

Decentralized Chance-Constrained Finite-Horizon Optimal Control for Multi-Agent Systems

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Abstract—This paper considers finite-horizon optimal control for multi-agent systems subject to additive Gaussian-distributed stochastic disturbance and a chance constraint. The problem is particularly difficult when agents are coupled through a *joint* chance constraint, which limits the probability of constraint violation by any of the agents in the system. Although prior approaches [1][2] can solve such a problem in a centralized manner, scalability is an issue.

We propose a dual decomposition-based algorithm, namely Market-based Iterative Risk Allocation (MIRA), that solves the multi-agent problem in a decentralized manner. The algorithm addresses the issue of scalability by letting each agent optimize its own control input given a fixed value of a dual variable, which is shared among agents. A central module optimizes the dual variable by solving a root-finding problem iteratively. MIRA gives exactly the same optimal solution as the centralized optimization approach since it reproduces the KKT conditions of the centralized approach. Although the algorithm has a centralized part, it typically uses less than 0.1% of the total computation time. Our approach is analogous to a price adjustment process in a competitive market called *tâtonnement* or Walrasian auction: each agent optimizes its *demand for risk* at a given price, while the central module (or the *market*) optimizes the *price of risk*, which corresponds to the dual variable. We give a proof of the existence and optimality of the solution of our decentralized problem formulation, as well as a theoretical guarantee that MIRA can find the solution. The empirical results demonstrate a significant improvement in scalability.

I. INTRODUCTION

We consider multi-agent systems under unbounded stochastic uncertainty, with state and control constraints. Stochastic uncertainty with a probability distribution, such as Gaussian, is a more natural model for exogenous disturbances, such as wind gusts and turbulence [3], than previously studied set-bounded models [4][5][6][7]. An effective framework to address robustness with stochastic unbounded uncertainty is optimization with a *chance constraint* [8]. A chance constraint requires that the probability of violating the state constraints (i.e., the probability of failure) is below a user-specified bound known as the *risk bound*. A substantial body of work has studied the optimization problems with chance constraint mainly for single-agent systems [1][2][9][10][11].

Users of multi-agent systems typically would like to bound the probability of *system failure* rather than the probabilities of individual agents' failure; in other words, we need to impose a *joint chance constraint*, which limits the probability

of having at least one agent failing to satisfy any of its state constraints. In such cases agents are coupled through the joint chance constraint even if they are not coupled through state constraints. It is then an important challenge to find globally optimal control inputs for the multi-agent system in a *decentralized* manner while guaranteeing the satisfaction of the joint chance constraint. There has been past research on the decentralized optimal control problem with *deterministic* plant model [12] or with *bounded* uncertainty [13]. As far as we know, no past work has solved the decentralized optimization problem under stochastic unbounded uncertainty with a coupling through a joint chance constraint.

We solve the problem in the following two steps. Firstly, we find a set of decomposed individual chance constraints that is a sufficient condition of the original joint chance constraint. This decomposition allows us to convert chance constraints into deterministic constraints, though agents are still coupled through the constraint on the sum of individual risk bounds. The resulting optimization problem is deterministic and convex, so it can be solved in a centralized manner. Secondly, we formulate a set of decomposed optimization problems that are solved by individual agents in a distributed manner. The decomposed problems share a fixed value of a dual variable of the constraint that couples agents. In order to be optimal, the dual variable must be a solution to a root-finding problem, which corresponds to the complementary slackness condition of the centralized optimization. The root-finding problem is solved by a central module. The solution obtained by this decentralized approach, which we call the decentralized solution, is exactly the same as the globally optimal solution of the centralized formulation. Moreover, if the centralized optimization problem has an optimal solution, a decentralized solution is guaranteed to exist.

Our algorithm, Market-based Iterative Risk Allocation (MIRA), finds the optimal solution by solving the decomposed optimization problem and the root-finding problem iteratively and alternatively. Although the root-finding problem is solved by a centralized module, it typically uses less than 0.1% of the total computation time.

We offer an economic interpretation of our decentralized approach. The MIRA algorithm can be viewed as a computational market for risk, where the agents' demand is for risk, while the user supplies a fixed amount of risk by specifying the risk bound. The dual value can be interpreted as the price of risk. The demand of each agent at a given price is obtained by solving the decomposed optimization problem. The algorithm looks for the equilibrium price, at which the supply and the aggregate demand are balanced, by

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iteratively solving the root-finding problem. This process is analogous to the price adjustment process called *tâtonnement* or *Walrasian auction* in general equilibrium theory [14]. Past work by Voos [15] has presented successful applications of the *tâtonnement* approach to distributed resource allocation optimization problems.

The major contributions of this paper are: i) the decentralized formulation of a finite-horizon optimal control problem with a joint chance constraint, ii) the proof of existence and optimality of the decentralized solution, iii) the proof of the continuity and monotonicity of the demand function, and iv) the development of the MIRA algorithm, which is guaranteed to find the optimal solution whenever it exists.

II. PROBLEM STATEMENT

The following notation is used throughout this paper:

S : Risk bound given by the user of the system

δ_n^i : Individual risk bound for the n 'th constraint of the i 'th agent

$\mathbf{x}_t^i$: State vector of the i 'th agent at the t 'th time step (random variable)

$\mathbf{u}_t^i$: Control input of the i 'th agent at the t 'th time step

$\mathbf{w}_t^i$: Additive disturbance on i 'th agent at the t 'th time step (random variable)

$\bar{\mathbf{x}}_t^i := E[\mathbf{x}_t^i]$: Nominal state of the i 'th agent at the t 'th time step

$\mathbf{X}^i := [\mathbf{x}_0^{iT} \cdots \mathbf{x}_T^{iT}]^T$, $\mathbf{U}^i := [\mathbf{u}_0^{iT} \cdots \mathbf{u}_{T-1}^{iT}]^T$, where the superscript T means transpose. A set of variables for all indices in their range is represented as $\delta_{1:N^i}^{1:I}$ and $\mathbf{U}^{1:I}$, for example.

