Massachusetts Institute of Technology

Finite Horizon Control Design for Optimal Model Discrimination

Lars Blackmore and Brian Williams

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Model-based Embedded & Robotic Systems

CSAIL

Context – Model Selection

Model Identification

– Which model best explains a given data set?

1. Parameter adaptation

2. **Selection from a finite set of models**
   - **Model Selection**

Example Application

- Aircraft fault diagnosis
  - Finite set of models for system dynamics
  - Given data, estimate most likely model

Gyros provide rotation rate data

Model 0: Working Elevator Actuator

Model 1: Faulty Elevator Actuator

Image courtesy of Aurora Flight Sciences

- Standard approach: Multiple Model fault detection[1]
  - Select between a **finite set** of **stochastic linear dynamic systems** using **Bayesian decision rule**

[1] *"Multiple-Model Adaptive Estimation Using a Residual Correlation Kalman Filter Bank"*, Hanlon, P. D. and Maybeck, P. S., IEEE Transactions on Aerospace and Electronic Systems, Vol. 36, No. 2, April 2000.

Control Design for Model Discrimination

- System inputs greatly affect performance of model selection algorithm
- **'Active'** model selection **designs system inputs** to discriminate optimally between models
- Previous approaches include (Esposito[2], Goodwin[3], Zhang[4])
  - Designed inputs have limited power to restrict effect on system
  - Maximization of information measure or minimization of detection delay
- We extend these approaches as follows:
  1. Design inputs with **explicit state and input constraints**
  2. Bayesian cost function: **probability of model selection error**
- We present novel method that uses **finite horizon constrained optimization approach** to design control inputs for optimal model discrimination
  - Key idea: Minimise **probability of model selection error** subject to explicit **input and state constraints**

[2] *"Probing Linear Filters – Signal Design for the Detection Problem"* Esposito, R. and Schumer, M. A., March 1970.
[3] *"Dynamic System Identification: Experiment Design and Data Analysis"* Goodwin, G. C. and Payne, R. L. 1977.
[4] *"Auxiliary Signal Design in Fault Detection and Diagnosis"* Zhang, X. J. 1989.

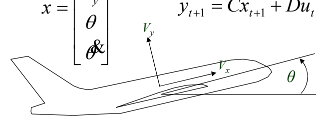
Problem Statement

- Design a finite sequence of control inputs $\mathbf{u}=[u_1 \ldots u_k]$ to minimize the probability of model selection error
  - Between **two** discrete-time, stochastic linear dynamic models
  - Subject to constraints on inputs and expected state

Example Experiment

- Linearised aircraft model
  - Longitudinal dynamics
- Elevator actuator
  - Model 0: Actuator functional, $B_0=[k\ 0]^T$
  - Model 1: Actuator failed, $B_1=[0\ 0]^T$

$$x = \begin{bmatrix} V_x \\ V_y \\ \theta \\ \dot{\theta} \end{bmatrix} \qquad \begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_{t+1} &= Cx_{t+1} + Du_t + v_t \end{aligned}$$



$$v_t \sim N(0,R) \qquad w_t \sim N(0,Q)$$

Slide 3

L3 link discrimination to diagnosis
Lars, 12/8/2005

Slide 4

LB7 link to aircraft eg
Lars Blackmore, 12/2/2005

L11 link to aircraft eg
Lars, 12/10/2005

L12 talk about going to hard limits like MPC
Lars, 12/10/2005

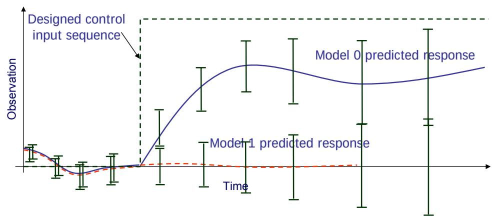
L13 talk about interpretation of information?
Lars, 12/10/2005

