FIRST-ORDER NECESSARY CONDITIONS FOR ROBUST OPTIMALITY*

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ABSTRACT

The first-order necessary conditions of optimality are extended to include information about robust design — cost insensitivity to model inaccuracies and changes in design specification, as well as the manufacturing tolerances treated in more traditional approaches. In these extended conditions, the Lagrangian is formulated as a tradeoff between cost and cost variability, where variability is measured as the flatness and curvature of the objective relative to local variations in design variables and constraints. During optimization these conditions allow cost and robustness to be considered simultaneously at each iteration.

1. INTRODUCTION

With international competition to produce quality products with rapid time to market, the field of cost optimization has become increasingly important for both research and practice. However, in addition to being optimal with respect to objectives like weight and value, quality products must be *robust*, *i.e.*, cost insensitive to variations in manufacturing, modeling and performance criteria. A robust design, then, is one that has minimal variation in performance as the design optimum shifts. In particular, a robust design is considered one that is insensitive to perturbations in design variables, design constraints (inequality constraints), performance criteria (the objective) and the model itself (equality constraints). Thus, existing optimization frameworks must be reformulated to account for robustness, and in particular, the Karush-Kuhn-Tucker (KKT) conditions of optimality must be re-thought.

The work presented here makes several novel contributions. First, while most work on robustness concentrates on minimizing the likelihood that process variations lead to infeasible designs (*e.g.*, faulty products) our approach concentrates on minimizing variations in cost. Second, our approach accounts for sensitivity to constraint variation, resulting from model approximations, experimental error, and changes in design specification. Third, our metric for robustness provides a conservative estimate of cost deviation based on the properties of the gradient and Hessian norms, corresponding to the tilt and curvature of the objective around the design solution. Fourth, the conditions developed here are formulated as the minimization of a *Robust Lagrangian* — a weighted

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combination of nominal cost and cost variation of the traditional Lagrangian. This allows a simultaneous tradeoff between cost and robustness which may be utilized concurrently to make design decisions. Finally, our formulation of robust optimality as an extension of the Lagrangian and KKT provides a vehicle for extending existing numerical optimization codes to move in optimally robust directions. The paper develops conditions for robust optimality by re-examining the KKT conditions, defining concepts of robustness in terms of the norms of slope and curvature, and presenting the first order necessary conditions of robust optimality.

2. EXAMPLE: PASSIVE FILTER NETWORK

Consider the following example taken from Suh (1990). Figure 1 illustrates a circuit diagram for a passive filter network where V_s is a time varying input and V_0 is the filter output. The network is required to obtain a full scale deflection (x) of $x_t = 3$ inches and a filter cutoff frequency (f) of $f_t = 6.84$ Hz. The design objective is to minimize the quality loss (Q) defined by:

$$Q = .5(f-f_t)^2 + .5(x-x_t)^2.$$

The frequency is defined as:

$$f = \frac{(R_2 + R_g)(R_s + R_3) + R_3 R_s}{2\pi(R_2 + R_g)R_3 R_s C},$$

and the deflection is defined as:

$$x = \frac{R_3 R_g V}{G\left[(R_2 + R_g)(R_s + R_3) + R_3 R_s\right]}.$$

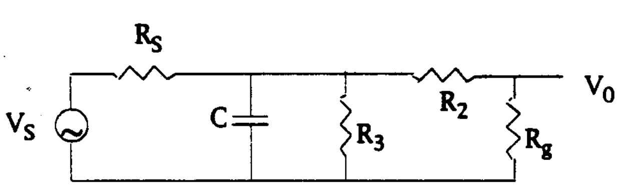

Figure 1. PASSIVE FILTER NETWORK.

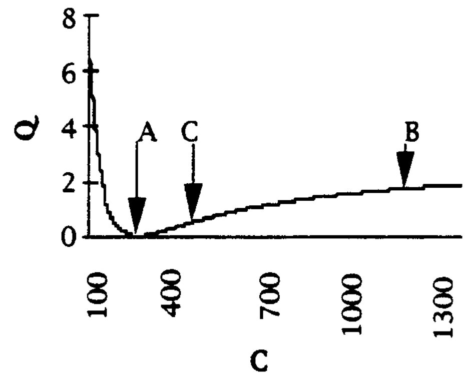

Figure 2. PLOT OF C VERSUS OBJECTIVE Q.

Two resistors (R_2 and R_3) and the capacitance (C) can each be adjusted. In this example, $R_s = 120\Omega$, $R_g = 98\Omega$, $G = 657.58\mu V/in.$, and $V = 15mV$. The cost optimal solution minimizing Q has $R_2 = 478\Omega$, and $R_3 = 1416\Omega$.

Figure 2 shows a plot of C versus Q along the cost optimal values of R_2 and R_3. Note that traditional optimization approaches would select point A as the optimum value of Q. Certainly the point represents the cost optimal solution; however, note how steep the curve climbs to the left of the point. A slight variation away from the solution, to the left, will significantly change the value of the objective function. Such a variation may occur because the models of the physical components are inaccurate, or the components vary in performance due to the allowable tolerance range.

A good design is robust in that it will not vary significantly away from the solution point due to slight variations in the physical state. Consider point B in Figure 2. The curve in the neighborhood of B is much more flat than that of point A, and thus is more robust. Note, however, that the cost of point B is also higher than that of point A; cost is sacrificed for robustness.

Finally consider point C. This solution is more costly than A and slightly more sensitive to variations than B. C is thus a good compromise between robustness and cost.

In this paper the tradeoffs between cost and robustness are defined through a mathematical formulation which extends the Karush-Kuhn-Tucker (KKT) conditions of optimality. After the KKT conditions are re-examined, robustness is defined and incorporated into the optimality condition. The resulting first order conditions can be used to solve problems such as this circuit, and is illustrated analytically on a simple objective with properties similar to the filter.

