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Active Estimation for Switching Linear Dynamic Systems

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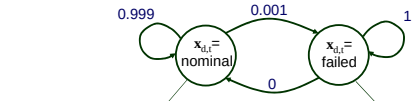
January 31, 2007

Model-based Embedded & Robotic Systems

CSAIL

Context – Hybrid Systems

- Hybrid discrete-continuous models are convenient for many systems
  - Failure-prone components *(Funiak03, Dearden02)*
  - Piloted aircraft *(Tomlin06)*
  - Insects *(Oh05)*
- Example: Switching Linear Dynamic Systems



$$\mathbf{x}_{c,t+1} = A_{\text{nominal}}\mathbf{x}_{c,t} + B_{\text{nominal}}\mathbf{u}_t + \upsilon_{\text{nominal}}$$
$$\mathbf{y}_t = C_{\text{nominal}}\mathbf{x}_{c,t} + D_{\text{nominal}}\mathbf{u}_t + \omega_{\text{nominal}}$$

$$\mathbf{x}_{c,t+1} = A_{\text{failure}}\mathbf{x}_{c,t} + B_{\text{failure}}\mathbf{u}_t + \upsilon_{\text{failure}}$$
$$\mathbf{y}_t = C_{\text{failure}}\mathbf{x}_{c,t} + D_{\text{failure}}\mathbf{u}_t + \omega_{\text{failure}}$$

Context – Hybrid State Estimation

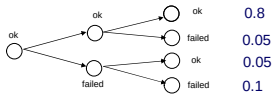
- Hybrid state estimation aims to determine:

$$p(\mathbf{x}_{c,t}, \mathbf{x}_{d,t} \mid \mathbf{y}_{1:T}, \mathbf{u}_{0:T-1})$$

  - Applications include fault detection, intent recognition…
- Exact hybrid estimation is intractable *(Lerner01)*
- Prior work has developed **approximate** approaches, for example:
  - Merging *(Lerner00)*
  - Pruning *(Hofbaur02)*
  - Sampling *(Doucet00)*

K-Best Hybrid State Estimation

- Full hybrid estimation considers all mode sequences



- K-best enumeration retains the k mode sequences with highest posterior probability
- Problem: losing true mode sequence
  - Fault detection particularly problematic

Active Hybrid Estimation

- System inputs greatly affect performance of hybrid estimator
- Prior work has used control inputs for optimal discrimination between linear dynamic models
- We present a novel method that uses control inputs to aid hybrid state estimation
  - Key idea: Minimize probability of losing true mode sequence subject to explicit input and state constraints

Problem Statement

- Design a finite sequence of control inputs $\mathbf{u}=[u_0 \ldots u_h]$ to minimize $p(loss)$, the probability of losing the true discrete mode sequence
  - Subject to constraints on inputs and expected state
    - For Switching Linear Dynamic Systems
    - Zero mean, Gaussian white process and observation noise
    - Assume pruning occurs at **end of horizon** L16

Slide 6

L16 mention can't calculate in closed form now
Lars, 12/12/2006

Summary of Technical Approach

1. Express future mode sequences as multiple known time-varying models
2. Bound *p(loss)* using Multiple-Model bound from *(Blackmore06)*
3. Make bound tractable using efficient pruning approach
4. Minimize bound using constrained optimization

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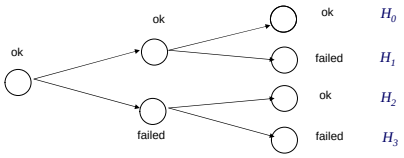
Mode Sequences as Models

- Each mode sequence corresponds to a known Linear Time Varying (LTV) model
  - Finite number of 'hypotheses'
  - Hybrid estimation picks k most likely hypotheses



- Idea: Use results from *(Blackmore06)* for multiple-model discrimination to upper-bound *p(loss)*

Bounding *p(loss)*

- Loss probability upper-bounded by probability that true mode sequence is not *most likely* sequence

$$p(loss) = p(\text{true sequence not in top } k) \le p(\text{true sequence not } \textit{most likely} \text{ hypothesis})$$

- Extend bound in *(Blackmore06)* to SLDS to give:

$$P(loss \mid \mathbf{u}) \le \sum_i \sum_{j>i} \sqrt{P(H_i)P(H_j)}\, e^{-k(i,j)}$$

$$k(i,j) = \frac{1}{4}(\boldsymbol{\mu}_j - \boldsymbol{\mu}_i)'\left[\Sigma_i + \Sigma_j\right]^{-1}(\boldsymbol{\mu}_j - \boldsymbol{\mu}_i) + \frac{1}{2}\ln \frac{\left|\Sigma_i + \Sigma_j\right|}{2\sqrt{\left|\Sigma_i\right|\left|\Sigma_j\right|}}$$

➢ Analytic upper bound on *p(loss)*
- Problem: exponential number of hypotheses
- Solution: find looser bound that considers subset

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Considering a Subset of Sequences