We consider a linear discrete dynamic system with I agents under Gaussian disturbance. Our problem is formulated as follows:

Problem 1: Multi-agent finite-horizon optimal control with a joint chance constraint

$$\min_{\mathbf{U}^{1:I}} \sum_{i=1}^I J^i(\mathbf{U}^i) \quad (1)$$

$$\text{s.t.} \quad \mathbf{x}_{t+1}^i = \mathbf{A}^i \mathbf{x}_t^i + \mathbf{B}^i \mathbf{u}_t^i + \mathbf{w}_t^i \quad (2)$$

$$\mathbf{u}_{\min}^i \leq \mathbf{u}_t^i \leq \mathbf{u}_{\max}^i \quad (3)$$

$$\Pr \left[\bigwedge_{i=1}^I \bigwedge_{n=1}^{N^i} h_n^{iT} \mathbf{X}^i \leq g_n^i \right] \geq 1 - S \quad (4)$$

$$\mathbf{w}_k^i \sim \mathcal{N}(\mathbf{0}, \Sigma_{w^i}), \mathbf{x}_0^i \sim \mathcal{N}(\bar{\mathbf{x}}_0, \Sigma_{x_0^i}) \quad (5)$$

($i = 1 \cdots I, t = 0 \cdots T-1, n = 1 \cdots N^i$)

We assume that $J^i(\cdot)$ is a proper convex function. $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is a Gaussian distribution with mean $\boldsymbol{\mu}$ and variance $\boldsymbol{\Sigma}$. Although we focus on Gaussian-distributed uncertainty in this paper, our algorithm can be extended to any additive stochastic uncertainties with quasi-concave probability distribution.

Our model considers that the multi-agent system has failed when at least one agent fails to satisfy any of its state constraints. Therefore, the joint chance constraint (4) requires

that the probability that all state constraints of *all* agents are satisfied must be more than $1 - S$, where S is the upper bound of the probability of failure (*risk bound*). The risk bound is specified by the user of the system. We assume that $0 \leq S \leq 0.5$ in order to guarantee the convexity of Problem 2 in Section III. This assumption is reasonable because acceptable risk of failure is much less than 50% in most practical cases. We assume no coupling through state constraints in this paper.

Given a risk bound S , the problem is to find the optimal control inputs $\mathbf{U}^i$ for all agents that minimize the system cost $\sum_{i=1}^I J^i(\mathbf{U}^i)$.

III. CENTRALIZED OPTIMIZATION

This section presents the centralized solution to Problem 1. It is, practically speaking, impossible to solve Problem 1 directly, since the joint chance constraint (4) is very hard to evaluate due to the difficulty of computing an integral of multi-variable probability distribution over an arbitrary region. It is also difficult to handle random variables $\mathbf{x}_t^i$ as it is. Building upon past work [1][16], we decompose the joint chance constraint into individual chance constraints that only involve single-dimensional distributions. Individual chance constraints can be turned into deterministic constraints defined on the nominal state $\bar{\mathbf{x}}_t^i$. As a result, we obtain the following approximated optimization problem that does not contain a joint chance constraint nor random variables:

Problem 2: Centralized optimization with decomposed chance constraints

$$\min_{\mathbf{U}^{1:I}, \delta_{1:N^i}^{1:I} \geq 0} \sum_{i=1}^I J^i(\mathbf{U}^i) \quad (6)$$

$$\text{s.t.} \quad \bar{\mathbf{x}}_{t+1}^i = \mathbf{A}^i \bar{\mathbf{x}}_t^i + \mathbf{B}^i \mathbf{u}_t^i \quad (7)$$

$$\mathbf{u}_{\min}^i \leq \mathbf{u}_t^i \leq \mathbf{u}_{\max}^i \quad (8)$$

$$h_n^{iT} \bar{\mathbf{X}}^i \leq g_n^i - m_n^i(\delta_n^i) \quad (9)$$

$$\sum_{i=1}^I \sum_{n=1}^{N^i} \delta_n^i \leq S \quad (10)$$

$$(i = 1 \cdots I, t = 0 \cdots T-1, n = 1 \cdots N^i)$$

Note that new decision variables $\delta_{1:N^i}^{1:I} \geq 0$ are introduced. They represent the risk bounds of individual chance constraints (9), whose sum of the individual risk bounds is bounded by the original risk bound S in (10). We interpret $\delta_{1:N^i}^{1:I}$ as *risk allocation* [17]: the total amount of risk S is allocated to each constraint, and the allocation is optimized in order to minimize the system cost.

In (9), $-m_n^i(\cdot)$ is the inverse of cumulative distribution function of univariate Gaussian distribution:

$$m_n^i(\delta_n^i) = -\sqrt{2\mathbf{h}_n^{iT} \Sigma_{X^i} \mathbf{h}_n^i} \text{erf}^{-1}(2\delta_n^i - 1) \quad (11)$$

where erf^{-1} is the inverse of the Gauss error function. Note that $m_n^i(\delta_n^i)$ is a convex, monotonically decreasing, and non-negative function for $\delta_n^i \in (0, 0.5]$. It is *strictly* convex for $\delta_n^i \in (0, 0.5)$. Our assumption that $S \leq 0.5$ guarantees $\delta_n^i \leq 0.5$.

Lemma 1: A feasible solution to Problem 2 is always a feasible solution to Problem 1.

Proof: It follows from Boole's inequality ($\Pr[\bigcup_i A_i] \leq \sum_i \Pr[A_i]$) that the condition (12), together with (10) and $\delta_t^n \geq 0$, implies the original chance constraint (4) [16]:

$$\forall_{(i,n)} \quad \Pr[h_n^{iT} \bar{\mathbf{X}}^i \leq g_n^i] \geq 1 - \delta_t^n \quad (12)$$

This probabilistic state constraint (12) is equivalent to the deterministic constraint (9) in Problem 2 [8]. Therefore any solution that satisfies (9)-(10) also satisfies (4). ■

Problem 2 has the following two important features:

a) *Convexity:* Problem 2 is a convex optimization problem since $m_n^i(\delta_n^i)$ is a convex function. Therefore, it can be solved by a convex optimization solver.

b) *Small conservatism:* Although the optimal solution of Problem 2 is not the optimal solution to Problem 1, our past work [1][2] showed that the conservatism is significantly smaller than past bounding approaches such as [11][18]. This is explained by the fact that the probability of violating more than two constraints is smaller than the probability of violating just one constraint by orders of magnitude in many practical cases, where $S \ll 1$.