3. KARUSH-KUHN-TUCKER CONDITIONS

The basis for theories of optimization stem from the Karush-Kuhn-Tucker (KKT) conditions, which model the first-order necessary conditions of optimality. Consider an optimization problem formulated as:

$$\begin{aligned} \text{min:} \quad & f(\mathbf{x}) \\ \text{s.t.:} \quad & \mathbf{g}(\mathbf{x}) \le \mathbf{0} \\ & \mathbf{h}(\mathbf{x}) = \mathbf{0}, \end{aligned}$$

where $\mathbf{x}$ is a vector of variables, $f(\mathbf{x})$ is the objective function, $\mathbf{g}(\mathbf{x})$ is a vector of inequality constraints, and $\mathbf{h}(\mathbf{x})$ is a vector of equality constraints. In an unconstrained optimization problem a necessary condition for optimality of $\mathbf{x}$ is $\nabla f(\mathbf{x}) = \mathbf{0}$ (*i.e.*, $\mathbf{x}$ is stationary). For constrained optimization the Lagrangian $L(\mathbf{x})$ extends the objective to account for these constraints:

$$\begin{aligned} & L(\mathbf{x}) = f(\mathbf{x}) + \lambda^T\mathbf{h}(\mathbf{x}) + \mu^T\mathbf{g}(\mathbf{x}), \\ & \mathbf{g}(\mathbf{x}) \le 0, \\ & \mathbf{h}(\mathbf{x}) = 0. \end{aligned}$$

where λ and μ are the Lagrange multipliers of the equality and inequality constraints. The KKT conditions for $\mathbf{x}$ being a constrained optimum (called $\mathbf{x}^*$), including constraints on μ, is stated as:

$$\nabla L(\mathbf{x}^*) = \nabla f(\mathbf{x}^*) + \lambda^T\nabla\mathbf{h}(\mathbf{x}^*) + \mu^T\nabla\mathbf{g}(\mathbf{x}^*) = \mathbf{0}, \qquad \text{(KKT-1)}$$

$$\mu_i g_i(\mathbf{x}) = 0, \qquad \text{(KKT-2)}$$

$$\mu \ge \mathbf{0}. \qquad \text{(KKT-3)}$$

Chapter 8 of Strang (1986) as well as chapter 4 of Papalambros and Wilde (1988) provide concise developments of these conditions, derived by Karush (1939), and Kuhn and Tucker (1951). The conditions state that a point is *stationary*, and possibly optimum, if the first derivative of the objective is zero in any direction feasible with respect to the equality and inequality constraints. The second condition states that either the inequality constraint is equal to zero, indicating that the constraint is active, or its corresponding Lagrange multiplier is zero, indicating that the constraint is inactive. The third condition states that the Lagrange multiplier for each inequality constraint must be positive or zero.

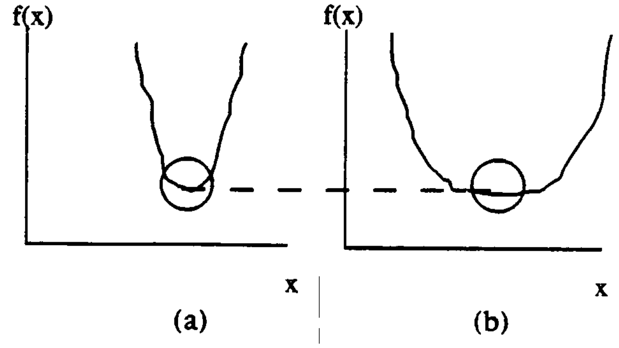

Figure 3. TWO EQUALLY OPTIMAL SOLUTIONS WHERE THE FIRST DERIVATIVE IS ZERO.

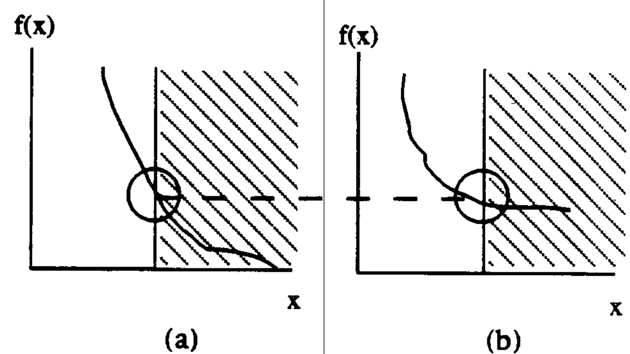

Figure 4. TWO EQUALLY OPTIMAL SOLUTIONS WHERE THE OBJECTIVE IS BOUNDED BY THE CONSTRAINTS.

Since KKT is a first order condition, it does not differentiate between Figures 3a and 3b, and Figures 4a and 4b. In each figure the minimum is found at the same value; however, in Figure 3b cost is less sensitive to variation about the minimum than in Figure 3a, and cost in Figure 4b is less sensitive to variation of the constraint than in Figure 4a. Thus KKT is not a measure of robustness.

4. ROBUST DESIGN

Optimization for nominal cost has been extensively studied. Nevertheless, this approach can be quite problematic if the design is slightly perturbed from the optimum. For example, a perturbation produced by manufacturing error may result in a product that fails to meet its specification — the perturbation results in an infeasible design. Techniques that reduce the likelihood that a perturbation produces infeasibility, such as statistical tolerancing, have been well examined (see Evans, 1974, 1975a, 1975b).

