

Is the solar system stable? and Can we use chaos to make measurements?

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Abstract. This talk addresses two separate questions: “Is the solar system stable?” and “Can we use chaos to make better measurements?” In the first part, a review is presented of the numerical experiments which indicate that the motion of Pluto, and indeed the whole solar system, is chaotic. The time scale for the exponential divergence of nearby trajectories is remarkably short compared to the age of the solar system. In the second part, numerical experiments are presented which indicate that the exponential sensitivity of trajectories to changes in initial conditions and parameters cannot be used to exponentially constrain initial conditions and parameters from trajectory measurements. It does appear though that parameters are better constrained by measurements of chaotic trajectories than might naively be expected.

Introduction

First, it is useful to remind ourselves of the reality of chaos, and just how much fun it is. I have a nice demonstration to show you, of a double pendulum (Fig. 1). The double pendulum is one of the simplest dynamical systems one can build after the pendulum: one pendulum supported at the end of another pendulum, constrained to move in a plane. This simple system exhibits outrageously complicated behavior. How could anyone watch the double pendulum and continue to assume that all solutions of Newton’s equations could be developed in quasiperiodic perturbation expansions? Did hundreds of years really go by without anyone looking at a dynamical system in action? The double pendulum can also be used to illustrate the divided phase-space characteristic of Hamiltonian systems (even though there is friction in the physical pendulum): trajectories with the same energy can be either chaotic or regular depending on the initial condition. Given that such a simple system as the double pendulum exhibits such complicated motion, it is hard to understand why it has taken so long for the importance of chaotic behavior to be realized. Chaos is not an irrelevant mathematical curiosity.

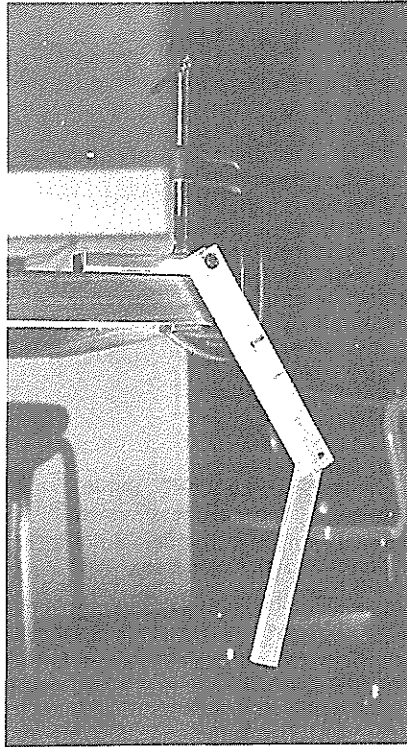


Fig. 1. The double pendulum provides a nice demonstration of chaotic behavior and the divided phase space in simple physical systems.

Question 1: Is the solar system stable?

Surely this is one of the oldest questions in modern science. As soon as Newton's equations of motion are written down one has to wonder about the long-term consequences. It has generally been assumed that our solar system is quasiperiodic and consequently stable, and that it is only a matter of time until a mathematical proof of this fact is given. Tremendous progress towards that goal has been made. Arnold (1963) has proven that solar systems are quasiperiodic in large measure provided that the masses, inclinations, and eccentricities of the planets are sufficiently small. On the other hand, we know that dynamical systems generally display chaotic behavior as well as regular behavior, and the solar system is, after all, just another dynamical system. The question of its stability should be approached with an open mind.

What we really would like to know is whether our solar system is on a chaotic or quasiperiodic trajectory. Since the physical experiment runs

too slowly for us to decide of the stability of the solar system, numerical experiments are necessary (Sussman and Wisdom,

1.1. Previous integration

Numerical integrations require a large amount of computer time, especially on time scales of interest. A direct calculation of the orbit of each planet around the sun is interesting on time scales of interest.

This time scale separates the two-body problem from the many-body problem. The frequency of the orbital precession is much smaller than the orbital frequency of the sun. The precession time scale is of the order of years to a couple of million years. The coupling between the planets is weak.

We have performed calculations (Applegate *et al.*, 1985) for 1000 years (Fig. 2), the computer designed specifically for this purpose runs at one-third the speed of a Cray. It consists of ten Cray computers because there are nine Cray computers: it can add, subtract, multiply, divide, integrate Newton's equations, and it has 100 bits on the Cray; it has 100 bits on the Cray.

In our first calculation we integrated the orbits of the planets (Sun, Jupiter, Saturn, Uranus, Neptune, Pluto) forward and backward in time. The time is longer than the classical time scale. Cohen, Hubbard, and Cohen (1985) have shown that the time scale is of the order of magnitude.

One result from that calculation is that the theory (Bretagnon, 1985) is numerically resolving the

too slowly for us to decide the matter, we have approached the question of the stability of the solar system through numerical experiments. Our numerical experiments indicate that, in fact, the solar system is chaotic (Sussman and Wisdom, 1988).

