CANONICAL SOLUTION OF THE TWO CRITICAL ARGUMENT PROBLEM

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Abstract. The solution by Sessin and Ferraz-Mello (*Celes. Mech.* **32**, 307–332) of the Hori auxiliary system for the motion of two planets with periods nearly commensurate in the ratio 2:1 is considerably simplified by the introduction of canonical variables. An analogous canonical transformation simplifies the elliptic restricted problem.

1. Introduction

It was Poincaré (1902) who first approximated the long term evolution near the 2:1 commensurability in the planar circular restricted problem with the single largest critical term in the disturbing function. The extension of Poincaré's method to the other principal commensurabilities was accomplished by Woltjer (1923). Higher order critical terms for the planar circular restricted problem were included by Message (1966). The long-term evolution of trajectories near commensurabilities in the elliptic restricted problem or the general three body problem is, however, a much more difficult problem. Basically, the difficulty arises because the disturbing function has two independent critical arguments of comparable importance. There are thus two non-trivial degrees of freedom; problems with two degrees of freedom are generally not solvable. The analytic solution by Sessin and Ferraz-Mello (1984) of a Hamiltonian approximating the motion of two planets with periods commensurate in the ratio 2:1 with two independent critical arguments is thus a significant achievement. Unfortunately, they achieve their analytic solution by a circuitous path involving non-canonical variables, which in turn necessitates a laborious enumeration of the various regions of parameter space. In this paper I give an alternate solution of this two critical argument problem which uses canonical variables throughout. Not only is the derivation much simpler, but the equivalence to the one critical argument problem becomes transparent.

2. Restatement of the Problem

I begin with the Hamiltonian F_1 (Equations (6)–(17)) from Sessin and Ferraz-Mello (1984):

$$F_1 = F_{02}\left(\frac{x}{x_{20}}\right)^2 + \frac{m'}{M}\left[P_{00} - P_{30}\left(\frac{-2C_2 y_1}{x_{20}}\right)^{1/2}\cos(\theta + \tilde{\omega}_1) + \right.$$
$$\left. + P_{40}\left(\frac{y_2}{x_{20}}\right)^{1/2}\cos(\theta + \tilde{\omega}_2)\right].$$

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Sessin and Ferraz-Mello use the following set of canonical variables (momenta and cordinates):

$$x_1 = L_1 + \tfrac{1}{2}L_2, \qquad \lambda_1 = l_1 + \tilde{\omega}_1,$$
$$x_2 = -\tfrac{1}{2}L_2, \qquad \theta = \lambda_1 - 2\lambda_2,$$
$$y_i = G_i - L_i = L_i(\sqrt{1-e_i^2} - 1), \qquad \tilde{\omega}_i = \tilde{\omega}_i + \Omega_i,$$

where $\lambda_2 = l_2 + \tilde{\omega}_2$ and $x = x_2 - x_{20}$. The variables L_i, G_i, l_i, ω_i and Ω_i are modified Delaunay variables:

$$L_i = m_i\sqrt{\mu_i a_i}, \qquad G_i = L_i\sqrt{1-e_i^2},$$

where

$$\mu_1 = \frac{k^2 M^3}{(M+m')^2}, \qquad \mu_2 = \frac{k^2(m+m'+M)M}{m'+M},$$

$$m_1 = \frac{m'(m'+M)}{M}, \qquad m_2 = \frac{m(m'+M)}{m+m'+M}.$$

The masses of the central body, and the inner and outer planets are M, and m' and m, respectively; the osculating Keplerian elements refer to the center of mass of M and m'. The time has been scaled by the factor $\mu_2^2 m_2^3/8x_{20}^2$. The other quantities are constants for which I refer the reader to Sessin and Ferraz-Mello (1984).

So that the solution will not be obscured by the notation, I rewrite the Hamiltonian in the following form:

$$H = \tfrac{1}{2}\alpha\Theta^2 - \beta\sqrt{2\rho_1}\cos(\theta - w_1) + \gamma\sqrt{2\rho_2}\cos(\theta - w_2). \tag{1}$$

The new momenta $\rho_i = -y_i$ are positive, and their conjugate coordinates are $w_i = -\tilde{\omega}_i$. The momentum conjugate to θ has been renamed Θ for clarity. The definition of the constants α, β and γ are readily deduced from Hamiltonaian F_1. The P_{00} term does not enter the equations of motion and has been omitted.