There are, however, other sources and effects of design perturbations that are not well understood. In terms of effects, although a perturbed design may remain feasible, its cost might change drastically, where cost may be a measure of dollar value, weight or performance. In terms of sources of perturbations, optimization codes necessarily operate on models and design specifications that are only approximations of the physical world. Discrepancies introduced through these idealizations may substantially affect cost. This paper focuses on robustness conditions that minimize cost variation due to modeling inaccuracies and manufacturing errors. Although not explicitly addressed, the conditions developed here for minimizing cost variation are complementary to techniques like statistical tolerancing which reduce the effects of variations on design feasibility.

Variations produced through manufacturing errors and environmental noise are modeled as perturbations $\Delta\mathbf{x}$ to the design variables $\mathbf{x}$. Intuitively a robust design $\mathbf{x}^*$ is one whose objective is least sensitive to small variations about the nominal solution, regardless of the cost objective. In the ideal case the objective is locally flat around $\mathbf{x}^*$. That is,

$$f(\mathbf{x}^*) = f(\mathbf{x}^* + \Delta\mathbf{x}),$$

for $|\Delta\mathbf{x}| < \varepsilon_{\mathbf{x}}$, where $\varepsilon_{\mathbf{x}}$ is a vector of positive perturbations in the neighborhood of $\mathbf{x}^*$. Figure 5 shows two designs that are equally robust, although not equally cost minimized. One indicator of cost sensitivity is the "tilt" of the hyperplane $h(\mathbf{x})$ tangent to f at $\mathbf{x}^*$,

$$\mathbf{h}(\mathbf{x}^*+\Delta\,\mathbf{x}) = f(\mathbf{x}^*)+\nabla f(\mathbf{x}^*)^T\ \Delta\mathbf{x}.$$

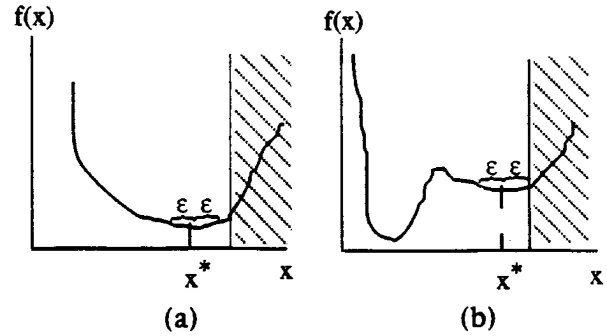

Figure 5. TWO EQUALLY ROBUST DESIGNS.

In our robustness metric tilt is measured using the gradient of the objective, $\nabla f(\mathbf{x}^*)$. A second indicator of cost sensitivity is the curvature of the objective around $\mathbf{x}^*$. For example, the points circled in Figures 3a and 3b both have zero tilt; however, the curvature of the point in Figure 3a is greater than that of Figure 3b, making it less robust. In our robustness metric curvature is measured from the Hessian of the objective,

$$\mathbf{H}(\mathbf{x}^*) = \nabla^2 f(\mathbf{x}^*).$$

The relative significance of tilt versus curvature depends on the degree to which $\mathbf{x}$ is perturbed. For small variations tilt dominates, but as the variations grow curvature becomes significant, and eventually dominates.

Inaccuracies in the model or design specification may result from measurement error, engineering approximation, or partial design knowledge. Incorporating these influences into our criteria, a robust design is one that is also insensitive to variations in inequality constraints $\mathbf{g}(\mathbf{x})$, equality constraints $\mathbf{h}(\mathbf{x})$ and the objective $f(\mathbf{x})$. The Lagrange multipliers λ and μ are indicators of the objective's sensitivity to variations in $\mathbf{h}$ and $\mathbf{g}$. They and their gradients are used in our robustness metric to measure tilt resulting from constraint variation.

Alternative approaches to robustness are found in Taguchi (*e.g.*, Taguchi and Wu, 1979) where signal (target) to noise (variation) ratios are calculated based on target variables (Taguchi's inner arrays) and external variations (Taguchi's outer arrays); however, variable interactions are limited in the modeling assumptions when utilizing these orthogonal arrays. Sundaresan, *et al.*, (1991) use Taguchi's orthogonal arrays to search for a robust design by calculating flatness of the objective through analysis of the statistical optimum. d'Entremont and Ragsdell (1988) also utilize Taguchi's orthogonal arrays within optimization code to minimize variance. In contrast, our theory simultaneously considers both cost optimality and robustness through a formal presentation of optimality conditions which, in theory, does not presuppose a limited model of variable independence.

5. FIRST-ORDER NECESSARY CONDITIONS OF ROBUST OPTIMALITY

5.1 Trading Cost for Robustness

We now develop a criteria for robust optimality based on the intuitions of the previous section. The resulting conditions determine solutions that optimize a weighted combination of cost and robustness objectives, and in their general form are an extension of KKT. A designer's intuitions about robust design are quantified through an objective $f_R(\mathbf{x})$. A design that optimally trades off cost with robustness is found by minimizing the weighted combination $f_{TOTAL}(\mathbf{x})$ of the cost and robustness objectives,

$$f_{TOTAL}(\mathbf{x}) = \alpha f(\mathbf{x}) + \beta f_R(\mathbf{x}),$$

where α and β are user supplied constants specifying the relative importance of robustness versus cost. The first order necessary condition for $\mathbf{x}^*$ being an unconstrained robust optimum is simply

$$\nabla f_{TOTAL}(\mathbf{x}^*) = \mathbf{0}.$$

The next subsection develops $f_R(\mathbf{x})$ and the robust optimality criteria for unconstrained optima, over perturbations in $\mathbf{x}$. This corresponds roughly to the influence of manufacturing tolerances on cost variation. The second subsection pursues a similar development for constrained optima, replacing $f(\mathbf{x})$ with its Lagrangian. The set of perturbations are expanded to those in the objective and the constraints. This corresponds to the effects of inaccuracies in modeling.