1.1. Previous integration

Numerical integrations of the solar system take an extraordinarily large amount of computer time. This is because there is a tremendous range of time scales. A direct calculation must take steps that are small enough to follow each planet around the sun, yet the motion of the planets is only interesting on time scales of millions of years.

This time scale separation is a consequence of the degeneracy of the two-body problem. The unperturbed two-body problem has only a single frequency, the orbital frequency, even though there are three degrees of freedom after elimination of the center of mass. The degeneracy is broken by planetary perturbations. The largest effect is to make the perihelia and orbital nodes precess. The frequency of these motions is of order the orbital frequency multiplied by the mass ratio of the perturbing planet to that of the sun. The precession time scales range from tens of thousands of years to a couple of million years. Only on a time scale longer than this precession time scale should we expect to find interesting dynamical coupling between the planets.

We have performed our numerical integrations on the Digital Orrery (Applegate *et al.*, 1985). Named after the orreries of the 18th and 19th centuries (Fig. 2), the Digital Orrery (Fig. 3) is a special purpose computer designed specifically for solar system dynamics. It runs at about one-third the speed of a Cray 1, but is smaller than the viewgraph projector. It consists of ten computers which run in parallel. There are ten because there are nine planets plus the sun. Each computer has limited capabilities: it can add, multiply, and take $-3/2$ powers—just enough to integrate Newton's equations. The mantissa has 56 bits compared to 48 bits on the Cray; it has turned out that this extra precision was crucial.

In our first calculation (Applegate *et al.*, 1985) we integrated the outer planets (Sun, Jupiter, Saturn, Uranus, Neptune, and several massless Plutos) forward and backward in time for about 100 million years. This is longer than the classic million year ($\pm 500\,000$ yr) integration of Cohen, Hubbard, and Oesterwinter (1973) by a factor of more than two orders of magnitude.

One result from that integration was that the best analytic perturbation theory (Bretagnon, 1982) for the solar system was inadequate. Numerically resolving the observed motions of the massive planets into a

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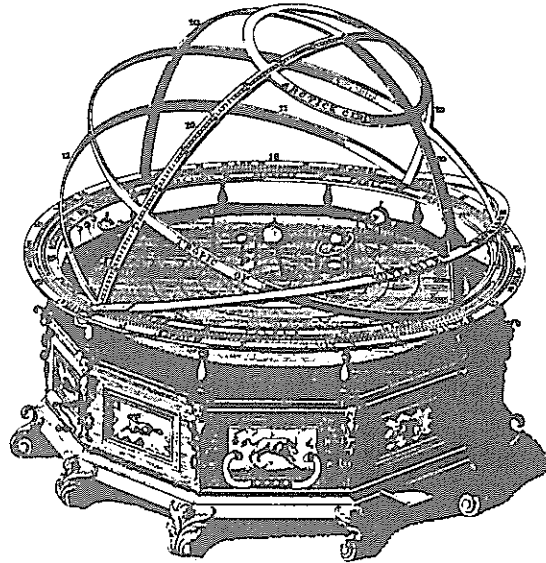


Fig. 2. The mechanical orreries are nice symbols of the apparent clockwork predictability of the motions of the planets.

quasiperiodic series, we found that the spectrum of Jupiter contained terms which were larger than all but seven of the 200 terms listed in

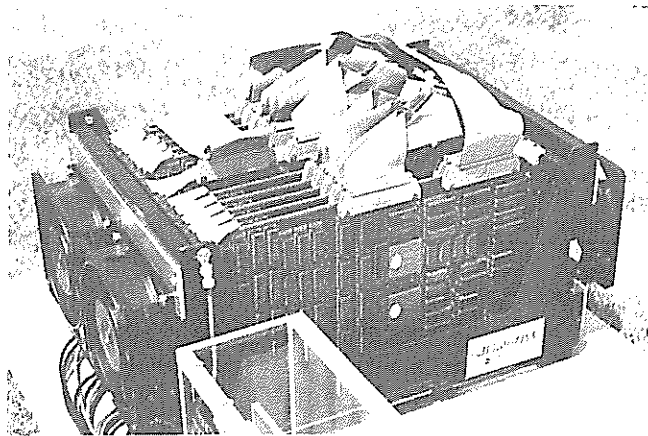


Fig. 3. The "Digital Orrery" is a computer which was specifically designed to investigate the dynamics of the solar system.