Therefore, by solving Problem 2, we can obtain a feasible, close-optimal solution of Problem 1. Note that agents are still coupled through (10).

IV. DECENTRALIZED OPTIMIZATION

Although the centralized method presented in the previous section can solve Problem 2 optimally, its scalability is an issue.

We propose a decentralized formulation of Problem 2 through dual decomposition, where each agent solves a decomposed convex optimization problem in a distributed manner while a central module solves a root-finding problem. Although this method has a centralized part, its computation time is negligible compared to the decentralized part. This claim is empirically validated in Section VII.

A. The Approach

Each individual agent solves the following Problem 3, which involves a convex optimization:

Problem 3: Decomposed optimization problem for i 'th agent

$$\min_{\mathbf{U}^i, \delta_{1:N^i}^i \geq 0} \quad J^i(\mathbf{U}^i) + p \sum_{n=1}^{N^i} \delta_n^i \quad (13)$$

$$\text{s.t.} \quad \bar{\mathbf{x}}_{t+1}^i = \mathbf{A}^i \bar{\mathbf{x}}_t^i + \mathbf{B}^i \mathbf{u}_t^i \quad (14)$$

$$\mathbf{u}_{\min}^i \leq \mathbf{u}_t^i \leq \mathbf{u}_{\max}^i \quad (15)$$

$$h_n^{iT} \bar{\mathbf{X}}^i \leq g_n^i - m_n^i(\delta_n^i) \quad (16)$$

($t = 0 \cdots T-1, n = 1 \cdots N^i$)

where the constant $p \geq 0$ is given by the central module and shared by all agents. We will show in Section VI that p can be interpreted as the *price of risk*. Problem 3 is completely

decoupled from other agents, since it does not include the coupling constraint (10).

Note that $\sum_{n=1}^{N^i} \delta_n^i$, the total amount of risk the i 'th agent takes, is not bounded. Instead, it penalizes the agent with a factor of p (13). Given p , each agent finds its optimal risk allocation $\delta_{1:N^i}^{i*}(p)$. Since the total amount of risk that each agent takes is also a function of p , we denote it by $D^i(p) := \sum_{n=1}^{N^i} \delta_n^{i*}(p)$. In Section IV-B, we will give a more formal definition of $D^i(p)$, and prove that it is a single-valued, continuous, monotonically decreasing function. We will show in Section VI that $D^i(p)$ can be interpreted as the i 'th agent's *demand for risk* given the price p .

The central module finds the optimal p by solving the following root-finding problem except in the case of $p = 0$:

Problem 4: Root-finding problem for the central module

$$\text{Find } p \geq 0 \quad \text{where} \quad \sum_{i=1}^I D^i(p) = S. \quad (17)$$

The central module plays the role of the *market of risk*, which decides the price of risk. An important fact is that the computational complexity of solving Problem 4 is *not* affected by the number of agents, since the input to the root-finding algorithm we use (Brent's method) is the difference of the left-hand side and the right-hand side of (17).

The following proposition holds:

Proposition 1: Optimality of decentralized solution

(a) If Problem 3 has an optimal solution $(\mathbf{U}^{i*}, \delta_{1:N^i}^{i*})$ for all $i = 1 \cdots I$ with $p^* > 0$ that is a root of Problem 4, then $(\mathbf{U}^{1:I*}, \delta_{1:N^I}^{1:I*})$ is a globally optimal solution for Problem 2.

(b) If Problem 3 has an optimal solution $(\mathbf{U}^{i*}, \delta_{1:N^i}^{i*})$ with $p = 0$ for all $i = 1 \cdots I$ that satisfies $\sum_{i=1}^I D^i(0) \leq S$, then $(\mathbf{U}^{1:I*}, \delta_{1:N^I}^{1:I*})$ is a globally optimal solution for Problem 2.

Proof: We prove by showing that $(\mathbf{U}^{1:I*}, \delta_{1:N^I}^{1:I*})$ and p^* satisfy the KKT conditions of Problem 2. Although Problem 3 shares most of the constraints of Problem 2, it misses (10). Therefore we need to pay attention to the KKT conditions that are related to (10) and δ_n^i :

$$\mu_n^i \frac{dm_n^i}{d\delta_n^i} + p = 0 \quad (18)$$

$$\sum_{i=1}^I \sum_{n=1}^{N^i} \delta_n^i \leq S \quad (19)$$

$$p \left(\sum_{i=1}^I \sum_{n=1}^{N^i} \delta_n^i - S \right) = 0 \quad (20)$$

$$p \geq 0 \quad (21)$$

where μ_n^i and p are the dual variables corresponding to (9) and (10), respectively. The KKT conditions are the necessary and *sufficient* conditions for global optimality of Problem 2 since J^i and m_n^i are convex functions, and the equality constraint (7) is linear. The optimal solution of Problem 3 $(\mathbf{U}^{i*}, \delta_{1:N^i}^{i*})$ satisfies (18) because it is also a part of the KKT

conditions of Problem 3. In the case of (a), $\delta_{1:N^i}^{1:I^*}$ satisfies (19) and (20) since p^* is a root of (17). In the case of (b), (19) and (20) are satisfied since $\sum_{i=1}^I D^i(0) \leq S$ and $p = 0$. Since the cost function of the centralized optimization (6) is a sum of the individual cost functions (13) and the constraints (7)-(9) are the same as (14)-(16), the partial derivatives of the Lagrangians of Problem 2 and 3 respect to $\mathbf{U}^i$ and $\mathbf{X}^i$ are the same. Therefore, their stationary constraints regarding to $\mathbf{U}^i$ and $\mathbf{X}^i$ are the same. Problem 2 and 3 also share the same primary feasibility, dual feasibility, and the complementary slackness conditions regarding to $\mathbf{U}^i$ and $\mathbf{X}^i$ since (7)-(9) and (14)-(16) are the same.