5.2 Robust Optimality for Unconstrained Optima

A robust design is one that has minimal cost variance ρ within a region, $|\Delta\mathbf{x}| < \varepsilon_{\mathbf{x}}$, about the design solution $\mathbf{x}^*$. We define variance as

$$\rho(\mathbf{x}+\Delta\mathbf{x}) = |f(\mathbf{x}+\Delta\mathbf{x}) - f(\mathbf{x})|.$$

For the robustness objective $f_R(\mathbf{x})$, we use a conservative estimate of the maximum cost variance throughout the region around a nominal point $\mathbf{x}$,

$$f_R(\mathbf{x}) \geq \text{Max}\ \rho(\mathbf{x}+\Delta\mathbf{x}) \text{ for } |\Delta\mathbf{x}| < \varepsilon_{\mathbf{x}}.$$

As was argued in the preceding section, the robustness objective f_R must include a measure of the objective's curvature, as well as its tilt, since near the traditional cost optimum tilt goes to zero, as shown in Figure 3. Tilt and curvature correspond to the first two terms in a Taylor series expansion of the objective around $\mathbf{x}$. Thus the objective is approximated using a second order Taylor series expansion, under the presumption that $\varepsilon_{\mathbf{x}}$ is sufficiently small that higher order terms are negligible. That is,

$$f(\mathbf{x}+\Delta\mathbf{x}) \approx f(\mathbf{x}) + \nabla f(\mathbf{x})^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T H(\mathbf{x})\Delta\mathbf{x},$$

where

$$H(\mathbf{x}) = \nabla^2 f(\mathbf{x}) = \left(\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}\right).$$

Substituting for f in the variance ρ results in

$$\rho(\mathbf{x}+\Delta\mathbf{x}) \approx |\nabla f(\mathbf{x})^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T H(\mathbf{x})\Delta\mathbf{x}|.$$

Returning to $f_R(\mathbf{x})$, computing the maximum of $\rho(\mathbf{x}+\Delta\mathbf{x})$ exactly for $|\Delta\mathbf{x}| < \varepsilon_x$ involves solving a constrained quadratic optimization problem for every application of the robustness objective, and can be computationally prohibitive. Instead, we construct a conservative estimate of the maximum using the norms of the gradient and Hessian, $\|\nabla f(\mathbf{x})\|$ and $\|\mathbf{H}(\mathbf{x})\|$, where the norms are measures of the objective's degree of tilt and curvature at x. By selecting *compatible* norms for the gradient and Hessian,

$\|Ax\| \le \|A\| \ \|x\|$ for vector **x** and matrix **A**,

and by using the triangular inequalities and related properties of both norms, the right hand side of the relation for $\rho(\mathbf{x}+\Delta\mathbf{x})$ is bounded above by

$$|\nabla f(\mathbf{x})^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T H(\mathbf{x})\Delta\mathbf{x}| \le$$

$$|\nabla f(\mathbf{x})\| \ \|\Delta\mathbf{x}\| + |\frac{1}{2}| \ \|H(\mathbf{x})\| \ \|\Delta\mathbf{x}\|^2 .$$

This upper bound reaches a maximum at $\Delta\mathbf{x} = \varepsilon_x$, thus defining a robust objective as

$$f_R(\mathbf{x}) = \|\nabla f(\mathbf{x})\| \ \|\varepsilon_x\| + \frac{1}{2} \|H(\mathbf{x})\| \ \|\varepsilon_x\|^2$$

provides a conservative estimate of the maximum cost variance in the neighborhood of **x**.

In the above equations any pair of compatible norms can be used for the gradient and Hessian. For the gradient we select the Euclidean norm (also called the 2-norm),

$$\|\nabla f(\mathbf{x})\| = \sqrt{\sum_i \left(\frac{\partial f(\mathbf{x})}{\partial x_i}\right)^2}$$

This measures the length of the gradient vector, and corresponds to the tilt of the plane tangent to $f(\mathbf{x})$.

For the Hessian we use the Frobenius norm, which generalizes the Euclidean norm to matrices,

$$\|\mathbf{H}(\mathbf{x})\|_F = \sqrt{\sum_i \sum_j \left(\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}\right)}.$$

This provides a measure of the typical curvature of f at **x**, and is compatible with the Euclidean norm. Note that an alternative satisfying compatibility is the *Spectral* norm,

$$\|\mathbf{H}(\mathbf{x})\|_2 = E_{max} [\mathbf{H}(\mathbf{x})^T \mathbf{H}(\mathbf{x})]^{1/2},$$

where E_{max} is the square root of the largest eigenvalue of $\mathbf{H}(\mathbf{x})^T \mathbf{H}(\mathbf{x})$. This norm provides a measure of worst case curvature.

Using the metric for robustness just developed, the combined objective becomes,

$$f_{TOTAL}(\mathbf{x}) = \alpha f(\mathbf{x}) + \beta\left[\|\nabla f(\mathbf{x})\| \ \|\varepsilon_x\| + \frac{1}{2}\|\mathbf{H}(\mathbf{x})\| \ \|\varepsilon_x\|^2\right]$$

where α , β, and $\|\varepsilon_x\|$ are supplied by the designer.

Finally, the first order necessary condition for unconstrained robust optimality at $\mathbf{x}^*$ results from setting the gradient of this combined objective to zero,

$$\alpha\nabla f(\mathbf{x}^*) + \beta \ \|\varepsilon_x\| \ \nabla\|\nabla f(\mathbf{x}^*)\| + \frac{1}{2}\beta \|\varepsilon_x\|^2 \ \nabla\|H(\mathbf{x}^*)\| = 0.$$

The first term measures absolute cost, the second tilt and the third curvature. The significance of curvature relative to tilt grows with the amount, $\|\varepsilon_x\|$, that **x** may be perturbed. Thus this condition captures the intuitions of section 4.

