Array Processing Problem

A number of sensors are sensing a scene.

A number of sources transmit in a scene.

Can we localize and reconstruct sources in the scene?

This talk: overview of basic models and methods
(Linearized) Wave Propagation

Distance: \( d_{k,m} \)

Sensor (array element) \( m \)

Propagation delay:
\( \tau_{k,m} = \frac{d_{k,m}}{c} \)

Source \( k \)

Signal delayed according to distance and speed of wave propagation in medium

\[
y_m(t) = x_k(t - \tau_{k,m})
\]

\( \iff \)

\[
Y_m(\omega) = e^{-i\omega\tau_{k,m}} X_k(\omega)
\]

Superposition: Signal at receiver sum of all transmitted signals

Narrowband approximation: Phase delay same for all \( \omega \)

Free space assumption: No secondary reflections
Wave Propagation, Far-field Approximation

Propagating waves are circular:
Same delay for same distance from source

Far field approximation
Sources located far relative to array size
Propagating waves become flat (planar)
Linear Array, Far Field Approximation

Delay to origin unknown:
\[ T = \frac{D}{c} \]

Signal at origin: \( X_k(\omega) \)

Signal at position \( p \) due to source \( k \):
\[ Y_p(\omega) = e^{-i\omega \frac{p \cos \theta_k}{c}} X_k(\omega) \]

Frequency: \( \omega = 2\pi f \), Wavelength: \( \lambda = c/f \)

Total signal at position \( p \)
\[ Y_p(\omega) = \sum_k e^{i2\pi \frac{p}{\lambda} \cos \theta_k} X_k(\omega) \]
Linear Array, Far Field Approximation

\[ Y_p(\omega) = \sum_{k} e^{i2\pi \frac{p}{\lambda} \cos \theta_k} X_k(\omega) \]

Drop \( \omega \) from notation
Substitute variable \( u_k = \cos \theta_k \)

We get a (spatial, inverse) Fourier transform!

\[ Y_p = \sum_{k} e^{i2\pi \frac{p}{\lambda} u_k} X_k \]

Received Signal  Sampling point In wavelengths  “Source Frequency” (cosine of angle)  Source Signal
Discretizing the (inverse) Fourier Transform

\[ Y_p = \sum_{k} e^{i2\pi \frac{p}{\lambda} u_k} X_k \]

Uniform linear array:
\[ p_m = mp_0, \quad m = 1, \ldots, M \]

Set \( u \in [-1,1] \) on a grid
\[ u_n = -1 + 2n/N \]

\[ Y_m = \sum_{n} e^{i2\pi \frac{m p_0}{\lambda} (-1 + \frac{2n}{N})} X_n \]

\[ = e^{-i2\pi \frac{mp_0}{\lambda}} \sum_{n} e^{i \frac{2\pi mn}{N} \frac{2p_0}{\lambda}} X_n \]
Discretizing the (inverse) Fourier Transform

\[ Y_m = e^{-i2\pi \frac{mp_0}{\lambda}} \sum_{n} e^{i \frac{2\pi mn}{N} \frac{2p_0}{\lambda}} X_n \]

Set this to one. We get the DFT!

Half wavelength spacing: \( P_m = \frac{\lambda}{2} \)
\[ M=N \] array elements.

For other spacing \( p_0 \), use DFT manipulations:
Zero padding and aliasing (folding)

\[ Y = FX \]
Inversion Problem

Y = FX

Inversion Problem: What X generated Y?

Classical approach:

\[ \hat{X} = F^\dagger Y = F^H Y \]

Common names: Beamforming, Backprojection, Matched Filter

Main design issue:
Given target at certain angle, what does the inversion look like?
Inversion Problem

\[ \hat{X} = \arg \min_X \| Y - FX \|_2 \quad \text{s.t.} \quad \| X \|_0 \leq K \]

Main design issue:
Given target at certain angle, what does the coherence look like?
Beampattern/coherence

Given target at certain angle, what does inversion look like?

\[
F = \begin{bmatrix}
    e^{j \frac{2\pi}{N} \frac{2p_0}{\lambda}} & \ldots & e^{j \frac{2\pi}{N} \frac{2p_0}{\lambda}} \\
    \vdots & \ddots & \vdots \\
    e^{j \frac{2\pi}{N} \frac{2p_0}{\lambda}} & \ldots & e^{j \frac{2\pi}{N} \frac{2p_0}{\lambda}} \\
\end{bmatrix}
\]

\[
\hat{X} = F^\dagger Y = F^H Y
\]

Narrow beampattern
Grating lobes $p_0 > \lambda/2$

Wider beampattern
No grating lobes $p_0 = \lambda/2$

Too wide beampattern
No grating lobes $p_0 < \lambda/2$

$u = \cos \theta$
**Beampattern/coherence**

Given target at certain angle, what does inversion look like?