Since all KKT conditions of Problem 2 are satisfied by $(\mathbf{U}^{1:I^*}, \delta_{1:N^i}^{1:I^*})$, which satisfies the KKT conditions of Problem 3 together with p^* , $(\mathbf{U}^{1:I^*}, \delta_{1:N^i}^{1:I^*})$ is an optimal solution of Problem 2. ■

Proposition 1 states that if a decentralized solution (i.e., solution for Problems 3 and 4) exists, then it is an optimal solution for the centralized problem (Problem 2). The following Proposition 2 guarantees the existence of a decentralized solution if there is an optimal solution for the centralized problem.

Proposition 2: Existence of decentralized solution

If Problem 2 has an optimal solution $(\mathbf{U}^{1:I^*}, \delta_{1:N^i}^{1:I^*})$,

- (a) $(\mathbf{U}^{i^*}, \delta_{1:N^i}^{i^*})$ is an optimal solution of Problem 3 for all $i = 1 \dots I$ given $p > 0$, which is a root of Problem 4, **or**
- (b) $(\mathbf{U}^{i^*}, \delta_{1:N^i}^{i^*})$ is an optimal solution of Problem 3 for all $i = 1 \dots I$ with $p = 0$ and $\sum_{i=1}^I D^i(0) \leq S$.

Proof: The KKT conditions of Problem 3 are the necessary and sufficient conditions for its optimality since J^i and m_n^i are convex functions, and the equality constraint (14) is linear. Since the KKT conditions of Problem 3 are the subset of the KKT conditions of Problem 2, the optimal solution of Problem 2 always satisfies all KKT conditions of Problem 3 for all $i = 1 \dots I$; hence it is an optimal solution of Problem 3. When $p > 0$, it is a root of Problem 4 since the second term of (20) must be zero. When $p = 0$, $\sum_{i=1}^I D^i(0) \leq S$ because (19) is satisfied. ■

Although the following Lemma 2 is just a contraposition of Proposition 2, it is useful when checking the feasibility of Problem 2.

Lemma 2: If both Proposition 2(a) and (b) do not apply, Problem 2 does not have an optimal solution.

B. Continuity and Monotonicity of Demand Function

Although the existence of a decentralized solution is established by Proposition 2, it does not tell how to find it. The objective of this subsection is to prove the continuity of the demand function $D^i(p)$ in order to guarantee that the root of Problem 4 can be found by a standard root-finding algorithm, Brent's method. We will also prove in this subsection that $D^i(p)$ is a monotonically decreasing function. This feature is very important since it allows us to find the

absence of a root by checking the feasibility conditions at the boundaries.

We first derive the optimal cost as a function of a risk bound. Observe that the following optimization problem gives the same solution as Problem 3:

$$\min_{\Delta^i \leq 0.5} \min_{\mathbf{U}^i, \delta_{1:N^i}^i \geq 0} J^i(\mathbf{U}^i) + p\Delta^i \quad (22)$$

s.t. (14) – (16)

$$\sum_{n=1}^{N^i} \delta_n^i \leq \Delta^i \quad (23)$$

Therefore, Problem 3 is equivalent to solving the following:

$$\min_{\Delta^i \leq 0.5} J^{i*}(\Delta^i) + p\Delta^i \quad (24)$$

where

$$J^{i*}(\Delta^i) = \min_{\mathbf{U}^i, \delta_{1:N^i}^i \geq 0} J^i(\mathbf{U}^i) \quad (25)$$

s.t. (14) – (16), (23)

The conditions (14)-(16), (23) define a compact space. If it is non-empty, (25) has a minimum since $J^i(\mathbf{U}^i)$ is a proper convex function by assumption. We denote by $\Delta_{\min}^i$ the smallest Δ^i that makes (14)-(16),(23) non-empty. Then it is non-empty for all $\Delta^i \geq \Delta_{\min}^i$ since $m_n^i(\delta_n^i)$ is a monotonically decreasing function. Therefore $J^{i*}(\Delta^i)$ is a single-valued function for all $\Delta^i \geq \Delta_{\min}^i$. Since $S \leq 0.5$, any feasible solution of Problem 2 has $\Delta^i \leq 0.5$. Therefore we limit the domain of $J^{i*}(\Delta^i)$ to $[\Delta_{\min}^i, 0.5]$ without the loss of generality. The convexity of $m_n^i(\delta_n^i)$ is implied by $\Delta^i \leq 0.5$.

Lemma 3: $J^{i*}(\Delta^i)$ is a convex, monotonically decreasing function for its entire domain.

Proof: Let Δ_1^i and Δ_2^i be real numbers that satisfy $\Delta_{\min}^i \leq \Delta_1^i < \Delta_2^i \leq 0.5$. Since larger Δ^i loosens the constraint (23) by allowing larger δ_n^i , $J^{i*}(\Delta_1^i) \geq J^{i*}(\Delta_2^i)$. Therefore J^{i*} is monotonically decreasing.

Let λ be a real scalar in $[0, 1]$. Let also $\mathbf{U}_1^{i*}$ and $\mathbf{U}_2^{i*}$ be optimal solutions of (25) with Δ_1^i and Δ_2^i , respectively. Since the feasible space defined by (14)-(16), (23) is convex, $\lambda\mathbf{U}_1^{i*} + (1-\lambda)\mathbf{U}_2^{i*}$ is a feasible (but not necessarily optimal) solution of (25) with $\lambda\Delta_1^i + (1-\lambda)\Delta_2^i$. Therefore,

$$\begin{aligned} J^{i*}(\lambda\Delta_1^i + (1-\lambda)\Delta_2^i) &\leq J^i(\lambda\mathbf{U}_1^{i*} + (1-\lambda)\mathbf{U}_2^{i*}) \\ &\leq \lambda J^i(\mathbf{U}_1^{i*}) + (1-\lambda) J^i(\mathbf{U}_2^{i*}) \\ &= \lambda J^{i*}(\Delta_1^i) + (1-\lambda) J^{i*}(\Delta_2^i). \end{aligned}$$

The second inequality holds since $J^i(\mathbf{U}^i)$ is a convex function of $\mathbf{U}^i$. ■

It immediately follows from Lemma 2 that $J^{i*}(\Delta^i)$ is continuous, and differentiable at all but countably many points.

