > \lambda/2$

$p_0 = \lambda/2$

$p_0 < \lambda/2$

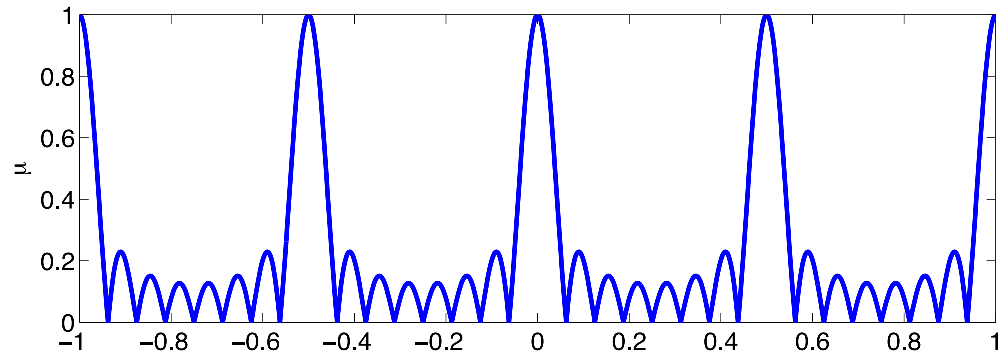


Larger aperture ⇒ Narrower main lobe
Large element spacing ⇒ Grating lobes

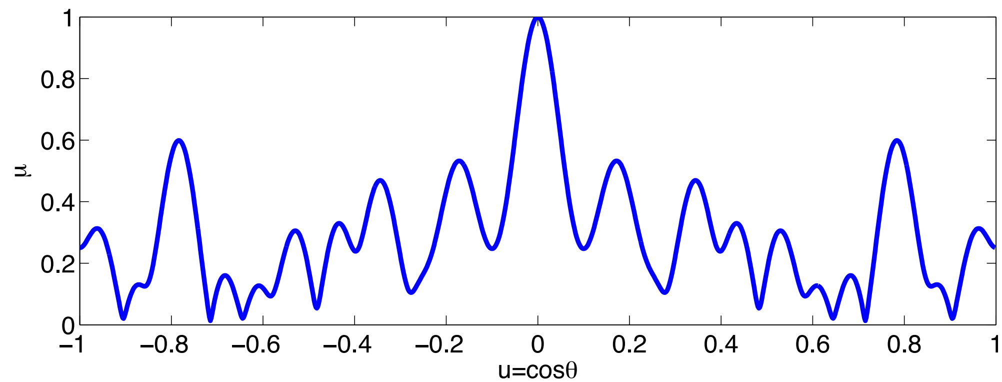
Narrow main lobe, no grating lobes ⇒ Many array elements?