5.3 __Robust Optimality [illegible] Constrained Optima__

A particularl[illegible] [illegible]gnificant influence on robustness, which [illegible] not been addressed in the literature, is that [illegible] variations in the design model and specifications, as modeled through the design constraints. Accounting for these requires a shift to constrained optimization problems:

min:	$f(\mathbf{x})$
s.t.:	$\mathbf{g}(\mathbf{x}) \le 0$
	$\mathbf{h}(\mathbf{x}) = 0,$

and the incorporation of variations in the objective and the equality and inequality constraints: Δf, $\Delta\mathbf{h}$, $\Delta\mathbf{g}$. Equality constraints model the physics, heuristics, and general design knowledge of the design problem. Model variations result from the engineering approximations selectively made during model formulation, as well as the inherently approximate nature of even the most thorough model, measurement inaccuracies, the absence of detailed knowledge during the early stages of the design process, or decisions to change technologies later on in the design process. Inequality constraint variations reflect variations in design specifications or manufacturing processes, where the inequality constraints represent acceptable performance (usually in the worst case), the precision of the manufacturing processes, or the limits and failure modes of the materials and devices used.

Recall from section 3 that, given the objective f of a constrained optimization problem, we reformulate this to an unconstrained problem with equivalent minima, whose objective is the Lagrangian of f,

$$L(\mathbf{x}) = f(\mathbf{x}) + \lambda^T \mathbf{h}(\mathbf{x}) + \mu^T \mathbf{g}(\mathbf{x}),$$

where

$$\mu_i g_i(\mathbf{x}) = 0,$$
$$\mu \geq 0.$$

Our approach to robust optimality for constrained problems is similar. We use the Lagrangian to map between constrained and unconstrained optima. We use a construction similar to the previous section to map the objective to one that measures the effects of perturbations on the objective, where perturbations are in f, **g**, and **h**[1]. The result we call the robust Lagrangian L_R.

Similar to f_R, we base L_R on a conservative estimate of cost variance in L, in response to perturbations. First, the effects of perturbations on the Lagrangian are described by:

$$L(\mathbf{x}+\Delta\mathbf{x}, f+\Delta f, \mathbf{g}+\Delta\mathbf{g}, \mathbf{h}+\Delta\mathbf{h}) = f(\mathbf{x}+\Delta\mathbf{x}) + \Delta f + \mu(\mathbf{x}+\Delta\mathbf{x})^T\left[\mathbf{g}(\mathbf{x}+\Delta\mathbf{x}) + \Delta\mathbf{g}\right] + \lambda(\mathbf{x}+\Delta\mathbf{x})^T\left[\mathbf{h}(\mathbf{x}+\Delta\mathbf{x}) + \Delta\mathbf{h}\right].$$

By distributing μ and λ over **g** and **h** respectively, and accumulating perturbations (Δf, Δ**g** and Δ**h**) to the model, this is equivalent to:

$$L(\mathbf{x}+\Delta\mathbf{x}, f+\Delta f, \mathbf{g}+\Delta\mathbf{g}, \mathbf{h}+\Delta\mathbf{h}) = L(\mathbf{x}+\Delta\mathbf{x}) + \Delta f + \mu(\mathbf{x}+\Delta\mathbf{x})^T \Delta\mathbf{g} + \lambda(\mathbf{x}+\Delta\mathbf{x})^T \Delta\mathbf{h}.$$

To estimate the first term, $L(\mathbf{x}+\Delta\mathbf{x})$, we need a second order approximation around **x**:

$$L(\mathbf{x}+\Delta\mathbf{x}) \approx L(\mathbf{x}) + \nabla L(\mathbf{x})^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T \nabla^2 L(\mathbf{x}) \Delta\mathbf{x}$$

As was the case for the estimate of $f(\mathbf{x}+\Delta\mathbf{x})$ in the unconstrained case, the robust optimum may lie near the traditional cost optimum, in which case $\nabla L(\mathbf{x}) \approx \mathbf{0}$. Thus the first order term approaches zero, and the second order term becomes significant. Mapping to our intuitions about robustness, the second and third terms of the approximation measure the tilt and curvature of the objective with respect to parameter variations.

For $\mu(\mathbf{x}+\Delta\mathbf{x})$, a first order approximation is sufficient,

$$\mu(\mathbf{x}+\Delta\mathbf{x}) \approx \mu(\mathbf{x}) + \nabla\mu(\mathbf{x})^T \Delta\mathbf{x}.$$

$\nabla\mu$ will typically not be zero when its corresponding constraint is active, and the first order term dominates the second. Additionally, the μ term is multiplied by a small Δ**g**, further reducing the significance of the second order term relative to L. A similar approximation is used for λ.

$$\lambda(\mathbf{x}+\Delta\mathbf{x}) \approx \lambda(\mathbf{x}) + \nabla\lambda(\mathbf{x})^T \Delta\mathbf{x}.$$

Mapping to our intuitions about robustness, λ and μ are, roughly speaking, measures of the objective's sensitivity ($\partial f/\partial\mathbf{g}$ and $\partial f/\partial\mathbf{h}$) to constraint variations. Thus the first and second terms of these approximations correspond to the tilt and curvature of the objective with respect to the constraints.

