Finite Horizon Control Design for Optimal Discrimination between Several Models
Lars Blackmore and Brian Williams
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Context – Model Selection

Model Identification
- Which model best explains a given data set?
1. Parameter adaptation

2. Selection from a finite set of models
- Model Selection

Example Application

- Aircraft fault diagnosis
  - Finite set of models for system dynamics
  - Given data, estimate most likely model

- Standard approach: Multiple Model fault detection
  - Select between a finite set of stochastic linear dynamic systems using Bayesian decision rule

Control Design for Model Discrimination

- System inputs greatly affect performance of model selection algorithm
- ‘Active’ model selection designs system inputs to discriminate optimally between models
- Previous approaches include (Esposito, Goodwin, Zhang)
  - Designed inputs have limited power to restrict effect on system
  - Maximization of information measure or minimization of detection delay

- We extend these approaches as follows:
  1. Design inputs with explicit state and input constraints
  2. Bayesian cost function: probability of model selection error

We present novel method that uses finite horizon constrained optimization approach to design control inputs for optimal model discrimination

- Key idea: Minimise probability of model selection error subject to explicit input and state constraints

Problem Statement

- Design a finite sequence of control inputs \( u = [u_1 \ldots u_k] \) to minimize the probability of model selection error
  - Between any number of discrete-time, stochastic linear dynamic models
  - Subject to constraints on inputs and expected state

Example Experiment

- Linearised aircraft model
  - Longitudinal dynamics
- Elevator actuator
  - Model 0: Actuator functional, \( B_0 = [k \ 0]^T \)
  - Model 1: Actuator failed, \( B_1 = [0 \ 0]^T \)

\[
\begin{align*}
    x_{1t+1} &= Ax_t + Bu_t + w_t \\
    y_{1t+1} &= Cx_{1t+1} + Du_t + v_t \\
    v_t &\sim N(0, R) \\
    w_t &\sim N(0, Q)
\end{align*}
\]

- Image courtesy of Aurora Flight Sciences

Slide 3

L3 link discrimination to diagnosis
Lars, 12/8/2005

Slide 4

LB7 link to aircraft eg
Lars Blackmore, 12/2/2005

L11 link to aircraft eg
Lars, 12/10/2005

L12 talk about going to hard limits like MPC
Lars, 12/10/2005

L13 talk about interpretation of information?
Lars, 12/10/2005

Slide 5

L9 put in a picture?
Lars, 12/8/2005
Example Experiment

1. Let transients decay to zero
2. Request a large elevator displacement
   - Model 0: Actuator is working, large response observed
   - Model 1: Actuator failed, no response

Key ideas

1. Separate predicted distribution of observations corresponding to different models
   - Planning distribution of future state
   - LP, MILP, SQP commonly used

2. Can view problem as finite horizon trajectory design

1. Define Bayesian cost function (probability of error)
2. Describe analytic upper bound to cost function
3. Show that finite horizon problem formulation can be solved using Sequential Quadratic Programming

Technical Approach: Assumptions

- Finite set of discrete-time, linear dynamic models, $H_0, H_1, \ldots, H_N$, can capture possible behaviors of system
  - One of models is true state of world for entire horizon
- Prior information about models:
  - Some prior distribution over the models
  - Distribution over initial state conditioned on model…may be viewed as current belief state from an estimator
- Gaussian process and observation noise
- Bayesian model selection used
  - Batch selection

Technical Approach: Outline

1. Define Bayesian cost function (probability of error)
2. Describe analytic upper bound to cost function
3. Show that finite horizon problem formulation can be solved using Sequential Quadratic Programming

Trajectory Design Formulation – Cost Function

- Bayesian decision rule:
  - Choose $H_i$ where: $i = \arg \max_i P(H_i | y, u)$
- $P(error|u) =$ probability wrong model is selected:

$$P(error|u) = \int p(y | H_i, u) p(H_i) dy$$

Trajectory Design Formulation – Cost Function

- The probability of model selection error is:
  $$p(error|u) = \sum_i \sum_j \int p(y | H_i, u) p(H_i) dy$$
- The integral does not have a closed form solution, but can derive an analytic upper bound
- For Gaussian distributions $p(y | H_0, u) \sim N(\mu_0, \Sigma_0)$:
  $$P(error|u) \leq \sum_{i,j} \sum_k P(H_i) P(H_j) e^{-k(i,j)}$$
  $$k(i,j) = \frac{1}{4} (\mu_j - \mu_i) (\Sigma_i + \Sigma_j)^{-1} (\mu_j - \mu_i) + \frac{1}{2} \ln \frac{\Sigma_i + \Sigma_j}{2\Sigma_i}$$

- Linear function of control inputs
- Not a function of control inputs
L2  mention what y, H and u are
   Lars, 12/8/2005