Random Element Spacing

$$p_0 = 4\lambda/2$$



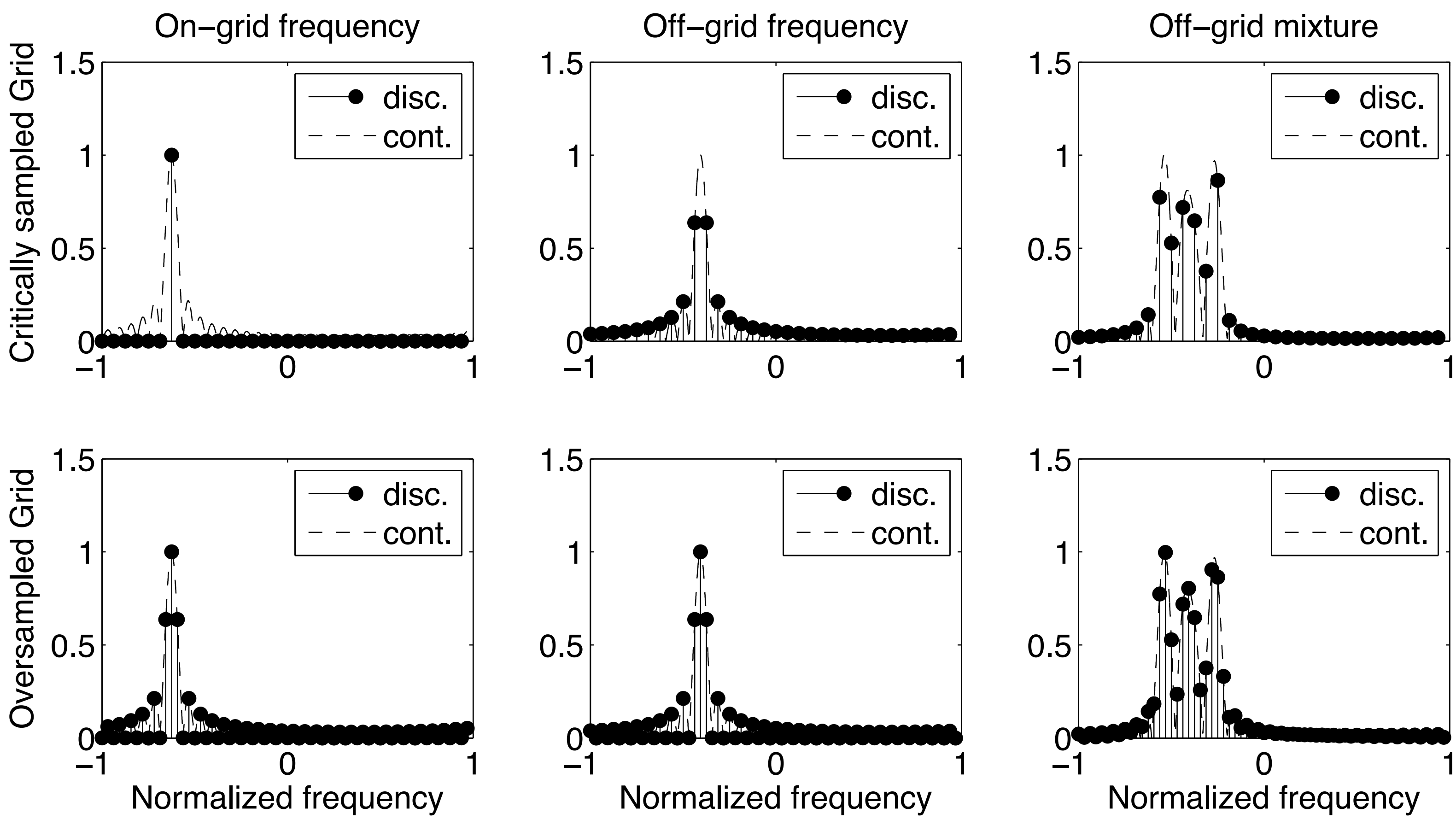
$$\bar{p} = 4\lambda/2$$



Solves the grating lobes problem!

Remaining Problem: Grid!