We then prove the *strict* convexity for a portion of the domain of J^{i*} where it is strictly decreasing. We define

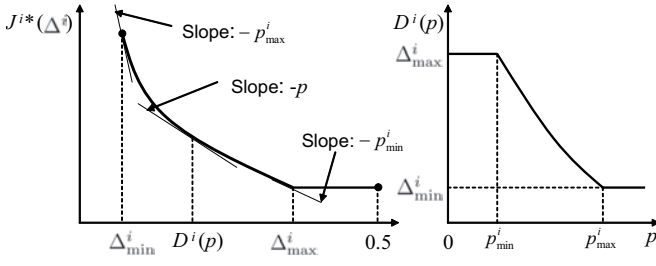


Fig. 1. Sketch of the functions $J^{i*}(\Delta^i)$ and $D^i(p)$ (demand function of i 'th agent). Note that in many practical cases $\Delta_{\max}^i = 0.5$.

$\Delta_{\max}^i$ as follows:

$$\Delta_{\max}^i = \min [0.5, \sup \{ \Delta^i \mid \partial J^{i*}(\Delta^i) < 0 \}]$$

where ∂J^{i*} is the subdifferential of J^{i*} . The inequality means that all subgradients are less than zero. See Fig. 1 for graphical interpretation.

Lemma 4: $J^{i*}(\Delta^i)$ is strictly convex for all $\Delta_{\min}^i \leq \Delta^i \leq \Delta_{\max}^i$.

Proof: Fix Δ_1^i, Δ_2^i , and λ such that $\Delta_{\min}^i \leq \Delta_1^i < \Delta_2^i \leq \Delta_{\max}^i$, $0 < \lambda < 1$. Let $(U_1^{i*}, \bar{X}_1^{i*}, \delta_{1:N_i,1}^{i*})$ and $(U_2^{i*}, \bar{X}_2^{i*}, \delta_{1:N_i,2}^{i*})$ be optimal solutions of (25) with Δ_1^i and Δ_2^i , respectively. Note that $\bar{X}_j^{i*}$ is linearly tied to U_1^{i*} by (14). Since $m_n^i(\delta_n^i)$ in (16) is strictly convex,

$$\begin{aligned} & h_n^{iT} \left(\lambda \bar{X}_1^i + (1-\lambda) \bar{X}_2^i \right) \\ & < g_n^i - m_n^i (\lambda \delta_{n,1}^{i*} + (1-\lambda) \delta_{n,2}^{i*}) \quad (n = 1 \cdots N_i). \end{aligned} \quad (26)$$

In other words, the constraints (16) are inactive for all $n = 1 \cdots N_i$ at $\lambda \bar{X}_1^i + (1-\lambda) \bar{X}_2^i$ (hence, at $\lambda U_1^{i*} + (1-\lambda) U_2^{i*}$) and $\lambda \delta_{n,1}^{i*} + (1-\lambda) \delta_{n,2}^{i*}$.

Also, the following inequality holds:

$$\begin{aligned} J^i(\lambda U_1^{i*} + (1-\lambda) U_2^{i*}) & \geq J^{i*}(\lambda \Delta_1^i + (1-\lambda) \Delta_2^i) \\ & > J^{i*}(\Delta_2^i) = J^i(U_2^{i*}). \end{aligned} \quad (27)$$

The second inequality follows from the mean-value theorem and the definition of $\Delta_{\max}^i$. Note that by assumption, $\lambda > 0$ (hence, $\lambda \Delta_1^i + (1-\lambda) \Delta_2^i < \Delta_2^i$) and $\Delta_2^i \leq \Delta_{\max}^i$. As for the first inequality, refer to the proof of Lemma 3. It is implied by (27) that $\lambda U_1^{i*} + (1-\lambda) U_2^{i*}$ is not a globally optimal solution; hence, it is not a local optimal solution either.

Therefore, there exists a non-zero perturbation δU^i to $\lambda U_1^{i*} + (1-\lambda) U_2^{i*}$ that satisfies the constraints (16) with $\lambda \delta_{1:N_i,1}^{i*} + (1-\lambda) \delta_{1:N_i,2}^{i*}$, and results in strictly less cost. Hence, we have:

$$\begin{aligned} J^{i*}(\lambda \Delta_1^i + (1-\lambda) \Delta_2^i) & \leq J^i(\lambda U_1^{i*} + (1-\lambda) U_2^{i*} + \delta U^i) \\ & < J^i(\lambda U_1^{i*} + (1-\lambda) U_2^{i*}) \leq \lambda J^i(U_1^{i*}) + (1-\lambda) J^i(U_2^{i*}) \\ & = \lambda J^{i*}(\Delta_1^i) + (1-\lambda) J^{i*}(\Delta_2^i). \end{aligned} \quad (28)$$

Therefore, $J^{i*}(\Delta^i)$ is strictly convex for all $\Delta_{\min}^i \leq \Delta^i < \Delta_{\max}^i$. ■

Since $J^{i*}(\Delta^i)$ is strictly convex and monotonically decreasing, there is a unique minimizer of (24) for $p > 0$. When $p = 0$, the optimal solution of (24) may not be unique.

Then, we define the demand function as follows so that $D^i(p)$ is a single-valued function for all $p \geq 0$:

Definition: Demand function

$$D^i(p) := \begin{cases} \arg \min_{\Delta^i \leq 0.5} J^{i*}(\Delta^i) + p \Delta^i & (p > 0) \\ \Delta_{\max}^i & (p = 0) \end{cases}$$

This definition is natural since $\Delta_{\max}^i$ is an optimal solution for $p = 0$ if it exists. Moreover, in such a case, $\Delta_{\max}^i$ is the smallest optimal solution. This feature is important since we need to check the condition (19), which is equivalent to $\sum_{i=1}^I D^i(0) \leq S$, against the smallest optimal solution in order to tell if Proposition 1(b) applies.

Proposition 3: Continuity and Monotonicity of Demand Function

(a) $D^i(p)$ is a continuous, monotonically decreasing function for $p \geq 0$.