Slide 5

L9 put in a picture?
Lars, 12/8/2005

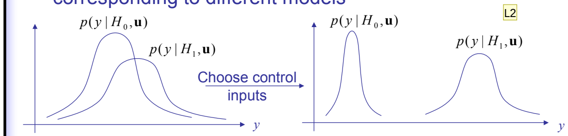
Example Experiment

1. Let transients decay to zero
2. Request a large elevator displacement
   - Model 0: Actuator is working, large response observed
   - Model 1: Actuator failed, no response



Key ideas

1. Separate predicted distribution of observations corresponding to different models



2. Can view problem as finite horizon trajectory design
   - Planning **distribution** of future state
   - LP, MILP, QP commonly used[5][6]
   - Can our cost function work with these formulations?

[5] *"Predictive Control with Constraints"*, Maciejowski, J. M., Prentice Hall, England, 2002.
[6] *"Mixed Integer Programming for Multi-Vehicle Path Planning"* Schouwenaars, T., Moor, B. D., Feron, E. and How, J. P. In Proceedings, European Control Conference, 2001.

Technical Approach: Assumptions

- **Two discrete-time, linear dynamic models,** H_0 and H_1, can capture possible behaviors of system
  - One of models is true state of world for entire horizon
- Prior information about models:
  - Some prior distribution over the two models
  - Distribution over initial state conditioned on model

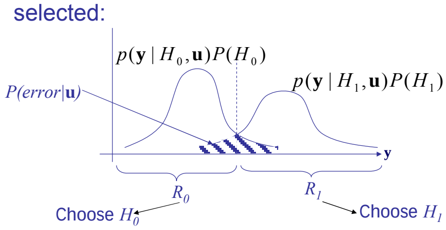
  …may be viewed as current belief state from an estimator
- Gaussian process and observation noise
- Bayesian model selection used
  - Batch selection

Technical Approach: Outline

1. Define Bayesian cost function (probability of error)
2. Describe analytic upper bound to cost function
3. Show that upper bound is quadratic in inputs
4. Show that finite horizon problem formulation can be solved using Quadratic Programming

Trajectory Design Formulation – Cost Function

- Bayesian decision rule:
  - Choose H_0 if: $P(H_0|\mathbf{y},\mathbf{u}) > P(H_1|\mathbf{y},\mathbf{u})$
- $P(error|\mathbf{u})$=probability that the wrong model is selected:



Trajectory Design Formulation – Cost Function

- The probability of model selection error is:

$$p(error|\mathbf{u}) = \int_{\Re_1} p(\mathbf{y}|H_0,\mathbf{u})P(H_0)d\mathbf{y} + \int_{\Re_0} p(\mathbf{y}|H_1,\mathbf{u})P(H_1)d\mathbf{y}$$

- The integral does not have a closed form solution, but an **upper bound** exists called the Battacharrya Bound[7]
- For Gaussian distributions $p(\mathbf{y}|H_0,\mathbf{u})\sim N(\boldsymbol{\mu}_0,\Sigma_0)$ and $p(\mathbf{y}|H_1,\mathbf{u})\sim N(\boldsymbol{\mu}_1,\Sigma_1)$:

$$P(error|\mathbf{u}) \le \sqrt{P(H_0)P(H_1)}\, e^{-\frac{1}{4}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)'[\Sigma_0+\Sigma_1]^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) - \ln\frac{\left|\frac{\Sigma_0+\Sigma_1}{2}\right|}{\sqrt{|\Sigma_0||\Sigma_1|}}}$$

- Take logarithm:

$$\ln(P(error)) \le f(\text{priors}) - \frac{1}{4}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)'[\Sigma_0+\Sigma_1]^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) + f(\Sigma_0,\Sigma_1)$$