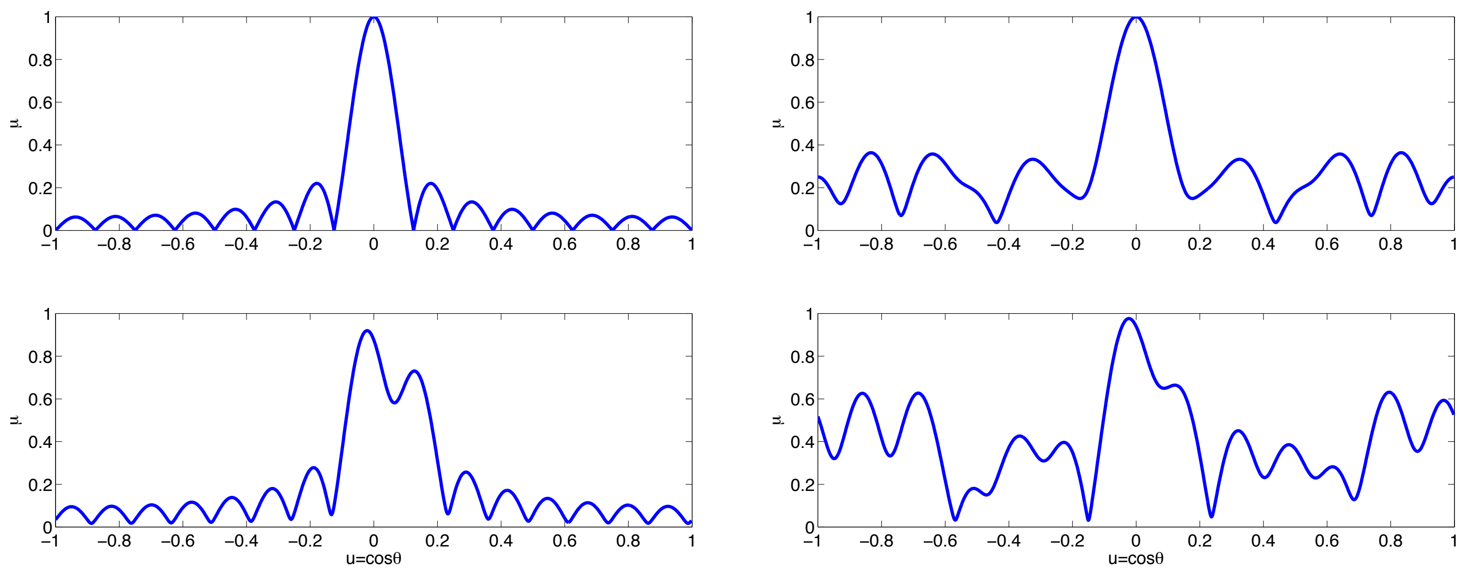
All this analysis has an implied angle (frequency) grid…



But sources are not always on the grid!!!

Solution (?) Make grid very fine

- Actual source closer to a grid point, “leakage” is smaller.
- Big problem: Computational complexity
  - Application of F is O(NlogN)
  - Sparse FFT could (maybe) help
- Bigger problem: Coherence!!!



Solution: Previous talk

- Off-the grid sampling
  - Previous talk (Yi Li)
  - Goal: identify *continuous* frequency components
  - Look ma no grid!

- Advantages:
  - Very efficient
  - No grid
  - Nice guarantees (robustness)

- Did we solve the coherence problem?
  - Partly: no leaking problem with off-grid frequencies
  - Partly NOT: sources should be separated by O(beamwidth)

Other solutions

- Finite Rate of Innovation (Vetterli et al.)
  - Advantages: Computationally very efficient
  - No robustness guarantees
  - Not very robust in practice
  - Newer results improving robustness (Eldar et al.)

- Atomic norm minimization (Recht et al.)
  - Advantages: Optimization-based principled approach, nice guarantees
  - Computationally very expensive
  - Also provides reconstruction guarantees for sparse minimization on a fine grid (less expensive than atomic norm minimization!)
  - Grid guarantees better than coherence/RIP-derived ones
  - Grid guarantees only in ℓ_2 sense (not on support estimation)

Broadband Processing [w/ Smaragdis, Raj]

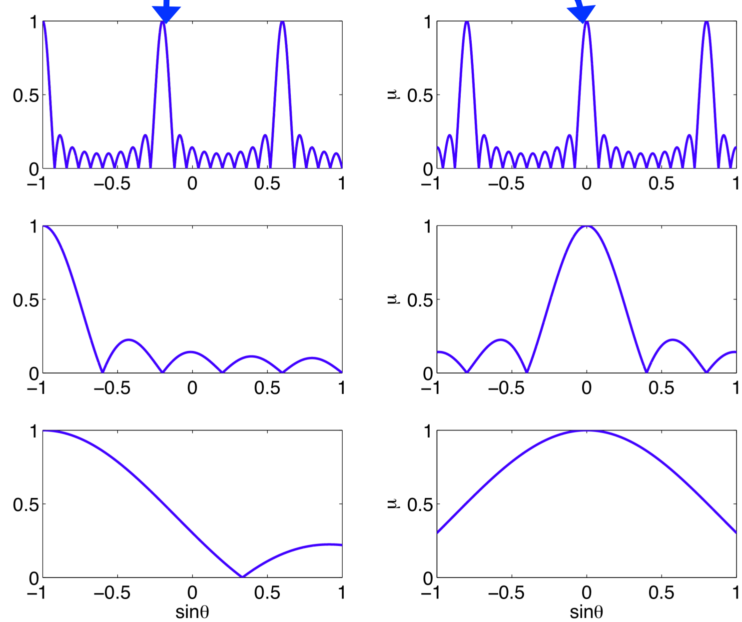
$$\mathbf{F}(\omega)=\begin{bmatrix} e^{j\frac{2\pi}{N}\frac{2p_0}{\lambda}} & \cdots & e^{j\frac{2\pi n}{N}\frac{2p_0}{\lambda}} & \cdots & e^{j\frac{2\pi N}{N}\frac{2p_0}{\lambda}} \\ \vdots & \ddots & \vdots & & \\ e^{j\frac{2\pi m}{N}\frac{2p_0}{\lambda}} & \cdots & e^{j\frac{2\pi mn}{N}\frac{2p_0}{\lambda}} & \cdots & e^{j\frac{2\pi mN}{N}\frac{2p_0}{\lambda}} \\ \vdots & \ddots & \vdots & & \\ e^{j\frac{2\pi M}{N}\frac{2p_0}{\lambda}} & \cdots & e^{j\frac{2\pi Mn}{N}\frac{2p_0}{\lambda}} & \cdots & e^{j\frac{2\pi MN}{N}\frac{2p_0}{\lambda}} \end{bmatrix}$$