(b) $D^i(p) = \Delta_{\max}^i$ for $0 \leq p \leq p_{\min}^i$, and $D^i(p) = \Delta_{\min}^i$ for $p \geq p_{\max}^i$.

Proof: We define $p_{\min}^i$ and $p_{\max}^i$ as follows:

$$\begin{aligned} p_{\max}^i & = \sup_{\Delta^i \in (\Delta_{\min}^i, \Delta_{\max}^i)} -\partial J^{i*}(\Delta^i) \\ p_{\min}^i & = \inf_{\Delta^i \in (\Delta_{\min}^i, \Delta_{\max}^i)} -\partial J^{i*}(\Delta^i), \end{aligned} \quad (29)$$

where $\partial J^{i*}(\Delta^i)$ is the subdifferential of J^{i*} .

We first prove the continuity and monotonicity in $(p_{\min}^i, p_{\max}^i)$. Since $D^i(p)$ is the optimal solution for (24), the following optimality condition is satisfied:

$$-p \in \partial J^{i*}(D^i(p)). \quad (30)$$

It follows from the Conjugate Subgradient Theorem (Proposition 5.4.3 of [19]) that

$$D^i(p) \in \partial(J^{i*})^*(-p)$$

where $(J^{i*})^*$ is the conjugate function of J^{i*} . Since the minimum of $J^{i*}(\Delta^i) + p \Delta^i$ is uniquely attained, $(J^{i*})^*$ is differentiable everywhere, and hence continuously differentiable, in $(p_{\min}^i, p_{\max}^i)$ (Proposition 5.4.4 of [19]). Therefore $D^i(p)$ is continuous in $(p_{\min}^i, p_{\max}^i)$. Also, since $(J^{i*})^*$ is a convex function (Ch. 1.6 of [19]), $D^i(p)$ is monotonically decreasing in $(p_{\min}^i, p_{\max}^i)$.

Next, we show that $D^i(p) = \Delta_{\max}^i$ for $0 \leq p \leq p_{\min}^i$. When $\Delta_{\max}^i < 0.5$, it follows from the definition of $\Delta_{\max}^i$ and $p_{\min}^i$ that $\partial J^{i*}(\Delta_{\max}^i) = [0, -p_{\min}^i]$. Therefore, $\Delta_{\max}^i$ is an optimal solution of (24) for $0 \leq p \leq p_{\min}^i$, since the optimality condition (30) is satisfied. This result agrees with the definition of $D^i(p)$ at $p = 0$. When $\Delta_{\max}^i = 0.5$, it follows from (29) that $p_{\min}^i < -\partial J^{i*}(\Delta^i)$ in $(\Delta_{\min}^i, 0.5)$. Therefore, the minimum of $J^{i*}(\Delta^i) + p \Delta^i$ is attained at the upper bound of Δ^i , which is $\Delta^i = 0.5 = \Delta_{\max}^i$, and hence $D^i(p) = \Delta_{\max}^i$. By definition, $D^i(0) = \Delta_{\max}^i$. The continuity at $p = 0$ is obvious.

Then we show that $D^i(p) = \Delta_{\min}^i$ for $p \geq p_{\max}^i$. It follows from (29) that $p_{\max}^i > -\partial J^{i*}(\Delta^i)$ in $(\Delta_{\min}^i, \Delta_{\max}^i)$. Therefore, the minimum of $J^{i*}(\Delta^i) + p\Delta^i$ is attained at the lower bound of the domain of J^{i*} , which is $\Delta^i = \Delta_{\min}^i$. Therefore, $D^i(p) = \Delta_{\min}^i$.

Finally, we prove that $D^i(p)$ is continuous at $p_{\min}^i$ and $p_{\max}^i$. Since $D^i(p)$ is constant for $0 \leq p \leq p_{\min}^i$ and $p \geq p_{\max}^i$, we only have to show that it is upper semi-continuous at $p_{\min}^i$ and lower semi-continuous at $p_{\max}^i$. Consider a sequence $p_k \in (p_{\min}^i, p_{\max}^i)$ with $p_k \rightarrow p_{\min}^i$. It follows from the definition of $p_{\min}^i$ (29) and the convexity of J^{i*} that we can find a sequence D_k^i such that

$$p_k \in -\partial J^{i*}(D_k^i), \quad D_k^i \rightarrow \Delta_{\max}^i.$$

Since D_k^i satisfies the condition for optimality for p_k , $D^i(p_k) = D_k^i$. Therefore,

$$\lim_{p_k \rightarrow p_{\min}^i} D^i(p_k) \rightarrow \Delta_{\max}^i.$$

Hence, $D(p)$ is continuous at $p_{\min}^i$. In the same way, it is lower semi-continuous at $p_{\max}^i$.

Therefore, $D^i(p)$ is a continuous, monotonically decreasing function for $p \geq 0$. ■

See Fig. 1 for the sketch of $D^i(p)$.

V. THE ALGORITHM

Now we present the Market-based Iterative Risk Allocation (MIRA), that finds the solution for Problems 3 and 4. Algorithm 1 (see the box below) shows the entire flow of the MIRA algorithm. Exploiting the fact that risk is a scalar, we use Brent's method to efficiently find the root of Problem 4.

A. Obtaining $D^i(0) (= \Delta_{\max}^i)$ (Algorithm 1, Line 1-3)

The algorithm first computes $D^i(0) (= \Delta_{\max}^i)$ in order to find if Proposition 1(b) applies. It is obtained through the following process:

Algorithm 1 Market-based Iterative Risk Allocation

- 1: Each agent computes $\Delta_{\max}^i$ and $\Delta_{\min}^i$;
 - 2: **if** $\sum_{i=1}^I \Delta_{\max}^i \leq S$ **then**
 - 3: $p = 0$ gives the optimal solution; terminate;
 - 4: **else if** $\sum_{i=1}^I \Delta_{\min}^i > S$ **then**
 - 5: There is no feasible solution; terminate;
 - 6: **else**
 - 7: **while** $|\sum_i D^i(p) - S| > \epsilon$ **do**
 - 8: The central module announces p to agents;
 - 9: Each agent computes $D^i(p)$ by solving Problem 3;
 - 10: Each agent submits $D^i(p)$ to the central module;
 - 11: The central module updates p by computing one step of Brent's method;
 - 12: **end while**
 - 13: **end if**
-

1) : Relax Problem 3 by fixing all risk bounds at $\delta_n^i = 0.5$. This relaxation makes $m_n^i(\delta_n^i) = 0$. The relaxed problem has only linear constraints, so it can be solved efficiently. If it does not have a feasible solution, Problem 3 is infeasible, and Problem 2 is also infeasible (Lemma 2). The algorithm terminates in this case.