3. Solution

The first step is to write the Hamiltonian in terms of the canonical momenta and coordinates $\xi_i = \sqrt{2\rho_i}\cos w_i$ and $\eta_i = \sqrt{2\rho_i}\sin w_i$:

$$H = \tfrac{1}{2}\alpha\Theta^2 - \beta(\xi_1\cos\theta + \eta_1\sin\theta) + \gamma(\xi_2\cos\theta + \eta_2\sin\theta).$$

A canonical transformation to the new set of canonical variables,

$$\mu_1 = \frac{\beta\xi_1 - \gamma\xi_2}{\sqrt{\beta^2+\gamma^2}}, \qquad v_1 = \frac{\beta\eta_1 - \gamma\eta_2}{\sqrt{\beta^2+\gamma^2}},$$

and

$$u_2 = \frac{\gamma\xi_1 + \beta\xi_2}{\sqrt{\beta^2+\gamma^2}}, \qquad v_2 = \frac{\gamma\eta_1 + \beta\eta_2}{\sqrt{\beta^2+\gamma^2}},$$

reduces the Hamiltonian to the form

$$H = \tfrac{1}{2}\alpha\Theta^2 - \sqrt{\beta^2+\gamma^2}(u_1\cos\theta + v_1\sin\theta).$$

The miracle is that the canonical pair u_2, v_2 no longer appears in the Hamiltonian; thus these two variables are constants of the motion.

The rest of the solution is trivial. Define the canonical pair Φ, ϕ implicitly by the equations

$$u_1 = \sqrt{2\Phi}\cos\phi, \qquad v_1 = \sqrt{2\Phi}\sin\phi,$$

and the Hamiltonian becomes

$$H = \tfrac{1}{2}\alpha\Theta^2 - \sqrt{\beta^2+\gamma^2}\sqrt{2\Phi}\cos(\theta - \phi). \tag{2}$$

This is the familiar form for the Hamiltonian when only one critical term is present. Finally, the canonical transformation induced by the generating function

$$W = (\phi - \theta)\Phi' + \theta\Theta',$$

giving the variables

$$\Phi' = \Phi, \qquad \phi' = \phi - \theta$$

and

$$\Theta' = \Theta + \Phi, \qquad \theta' = \theta,$$

reduces the Hamiltonian to

$$H = \tfrac{1}{2}\alpha(\Theta' - \Phi')^2 - \sqrt{\beta^2+\gamma^2}\sqrt{2\Phi'}\cos\phi'.$$

The Hamiltonian is cyclic in θ', thus Θ' is an integral. The problem is reduced to quadrature.

Of course Hamiltonian (2) has been solved numerous times (e.g. Hagihara 1971), so there is no need to carry it out explicitly here. Sessin and Ferraz-Mello (1984) were forced to laboriously determine a new solution because they did not see how to reduce their statement of the problem to the familiar one.

4. Application to the Planar-Elliptic Restricted Problem

It is instructive to carry out the analogous transformations for motion near the first-order mean-motion commensurabilities of the planar-elliptic restricted three-body problem. The Hamiltonian for motion near the inner $p+1$ to p mean-motion commensurability is

$$H = -\frac{\mu_1^2}{2L^2} - A(L)\sqrt{2G}\cos[pl - (p+1)l' - g] -$$
$$- B(L)e' \cos[pl - (p+1)l' + \tilde{\omega}'].$$

Here l is the mean longitude, which is conjugate to the momentum $L = \sqrt{\mu_1 a}$, and the momentum $G = L(\sqrt{1-e^2} - 1)$ is conjugate to $g = -\tilde{\omega}$, where $\tilde{\omega}$ is the longitude of perihelion. These elliptic elements are now the usual osculating heliocentric Keplerian elements. The coefficients may be found in Leverrier (1855), and in the first order approximation can simply be evaluated at the exact resonance, $L_p = (p\mu_1^2/(p+1))^{1/3}$. The ratio of the mass of the secondary to that of the primary is μ and $\mu_1 = 1 - \mu$. Units are chosen so that the period of the secondary is equal to 2π and its semimajor axis is unity. The longitude of perihelion of the secondary is taken to be the origin of longitudes.