vs. $\theta=-\pi/2$

vs. $\theta=0$

High resolution, Ambiguity

High Frequency (Large ω)



Just right
$d=\lambda/2=c/4\pi\omega$

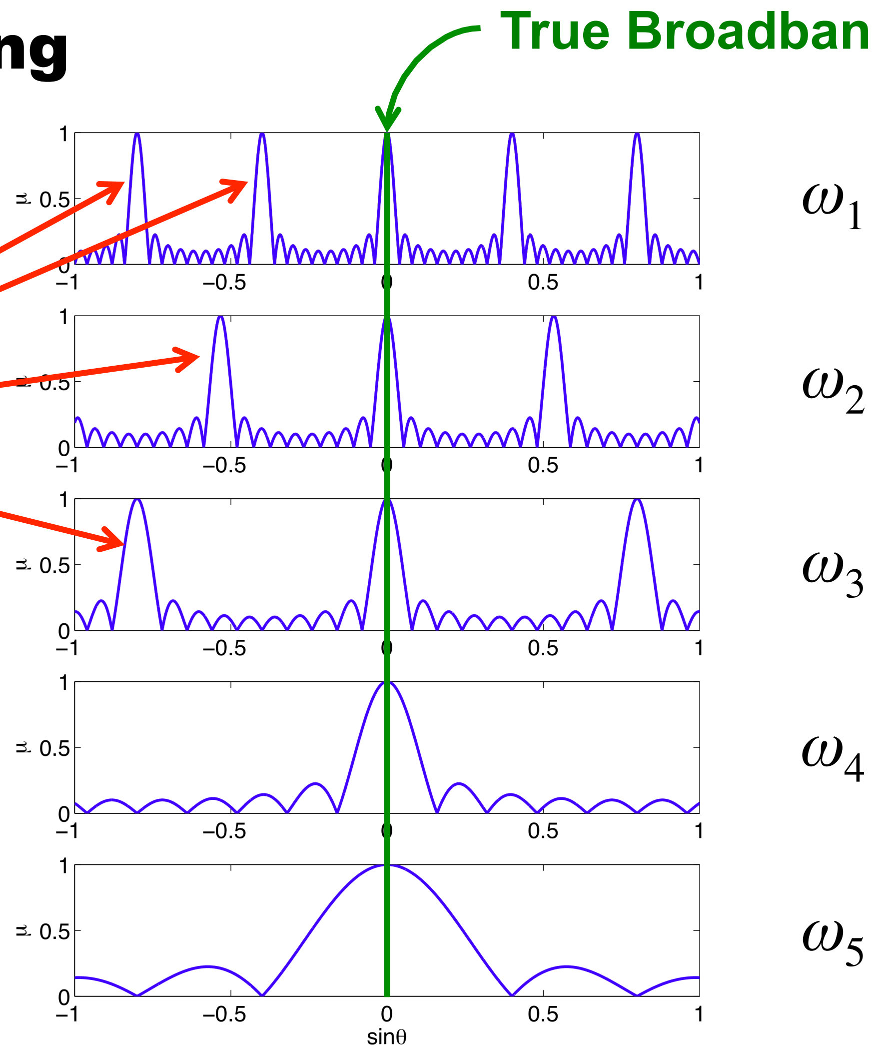
Low resolution, No ambiguity

Low Frequency (Small ω)

Sensor location is fixed. Can we exploit bandwidth?