- Full bound with all terms:

$$P(loss) \le \sum_i \sum_{j>i} F_{ij}(\mathbf{u}_{0:T-1})$$

- Individual terms can be replaced by looser bound:

$$G_{ij} = P(H_i)^{1/2} P(H_j)^{1/2} \ge F_{ij}(\mathbf{u}_{0:T-1})$$

- These terms do not depend on $\mathbf{u}_{0:T-1}$
  - Need not be considered in optimization
- Challenge:
  - Replace terms so resulting bound is as tight as possible

$$P(loss) \le \sum_{i \in S} \sum_{j>i,\, j \in S} F_{ij}(\mathbf{u}_{0:T-1}) + \sum_{i \notin S} \sum_{j>i,\, j \notin S} G_{ij}$$

Considering a Subset of Sequences

- Worst-case difference between $F_{ij}(\mathbf{u}_{0:T-1})$ and G_{ij} is:

$$P(H_i)^{1/2} P(H_j)^{1/2}$$

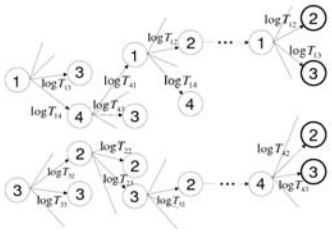
- Include in S the mode sequences that maximize:

$$\sum_{i \in S} \sum_{j>i, j \in S} P(H_i)^{1/2} P(H_j)^{1/2}$$

- S must include hypotheses with highest prior $p(H_i)$
- **Challenge:** efficient enumeration of mode sequences with highest prior probability

Considering a Subset of Sequences

- Idea: Use graph search to find highest priors



$x_{d,0}$ $x_{d,1}$ $x_{d,2}$ $x_{d,3}$ … $x_{d,h-1}$ $x_{d,h}$

$$\max p(x_{d,0} : x_{d,t}) = p(x_{d,0}) \prod_{i=1}^{h} T_{x_{d,i} x_{d,i-1}}$$

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Optimization: Constraints

- As in many trajectory design problems, we may want to:
  - Ensure fulfillment of task defined in terms of expected state $E[x_i] = x_{\text{task}}$
  - Bound expected state of the system $|E[x_i]| \le x_{\max}$
  - Model actuator saturation $|u_i| \le u_{\max}$
  - Restrict total fuel usage $\sum_{i=1}^{k} |u_i| \le fuel$
- All of these are **linear constraints**

Optimization: Overall Formulation

- Resulting nonlinear optimization
  1. Cost function that is nonlinear, nonconvex

$$P(loss \mid \mathbf{u}) \le \sum_i \sum_{j>i} \sqrt{P(H_i) P(H_j)}\, e^{-k(i,j)}$$

  2. Constraints that are linear in the control inputs
  - E.g. $|E[x_k]| \le x_{\max} \qquad |u_i| \le u_{\max} \;\; \forall i$
- Can solve using Sequential Quadratic Programming
  - Local optimality

- Now **constrained** active hybrid estimation possible:
  - Use constraints for control, optimization for discrimination

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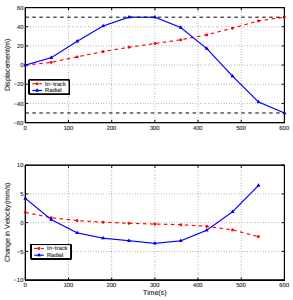
Simulation Results – Active Approach

- Satellite dynamics linearized about nominal circular orbit (Hill's equations)
- Motion in two dimensions considered (in-track and radial)
- Sensors:
  - Radial and in-track velocity
- Actuators
  - Radial and in-track thrusters
- Hybrid model has 4 discrete modes:

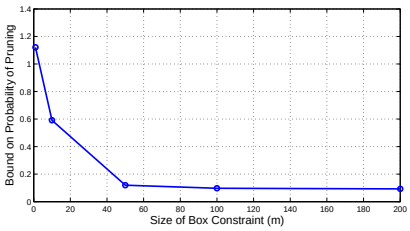
  Mode 0: Nominal (no faults)
  Mode 1: Radial velocity sensor failure (zero mean noise observed)
  Mode 2: In-track velocity sensor failure (zero mean noise observed)
  Mode 3: Radial thruster failure (no response)
- Horizon of 10 time steps, dt = 60s

Results: Box-Constrained Maneuver



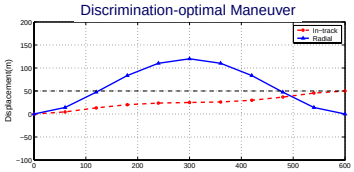
$p(loss) \le 0.12$

Results: Box-Constrained Maneuver



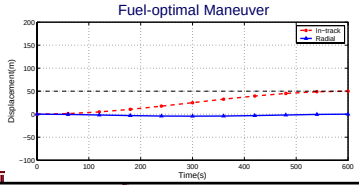
Results: Displacement Maneuver

Discrimination-optimal Maneuver



$p(loss) \le 0.10$

Fuel-optimal Maneuver



$p(loss) \le 0.87$

Conclusion

- A novel approach for **active** hybrid estimation
  - Minimize upper bound on probability of losing true mode sequence, subject to constraints on inputs and state

Questions?

Slide 20

L7 mention constraints explicitly
Lars, 12/8/2005

Summary of Approach

1. Hybrid Estimation calculates approximate belief state
   - Distribution over k mode sequence
   - Continuous distribution conditioned on mode sequence
2. Best first search enumerates s most likely future mode sequences
3. Form cost function with s most likely sequences
4. Optimize subject to constraints, using SQP
5. Execute control inputs, while estimating hybrid state

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