The Hamiltonian is first transformed to a form similar to Hamiltonian (1) by the generating function

$$W = [pl - (p+1)t]\Psi + g\Phi$$

which gives the canonical variables

$$\psi = pl - (p+1)t, \qquad \Psi = L/p$$

and

$$\phi = g, \qquad \Phi = G.$$

This is followed by the canonical translation $\Theta = \Psi - \Psi_p$, $\theta = \psi$ (defining $\Psi_p = L_p/p$), and the retention of only the quadratic terms in Θ. The new Hamiltonian is

$$H' = \tfrac{1}{2}\alpha\Theta^2 - A\sqrt{2\Phi}\cos(\theta - \phi) - Be'\cos\theta.$$

Proceeding now as in the last section I define the new canonical variables

$$\xi = \sqrt{2\Phi}\cos\phi, \qquad \eta = \sqrt{2\Phi}\sin\phi.$$

The Hamiltonian becomes

$$H' = \tfrac{1}{2}\alpha\Theta^2 - A(\xi\cos\theta + \eta\sin\theta) - Be'\cos\theta.$$

A canonical rotation was used to simplify the commensurate two planet problem in the last section. Here the elimination of the e' term is accomplished by a canonical translation. Introducing the canonical variables

$$u = \xi + Be'/A, \qquad v = \eta, \tag{3}$$

the Hamiltonian is

$$H' = \tfrac{1}{2}\alpha\Theta^2 - A(u\cos\theta + v\sin\theta),$$

and the term proportional to e' has disappeared. The rest of the solution is completely analogous to that following Equation (2).

It is quite interesting to note that the transformation (3) was already discovered empirically by Schubart (1968) during his study of the evolution of the Hilda asteroids with the numerically averaged equations of motion for the planar elliptic problem. Evaluating Leverrier's expressions for the coefficients I find that the constant in the definition of u has the magnitude $Be'/A = 0.055$, for a Jupiter eccentricity of 0.048. Now ξ is porportional to $\sqrt{2G} \approx e\sqrt{L} \approx e\sqrt{L_2}$. Consequently to compare the magnitude of the translation of the canonical variables with the eccentricity values obtained by Schubart, the magnitude of the translation must be divided by $\sqrt{L_2}$ giving 0.059. Empirically Schubart found values ranging from 0.062 to 0.072 for the Hildas. This agreement is quite satisfactory, considering that the analytic solution has considered only first order terms in the disturbing function. A theoretical justification for Schubart's empirical result has thus been provided.

5. Summary

Sessin and Ferraz-Mello have shown that an approximate Hamiltonian for the motion of two planets with nearly commensurate mean-motions, including terms which are first order in the eccentricity, is analytically solvable. I have shown that this problem involving two critical arguments is more easily solved using canonical variables, and that the solution reduces to the standard one critical argument problem, obviating the need for extensive new solutions. Of course, this does not detract from the accomplishment of Sessin and Ferraz-Mello. The canonical solution of the first order problem should facilitate the extension of the solution to higher orders.

I would like to point out that a similar approach to the solution of the motion of planets near the 3:1 commensurability is bound to fail since the lowest order Hamiltonian gives rise to chaotic behavior (Wisdom, 1983). Furthermore, it is known that the 2:1 mean-motion commensurability with Jupiter is accompanied by a large chaotic zone (Giffen, 1973). Evidently, the integrability of the first order Hamiltonian is spoiled by terms of higher order. The solution of such problems where chaotic behavior is widespread requires a new approach to perturbation theory. In Wisdom (1985) I derive a perturbation theory which successfully explains the qualitative features of the phase space near the 3:1 commensurability which were previously reported (Wisdom 1983). The theory not only describes the quasiperiodic solutions, but the chaotic ones as well. Indeed, the extent and shape of the chaotic zones are now predictable!

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