LB25 cut this?
   Lars Blackmore, 12/5/2005

L14 mention what y, H and u are
   Lars, 12/8/2005
Trajectory Design Formulation - Constraints

- As in many trajectory design problems, we may want to:
  - Ensure fulfillment of task defined in terms of expected state
  - Bound expected state of the system
  - Model actuator saturation
  - Restrict total fuel usage

- All of these are linear constraints

\[
\begin{align*}
\sum_{k} x_k & \leq x_{\text{max}} \\
\sum_{k} u_k & \leq u_{\text{max}} \\
\end{align*}
\]

Trajectory Design Formulation - Summary

- Resulting nonlinear optimization
  1. Cost function that is nonlinear, nonconvex
  2. Constraints that are linear in the control inputs
     - E.g.

- Can solve using Sequential Quadratic Programming
  - Local optimality

- Now constrained active model discrimination possible:
  - Use constraints for control, optimization for discrimination

Simulation Results – Active Approach

- Linearized aircraft discrete-time longitudinal dynamics
- Pitch rate, vertical velocity observed
- Consider 3 single-point failures and nominal model:
  - \( H_0 \): Nominal (no faults)
  - \( H_1 \): Faulty pitch rate sensor (zero mean noise observed)
  - \( H_2 \): Faulty vertical velocity sensor (zero mean noise observed)
  - \( H_3 \): Faulty elevator actuator (no response)

- Horizon of 30 time steps, \( dt = 0.5s \)

\[
\begin{align*}
x_{k+1} &= A x_k + B u_k + w_k \\
y_{k+1} &= C x_k + D u_k + v_k \\
  &\quad \text{where } x = [x', y', \theta, \phi, \psi] \\
  &\quad v_i \sim N(0, R) \\
  &\quad w_i \sim N(0, Q) \\
\end{align*}
\]

Results: Constrained Input and State

- Discrimination-optimal sequence: \( p(\text{err}) \leq 0.0013 \)
- Pilot-generated identification sequence: \( p(\text{err}) \leq 0.063 \)

Expected Observations

<table>
<thead>
<tr>
<th>Model 0</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected altitude (working actuator, discrimination optimal)</td>
<td>Expected altitude (working actuator, fuel optimal)</td>
<td>Optimized control input (discrimination optimal)</td>
<td>Optimized control input (fuel optimal)</td>
</tr>
</tbody>
</table>

Results: Altitude Change Maneuver

- Discrimination-optimal sequence: \( p(\text{err}) \leq 0.0011 \)
- Fuel-optimal sequence: \( p(\text{err}) \leq 0.12 \)
L8 explain what I mean by safety
Lars, 12/8/2005

L7 mention constraints explicitly
Lars, 12/8/2005

LB12 up to now, plan is safe but now go to task fulfillment
Lars Blackmore, 12/2/2005
Limitations

- Linear systems only
  - Linearize about an equilibrium point
  - Feedback linearization
- Not directly minimizing the probability of error
  - No guarantees about tightness of bound
  - Empirical results show probability of error dramatically reduced
- Local optimality only
  - Comparison with fuel-optimal and manually generated sequences show optimization for discrimination has large impact

Conclusion

- Novel algorithm for model discrimination between arbitrary number of linear models
- Arbitrary linear state and control constraints can be incorporated
  - Fulfill specified task defined in terms of system state
  - Guarantee safe execution
  - Maintain state within linearisation region
  … while optimally detecting failures

Questions?

Results: Constrained Elevator Angle

- Optimized sequence drives aircraft at Short Period Oscillation (SPO) mode

Battacharyya bound: 0.0021
QP solution time: 0.19s

Alternative Criteria

- **Battacharyya bound**
  \[ \frac{(\mu_1 - \mu_0)(\Sigma_0 + \Sigma_i)^{-1}(\mu_1 - \mu_0)}{\log(2\pi e) + \log(\det(\Sigma + \Sigma_i))} \]
- Baram’s Distance
- KL divergence
- ‘Symmetric’ KL divergence
- Information

\[ f(M) \]
Concave Quadratic Programming

- O(N) Linear Programs must be solved
- Each LP typically O(NM) number of simplex ops
- M = # constraints
- N = size of QP = (# output variables) x (horizon length)

Open-loop vs Closed-loop

- Design is open loop
- But can be used within an MPC closed-loop framework
- Efficient QP solution makes this possible

Cost Criterion

- Can be handled in very similar manner, assuming detector is cost-optimal

Unbounded Objective Function

- An optimal solution of negative infinity cannot occur with bounded u if either covariance > 0
- We can get a p(error) of zero for bounded u if:
  - One of the priors is zero
  - One of the covariances has zero determinant
- Otherwise for bounded u we cannot.