[7] *"Pattern Classification"*, Duda, R., Hart, P., Stork, D., 2000

Slide 8

L2 mention what y, H and u are
Lars, 12/8/2005

Slide 10

LB21 cut this?
Lars Blackmore, 12/5/2005

Slide 11

L1 mention what y,H and u are
Lars, 12/8/2005

Slide 12

LB9 Go through each term
Lars Blackmore, 12/2/2005

Trajectory Design Formulation – Cost Function

- Important analytic properties:
  1. Expected observations μ_0 and μ_1 are known **linear** functions of the inputs $\mathbf{u}$
  2. The covariances Σ_0 and Σ_1 are **not** functions of the inputs
  3. Covariances and priors **do not affect optimization**
- Cost function for trajectory design problem:

$$J = f(\text{priors}) - \frac{1}{4}(\mu_1 - \mu_0)'[\Sigma_0 + \Sigma_1]^{-1}(\mu_1 - \mu_0) + f(\Sigma_0, \Sigma_1)$$

- **Cost function is quadratic in the inputs**

LB18

Trajectory Design Formulation - Constraints

- As in many trajectory design problems, we may want to:
  - Ensure fulfillment of task defined in terms of expected state $E[x_i] = x_{task}$
  - Bound expected state of the system $|E[x_i]| \le x_{max}$
  - Model actuator saturation $|u_i| \le u_{max}$
  - Restrict total fuel usage $\sum_{i=1}^{k} |u_i| \le fuel$
- All of these are **linear constraints**

Trajectory Design Formulation - Summary

- What we have is a **Quadratic Program**:
  1. Cost function that is quadratic in the control inputs

$$J = [u_1 \ \cdots \ u_k]\mathbf{H}\begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix} + \mathbf{f}\begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}$$

  2. Constraints that are linear in the control inputs
  - E.g. $|E[x_k]| \le x_{max} \qquad |u_i| \le u_{max} \quad \forall i$
- Quadratic Programs can be solved efficiently

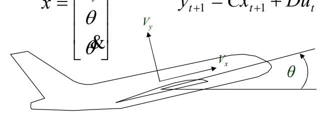
L8

- Now **constrained** active model discrimination possible:
  - **On-line,** while **fulfilling defined task** and **ensuring safety**

Simulation Results – Active Approach

- Elevator failure scenario
- Linearised, discrete-time longitudinal dynamics
- Pitch rate observed
- Horizon of 40 time steps

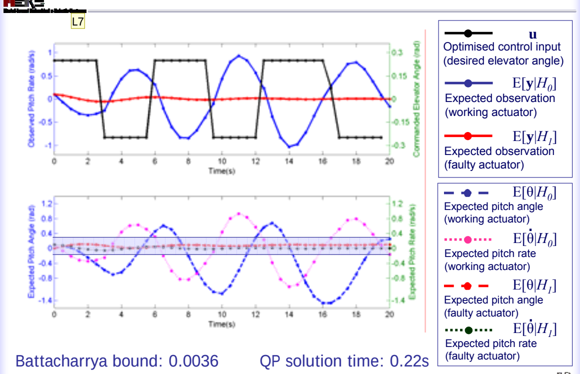
$$x = \begin{bmatrix} V_x \\ V_y \\ \theta \\ \dot{\theta} \end{bmatrix} \qquad x_{t+1} = Ax_t + Bu_t + w_t \qquad y_{t+1} = Cx_{t+1} + Du_t + v_t \qquad \Delta t = 0.5s$$