Broadband Processing

Aliased images in other bands

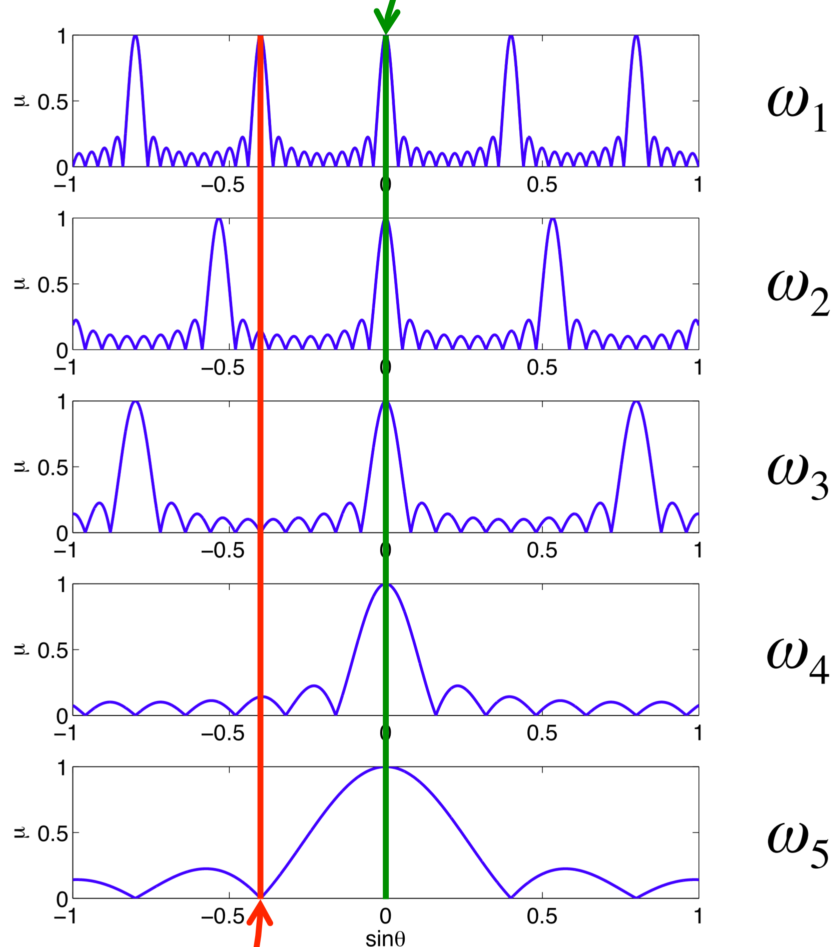


Joint sparsity across bands selects correct location!