Incorporating these approximations results in,

$$L(\mathbf{x}+\Delta\mathbf{x}, f+\Delta f, \mathbf{g}+\Delta\mathbf{g}, \mathbf{h}+\Delta\mathbf{h}) \approx L(\mathbf{x}) + \nabla L(\mathbf{x})^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T \nabla^2 L(\mathbf{x}) \Delta\mathbf{x} +$$

[1] We drop the argument **x** where unambiguous, thus f, **g**, **h**, λ, μ and L are used to represent f(**x**), **g**(**x**), **h**(**x**), λ(**x**), μ(**x**) and L(**x**), respectively.

$$\Delta f + \left(\mu(\mathbf{x}) + \nabla\mu(\mathbf{x})^T \Delta\mathbf{x}\right)^T \Delta\mathbf{g} +$$
$$\left(\lambda(\mathbf{x}) + \nabla\lambda(\mathbf{x})^T \Delta\mathbf{x}\right)^T \Delta\mathbf{h}.$$

Analogous to section 5.2, we define cost variance of L as

$$\zeta(\mathbf{x}+\Delta\mathbf{x}, \mathbf{g}+\Delta\mathbf{g}, \mathbf{h}+\Delta\mathbf{h}) = |L(\mathbf{x}+\Delta\mathbf{x}, \mathbf{g}+\Delta\mathbf{g}, \mathbf{h}+\Delta\mathbf{h}) - L(\mathbf{x})|.$$

A robust design is taken to be one that has minimal worst case variance in L throughout the region, $|\Delta\mathbf{x}| < \varepsilon_x$, $|\Delta f| < \varepsilon_f$, $|\Delta\mathbf{g}| < \varepsilon_g$, $|\Delta\mathbf{h}| < \varepsilon_h$, about the design solution **x**, where ε_x, ε_g, and ε_h are vectors of maximum positive deviations in **x**, **g**, and **h**, respectively.

Substituting our approximation of L in ζ results in

$$\zeta(\mathbf{x}+\Delta\mathbf{x}, \mathbf{g}+\Delta\mathbf{g}, \mathbf{h}+\Delta\mathbf{h}) \approx \left| \begin{array}{l} \nabla L^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T \nabla^2 L \Delta\mathbf{x} + \Delta f + \\ \left(\mu + \nabla\mu^T \Delta\mathbf{x}\right)^T \Delta\mathbf{g} + \left(\lambda + \nabla\lambda^T \Delta\mathbf{x}\right)^T \Delta\mathbf{h} \end{array} \right|$$

where note that the parameters **x** of each function have been left implicit. The first two terms measure the effect on variance of the tilt and curvature of the Lagrangian with respect to **x**. The third term, Δf, measures the effect of variations in the objective. The fourth and fifth terms measure the effect of variations in the constraints on robustness, where μ and λ are approximately the sensitivities $\frac{\partial f}{\partial \mathbf{g}}$ and $\frac{\partial f}{\partial \mathbf{h}}$, respectively.

Using the definition of Lagrangian the gradient in the first term becomes

$$\begin{aligned} \nabla L &= \nabla\left(f + \mu^T\mathbf{g} + \lambda^T\mathbf{h}\right) \\ &= \nabla f + \nabla\left(\mu^T\mathbf{g}\right) + \nabla\left(\lambda^T\mathbf{h}\right) \\ &= \nabla f + \nabla\mathbf{g}\mu + \nabla\mu\mathbf{g} + \nabla\mathbf{h}\lambda + \nabla\lambda\mathbf{h} \\ &= \nabla f + \nabla\mathbf{g}\mu + \nabla\mathbf{h}\lambda, \end{aligned}$$

which is the familiar right hand side of KKT. Note that on the third line the fifth term is eliminated since **h** is always zero, and the third term is eliminated since μ, and thus $\nabla\mu$, is zero whenever **g** is non-zero. The Hessian in the second term of ζ simplifies to

$$\begin{aligned} \nabla^2 L &= \nabla\left(\nabla f + \nabla\mathbf{g}\mu + \nabla\mathbf{h}\lambda\right) \\ &= \nabla^2 f + \nabla\left(\nabla\mathbf{g}\mu\right) + \nabla\left(\nabla\mathbf{h}\lambda\right) \\ &= \nabla^2 f + \nabla^2\mathbf{g}\mu + \nabla\mu\nabla\mathbf{g}^T + \nabla^2\mathbf{h}\lambda + \nabla\lambda\nabla\mathbf{h}^T. \end{aligned}$$

Substituting these observations for the gradient and Hessian of L in ζ results in

$$\zeta(\mathbf{x}+\Delta\mathbf{x}, \mathbf{g}+\Delta\mathbf{g}, \mathbf{h}+\Delta\mathbf{h}) \approx \left| \begin{array}{l} \left(\nabla f + \nabla\mathbf{g}\mu + \nabla\mathbf{h}\lambda\right)^T \Delta\mathbf{x} + \\ \frac{1}{2}\Delta\mathbf{x}^T\left(\nabla^2 f + \nabla^2\mathbf{g}\mu + \nabla\mu\nabla\mathbf{g}^T + \nabla^2\mathbf{h}\lambda + \nabla\lambda\nabla\mathbf{h}^T\right)\Delta\mathbf{x} + \\ \Delta f + \left(\mu + \nabla\mu^T\Delta\mathbf{x}\right)^T \Delta\mathbf{g} + \left(\lambda + \nabla\lambda^T\Delta\mathbf{x}\right)^T \Delta\mathbf{h} \end{array} \right|.$$

The conservative estimate used for the robust Lagrangian, $L_R(\mathbf{x})$, is:

$$L_R(\mathbf{x}) \geq \mathrm{Max}\ \zeta(\mathbf{x}+\Delta\mathbf{x}, \mathbf{g}+\Delta\mathbf{g}, \mathbf{h}+\Delta\mathbf{h})$$
$$\text{for } |\Delta\mathbf{x}| < \varepsilon_x, |\Delta f| < \varepsilon_f, |\Delta\mathbf{g}| < \varepsilon_g, |\Delta\mathbf{h}| < \varepsilon_h$$