$$v_t \sim N(0, R) \qquad w_t \sim N(0, Q)$$

Results: Constrained Input and State

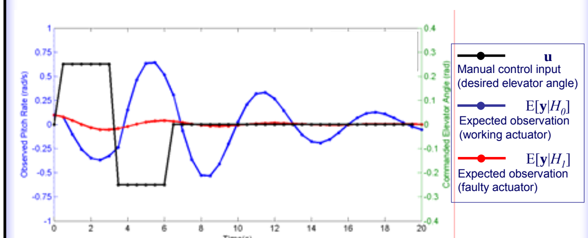
L7



Battacharrya bound: 0.0036 QP solution time: 0.22s

Results: Manually Generated Sequence

- Typical pilot-generated identification sequence



Battacharrya bound: 0.16

Slide 13

LB18 Also mention concave
Lars Blackmore, 12/2/2005

Slide 15

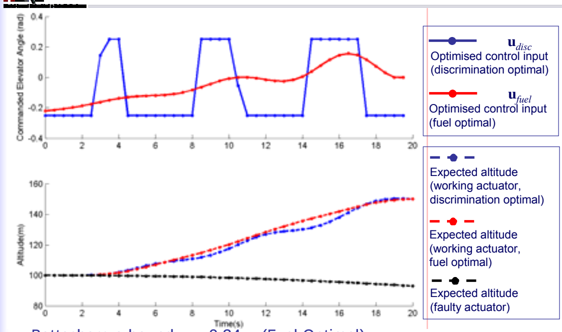
L8 explain what I mean by safety
Lars, 12/8/2005

Slide 17

L7 mention constraints explicitly
Lars, 12/8/2005

Results: Altitude Change Maneuver

LB12



Battacharrya bound: 0.34 (Fuel Optimal)
0.0046 (Discrimination Optimal)

Limitations

- Linear systems only
  - Linearise about an equilibrium point
  - Feedback linearization
- Not **directly** minimising the probability of error
  - No guarantees about tightness of bound
  - Low upper bound sufficient in many cases
- **Two models only**
  - Battacharrya bound does not apply to more than two hypotheses
  - Submission soon on generalized approach

Conclusion

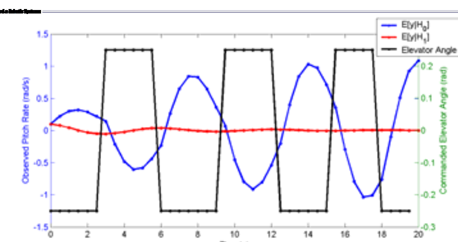
- Novel algorithm for model discrimination between any two linear systems
- On-line solution possible due to efficient Quadratic Programming formulation
- Arbitrary linear state and control constraints can be incorporated
  - Fulfill specified task defined in terms of system state
  - Guarantee safe execution
  - Maintain state within linearisation region

… while optimally detecting failures

Questions?

Backup

Results: Constrained Elevator Angle



- Optimized sequence drives aircraft at Short Period Oscillation (SPO) mode

Battacharrya bound: 0.0021 QP solution time: 0.19s

Slide 19

LB12 up to now, plan is safe but now go to task fulfillment
Lars Blackmore, 12/2/2005

Alternative Criteria

- **Battacharyya bound**

$$(\mu_1 - \mu_0)'[\Sigma_0 + \Sigma_1]^{-1}(\mu_1 - \mu_0)$$

- Baram's Distance
- KL divergence

$$(\mu_1 - \mu_0)'\Sigma_0^{-1}(\mu_1 - \mu_0)$$

- 'Symmetric' KL divergence

$$(\mu_1 - \mu_0)'[\Sigma_0^{-1} + \Sigma_1^{-1}](\mu_1 - \mu_0)$$

- Information

$$f(\overline{M}_\theta)$$

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Concave Quadratic Programming

- "*An Algorithm for Global Minimization of Linearly Constrained Concave Quadratic Functions*" Kalantari, B. and Rosen, J. B. Mathematics of Operations Research, Vol. 12, No. 3. August 1987
- O(N) Linear Programs must be solved
- Each LP typically O(NM) number of simplex ops
- M = # constraints
- N = size of QP = (# output variables) x (horizon length)

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Open-loop vs Closed-loop

- Design is open loop
- But can be used within an MPC closed-loop framework
- Efficient QP solution makes this possible

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Cost Criterion

- Can be handled in very similar manner, assuming detector is cost-optimal

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Unbounded Objective Function

- An optimal solution of negative infinity cannot occur with bounded $\mathbf{u}$ if either covariance > 0
- We can get a p(error) of zero for bounded $\mathbf{u}$ if:
  - One of the priors is zero
  - One of the covariances has zero determinant
- Otherwise for bounded $\mathbf{u}$ we cannot.

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