2) : Compute δ_n^{i*} using the following equation with the optimal solution $\bar{\mathbf{X}}^{i*}$ obtained from the relaxed problem:

$$\delta_n^{i*} = \text{cdf}_n^i(h_n^{iT} \bar{\mathbf{X}}^{i*} - g_n^i)$$

where $\text{cdf}_n^i(\cdot)$ is a cumulative distribution function of univariate Gaussian distribution with variance $\mathbf{h}_n^{iT} \Sigma_{X^i} \mathbf{h}_n^i$, or in other words, the inverse of $-m_n^i(\cdot)$.

3) : Obtain $D^i(0)$ by:

$$D^i(0) = \min \left[0.5, \sum_{n=1}^{N^i} \delta_n^{i*} \right]$$

Each agent computes $D^i(0)$ and sends it to the central module. The central module checks if $\sum_{i=1}^I D^i(0) \leq S$ holds. If so, the optimal solution of the relaxed problem is the solution of Problem 2 by Proposition 1(b). Proposition 3 guarantees that there is no positive root of Problem 4, and hence, the algorithm terminates. Otherwise the algorithm looks for a solution with $p > 0$.

B. Obtaining $\Delta_{\min}^i$ (Algorithm 1, Line 1, 4, and 5)

Each agent also computes $\Delta_{\min}^i$ by solving the following convex optimization problem:

$$\begin{aligned} \Delta_{\min}^i &= \min_{\Delta^i \leq 0.5, \mathbf{U}^i, \delta_{1:N^i}^i \geq 0} \Delta^i \\ \text{s.t.} \quad &(14) - (16), (23) \end{aligned} \quad (31)$$

If $\sum_{i=1}^I \Delta_{\min}^i > S$, then it follows from Proposition 3 that $\sum_{i=1}^I D^i(p) > S$ for all $p \geq 0$. In this case, since both Proposition 1(a) and (b) do not apply, there is no feasible solution (Lemma 2), and the algorithm terminates. Otherwise, Proposition 3 guarantees that a solution exists in $(0, \max_i p_{\max}^i]$. $p_{\max}^i$ can be computed from the solution of the optimization problem (31).

C. Finding a root for Problem 4 (Algorithm 1, Line 7-12)

If the algorithm has not terminated in the previous steps, we have $\sum_{i=1}^I D^i(0) > S$ and $\sum_{i=1}^I D^i(p_{\max}^i) \leq S$. Therefore, the continuity of $D^i(p)$ (Proposition 3) guarantees that Brent's method can find a root between $(0, \max_i p_{\max}^i]$. Brent's method provides superlinear rate of convergence [20]. It is suitable for our application since it does not require the derivative of $D^i(p)$, which is generally hard to obtain. In many practical cases, it is more efficient to incrementally search p that is large enough to make $\sum_{i=1}^I D^i(p_{\max}^i) \leq S$, rather than initializing Brent's method with $[0, \max_i p_{\max}^i]$.

The algorithm updates $(\mathbf{U}^{i*}, \delta_{1:N^i}^{i*})$ (hence, $D^i(p)$) and p alternatively and iteratively. In each iteration, each agent computes $D^i(p)$ (Line 9) in a distributed manner by solving Problem 3 with p , which is given by the central module (Line 8). Since Problem 3 is a convex optimization problem,

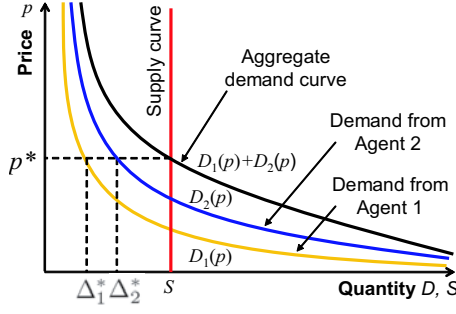


Fig. 2. Economic interpretation of the decentralized optimization approach in a system with two agents. Note that we followed the economics convention of placing the price on the vertical axis. The equilibrium price is p^* , and the optimal risk allocation is $\Delta_1^* = D_1(p^*)$ for Agent 1 and $\Delta_2^* = D_2(p^*)$ for Agent 2.

it can be solved efficiently using interior-point methods. The central module collects $D^i(p)$ from all agents (Line 10) and updates p by computing one step of Brent's method (Line 11).

The communication requirements between agents are small: in each iteration, each agent receives p (Line 8) and transmits $D^i(p)$ (Line 10), both of which are scalars.

The central module can be removed by letting all individual agents solve Problem 4 to update p simultaneously. However, since the computation of p is duplicated among the agents, there is no advantage of doing so in terms of computation time.

VI. ECONOMIC INTERPRETATION

The economic interpretation of the distributed optimization becomes clear by regarding the dual variable p as the *price of risk*.

Each agent takes risk Δ_n^i by paying $p\Delta_n^i$ as an additional cost (see (13) or (24)). It optimizes the amount of risk it wants to take $D(p)$, as well as the control sequence U^i , by solving Problem 3 with a given price of risk p . Therefore $D^i(p)$ can be interpreted as the *demand for risk* of the i 'th agent. On the other hand, the upper bound on the total amount of risk S , which is a constant specified by the user of the system, can be interpreted as the *supply of risk*.

The optimal price p^* must satisfy the complementary slackness condition (20). In the usual case where the optimal price is positive $p^* > 0$, the aggregate demand $\sum_i D^i(p^*)$ must be equal to the supply S at p^* . Such a price is called the *equilibrium price*. In a special case where the supply always exceeds the demand for all $p \geq 0$, the optimal price is zero $p^* = 0$. If the aggregate demand always exceeds the supply for all $p \geq 0$, there is no solution that satisfies the primal feasibility condition (19), and hence the problem is infeasible. See Fig. 2 for the graphical interpretation.