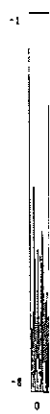


Fig. 4. The power spectrum common logarithm of the p

Bretagnon's solution (perturbation theory wa

The motion of Pluto crosses the orbit of Neptune are in an orbital resonance orbital period of Neptune Pluto. With the system



Fig. 5. The power spectrum perturbation theories. Lines which are the order in the planetary m

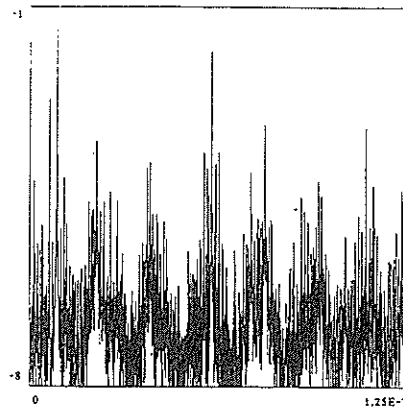


Fig. 4. The power spectrum of a variable related to the eccentricity of Jupiter's orbit. The common logarithm of the power is plotted vs frequency (in cycles per day).

Bretagnon's solution (see Figs. 4 and 5). The problem was that the perturbation theory was not carried to high enough order.

The motion of Pluto is particularly complicated. The orbit of Pluto crosses the orbit of Neptune. This is only possible because the two planets are in an orbital resonance (Cohen and Hubbard, 1965): three times the orbital period of Neptune is approximately two times the orbital period of Pluto. With the system in this resonance close encounters do not occur

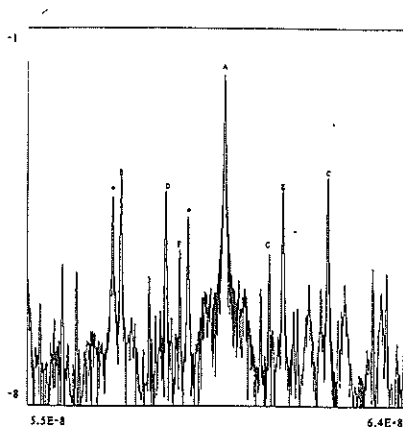
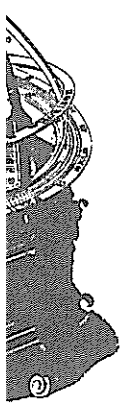
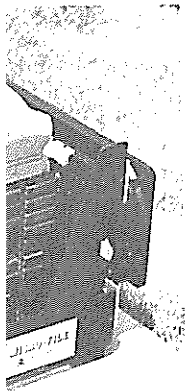


Fig. 5. The power spectrum of Jupiter was not adequately represented by the best analytic perturbation theories. Lines marked D and E are not recovered in perturbation theories which are third order in the eccentricities and inclinations and second order in the planetary masses.



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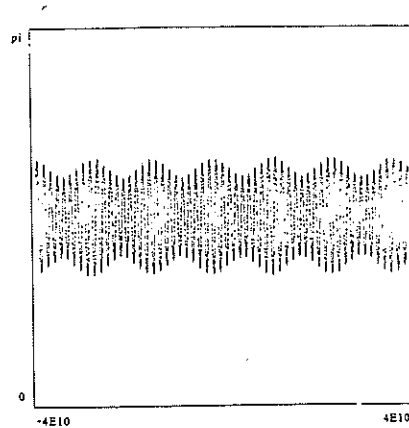


Fig. 6. The argument of perihelion of Pluto had a strong long-period modulation, with a period of 34 million years.

even though the orbits cross. Integrations by Williams and Benson (1973) showed that Pluto was involved in yet another resonance: Pluto's perihelion (the longitude at which the planet is closest to the sun) and its ascending node (the longitude at which the orbit plane crosses the plane perpendicular to the angular momentum of the solar system) are locked together. The regression of the perihelion and the ascending node have precisely the same periods. The difference between the two angles, which is called the argument of perihelion, oscillates about $\pi/2$ with a period of about 3.8 million years.

Several new features in the motion of Pluto were revealed by our calculation. We found a surprisingly large number of strong, long-period variations. The argument of perihelion showed a strong modulation with a period of 34 million years (Fig. 6). The variable $h = e \sin \varpi$, where e is the orbital eccentricity and ϖ is the longitude of perihelion, had a strong component with a period near 137 million years (Fig. 7). (This variable plays an important role in analytic theories.) This frequency may be associated with a near resonance between one of the fundamental frequencies associated with Pluto and one of the fundamental frequencies of the system of massive planets. The inclination of Pluto showed some evidence of a period longer than our integration, or perhaps even a secular decline (Fig. 8). The most suspicious bit of evidence was the very noisy power spectrum of the resonance variable associated with the basic 3:2 commensurability of the orbital periods (Fig. 9). The power spectrum of a quasiperiodic trajectory should have no more independent frequencies than the number of degrees of freedom. Noisiness of power



Fig. 7. There was a very long period modulation associated with Pluto's eccentricity.

spectra or the presence of chaos associated with chaotic motion is difficult to detect.

The proper thing to do in this case is to calculate whether or not neighboring trajectories diverge. A calculation of the Lyapunov exponent γ displays $\log_{10} \gamma$ versus

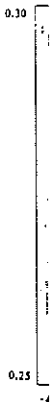


Fig. 8. The inclination of Pluto showed some evidence of a period longer than our integration, or perhaps even a secular decline.