As in section 5.2, we construct an upper bound of $\zeta(\mathbf{x}+\Delta\mathbf{x}, \mathbf{g}+\Delta\mathbf{g}, \mathbf{h}+\Delta\mathbf{h})$, by selecting the Euclidean and Frobenius norms as compatible norms for the first and second derivatives, by applying the triangular inequalities and related properties of both norms, producing

$$\begin{aligned} \zeta(\mathbf{x}+\Delta\mathbf{x}, \mathbf{g}+\Delta\mathbf{g}, \mathbf{h}+\Delta\mathbf{h}) \leq\ & \left\|\nabla f + \nabla\mathbf{g}\mu + \nabla\mathbf{h}\lambda\right\|\left\|\Delta\mathbf{x}\right\| + \\ & \frac{1}{2}\left\|\nabla^2 f + \nabla^2\mathbf{g}\mu + \nabla\mu\nabla\mathbf{g}^T + \nabla^2\mathbf{h}\lambda + \nabla\lambda\nabla\mathbf{h}^T\right\|\left\|\Delta\mathbf{x}\right\|^2 + \\ & |\Delta f| + \|\mu\|\|\Delta\mathbf{g}\| + \|\lambda\|\|\Delta\mathbf{h}\| + \\ & \left(\|\nabla\mu\|\|\Delta\mathbf{g}\| + \|\nabla\lambda\|\|\Delta\mathbf{h}\|\right)\|\Delta\mathbf{x}\|. \end{aligned}$$

The maximum of this upper bound on ζ occurs at the extremes of the variations, $|\Delta\mathbf{x}| = \varepsilon_x$, $|\Delta f| = \varepsilon_f$, $|\Delta\mathbf{g}| = \varepsilon_g$, and $|\Delta\mathbf{h}| = \varepsilon_h$, thus

$$L_R(\mathbf{x}) =$$
$$\left\|\nabla f + \nabla \mathbf{g}\mu + \nabla \mathbf{h}\lambda\right\|\left\|\varepsilon_{\mathbf{x}}\right\| +$$
$$\frac{1}{2}\left\|\nabla^2 f + \nabla^2 \mathbf{g}\mu + \nabla\mu\nabla\mathbf{g}^T + \nabla^2\mathbf{h}\lambda + \nabla\lambda\nabla\mathbf{h}^T\right\|\left\|\varepsilon_{\mathbf{x}}\right\|^2 +$$
$$\left|\Delta f\right| + \left\|\mu\right\|\left\|\varepsilon_{\mathbf{g}}\right\| + \left\|\lambda\right\|\left\|\varepsilon_{\mathbf{h}}\right\| +$$
$$\left(\left\|\nabla\mu\right\|\left\|\varepsilon_{\mathbf{g}}\right\| + \left\|\nabla\lambda\right\|\left\|\varepsilon_{\mathbf{h}}\right\|\right)\left\|\varepsilon_{\mathbf{x}}\right\|.$$

The total Lagrangian is now approximated as:

$$L_{TOTAL}(\mathbf{x}) = \alpha\left\{f + \lambda^T\mathbf{h} + \mu^T\mathbf{g}\right\} + \beta\left\{\begin{array}{l}\left\|\varepsilon_{\mathbf{x}}\right\|\left\|\nabla f + \nabla\mathbf{g}\mu + \nabla\mathbf{h}\lambda\right\| + \\ \frac{1}{2}\left\|\varepsilon_{\mathbf{x}}\right\|^2\left\|\nabla^2 f + \nabla^2\mathbf{g}\mu + \nabla\mu\nabla\mathbf{g}^T + \nabla^2\mathbf{h}\lambda + \nabla\lambda\nabla\mathbf{h}^T\right\| + \\ \left\|\varepsilon_{\mathbf{f}}\right\| + \left\|\varepsilon_{\mathbf{g}}\right\|\left\|\mu\right\| + \left\|\varepsilon_{\mathbf{h}}\right\|\left\|\lambda\right\| + \\ \left\|\varepsilon_{\mathbf{x}}\right\|\left(\left\|\varepsilon_{\mathbf{g}}\right\|\left\|\nabla\mu\right\| + \left\|\varepsilon_{\mathbf{h}}\right\|\left\|\nabla\lambda\right\|\right)\end{array}\right\}.$$

Note that the calculation of some of these terms, such as the derivatives of the Lagrange multipliers, may be difficult analytically; however, they are computable numerically.

Finally, the first order necessary conditions of robust optimality for constrained problems results from setting the gradient of the combined Lagrangian to zero

$$\nabla L_{TOTAL}(\mathbf{x}^*) = 0$$

and thus

$$\alpha\left\{\nabla f + \nabla\mathbf{h}\lambda + \nabla\mathbf{g}\mu\right\} + \beta\left\{\begin{array}{l}\left\|\varepsilon_{\mathbf{x}}\right\|\nabla\left\|\nabla f + \nabla\mathbf{g}\mu + \nabla\mathbf{h}\lambda\right\| + \\ \frac{1}{2}\left\|\varepsilon_{\mathbf{x}}\right\|^2\nabla\left\|\begin{array}{l}\nabla^2 f + \nabla^2\mathbf{g}\mu + \nabla\mu\nabla\mathbf{g}^T \\ +\nabla^2\mathbf{h}\lambda + \nabla\lambda\nabla\mathbf{h}^T\end{array}\right\| + \\ \left\|\varepsilon_{\mathbf{g}}\right\|\nabla\left\|\mu\right\| + \left\|\varepsilon_{\mathbf{h}}\right\|\nabla\left\|\lambda\right\| + \\ \left\|\varepsilon_{\mathbf{x}}\right\|\left(\left\|\varepsilon_{\mathbf{g}}\right\|\nabla\left\|\nabla\mu\right\| + \left\|\varepsilon_{\mathbf{h}}\right\|\nabla\left\|\nabla\lambda\right\|\right)\end{array}\right\} = 0$$

subject to:

$$\mu_i g_i(\mathbf{x}^*) = 0,$$
$$\mu \geq 0.$$

The first line is a weighted measure of nominal cost, The second and third lines measure the tilt and curvature, respectively, of the objective with respect to **x**, biased by the active constraints within **h** and **g**. The fourth and fifth lines measure the tilt and curvature, respectively, with respect to the constraints. Finally, note that the term involving ε_x has been eliminated. Since variation in f is modeled as independent of **x**, it changes the robustness of all points uniformly, and thus does not influence which point is most robust.

