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Finite Horizon Control Design for Optimal Discrimination between Several Models

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Model-based Embedded & Robotic Systems

CSAIL

Context – Model Selection

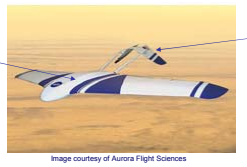
Model Identification
- Which model best explains a given data set?

1. Parameter adaptation
2. **Selection from a finite set of models**
   - **Model Selection**

Example Application

- Aircraft fault diagnosis L3
  - Finite set of models for system dynamics
  - Given data, estimate most likely model

Gyros provide rotation rate data



Model 0: Working Elevator Actuator

Model 1: Faulty Elevator Actuator

Image courtesy of Aurora Flight Sciences

- Standard approach: Multiple Model fault detection[1]
  - Select between a **finite set** of **stochastic linear dynamic systems** using **Bayesian decision rule**

[1] *"Multiple-Model Adaptive Estimation Using a Residual Correlation Kalman Filter Bank"*, Hanlon, P. D. and Maybeck, P. S., IEEE Transactions on Aerospace and Electronic Systems, Vol. 36, No. 2, April 2000.

Control Design for Model Discrimination

- System inputs greatly affect performance of model selection algorithm
- **'Active'** model selection **designs system inputs** to discriminate optimally between models L11
- Previous approaches include (Esposito[2], Goodwin[3], Zhang[4])
  - Designed inputs have limited power to restrict effect on system
  - Maximization of information measure or minimization of detection delay
- We extend these approaches as follows:
  1. Design inputs with **explicit state and input constraints** L12 LB7
  2. Bayesian cost function: **probability of model selection error** L13

- We present novel method that uses **finite horizon constrained optimization approach** to design control inputs for optimal model discrimination
  - Key idea: Minimise **probability of model selection error** subject to explicit **input and state constraints**

[2] *"Probing Linear Filters – Signal Design for the Detection Problem"* Esposito, R. and Schumer, M. A., March 1970.
[3] *"Dynamic System Identification: Experiment Design and Data Analysis"* Goodwin, G. C. and Payne, R. L. 1977.
[4] *"Auxiliary Signal Design in Fault Detection and Diagnosis"* Zhang, X. J. 1989.

Problem Statement

L9

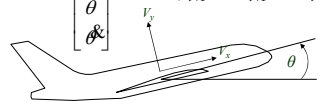
- Design a finite sequence of control inputs $\mathbf{u}=[u_1 \ldots u_k]$ to minimize the probability of model selection error
  - Between **any number** of discrete-time, stochastic linear dynamic models
  - Subject to constraints on inputs and expected state

Example Experiment

- Linearised aircraft model
  - Longitudinal dynamics
- Elevator actuator
  - Model 0: Actuator functional, $B_0=[k\ 0]^T$
  - Model 1: Actuator failed, $B_1=[0\ 0]^T$

$$x = \begin{bmatrix} V_x \\ V_y \\ \theta \\ \dot{\theta} \end{bmatrix}$$

$$x_{t+1} = Ax_t + Bu_t + w_t$$
$$y_{t+1} = Cx_{t+1} + Du_t + v_t$$



$$v_t \sim N(0,R)$$
$$w_t \sim N(0,Q)$$

Slide 3

L3 link discrimination to diagnosis
Lars, 12/8/2005

Slide 4

LB7 link to aircraft eg
Lars Blackmore, 12/2/2005

L11 link to aircraft eg
Lars, 12/10/2005

L12 talk about going to hard limits like MPC
Lars, 12/10/2005

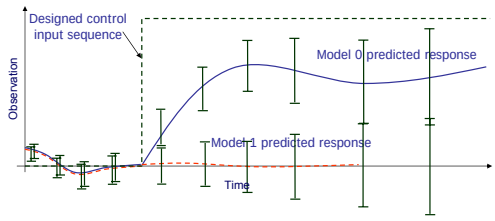
L13 talk about interpretation of information?
Lars, 12/10/2005

Slide 5

L9 put in a picture?
Lars, 12/8/2005

Example Experiment

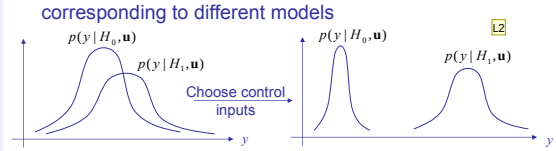
1. Let transients decay to zero
2. Request a large elevator displacement
   - Model 0: Actuator is working, large response observed
   - Model 1: Actuator failed, no response



Key ideas

1. Separate predicted distribution of observations corresponding to different models

L2



2. Can view problem as finite horizon trajectory design
   - Planning **distribution** of future state
   - LP, MILP, SQP commonly used[5][6]
   - Can our cost function work with these formulations?

[5] *"Predictive Control with Constraints"*, Maciejowski, J. M., Prentice Hall, England, 2002.
[6] *"Mixed Integer Programming for Multi-Vehicle Path Planning"* Schouwenaars, T., Moor, B. D., Feron, E. and How, J. P. In Proceedings, European Control Conference, 2001.