Localization vs. Recovery

Signal recovery: Invert system on detected locations

Broadband Source



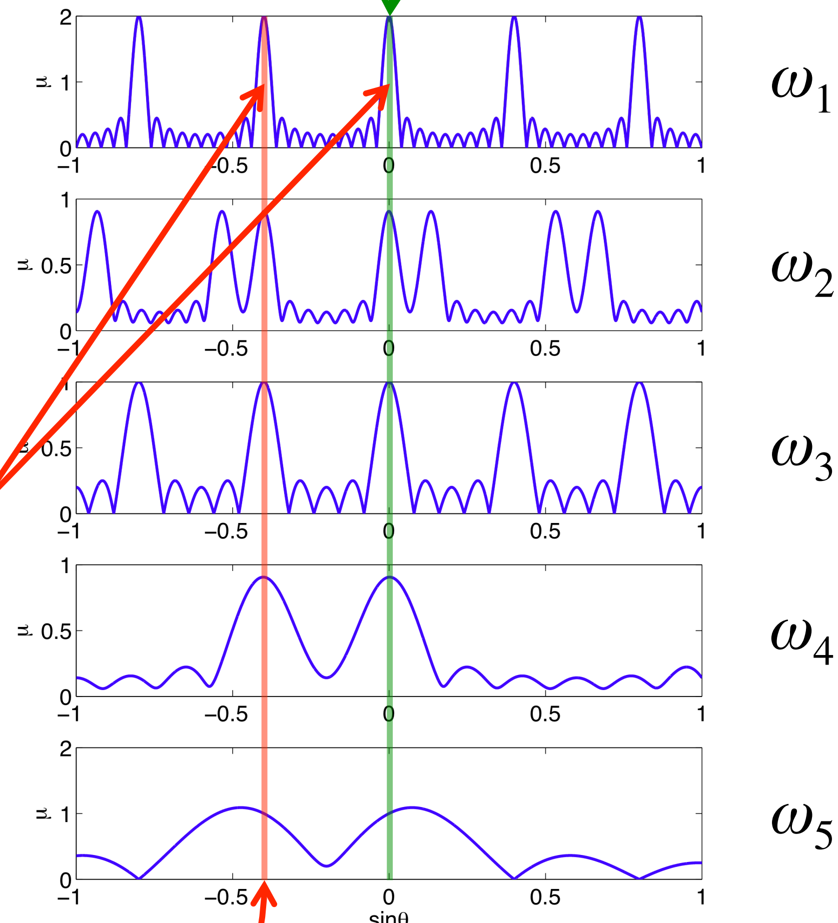
What if we have a second source?

Localization vs. Recovery

Broadband Source

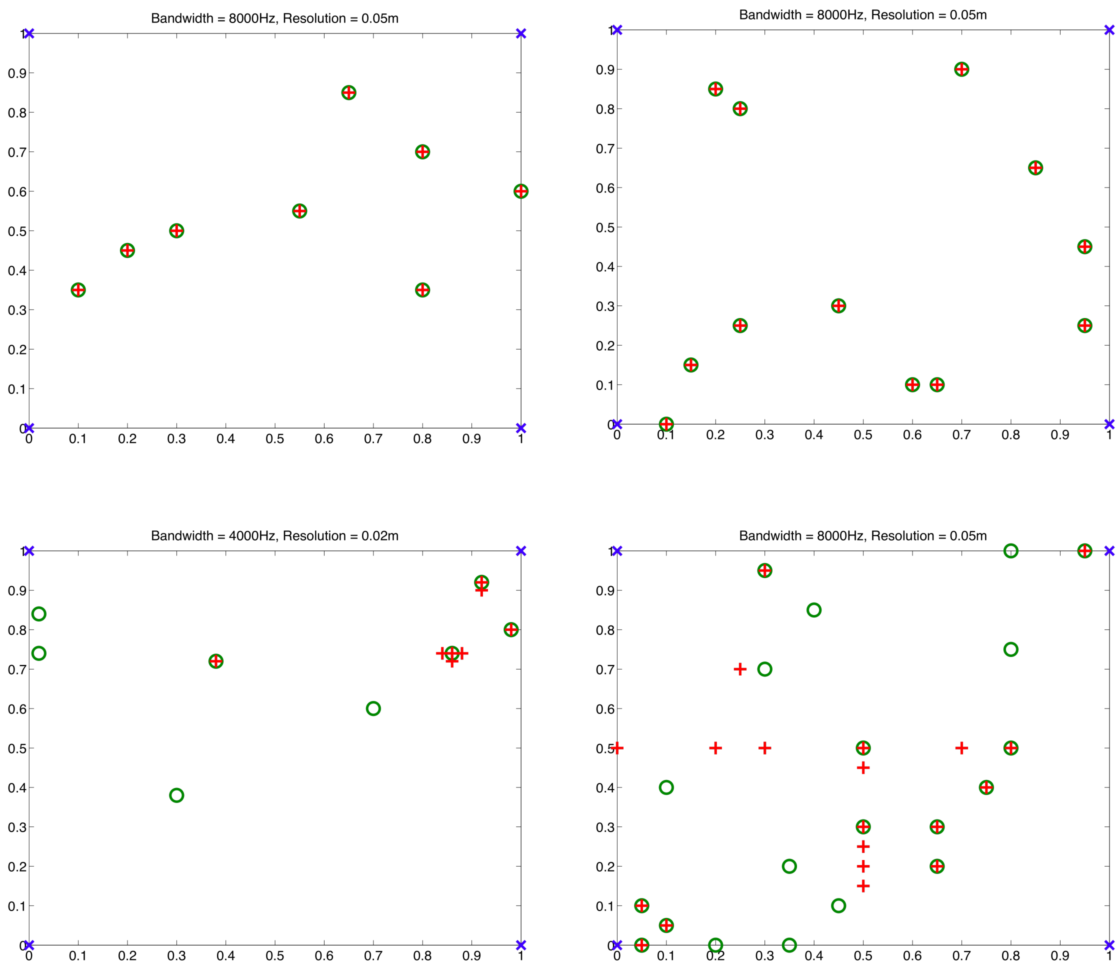
Signal recovery: Invert system on detected locations