With these conditions, the designer chooses the relative weightings between cost optimality (through α) and robustness (through β). Further, the designer chooses the neighborhood over which sensitivities will be considered (through ε_x, ε_g, and ε_h).

6. EXAMPLE: SIGMOID FUNCTION

Recall the example of the passive filter network described in section 1. Although the problem illustrates the concepts of robustness versus cost optimality, the model leads to a first order necessary condition of robust optimality that is too complicated to solve analytically in a short presentation. Instead, we apply the conditions of robust optimality to a simpler problem which maintains the basic properties of separate optimal and robust regions of the objective, with the addition of simple inequality constraints. A constrained sigmoid function is formulated as:

min:	$f(x) = 8 - (x-2)^3$	
s.t.:	$.5 - x \leq 0$	$g_1(x)$
	$x - 3.5 \leq 0.$	$g_2(x)$

Figure 6 shows this sigmoid function and the constraints. In this example we assume that μ dominates $\nabla\mu$ and thus terms with $\nabla\mu$ are neglected. A more accurate solution would include these terms, computing $\nabla\mu$ numerically.

The total Lagrangian is formulated as:

$$L_{TOTAL}(x) = \alpha\{(8 - (x-2)^3) + \mu_1(.5 - x) + \mu_2(x - 3.5)\} + \beta\{\|(-3(x-2)^2 - \mu_1 + \mu_2)\|\,\|\varepsilon_x\| + \|-3(x-2)\|\,\|\varepsilon_x\|^2 + \|\varepsilon_f\| + \|\mu\|\,\|\varepsilon_g\|\}.$$

The first order necessary conditions of robust optimality are found by differentiating the total Lagrangian and setting it equal to zero:

$$\alpha\{-3(x-2)^2 - \mu_1 + \mu_2\} +$$
$$\beta\{|6(x-2)|\, \|\varepsilon_x\| + 3\|\varepsilon_x\|^2\} = 0,$$

subject to:

$$\mu_1(.5 - x) = 0,$$
$$\mu_2(x - 3.5) = 0,$$

where the $\nabla\mu$ term is dropped. Let $\varepsilon_x = .25$.

Let us consider the tradeoff between robustness and cost. For cost optimization only ($\alpha = 1$; $\beta = 0$) the solution falls on constraint g_2 ($\mu_1 = 0$ and $\mu_2 \neq 0$ implying g_2 active) with $x = 3.5$. For robustness only ($\alpha = 0$; $\beta = 1$), neither constraint is active ($\mu_1 = 0$ and $\mu_2 = 0$) and the robust solution occurs at the point of inflection at $x = 2$. Finally, considering a skewed tradeoff between robustness ($\beta = .75$) and cost (3: $\alpha = .25$ and $\beta = .75$, the solution lies at $x = 1.884$.

Note that for the uncosntrained cases, the first order necessary conditions become:

$$\alpha\{-3(x-2)^2\} + \beta\{1.5|(x-2)| + .1875\} = 0.$$

For $\alpha \neq 0$, the x value for the minimum is:

$$x = \frac{-1.5\beta \pm \sqrt{2.25(\beta+\alpha)}}{-6\alpha} + 2;$$

for $\alpha = 0$, the x value for the minimum is $x = 2$.

In this problem three points are of interest (see Figure 6). Point A ($x = 3.5$) is the minimum for the constrained cost optimum; point B ($x = 2.0$) illustrates the optimum robustness; point C ($x = 1.884$) models a weighted tradeoff between robustness (75%) and cost optimum (25%).

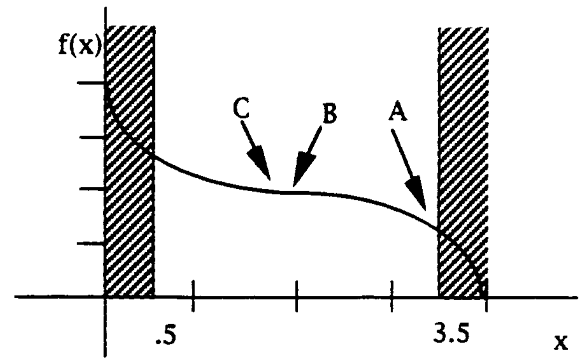

Figure 6. SIGMOID FUNCTION.

7. CONCLUSIONS

The first-order necessary conditions of robust optimality are presented as an extension to the Karush-Kuhn-Tucker conditions. These conditions are novel in that they account for the influence on robustness of model variations, as well as parameter variations, and they provide a simultaneous tradeoff of performance for robustness. The conditions dictate that the robust optimum will include terms that minimize cost and variation about the optimum; the tradeoff between these terms is defined by the designer. The criteria define robustness as a measure of tilt and curvature in a neighborhood about the optimum; our metric provides a conservative estimate of cost deviation based on the properties of the gradient and Hessian norms. The robust Lagrangian, a weighted combination of nominal cost and cost variation of the traditional Lagrangian, is approximated as a second order Taylor series modeling both cost variation and constraint variation. This allows a simultaneous tradeoff between cost and robustness which may be utilized concurrently to make design decisions. Finally, because our formulation of robust optimality is an extension of the Lagrangian and KKT, existing numerical optimization codes may be extended to converge on optimally robust minima. Such an implementation is necessary to examine more complicated, industrially relevant problems.

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