Technical Approach: Assumptions

- **Finite set of discrete-time, linear dynamic models,** $H_0 \ldots H_N$, can capture possible behaviors of system
  - One of models is true state of world for entire horizon
- Prior information about models:
  - Some prior distribution over the models
  - Distribution over initial state conditioned on model

  …may be viewed as current belief state from an estimator
- Gaussian process and observation noise
- Bayesian model selection used
  - Batch selection

Technical Approach: Outline

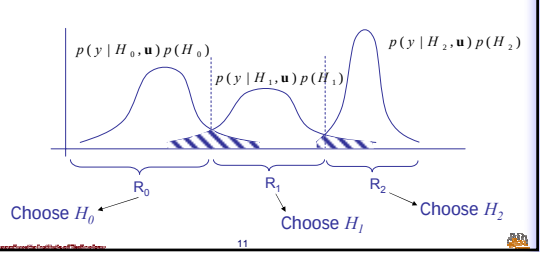
LB25

1. Define Bayesian cost function (probability of error)
2. Describe analytic upper bound to cost function
3. Show that finite horizon problem formulation can be solved using Sequential Quadratic Programming

Trajectory Design Formulation – Cost Function

L14

- Bayesian decision rule:
  - Choose H_i where: $i = \arg\max_i P(H_i \mid \mathbf{y}, \mathbf{u})$
- $P(error|\mathbf{u})$=probability wrong model is selected:



Trajectory Design Formulation – Cost Function

- The probability of model selection error is:

$$p(error \mid \mathbf{u}) = \sum_i \sum_{j \neq i} \int_{\Re_j} p(\mathbf{y} \mid \mathbf{u}, H_i)\, p(H_i)\, d\mathbf{y}$$

- The integral does not have a closed form solution, but can derive an analytic upper bound
- For Gaussian distributions $p(\mathbf{y}|H_i,\mathbf{u}) \sim N(\boldsymbol{\mu}_i, \Sigma_i)$:

$$P(error \mid \mathbf{u}) \leq \sum_i \sum_{j>i} \sqrt{P(H_i)P(H_j)}\, e^{-k(i,j)}$$

$$k(i,j) = \frac{1}{4}(\boldsymbol{\mu}_j - \boldsymbol{\mu}_i)' \left[\Sigma_i + \Sigma_j\right]^{-1} (\boldsymbol{\mu}_j - \boldsymbol{\mu}_i) + \frac{1}{2} \ln \frac{\left|\Sigma_i + \Sigma_j\right|}{2\sqrt{\left|\Sigma_i\right|\left|\Sigma_j\right|}}$$

Linear function of control inputs

Not a function of control inputs

Slide 8

L2 mention what y, H and u are
Lars, 12/8/2005

Slide 10

LB25 cut this?
Lars Blackmore, 12/5/2005

Slide 11

L14 mention what y,H and u are
Lars, 12/8/2005

Trajectory Design Formulation - Constraints

- As in many trajectory design problems, we may want to:
  - Ensure fulfillment of task defined in terms of expected state $E[x_i] = x_{\text{task}}$
  - Bound expected state of the system $|E[x_i]| \le x_{\max}$
  - Model actuator saturation $|u_i| \le u_{\max}$
  - Restrict total fuel usage $\sum_{i=1}^{k} |u_i| \le fuel$
- All of these are **linear constraints**

Trajectory Design Formulation - Summary

- Resulting nonlinear optimization
  1. Cost function that is nonlinear, nonconvex

$$P(error \mid \mathbf{u}) \le \sum_i \sum_{j>i} \sqrt{P(H_i)P(H_j)}\, e^{-k(i,j)}$$

  2. Constraints that are linear in the control inputs
  - E.g. $|E[x_k]| \le x_{\max} \qquad |u_i| \le u_{\max} \quad \forall i$
- Can solve using Sequential Quadratic Programming
  - Local optimality

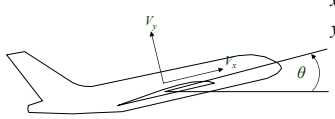
L8

- Now **constrained** active model discrimination possible:
  - Use constraints for control, optimization for discrimination

Simulation Results – Active Approach

- Linearized aircraft discrete-time longitudinal dynamics
- Pitch rate, vertical velocity observed
- Consider 3 single-point failures and nominal model:

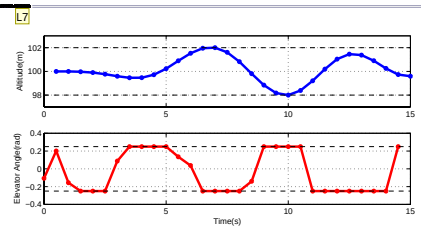
  H_0: Nominal (no faults)
  H_1: Faulty pitch rate sensor (zero mean noise observed)
  H_2: Faulty vertical velocity sensor (zero mean noise observed)
  H_3: Faulty elevator actuator (no response)
- Horizon of 30 time steps, dt = 0.5s



$$x_{t+1} = Ax_t + Bu_t + w_t$$
$$y_{t+1} = Cx_{t+1} + Du_t + v_t$$
$$v_t \sim N(0,R) \qquad x = \begin{bmatrix} V_x \\ V_y \\ \theta \\ \dot{\theta} \end{bmatrix}$$
$$w_t \sim N(0,Q)$$