We can **localize** the sources
Can not invert in all frequencies



What if we have a second source?

Simulation Examples

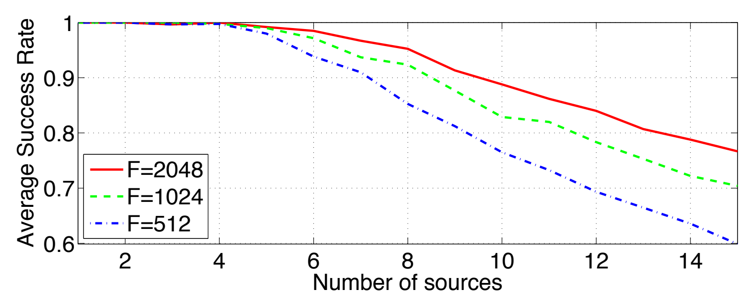


× Sensor

○ Actual Source

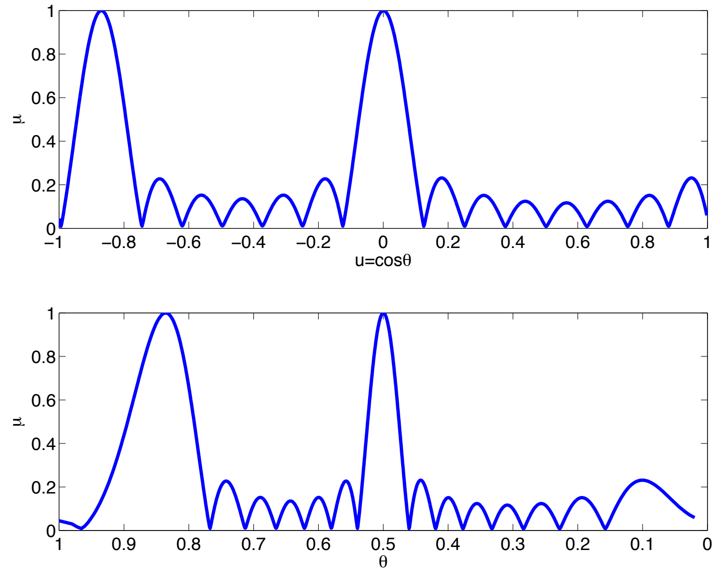
+ Estimated Source

SNR ~5dB



Discussion and open questions

- Relationship with FRI and Atomic Norm
- Can we identify sources closer than $\mathrm{O}(\text{beamwidth})$?
  - There is a resolution limit (can be proven by the nullspace of $\mathbf{F}$) but can we improve the constant in $\mathrm{O}(.)$
- Related issue: remember that we are operating in $u = \cos\theta$
  - Ambiguity in θ different on sides
  - Can we resolve that? (maybe not)
- 2D-versions?
- Off-grid joint sparsity?
- Aperture size ⇒ Resolution limit
  - Can it be improved with signal models?



Questions/Comments?

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