The iterative optimization process of MIRA is analogous to the price adjustment process called *tâtonnement* or *Walrasian auction* in the general equilibrium theory [14]. Intuitively, the price is raised when there is excess demand $\sum_i D^i(p) - S > 0$, and it is lowered when there is excess

supply $\sum_i D^i(p) - S < 0$ until the supply and the aggregate demand are balanced.

VII. SIMULATION

A. Result

Fig. 3 shows the average computation times of 100 runs of the MIRA algorithm with 2 to 128 agents, compared against the centralized approach that directly solves Problem 2. Demands for risk are computed parallelly in each agent.

The computation time of the centralized algorithm quickly grows as the problem size increases. Although MIRA, the proposed algorithm, is slower for the problems with less than eight agents, it outperforms the centralized algorithm when the number of agents is more than eight. The exponential fits to the average computation time of MIRA and the centralized approach are $15.2e^{0.0160n}$ and $0.886e^{0.328n}$, respectively, where n is the number of agents. MIRA has a 20 times smaller exponent than the centralized approach, which means a significant improvement in scalability.

A counterintuitive phenomenon observed in the result is that MIRA also slows down for large problems, although not as significantly as the centralized algorithm. This is because iterations must be synchronized among all agents. When each agent computes its demand for risk by solving the non-linear optimization problem, the computation time diverges from agent to agent. In each iteration, all agents must wait until the slowest agent finishes computing its demand. As a result, MIRA slows down for large problems, as the expected computation time of the slowest agent grows. Our future work is to develop an asynchronous algorithm to improve scalability.

The computation time of the central module (CM), which is shown in Figure 4, is at most 0.1% of the total computation time of MIRA. Figure 4 also shows that the number of iterations are almost constant. Moreover, the computational complexity of the root finding algorithm used for the CM does not increase with the number of agents. As a result, the computation time of the CM (Figure 4) grows less significantly than the computation time of the entire algorithm (Figure 4). Therefore, the existence of the central module does not harm the scalability of MIRA. The growth in the computation time of the CM is mainly due to computational overhead of handling the data from multiple agents.

B. Implementation and Setting

In the decentralized approach (MIRA), a convex optimization solver SNOPT is used to solve Problem 3 (computation of $D^i(p)$), and the Matlab implementation of Brent's method (fzero) is used to find the root p^* of Problem 4. In the centralized approach, Problem 2 is solved by SNOPT. Since it is hard to set exactly the same optimality tolerances for centralized and decentralized approaches, we set a *stricter* tolerance for the decentralized approach than the centralized approach. Specifically, the optimality tolerance of SNOPT is defined in terms of the complementary slackness normalized by dual variables $\frac{p}{\|\pi\|} \left| \sum_{i=1}^I D^i(p) - S \right| < \epsilon$, where π is the vector of all dual variables. In the decentralized approach, we

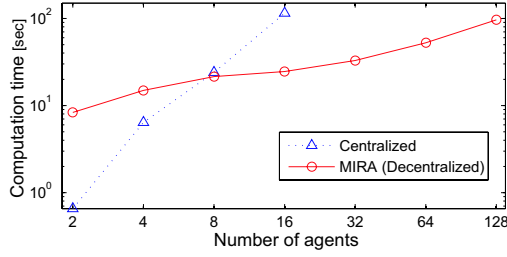


Fig. 3. Computation time of MIRA compared to the centralized method. Plotted values are the average of 100 runs with randomly generated constraints. Note that plot is in log-log scale.

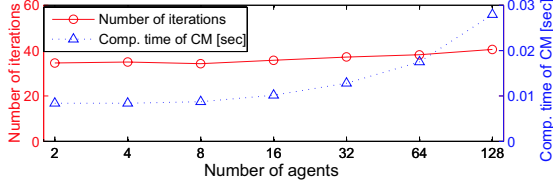


Fig. 4. Number of iterations and the computation time of the central module (CM) of MIRA. Plotted values are the average of 100 runs with randomly generated constraints.

set the tolerance of fzero as $\left| \sum_{i=1}^I D^i(p) - S \right| < \epsilon$. This tolerance is stricter than the previous one since $p/\|\pi\| \leq 1$. We set $\epsilon = 10^{-6}$. Simulations were conducted on a machine with Intel(R) Core(TM) i7 CPU clocked at 2.67 GHz and 8GB RAM. MIRA is simulated by a single processor; however, we counted the computation time of the agent that is slowest to compute the demand in each iteration, so that the result shown in Fig. 3 corresponds to the computation time when running MIRA with parallel computing. Communication delay is not simulated in our result.

C. Parameters used

The planning window is $1 \leq t \leq 5$. The Parameters are set as follows:

$$A^i = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B^i = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, x_0^i = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \\ u_{\min} = -0.2, u_{\max} = 0.2, h_n^i = -[1 \ 0] \\ \Sigma_{w^i} = \begin{bmatrix} 0.001 & 0 \\ 0 & 0 \end{bmatrix}, J^i = \sum_{t=1}^5 (u_t^i)^2$$

The bound of the state constraints g_n^i is generated by a random walk starting from $g_0^i = 0$, and $g_{n+1}^i - g_n^i$ is sampled from a uniform distribution in $[-0.3, 0.3]$.

VIII. CONCLUSION

We presented a decentralized approach to a finite-horizon optimal control problem with a joint chance constraint, where each agent solves a decomposed convex optimization problem (Problem 3) in a distributed manner while a central module solves a root-finding problem (Problem 4). The proofs of the existence and optimality of the decentralized solution (Propositions 1 and 2), as well as the continuity

and monotonicity of the demand function (Proposition 3), were given. We developed the Market-based Iterative Risk Allocation (MIRA) algorithm, which is guaranteed to find the optimal solution whenever it exists. The empirical results demonstrated a significant improvement in scalability compared to the centralized algorithm.

IX. ACKNOWLEDGMENTS

Thanks to Michael Kerstetter, Scott Smith, and the Boeing Company for their support. Hirokazu Ishise and Yuichi Yamamoto provided valuable comments on economics.

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