- $p_0 > \lambda/2$
  - Larger aperture ⇒ Narrower main lobe
  - Large element spacing ⇒ Grating lobes

- $p_0 = \lambda/2$

- $p_0 < \lambda/2$
  - Narrow main lobe, no grating lobes ⇒ Many array elements?
Random Element Spacing

\[ p_0 = 4\lambda/2 \]

\[ \bar{p} = 4\lambda/2 \]

Solves the grating lobes problem!
Remaining Problem: Grid!

All this analysis has an implied angle (frequency) grid…

But sources are not always on the grid!!!
Solution (?) Make grid very fine

- Actual source closer to a grid point, “leakage” is smaller.
- Big problem: Computational complexity
  - Application of F is $O(N \log N)$
  - Sparse FFT could (maybe) help
- Bigger problem: Coherence!!!
Solution: Previous talk

• Off-the grid sampling
  – Previous talk (Yi Li)
  – Goal: identify *continuous* frequency components
  – Look ma no grid!

• Advantages:
  – Very efficient
  – No grid
  – Nice guarantees (robustness)

• Did we solve the coherence problem?
  – Partly: no leaking problem with off-grid frequencies
  – Partly NOT: sources should be separated by $O(\text{beamwidth})$
Other solutions

• Finite Rate of Innovation (Vetterli et al.)
  – Advantages: Computationally very efficient
  – No robustness guarantees
  – Not very robust in practice
  – Newer results improving robustness (Eldar et al.)

• Atomic norm minimization (Recht et al.)
  – Advantages: Optimization-based principled approach, nice guarantees
  – Computationally very expensive
  – Also provides reconstruction guarantees for sparse minimization on a fine grid (less expensive than atomic norm minimization!)
  – Grid guarantees better than coherence/RIP-derived ones
  – Grid guarantees only in $\ell_2$ sense (not on support estimation)
Broadband Processing [w/ Smaragdis, Raj]

\[ F(\omega) = \begin{bmatrix}
  e^{j \frac{2\pi}{N} p_0} & \cdots & e^{j \frac{2\pi}{N} N p_0} \\
  \vdots & \ddots & \vdots \\
  e^{j \frac{2\pi}{N} m N p_0} & \cdots & e^{j \frac{2\pi}{N} m N N p_0}
\end{bmatrix} \]

vs. \( \theta = -\frac{\pi}{2} \)

High resolution, Ambiguity

High Frequency (Large \( \omega \))

Just right

\[ d = \frac{\lambda}{2} = \frac{c}{4\pi\omega} \]

Low resolution, No ambiguity

Low Frequency (Small \( \omega \))

Sensor location is fixed. Can we exploit bandwidth?
Broadband Processing

Aliased images in other bands

Joint sparsity across bands selects correct location!
Localization vs. Recovery

Signal recovery: Invert system on detected locations

What if we have a second source?
Localization vs. Recovery

Signal recovery: Invert system on detected locations

We can **localize** the sources

**Can not invert** in all frequencies

What if we have a second source?
Simulation Examples

- **Sensor**
- **Actual Source**
- **Estimated Source**

SNR ~5dB

- Bandwidth = 8000Hz, Resolution = 0.05m
- Bandwidth = 4000Hz, Resolution = 0.02m
- Bandwidth = 8000Hz, Resolution = 0.05m

Average Success Rate

- F=2048
- F=1024
- F=512

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Discussion and open questions

• Relationship with FRI and Atomic Norm
  • Can we identify sources closer than \( O(\text{beamwidth}) \)?
    – There is a resolution limit (can be proven by the nullspace of \( F \)) but can we improve the constant in \( O(\cdot) \)
• Related issue: remember that we are operating in \( u = \cos \theta \)
  – Ambiguity in \( \theta \) different on sides
  – Can we resolve that? (maybe not)
• 2D-versions?
• Off-grid joint sparsity?
• Aperture size \( \Rightarrow \) Resolution limit
  – Can it be improved with signal models?
Questions/Comments?

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