Results: Constrained Input and State

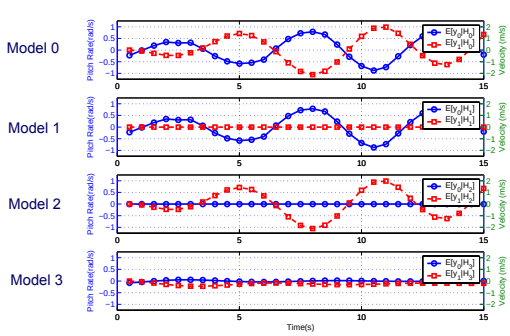
L7



Discrimination-optimal sequence: $p(err) \le 0.0013$

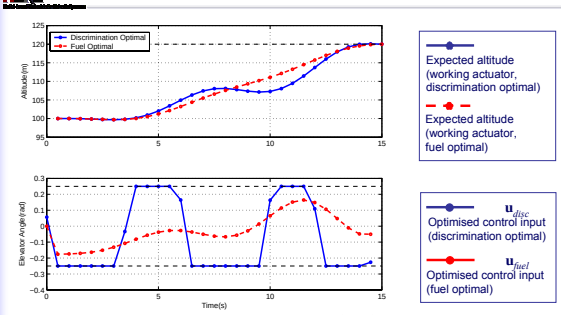
Pilot-generated identification sequence: $p(err) \le 0.063$

Expected Observations



Results: Altitude Change Maneuver

LB12



Discrimination-optimal sequence: $p(err) \le 0.0011$

Fuel-optimal sequence: $p(err) \le 0.12$

Slide 14

L8 explain what I mean by safety
Lars, 12/8/2005

Slide 16

L7 mention constraints explicitly
Lars, 12/8/2005

Slide 18

LB12 up to now, plan is safe but now go to task fulfillment
Lars Blackmore, 12/2/2005

Limitations

- Linear systems only
  - Linearize about an equilibrium point
  - Feedback linearization
- Not **directly** minimizing the probability of error
  - No guarantees about tightness of bound
  - Empirical results show probability of error dramatically reduced
- Local optimality only
  - Comparison with fuel-optimal and manually generated sequences show optimization for discrimination has large impact

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Conclusion

- Novel algorithm for model discrimination between arbitrary number of linear models
- Arbitrary linear state and control constraints can be incorporated
  - Fulfill specified task defined in terms of system state
  - Guarantee safe execution
  - Maintain state within linearisation region

  … while optimally detecting failures

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Questions?

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Backup

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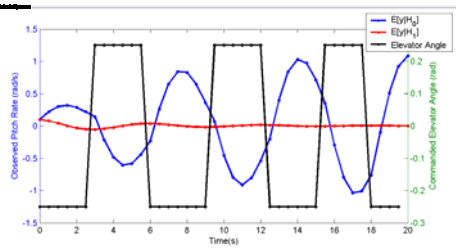
Results: Constrained Elevator Angle



- Optimized sequence drives aircraft at Short Period Oscillation (SPO) mode

Battacharrya bound: 0.0021 QP solution time: 0.19s

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Alternative Criteria

- **Battacharyya bound**

$$(\mu_1-\mu_0)'[\Sigma_0+\Sigma_1]^{-1}(\mu_1-\mu_0)$$

- Baram's Distance
- KL divergence

$$(\mu_1-\mu_0)'\Sigma_0^{-1}(\mu_1-\mu_0)$$

- 'Symmetric' KL divergence

$$(\mu_1-\mu_0)'[\Sigma_0^{-1}+\Sigma_1^{-1}](\mu_1-\mu_0)$$

- Information

$$f(\overline{M}_\theta)$$

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Concave Quadratic Programming

- "*An Algorithm for Global Minimization of Linearly Constrained Concave Quadratic Functions*" Kalantari, B. and Rosen, J. B. Mathematics of Operations Research, Vol. 12, No. 3. August 1987
- O(N) Linear Programs must be solved
- Each LP typically O(NM) number of simplex ops
- M = # constraints
- N = size of QP = (# output variables) x (horizon length)

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Open-loop vs Closed-loop

- Design is open loop
- But can be used within an MPC closed-loop framework
- Efficient QP solution makes this possible

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Cost Criterion

- Can be handled in very similar manner, assuming detector is cost-optimal

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Unbounded Objective Function

- An optimal solution of negative infinity cannot occur with bounded $\mathbf{u}$ if either covariance > 0
- We can get a p(error) of zero for bounded $\mathbf{u}$ if:
  - One of the priors is zero
  - One of the covariances has zero determinant
- Otherwise for bounded $\mathbf{u}$ we cannot.

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